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Abstract. In this work we develop an adaptive algorithm for solving elliptic optimal control problems with simultaneously appearing state and control constraints. Building upon the concept proposed in [9] the algorithm applies a Moreau-Yosida regularization technique for handling state constraints. The state and co-state variables are discretized using continuous piecewise linear finite elements while a variational discretization concept is applied for the control. To perform the adaptive mesh refinement cycle we derive local error representations which extend the goal-oriented error approach to our setting. The performance of the overall adaptive solver is demonstrated by a numerical example.

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1. Introduction

Optimal control problems with state constraints have been the topic of an increasing number of theoretical and numerical studies. The challenging character of these problems has its origin in the fact that state constraints feature Lagrange multipliers of low regularity only [4, 7]. In the presence of additional control constraints, the solution may exhibit subsets where both control and state are active simultaneously. In this case, the Lagrange multipliers associated to the control and state constraints may not be unique [20]. To overcome this difficulty several techniques in the literature have been proposed. Very popular are relaxation concepts for state constraints such as Lavrentiev, interior point and Moreau-Yosida regularization. The former one is investigated in [21] and [17]. Barrier methods

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in function space [24] applied to state constrained optimal control problems are considered in [18]. Relaxation by Moreau-Yosida regularization is considered for the fully discrete case in [3, 5], and in function space in [13]. Residual-type a posteriori error estimators for mixed control-state constrained problems are derived in [16]. The dual weighted residual method proposed in [1] is investigated in [11, 22] the presence of control constraints and for state constraints in [2, 9]. Within the framework of goal-oriented adaptive function space algorithms a Lavrentiev regularization approach is considered in [16], and an adaptive interior point method is proposed in [19, 25].

In this note we combine results from [9] and [10] to design an adaptive finite element algorithm for solving elliptic optimal control problems with pointwise control and state constraints. Following [14], our algorithm combines a Moreau-Yosida regularization approach with a semi-smooth Newton solver [12]. We apply variational discretization [8, 15] to the regularized optimal control problem. For a fixed regularization parameter, we develop a goal-oriented a posteriori error estimate by extending the error representation obtained in [9] to the control and state constrained case. Let us note that in this work we are not interested in controlling the error contribution stemming from the regularization.

The rest of this paper is organized as follows: In the next section we present the optimal control problem under consideration and recall its first order necessary optimality system. Section 3 is devoted to the purely state-constrained case and collects results from [9] which in Section 4 are extended to simultaneously appearing control and state constraints. We introduce a regularized version of the original problem and derive an error representation in terms of the objective functional. Finally, a numerical experiment is reported in Section 5.

2. Optimal Control Problem

Let Ω be a bounded polygonal and convex domain in $\mathbb{R}^d$ $(d = 2, 3)$ with boundary $\partial\Omega$. We consider the general elliptic partial differential operator $\mathcal{A} : H^1(\Omega) \longrightarrow H^1(\Omega)^*$ defined by

$$\mathcal{A}y := \sum_{i,j=1}^{d} \partial_{x_j}(a_{ij} y_{x_i}) + \sum_{i=1}^{d} b_i y_{x_i} + cy$$

along with its formal adjoint operator $\mathcal{A}^*$

$$\mathcal{A}^* y = \sum_{i=1}^{d} \partial_{x_i}\Big(\sum_{j=1}^{d} a_{ij} y_{x_j} + b_i y\Big) + cy.$$

We subsequently assume the coefficients a_{ij}, b_i and c $(i, j = 1, \ldots, d)$ to be sufficiently smooth functions on $\bar{\Omega}$. Moreover we suppose that there exists $c_0 > 0$ such that $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq c_0$ for almost all x in Ω and all ξ in $\mathbb{R}^d$. Corresponding to the operator $\mathcal{A}$ we associate the bilinear form $a(\cdot,\cdot) : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ with

$$a(y,v) := \int_\Omega \Big(\sum_{i,j=1}^d a_{ij} y_{x_i} v_{x_j} + \sum_{i=1}^d b_i y_{x_i} v + cyv \Big).$$

Suppose that the form a is coercive on $H^1(\Omega)$, i.e. there exists $c_1 > 0$ such that $a(v,v) \geq c_1 \|v\|^2_{H^1(\Omega)}$ for all v in $H^1(\Omega)$. This follows for instance if

$$\inf \text{ess}_{x\in\Omega} \Big(c - \frac{1}{2} \sum_{i=1}^d \partial_{x_i} b_i \Big) > 0 \text{ and } \inf \text{ess}_{x\in\partial\Omega} \Big(\sum_{i=1}^d b_i \nu_i \Big) \geq 0$$

holds. Here ν denotes the unit outward normal at $\partial\Omega$.

