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THE NONEXISTENCE OF PSEUDOQUATERNIONS IN $\mathbb{C}^{2\times 2}$

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Abstract. The field of quaternions, denoted by $\mathbb{H}$ can be represented as an isomorphic four dimensional subspace of $\mathbb{R}^{4\times 4}$, the space of real matrices with four rows and columns. In addition to the quaternions there is another four dimensional subspace in $\mathbb{R}^{4\times 4}$ which is also a field and which has in - connection with the quaternions - many pleasant properties. This field is called *field of pseudoquaternions*. It exists in $\mathbb{R}^{4\times 4}$ but not in $\mathbb{H}$. It allows to write the quaternionic linear term axb in matrix form as $\mathbf{M}\mathbf{x}$ where $\mathbf{x}$ is the same as the quaternion x only written as a column vector in $\mathbb{R}^4$. And $\mathbf{M}$ is the product of the matrix associated with the quaternion a with the matrix associated with the pseudoquaternion b.

Now, the field of quaternions can also be represented as an isomorphic four dimensional subspace of $\mathbb{C}^{2\times 2}$ over $\mathbb{R}$, the space of complex matrices with two rows and columns. We show that in this space pseudoquaternions with all the properties known from $\mathbb{R}^{4\times 4}$ do not exist. However, there is a subset of $\mathbb{C}^{2\times 2}$ for which some of the properties are still valid. By means of the Kronecker product we show that there is a matrix in $\mathbb{C}^{4\times 4}$ which has the properties of the pseudoquaternionic matrix.

Keywords: Quaternions, Pseudoquaternions
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1. Introduction.

Let us denote by $\mathbb{R}$, $\mathbb{C}$ the fields of real and complex numbers, respectively, and by $\mathbb{H}$ the field of quaternions, which is $\mathbb{R}^4$ equipped with a special multiplication rule which makes $\mathbb{R}^4$ a skew field. In order to explain that let $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ be the four standard basis elements in $\mathbb{H}$. They obey the following multiplication rules:

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1; \quad \mathbf{ij} = \mathbf{k},\ \mathbf{jk} = \mathbf{i},\ \mathbf{ki} = \mathbf{j}. \tag{1.1}$$

Instead of $a := a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}$ we write equivalently also $a = (a_1, a_2, a_3, a_4)$. Let $a := (a_1, a_2, a_3, a_4),\ b := (b_1, b_2, b_3, b_4)$. Then, the multiplication rules (1.1) imply

$$\begin{aligned} ab := (&a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4,\ a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3,\\ &a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2,\ a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1). \end{aligned} \tag{1.2}$$

The first component of a quaternion a will be called *real part* of a, denoted by $\Re a$. A real number, a_1, will be identified with the quaternion $a := (a_1, 0, 0, 0)$. A complex number $a_1 + a_2\mathbf{i}$ will be identified with $a := (a_1, a_2, 0, 0)$. And we see from the above multiplication rule, that the set of quaternions of the form $a := (a_1, 0, 0, 0)$ is isomorphic to the field of real numbers $\mathbb{R}$, and the set of quaternions of the form $a := (a_1, a_2, 0, 0)$ is isomorphic to the field of complex numbers $\mathbb{C}$. Let $a := (a_1, a_2, a_3, a_4)$. Then, $\overline{a} := (a_1, -a_2, -a_3, -a_4)$ will be called *conjugate* of a. The *absolute value* of a is denoted by $|a|$ and defined by $|a| := \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$. And for all $a, b \in \mathbb{H}$ there are the rules

$$|a|^2 = a\overline{a} = \overline{a}a,\ |ab| = |ba| = |a||b|,\ \overline{ab} = \overline{b}\,\overline{a},\ \Re(ab) = \Re(ba),\ a^{-1} = \frac{\overline{a}}{|a|^2}, \tag{1.3}$$

where the last rule applies only for $a \neq 0$. The field $\mathbb{H}$ is isomorphic to a certain set of complex (2×2) matrices and also isomorphic to a certain set of real (4×4) matrices. This will be explained and used in the next sections. Pseudoquaternions appear only in matrix spaces and they are useful when treating equations which contain terms of the type axb, where all three quantities a, b, x represent quaternions. So far, pseudoquaternions - without

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using that name - were known only in $\mathbb{R}^{4\times 4}$ by the work of Aramanovitch, [1]. We shall show here that one cannot define pseudoquaternions in $\mathbb{C}^{2\times 2}$ with all the properties known from the matrix space $\mathbb{R}^{4\times 4}$. This will be the topic of the next two sections.