For given $u \in L^2(\Omega)$ and fixed $f \in L^2(\Omega)$ the homogeneous Neumann boundary value problem

$$\begin{aligned} \mathcal{A}y &= u + f \qquad && \text{in } \Omega \\ \partial_{\nu_\mathcal{A}} y := \textstyle\sum_{i,j=1}^d a_{ij} y_{x_i} \nu_j &= 0 && \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

has a unique solution $y =: \mathcal{G}(u) \in H^2(\Omega)$. Moreover, there exists a constant C depending on the domain Ω such that

$$\|\mathcal{G}(u)\|_{H^2(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}).$$

The weak form of (2.1) is given by

$$a(y,v) = (u+f, v) \qquad \forall\, v \in H^1(\Omega), \tag{2.2}$$

where $(\,\cdot,\cdot\,)$ denotes the standard inner product in $L^2(\Omega)$.

For given $u_d, y_d \in L^2(\Omega)$, $\alpha > 0$, u_a, $u_b \in \mathbb{R}$ with $u_a < u_b$, y_a and $y_b \in C(\Omega)$ with $y_a < y_b$ we focus on the optimal control problem

$$\begin{aligned} & J(y,u) := \tfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \tfrac{\alpha}{2}\|u - u_d\|^2_{L^2(\Omega)} \to \min \\ & \text{s.t.} \quad y = \mathcal{G}(u), \quad u \in U_{ad}, \quad \text{and} \quad y_a \leq y \leq y_b \quad \text{a.e. in } \bar{\Omega}, \end{aligned} \tag{2.3}$$

where U_{ad} is the set of admissible controls given by

$$U_{ad} = \big\{ u \in L^2(\Omega) \,:\, u_a \leq u \leq u_b \quad \text{in } \Omega \big\}\,.$$

We require the Slater condition

$$\exists\, u_s \in U_{ad} : \qquad y_a < \mathcal{G}(u_s) < y_b \quad \text{in } \Omega.$$

The proof of the following theorem follows from [6, 7].

Theorem 2.1. *The optimal control problem* (2.3) *has a unique solution* $(y,u) \in H^2(\Omega) \times U_{ad}$. *Moreover there exist* $p \in W^{1,s}(\Omega)$ *for all* $1 \leq s < d/(d-1)$, λ_a,

$\lambda_b \in L^2(\Omega)$ *and* μ_a, $\mu_b \in \mathcal{M}(\bar\Omega)$ *satisfying the optimality system*

$$
\begin{aligned}
y &= \mathcal{G}(u),\\
(p, \mathcal{A}v) = (y - y_d, v) + \langle \mu_a + \mu_b, v\rangle \quad \forall\, v \in W^{1,1-\frac{1}{s}}(\Omega) \ \textit{with}\ \partial_{\nu_{\mathcal{A}}} v|_{\partial\Omega} &= 0,\\
\alpha(u - u_d) + p + \lambda_a + \lambda_b &= 0,\\
\lambda_a \le 0, \quad u \ge u_a, \quad (\lambda_a, u - u_a) &= 0,\\
\lambda_b \ge 0, \quad u \le u_b, \quad (\lambda_b, u - u_b) &= 0,\\
\mu_a \le 0, \quad y \ge y_a, \quad \langle \mu_a, y - y_a\rangle &= 0,\\
\mu_b \ge 0, \quad y \le y_b, \quad \langle \mu_b, y - y_b\rangle &= 0.
\end{aligned}
\tag{2.4}
$$

3. The Purely State-constrained Case

First let us consider problem (2.3) without control constraints, i.e. $U_{ad} = U = L^2(\Omega)$. Then clearly $\lambda_a = \lambda_b = 0$, $p = -\alpha(u - u_d)$ ans μ_a, μ_b in (2.4) are uniquely determined.

3.1. Finite Element Discretization

In the sequel we consider a shape-regular simplicial triangulation $\mathcal{T}_h$ of Ω. Since Ω is assumed to be a polyhedral, the boundary $\partial\Omega$ is exactly represented by the boundaries of simplices $T \in \mathcal{T}_h$. We refer to $\mathcal{N}_h = \cup_{i=1}^{np}\{x_i\}$ as the set of nodes of $\mathcal{T}_h$. The overall mesh size is defined by $h := \max_{T\in\mathcal{T}_h} \operatorname{diam} T$. Further, we associate with $\mathcal{T}_h$ the continuous piecewise linear finite element space

$$V_h = \{v \in C_0(\bar\Omega) \,:\, v|_T \in P_1(T),\ \forall\, T \in \mathcal{T}_h\},$$

where $P_1(T)$ is the space of first-order polynomials on T. The standard nodal basis of V_h denoted by $\{\phi_i\}_{i=1}^{np}$ satisfies $\phi_i(x_j) = \delta_{ij}$ for all x_j in $\mathcal{N}_h$ and $i, j \in \{1, \dots, np\}$. Furthermore for all $v \in C^0(\bar\Omega)$ we denote by $I_h v := \sum_{i=1}^{np} v(x_i)\phi_i$ the Lagrange interpolation of v. In analogy to (2.2) we define for given $u \in L^2(\Omega)$ the discrete solution operator $\mathcal{G}_h$ by

$$y_h =: \mathcal{G}_h(u) \Longleftrightarrow y_h \in V_h \text{ and } a(y_h, v_h) = (u + f, v_h) \quad \forall\, v_h \in V_h.$$

Problem (2.3) is now approximated by the following sequence of variational discrete control problems depending on the mesh parameter h:

$$
\begin{aligned}
&\min_{u\in U} J_h(y_h, u) := \frac{1}{2}\|y_h - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u - u_{d,h}\|^2_U\\
&\text{subject to } \ y_h = \mathcal{G}_h(u) \text{ and } y_a(x_j) \le y_h(x_j) \le y_b(x_j) \text{ for } j = 1, \dots, np.
\end{aligned}
\tag{3.1}
$$

Here, $u_{d,h}$ denotes an approximation to u_d which is assumed to satisfy

$$\|u_d - u_{d,h}\| \le Ch.$$

Problem (3.1) represents a convex infinite-dimensional optimization problem of similar structure as problem (2.3), but with only finitely many equality and inequality constraints for the state, which define a convex set of admissible functions. Since for $h > 0$ small enough we have $y_a < \mathcal{G}_h(u_s) < y_b$ such that [6, 7] can again be applied to obtain

Lemma 3.1. *Problem* (3.1) *has a unique solution* $u_h \in U$. *There exist unique* $\mu_1^a, \dots, \mu_{np}^a, \mu_1^b, \dots, \mu_{np}^b \in \mathbb{R}$ *and a unique function* $p_h \in V_h$ *such that with* $y_h = \mathcal{G}_h(u_h)$, $\mu_{a,h} = \sum_{j=1}^{np} \mu_j^a \delta_{x_j}$ *and* $\mu_{b,h} = \sum_{j=1}^{np} \mu_j^b \delta_{x_j}$ *we have*

$$a(v_h, p_h) = \int_\Omega (y_h - y_d) v_h + \int_{\bar\Omega} v_h \, d(\mu_{a,h} + \mu_{b,h}) \qquad \forall\, v_h \in V_h, \quad (3.2)$$

$$p_h + \alpha(u_h - u_{d,h}) = 0, \quad (3.3)$$

$$\mu_j^a \le 0, \quad y_h(x_j) \ge y_a(x_j), \quad j = 1, \dots, np \text{ and } \int_{\bar\Omega} \big(y_h - I_h y_a\big)\, d\mu_{a,h} = 0, \quad (3.4)$$

$$\mu_j^b \ge 0, \quad y_h(x_j) \le y_b(x_j), \quad j = 1, \dots, np \text{ and } \int_{\bar\Omega} \big(y_h - I_h y_b\big)\, d\mu_{b,h} = 0. \quad (3.5)$$

Here, δ_x denotes the Dirac measure concentrated at x.

Remark 3.2. Problem (3.1) is still an infinite–dimensional optimization problem, but with finitely many state constraints. By (3.3) it follows that $u_h \in V_h$, i.e. the optimal discrete solution is discretized implicitly through the optimality condition of the discrete problem. Hence in (3.1) U may be replaced by V_h to obtain the same discrete solution u_h, which results in a finite–dimensional discrete optimization problem instead.

3.2. Local error indicators

From here onwards we assume $u_d = u_{d,h}$, so that $J = J_h$ holds. This assumption is fulfilled by affine linear functions u_d. Including more general desired controls u_d would lead to additional weighted data oscillation contributions $(u_d - u_{d,h}, \cdot)$ in the error representation (4.5). For their treatment in the context of residual type a posteriori estimators we refer to [16].

Let us abbreviate

$$\mu := \mu_a + \mu_b, \quad \mu_h := \mu_{a,h} + \mu_{b,h}.$$

Following [1] we introduce the dual, control and primal residual functionals determined by the discrete solution y_h, u_h, p_h, μ_h^a and μ_h^b of (3.2)-(3.5) by

$$\rho^p(\cdot) := J_y(y_h, u_h)(\cdot) - a(\cdot, p_h) + \langle \mu_h, \cdot \rangle,$$
$$\rho^u(\cdot) := J_u(y_h, u_h)(\cdot) + (\cdot, p_h) \text{ and}$$
$$\rho^y(\cdot) := -a(y_h, \cdot) + (u_h + f, \cdot).$$

In addition we set

$$e^\mu(y) := \langle \mu + \mu_h, y_h - y \rangle.$$

It follows from (3.3) that $\rho^u(\cdot) \equiv 0$. This is due to variational discretization, i.e. we do not discretize the control, so that the discrete structure of the solution u_h of problem (3.1) is induced by the optimality condition (3.3).

The proof of the following theorem can be found in [9].

Theorem 3.3. *There hold the error representations*

$$2(J(y,u) - J(y_h, u_h)) = \rho^p(y - i_h y) + \rho^y(p - i_h p) + e^\mu(y), \quad (3.6)$$

and

$$\begin{aligned} 2(J(y,u)-J(y_h,u_h)) \quad = \quad & J_y(y_h,u_h)(y-y_h)-a(y-y_h,p_h) \\ & -a(y_h,p-i_hp)+(u_h+f,p-i_hp) \\ & +(y-y_d,y-y_h)-a(y-y_h,p) \end{aligned} \tag{3.7}$$

with arbitrary quasi–interpolants i_hy and $i_hp \in V_h$.