2. Quaternions and pseudoquaternions in the matrix space $\mathbb{R}^{4\times 4}$. Let $a := (a_1, a_2, a_3, a_4)$ be a quaternion. We define two mappings $\imath_j : \mathbb{H} \to \mathbb{R}^{4\times 4}, j = 1, 2$ by

$$\imath_1(a) := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \tag{2.1}$$

$$\imath_2(a) := \begin{pmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{pmatrix} \in \mathbb{R}^{4\times 4}. \tag{2.2}$$

We use the notation

$$\mathbb{H}_{\mathbb{R}} := \imath_1(\mathbb{H}) \subset \mathbb{R}^{4\times 4}, \quad \mathbb{H}_{\mathbb{P}} := \imath_2(\mathbb{H}) \subset \mathbb{R}^{4\times 4}. \tag{2.3}$$

The first mapping, $\imath_1$, maps $\mathbb{H}$ isomorphically onto $\mathbb{H}_{\mathbb{R}}$ which means that for all $a, b \in \mathbb{H}$ we have

$$\imath_1(a+b) = \imath_1(a) + \imath_1(b); \quad \imath_1(\alpha a) = \alpha\imath_1(a),\ \alpha \in \mathbb{R}; \quad \imath_1(ab) = \imath_1(a)\imath_1(b). \tag{2.4}$$

The first two properties are obvious. For the third see Gürlebeck and Sprössig, Chapter 1, [3]. Let $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}$. We can write $\imath_1$ in the form

$$\imath_1(a) = a_1\mathbf{I}_1 + a_2\mathbf{I}_2 + a_3\mathbf{I}_3 + a_4\mathbf{I}_4,$$

where $\mathbf{I}_1$ is the identity matrix in $\mathbb{R}^{4\times 4}$ and

$$\mathbf{I}_2 := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathbf{I}_3 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \mathbf{I}_4 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

These matrices obey the same multiplication rules as the standard units $1, \mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$, namely

$$\mathbf{I}_2^2 = \mathbf{I}_3^2 = \mathbf{I}_4^2 = -\mathbf{I}_1; \quad \mathbf{I}_2\mathbf{I}_3 = \mathbf{I}_4,\ \mathbf{I}_3\mathbf{I}_4 = \mathbf{I}_2,\ \mathbf{I}_4\mathbf{I}_2 = \mathbf{I}_3. \tag{2.6}$$

The second mapping $\imath_2$ looks very much alike $\imath_1$. It has the following basis representation:

$$\imath_2(a) = a_1\mathbf{I}_1 + a_2\mathbf{J}_2 + a_3\mathbf{J}_3 + a_4\mathbf{J}_4,$$

where

$$\mathbf{J}_2 := \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \mathbf{J}_3 := \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{J}_4 := \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

These basis elements obey the following set of equations:

$$\mathbf{J}_2^2 = \mathbf{J}_3^2 = \mathbf{J}_4^2 = -\mathbf{I}_1; \quad \mathbf{J}_2\mathbf{J}_3 = -\mathbf{J}_4,\ \mathbf{J}_3\mathbf{J}_4 = -\mathbf{J}_2,\ \mathbf{J}_4\mathbf{J}_2 = -\mathbf{J}_3, \tag{2.7}$$

which differ from those given in (2.6). Though the matrices, $\imath_1(a), \imath_2(a)$ look almost alike, they coincide, however, if and only if $a \in \mathbb{R}$ or in other words

$$\mathbb{H}_{\mathbb{R}} \cap \mathbb{H}_{\mathbb{P}} = \{a\mathbf{I} : a \in \mathbb{R},\ \mathbf{I} \text{ is the identiy matrix in } \mathbb{R}^{4\times 4}\}.$$

This set is the center of $\mathbb{R}^{4\times 4}$ where, by definition, the *center* of $\mathbb{R}^{4\times 4}$ is the subset of all matrices in $\mathbb{R}^{4\times 4}$ which commute with all matrices in $\mathbb{R}^{4\times 4}$. The mapping $\imath_2$ has the following interesting properties for all $a, b \in \mathbb{H}$:

$$\imath_2(ab) = \imath_2(b)\imath_2(a), \tag{2.8}$$

$$\imath_1(a)\imath_2(b) = \imath_2(b)\imath_1(a). \tag{2.9}$$

See Aramanovitch, [1] and Janovská and Opfer, [6]. The first property means that $\mathbb{H}_{\mathbb{P}}$ is also a field, only the multiplication rule is reversed. By putting $b := a^{-1}$ for $a \neq 0$ in (2.8) we obtain