Following the lines of Remark 3.5 in [1] we split the above equation into a cellwise representation and integrate by parts. This gives rise to define

$$\begin{aligned} R^{y_h}_{|T} &= u_h+f-\mathcal{A}y_h \\ R^{p_h}_{|T} &= y_h-y_d-\mathcal{A}^*p_h \\ R^{p}_{|T} &= y-y_d-\mathcal{A}^*p \\ r^{y_h}_{|\Gamma} &= \begin{cases} \frac{1}{2}\nu\cdot[\nabla y_h\cdot(a_{ij})] & \text{for } \Gamma\subset\partial T\setminus\partial\Omega \\ \nu\cdot(\nabla y_h\cdot(a_{ij})) & \text{for } \Gamma\subset\partial\Omega \end{cases} \\ r^{p_h}_{|\Gamma} &= \begin{cases} \frac{1}{2}\nu\cdot[(a_{ij})\nabla p_h] & \text{for } \Gamma\subset\partial T\setminus\partial\Omega \\ \nu\cdot((a_{ij})\nabla p_h+p_hb) & \text{for } \Gamma\subset\partial\Omega \end{cases} \\ r^{p}_{|\Gamma} &= \begin{cases} \frac{1}{2}\nu\cdot[(a_{ij})\nabla p] & \text{for } \Gamma\subset\partial T\setminus\partial\Omega \\ \nu\cdot((a_{ij})\nabla p+pb) & \text{for } \Gamma\subset\partial\Omega \end{cases}, \end{aligned}$$

where $[\cdot]$ defines the jump across the inter-element edge Γ. Now (3.7) can be rewritten in the form

$$\begin{aligned} 2(J(y,u)-J(y_h,u_h)) = \sum_{T\in\mathcal{T}_h} & (y-y_h,R^{p_h}_{|T})_T-(y-y_h,r^{p_h}_{|\partial T})_{\partial T} \\ & +(R^{y_h}_{|T},p-i_hp)_T-(r^{y_h}_{|\partial T},p-i_hp)_{\partial T} \\ & +(y-y_h,R^{p}_{|T})_T-(y-y_h,r^{p}_{|\partial T})_{\partial T}. \end{aligned}$$

Since this localized sum still contains unknown quantities, we make use of local higher order approximation ([1, Sec. 5.1]) which has shown to be a successful heuristic technique for a posteriori error estimation. More precisely we take the local higher order quadratic interpolant operator $i^{(2)}_{2h} : V_h \to P_2(T)$ for some $T \in \mathcal{T}_h$. In detail for $d = 2$ the local interpolant $i^{(2)}_{2h}v_h$ for an arbitrary function $v_h \in V_h$ on a triangle T is defined by

$$(i^{(2)}_{2h}v_h)(x_1,x_2) := a+bx_1+cx_2+dx_1x_2+ex_1^2+fx_2^2, \qquad (x_1,x_2)^T\in\Omega,$$

where the coefficients $a,b,c,d,e,f \in \mathbb{R}$ are obtained by the solution of a linear system demanding the exact interpolation in the sampling nodes shown in Figure 1. The technique for computing $i^{(2)}_{2h}v_h$ for some $v_h \in V_h$ can easily be carried over to three space dimensions.

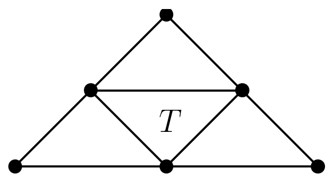

FIGURE 1. Used sampling nodes for $i_{2h}^{(2)}$, $d = 2$.

We substitute $R_{|T}^{p}$ and $r_{|\Gamma}^{p}$ by

$$R_{|T}^{i_{2h}^{(2)} p_h} = i_{2h}^{(2)} y_h - y_d - \mathcal{A}^* i_{2h}^{(2)} p_h$$

$$r_{|\Gamma}^{i_{2h}^{(2)} p_h} = \begin{cases} 0 & \text{for } \Gamma \subset \partial T \setminus \partial\Omega \\ \nu \cdot ((a_{ij}) \nabla i_{2h}^{(2)} p_h + i_{2h}^{(2)} p_h b) & \text{for } \Gamma \subset \partial\Omega \end{cases}$$

and define

$$\begin{aligned} \eta := \frac{1}{2} \sum_{T \in \mathcal{T}_h} & (i_{2h}^{(2)} y_h - y_h, R_{|T}^{p_h})_T - (i_{2h}^{(2)} y_h - y_h, r_{|\partial T}^{p_h})_{\partial T} \\ & + (R_{|T}^{y_h}, i_{2h}^{(2)} p_h - p_h)_T - (r_{|\partial T}^{y_h}, i_{2h}^{(2)} p_h - p_h)_{\partial T} \\ & + (i_{2h}^{(2)} y_h - y_h, R_{|T}^{i_{2h}^{(2)} p_h})_T - (i_{2h}^{(2)} y_h - y_h, r_{|\partial T}^{i_{2h}^{(2)} p_h})_{\partial T}. \end{aligned}$$

For numerical experiments we refer to [9] and Section 5.

4. The Control- and State-constrained Case

Let us now consider problem (2.3) with control and state constraints. The situation then becomes more involved due to the fact that Lagrange multipliers may not be unique if control and state active sets intersect [20]. To overcome difficulties arising from this fact we consider

4.1. Moreau-Yosida Regularization

In Moreau-Yosida regularization the state constraints $y_a \le y \le y_b$ are substituted by appropriate regularization terms which are added to the objective functional J. The corresponding regularized optimal control problem reads

$$\begin{aligned} J^\gamma(y,u) := J(y,u) + \tfrac{\gamma}{2} \|\min(0, y - y_a)\|^2 + \tfrac{\gamma}{2} \|\max(0, y - y_b)\|^2 \to \min \\ \text{s.t.} \quad y = \mathcal{G}(u) \quad \text{and} \quad u \in U_{ad}, \end{aligned} \tag{4.1}$$

where $\gamma > 0$ denotes a regularization parameter tending to $+\infty$ later on. The max- and min-expressions in the regularized objective functional J^γ arise from regularizing the indicator function corresponding to the set of admissible states. Notice that (4.1) is a purely control constrained optimal control problem that has a unique solution $(y^\gamma, u^\gamma) \in H^2(\Omega) \times U_{ad}$. Furthermore, we can prove the existence of Lagrange multipliers $(p^\gamma, \lambda_a^\gamma, \lambda_b^\gamma) \in L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ using standard theory of mathematical programming in Banach spaces such that

$$\begin{aligned} y^\gamma &= \mathcal{G}(u^\gamma) \\ (p^\gamma, \mathcal{A}v) = (y^\gamma - y_d, v) + (\mu_a^\gamma + \mu_b^\gamma, v) \quad \forall\, v \in H^2(\Omega) \text{ with } \partial_{\nu_{\mathcal{A}}} v|_{\partial\Omega} &= 0, \\ \alpha(u^\gamma - u_d) + p^\gamma + \lambda_a^\gamma + \lambda_b^\gamma &= 0, \\ \lambda_a^\gamma \le 0, \quad u^\gamma \ge u_a, \quad (\lambda_a^\gamma, u^\gamma - u_a) &= 0, \\ \lambda_b^\gamma \ge 0, \quad u^\gamma \le u_b, \quad (\lambda_b^\gamma, u^\gamma - u_b) &= 0 \end{aligned} \tag{4.2}$$

holds, where

$$\mu_a^\gamma = \gamma \min(0, y^\gamma - y_a) \quad \text{and} \quad \mu_b^\gamma = \gamma \max(0, y^\gamma - y_b).$$

For the convergence of the solutions of the regularized problems to the solution of the limit problem (2.3) we refer the reader to [10, Thm. 3.1]. To recover the solution of the optimal control problem (2.3) an overall algorithm can be designed by solving (4.1) for a sequence $\gamma \to \infty$. For (4.1) with γ fixed, a locally superlinear semi-smooth Newton method can be applied (see [14]).

4.2. Finite Element Discretization

For the convenience of the reader we assume y_a and y_b to be real numbers. Again we apply variational discretization [15], but now to problem (4.1). We therefore consider

$$\begin{aligned} J_h^\gamma(y_h, u_h) := J_h(y_h, u_h) + \tfrac{\gamma}{2}\|\min(0, y_h - y_a)\|^2 + \tfrac{\gamma}{2}\|\max(0, y_h - y_b)\|^2 \to \min \\ \text{s.t.} \quad y_h = \mathcal{G}_h(u_h) \quad \text{and} \quad u_h \in U_{ad}. \end{aligned} \tag{4.3}$$

The existence of a solution of (4.3) as well as of Lagrange multipliers again follows from standard arguments. The corresponding first order optimality system associated to (4.3) leads to the variationally discretized counterpart of (4.2)

$$\begin{aligned} y_h^\gamma &= \mathcal{G}_h(u_h^\gamma), \\ a(v_h, p_h^\gamma) &= (v_h, y_h^\gamma - y_d + \mu_{a,h}^\gamma + \mu_{b,h}^\gamma) \quad \forall\, v_h \in V_h, \\ \alpha(u_h^\gamma - u_{d,h}) + p_h^\gamma + \lambda_{a,h}^\gamma + \lambda_{b,h}^\gamma &= 0, \\ \lambda_{a,h}^\gamma \le 0, \quad u_h^\gamma \ge u_a, \quad (\lambda_{a,h}^\gamma, u_h^\gamma - u_a) &= 0, \\ \lambda_{b,h}^\gamma \ge 0, \quad u_h^\gamma \le u_b, \quad (\lambda_{b,h}^\gamma, u_h^\gamma - u_b) &= 0, \end{aligned} \tag{4.4}$$

where y_h^γ, $p_h^\gamma \in V_h$ and u_h^γ, $\lambda_{a,h}^\gamma$, $\lambda_{b,h}^\gamma \in L^2(\Omega)$. The multipliers corresponding to the regularized state constraints $\mu_{a,h}^\gamma$ and $\mu_{b,h}^\gamma$ are given by

$$\mu_{a,h}^\gamma = \gamma \min(0, y_h^\gamma - y_a) \quad \text{and} \quad \mu_{b,h}^\gamma = \gamma \max(0, y_h^\gamma - y_b).$$