$$(\imath_2(a))^{-1} = \imath_2(a^{-1}) = \frac{1}{|a|^2}\imath_2(\overline{a}) = \frac{1}{|a|^2}\big(\imath_2(a)\big)^{\mathrm{T}},$$

where $^{\mathrm{T}}$ indicates transposition. Let us note that property (2.8) alone is not characteristic for $\imath_2$. Let $\widetilde{\imath_1} := \imath_1^{\mathrm{T}}$, then $\widetilde{\imath_1}(ab) = \widetilde{\imath_1}(b)\widetilde{\imath_1}(a)$. Thus, the different mappings $\imath_2$ and $\widetilde{\imath_1}$ share property (2.8). However, equation (2.9) is not valid if we would replace $\imath_2$ by $\imath_1^{\mathrm{T}}$.

DEFINITION 2.1. The field $\mathbb{H}_{\mathbb{P}} := \imath_2(\mathbb{H})$ will be called the field of *pseudoquaternions* in $\mathbb{R}^{4\times 4}$.

The mapping $\imath_2$ has more interesting properties, which are called *good relations* by Gürlebeck and Sprössig, p. 6, [3]. For $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}$ we introduce the *column operator*

$$\mathrm{col}(a) := \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \tag{2.10}$$

which is useful in connection with matrix operations. Note that $\mathrm{col}(a)$ is the first column of $\imath_1(a)$ and also the first column of $\imath_2(a)$.

LEMMA 2.2. *For all $a, b, c \in \mathbb{H}$ we have*

$$\begin{aligned}
\mathrm{col}(ab) &= \imath_1(a)\mathrm{col}(b), && (2.11)\\
&= \imath_2(b)\mathrm{col}(a), && (2.12)\\
\mathrm{col}(abc) &= \imath_2(c)\imath_2(b)\mathrm{col}(a), && (2.13)\\
&= \imath_1(a)\imath_2(c)\mathrm{col}(b), && (2.14)\\
&= \imath_1(a)\imath_1(b)\mathrm{col}(c). && (2.15)
\end{aligned}$$

Proof: Aramanovitch, Appendix A No. 8, p. 1252, [1]. □

THEOREM 2.3. *The two mappings $\imath_1$, $\imath_2$ are uniquely defined by the two properties* $\mathrm{col}(ab) = \imath_1(a)\mathrm{col}(b)$, $\mathrm{col}(ab) = \imath_2(b)\mathrm{col}(a)$, *respectively, for all $a, b \in \mathbb{H}$.*

Proof: Let $a := (a_1, a_2, a_3, a_4), b := (b_1, b_2, b_3, b_4) \in \mathbb{H}$ and $\mathbf{M} := (m_{jk}) \in \mathbb{R}^{4\times 4}$ be an arbitrary matrix, $j, k = 1, 2, 3, 4$. We first show the uniqueness of $\imath_1$. Let $\mathbf{M}\mathrm{col}(b) = \mathrm{col}(ab)$. We compare the four columns of $\mathbf{M}\mathrm{col}(b)$ with the four columns of $\mathrm{col}(ab)$ which we can find in (1.2). The jth column of $\mathbf{M}\mathrm{col}(b)$ is

$$b_1 m_{j1} + b_2 m_{j2} + b_3 m_{j3} + b_4 m_{j4},\ j = 1, 2, 3, 4.$$

The comparison with $\mathrm{col}(ab)$ yields four equations

$$\begin{aligned}
b_1m_{11}+b_2m_{12}+b_3m_{13}+b_4m_{14} &= a_1b_1-a_2b_2-a_3b_3-a_4b_4,\\
b_1m_{21}+b_2m_{22}+b_3m_{23}+b_4m_{24} &= a_2b_1+a_1b_2-a_4b_3+a_3b_4,\\
b_1m_{31}+b_2m_{32}+b_3m_{33}+b_4m_{34} &= a_3b_1+a_4b_2+a_1b_3-a_2b_4,\\
b_1m_{41}+b_2m_{42}+b_3m_{43}+b_4m_{44} &= a_4b_1-a_3b_2+a_2b_3+a_1b_4.
\end{aligned}$$

One obvious solution is $\mathbf{M}=\imath_1(a)$. The above system can be written in the form

$$\sum_{j=1}^{4} b_j x_{kj}=0,\ k=1,2,3,4 \text{ for all } b\in\mathbb{H},\ x_{11}:=m_{11}-a_1, x_{12}:=m_{12}+a_2,\ldots$$