We mention here that (4.3) is a function space optimization problem and the optimal control u_h^γ is not necessarily an element of the finite element space. However, regarding (4.4), u_h^γ corresponds to the projection of a finite element function onto the admissible set U_{ad}, namely

$$u_h^\gamma = \Pi_{[u_a, u_b]}\left(-\frac{1}{\alpha} p_h^\gamma + u_{d,h}\right),$$

where $\Pi_{[u_a,u_b]}$ is the orthogonal projection onto U_{ad}.

4.3. Error Representation and Estimator

For a fixed regularization parameter γ we now derive an error representation in J for the solutions of (4.1) and (4.3) respectively.

Similar as before we define the following residuals

$$\rho^{p^\gamma}(\cdot) := J_y(y_h^\gamma, u_h^\gamma)(\cdot) - a(\cdot, p_h^\gamma) + (\mu_h^\gamma, \cdot), \text{ and}$$
$$\rho^{y^\gamma}(\cdot) := -a(y_h^\gamma, \cdot) + (u_h^\gamma + f, \cdot)$$

with

$$\mu^\gamma := \gamma \min(0, y^\gamma - y_a) + \gamma \max(0, y^\gamma - y_b), \text{ and}$$
$$\mu_h^\gamma := \gamma \min(0, y_h^\gamma - y_a) + \gamma \max(0, y_h^\gamma - y_b).$$

The functions μ^γ and μ_h^γ play the role of the Lagrange multipliers μ, μ_h corresponding to state constraints in the limit problem (2.3) (compare with [9, Thm. 4.1, Rem. 4.1]). Furthermore we abbreviate

$$\lambda^\gamma := \lambda_a^\gamma + \lambda_b^\gamma \quad \text{and} \quad \lambda_h^\gamma := \lambda_{a,h}^\gamma + \lambda_{b,h}^\gamma.$$

Theorem 4.1. *Let (u^γ, y^γ) and (u_h^γ, y_h^γ) be the solutions of the optimal control problems (4.1) and (4.3) with corresponding adjoint states p^γ, p_h^γ and multipliers $\lambda^\gamma, \lambda_h^\gamma, \mu^\gamma, \mu_h^\gamma$ associated to control and state constraints respectively. Then*

$$\begin{aligned} &2(J(y^\gamma, u^\gamma) - J_h(y_h^\gamma, u_h^\gamma)) = \\ &\quad \rho^{p^\gamma}(y^\gamma - i_h y^\gamma) + \rho^{y^\gamma}(p^\gamma - i_h p^\gamma) + (\mu^\gamma + \mu_h^\gamma, y_h^\gamma - y^\gamma) + (\lambda^\gamma + \lambda_h^\gamma, u_h^\gamma - u^\gamma). \end{aligned} \tag{4.5}$$

For the proof we refer the reader to [10]. Applying integration by parts the error representation (4.5) can be localized as follows,

$$\begin{aligned} 2(J(y^\gamma, u^\gamma) - J_h(y_h^\gamma, u_h^\gamma)) = \sum_{T \in \mathcal{T}_h} &(y^\gamma - y_h^\gamma, R_{|T}^{p_h^\gamma})_T - (y^\gamma - y_h^\gamma, r_{\partial T}^{p_h^\gamma})_{\partial T} \\ &+ (R_{|T}^{y_h^\gamma}, p^\gamma - i_h p^\gamma)_T - (r_{|\partial T}^{y_h^\gamma}, p^\gamma - i_h p^\gamma)_{\partial T} \\ &+ (y^\gamma - y_h^\gamma, R_{|T}^{p^\gamma})_T - (y^\gamma - y_h^\gamma, r_{|\partial T}^{p^\gamma})_{\partial T} \\ &+ (\lambda^\gamma + \lambda_h^\gamma, u_h^\gamma - u^\gamma)_T. \end{aligned}$$

Here the interior and edge residuals $R_{|T}^{\bullet}, r_{|\partial T}^{\bullet}$ are similarly defined as in the purely state constrained case. In order to derive a computable estimator we again replace the unknown functions y^γ and p^γ in (4.5) by $i_{2h}^{(2)} y_h^\gamma$ and $i_{2h}^{(2)} p_h^\gamma$. Since $u^\gamma = \Pi_{[u_a,u_b]}\left(-\frac{1}{\alpha} p^\gamma + u_d\right)$ holds, a reasonable locally computable approximation then is given by

$$\tilde{u}^\gamma = \Pi_{[u_a,u_b]}\left(-\frac{1}{\alpha} i_{2h}^{(2)} p_h^\gamma + u_d\right),$$

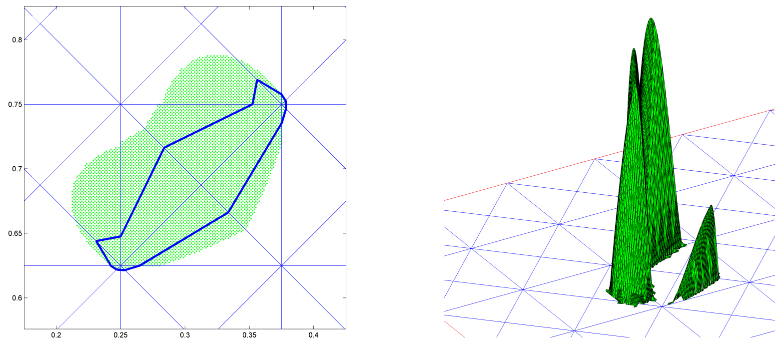

FIGURE 2. u_a active set: blue by u_h^γ, green by $\tilde{u}^\gamma$ (left), integrand $(\tilde{\lambda}^\gamma + \lambda_h^\gamma)(u_h^\gamma - \tilde{u}^\gamma)$ with support on symmetric difference of active sets (right).

as is already suggested in [22]. Similarly for $\lambda^\gamma = -p^\gamma - \alpha(u^\gamma - u_d)$ we locally compute

$$\tilde{\lambda}^\gamma = -i_{2h}^{(2)} p_h^\gamma - \alpha\,(\tilde{u}^\gamma - u_d)$$

instead. The estimator η^γ now reads

$$\eta^\gamma = \sum_{T \in \mathcal{T}_h} \eta_T^\gamma,$$

where

$$\begin{aligned} 2\eta_T^\gamma =& (i_{2h}^{(2)} y_h^\gamma - y_h^\gamma, R_{|T}^{p_h^\gamma})_T - (i_{2h}^{(2)} y_h^\gamma - y_h^\gamma, r_{|\partial T}^{p_h^\gamma})_{\partial T} \\ &+ (R_{|T}^{y_h^\gamma}, i_{2h}^{(2)} p_h^\gamma - p_h^\gamma)_T - (r_{|\partial T}^{y_h^\gamma}, i_{2h}^{(2)} p_h^\gamma - p_h^\gamma)_{\partial T} \\ &+ (i_{2h}^{(2)} y_h^\gamma - y_h^\gamma, R_{|T}^{i_{2h}^{(2)} p_h^\gamma})_T - (i_{2h}^{(2)} y_h^\gamma - y_h^\gamma, r_{|\partial T}^{i_{2h}^{(2)} p_h^\gamma})_{\partial T} \\ &+ (\tilde{\lambda}^\gamma + \lambda_h^\gamma, u_h^\gamma - \tilde{u}^\gamma)_T. \end{aligned}$$

While for most quantities in η_T^γ quadrature rules of moderate order are well suited, one has to take care for the last term

$$(\tilde{\lambda}^\gamma + \lambda_h^\gamma, u_h^\gamma - \tilde{u}^\gamma)_T = \int_T (\tilde{\lambda}^\gamma + \lambda_h^\gamma)(u_h^\gamma - \tilde{u}^\gamma). \tag{4.6}$$

The integrand is continuous but has a support within the symmetric difference of the control active set of the variational discrete solution and the locally improved quantities. Such a situation is depicted in Figure 2. One recognizes that $\tilde{u}^\gamma$ captures the activity structure of u_h^γ but smoothes out the control active boundary towards the exact control active boundary. The kidney-shaped green area resolves the true control active set from the example of Section 5 already very well even on a coarse mesh (compare also Figure 3 (right)). Finally for computing (4.6) we just provide the integrand and a desired tolerance and apply the adaptive quadrature routine of [23, Algo. 31] for triangles containing the boundary of the control active set.

In order to study the effectivity of our implemented estimator, we define

$$I_{\text{eff}} := \frac{J(y^\gamma, u^\gamma) - J_h(y_h^\gamma, u_h^\gamma)}{\eta^\gamma}.$$

Remark 4.2. The adjoint variable p admits less regularity at state active sets so that higher order interpolation with regard to the adjoint variable is not completely satisfying. However this circumstance only leads to local higher weights in the estimator and therefore reasonably suggests to refine the corresponding regions. The effectivity of the estimator is not affected as we are going to see in the numerical experiment. Another possible technique to derive a computable approximation for $p^\gamma - i_h p^\gamma$ is to substitute p_h^γ for p^γ and compute $p_h^\gamma - p_h^\gamma(x_T)$, where x_T denotes the barycenter of the element T.

5. Numerical Experiment

We consider problem (2.3) with data

$$\Omega = (0,\, 1)^2, \quad \mathcal{A} = -\Delta + \mathrm{Id}, \quad y_d = \sin(2\pi x_1)\sin(2\pi x_2), \quad f = u_d = 0,$$
$$u_a = -30, \quad u_b = 30, \quad y_a = -0.55, \quad y_b = 0.55, \quad \alpha = 10^{-4}.$$

Its analytic solution is not known, so for obtaining the effectivity index we compute a reference solution on a uniform grid (level $l = 14$, $np = 525313$, $h = 0.00195$) which delivers an approximation of $J(y^\gamma, u^\gamma)$. The numerical solution in terms of $-\frac{1}{\alpha}p_h^\gamma$ as well as the optimal state y_h^γ is displayed in Figure 3 for $\gamma = 10^{14}$ on the mesh $l = 14$. The projection of $-\frac{1}{\alpha}p_h^\gamma$ onto $[u_a, u_b]$ corresponds to the optimal control u_h^γ which together with y_h^γ represents our best approximation to the solution of (4.1). The boundaries of the control active sets are depicted as solid lines, while the state active sets are displayed as star and cross markers. The color blue corresponds to the lower bound while the color red highlights the upper bound. We numerically approximate $J(y^\gamma, u^\gamma)$ by 0.0375586175. In Table 1 we depict the effectivity coefficient and the convergence history of the goal. Notice that the values of the effectivity coefficient are close to 1 which illustrates the good performance of our error estimator. A comparison between our adaptive finite element algorithm and a uniform mesh refinement in terms of number of degrees of freedom $N_{dof} := np$ is reported in Figure 4. The adaptive refinement process performs well even though the benefit in this example is not big since the characteristic features of the optimal solution already occupy a considerable area of the computational domain, as is illustrated by the adapted grid in Figure 4. Our motivation for including this example is to illustrate the variational discretization effect on the mesh refinement process. If variational discretization for the control would not have been used, also some refinement at the boundary of the control active set would be expected. For the details in terms of our marking strategy,

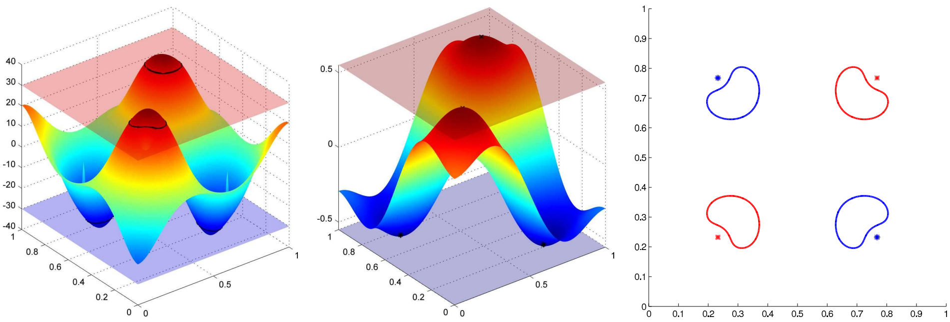

FIGURE 3. $u_a, -\frac{1}{\alpha}p_h^\gamma, u_b$ (left), $y_a \leq y_h^\gamma \leq y_b$ (middle) and active sets (right) for $l = 14$.

k	np	$J(y^\gamma, u^\gamma) - J_h(y_h^\gamma, u_h^\gamma)$	I_{eff}
1	81	$4.275 \cdot 10^{-3}$	1.622
10	3123	$7.148 \cdot 10^{-5}$	1.144
18	63389	$3.996 \cdot 10^{-6}$	1.290

TABLE 1. Adaptive refinement.

stopping criterion of the adaptive finite element algorithm and one more example we refer the reader to [10].

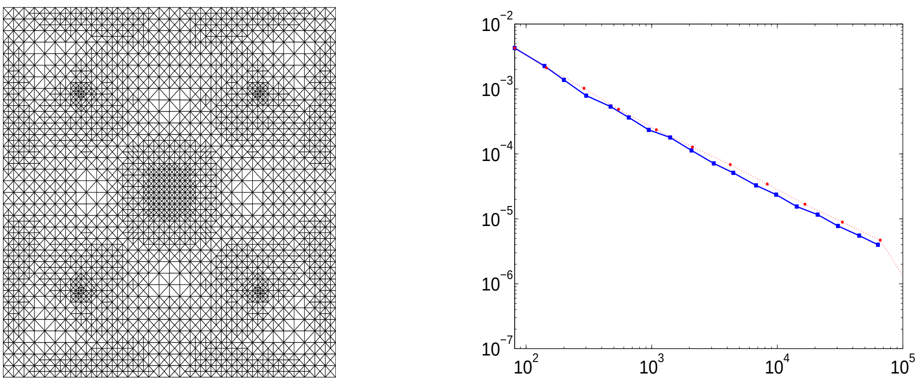

FIGURE 4. Adaptive mesh for $k = 10$ (left), comparison of error decrease in the goal (right).

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