This leaves only the possibility $x_{kj}=0$ for all $j,k=1,2,3,4$, which is equivalent to $\mathbf{M}=\imath_1(a)$. A very similar proof works for $\imath_2$. □

The uniqueness result in Theorem 2.3 does not imply that $\imath_1$ is the only mapping, which represents the isomorphism $\mathbb{H}\to\mathbb{R}^{4\times 4}$. E. g. Gürlebeck and Sprössig, p. 5/6 in [3] define this isomorphism by

$$\widehat{\imath_1}(a)=\begin{pmatrix} a_1 & -a_2 & -a_3 & a_4\\ a_2 & a_1 & -a_4 & -a_3\\ a_3 & a_4 & a_1 & a_2\\ -a_4 & a_3 & -a_2 & a_1\end{pmatrix}.$$

However, in order that the property (2.11) (and (2.12) as well) remains valid, one has to change the definition of $\mathrm{col}(a)$. In this case $\mathrm{col}(a)$ must be defined as the first column of $\widehat{\imath_1}(a)$. In a paper by Farebrother, Groß, and Troschke, [2], these authors in 2003 have made a systematic search for all matrix representations of $\mathbb{H}$ in $\mathbb{R}^{4\times 4}$.

All rules (2.13) to (2.15) are immediate consequences of (2.11), (2.12). The most important rule is rule (2.14). It allows to write $\mathrm{col}(axb)=\imath_1(a)\imath_2(b)\mathrm{col}(x)$, which means that the linear mapping

$$l:\mathbb{R}^4\to\mathbb{R}^4,\quad l(x):=axb$$

can be put into the explicit form

$$l(x)=\mathbf{M}x,\quad \mathbf{M}:=\imath_1(a)\imath_2(b).$$

This was successfully applied to the solution of quaternionic, linear systems, and to finding zeros of certain quaternionic polynomials, see Janovská and Opfer, [6, 7, 8].

Property (2.9) says in algebraic terms that $\imath_2(b)$ belongs to the *centralizer of* $\imath_1(a)$ for all $b\in\mathbb{H}$ and also for all $a\in\mathbb{H}$. The centralizer for the fixed matrix $\imath_1(a)$, denoted by $\mathcal{C}(\imath_1(a))$, is the set of all matrices in $\mathbb{R}^{4\times 4}$ which commute with $\imath_1(a)$. It is clear that the set of all polynomials in $\imath_1(a)$ with real coefficients, denoted by $\mathcal{P}(\imath_1(a))$, belongs to $\mathcal{C}(\imath_1(a))$. However, in this case, the centralizer $\mathcal{C}(\imath_1(a))$ does contain elements that are not belonging to $\mathcal{P}(\imath_1(a))$. This is a consequence of the fact that the characteristic and the minimal polynomial of $\imath_1(a)$ are different. More details are given by Horn and Johnson, p. 274–276, [5]. Let $a:=(a_1,a_2,a_3,a_4)\in\mathbb{H}$. Both matrices $\imath_1(a)$ and $\imath_2(a)$ have the same minimal polynomial

$$\mu(z):=z^2-2a_1z+|a|^2. \tag{2.16}$$

And both matrices are normal, with the consequence that they are similar. That means there is a nonsingular matrix $H\in\mathbb{R}^{4\times 4}$ such that $H\imath_1(a)=\imath_2(a)H$. If we would assume, that

$H \in \mathbb{H}_{\mathbb{R}}$ or $H \in \mathbb{H}_{\mathbb{P}}$, then it would follow that $a \in \mathbb{R}$. In other words, if a is not real, then H is neither in $\mathbb{H}_{\mathbb{R}}$ nor in $\mathbb{H}_{\mathbb{P}}$.

As a consequence of (2.16), there is the following formula valid for both matrices:

$$(\imath_k(a))^j = \alpha_j\, \imath_k(a) + \beta_j,\ j = 0, 1, \ldots;\ k = 1, 2. \tag{2.17}$$

A formula for the sequences $\{\alpha_j\}, \{\beta_j\}, j \geq 0$, is given in Section 3 of [7]. The two sequences $\{\alpha_j\}, \{\beta_j\}$ are the same for all matrices of the same similarity class. This implies that

$$z^j = \alpha_j z + \beta_j, \quad j = 0, 1, \ldots$$

for all quaternions $z \in \mathbb{H}$. This was used by Pogorui and Shapiro, 2004, [9] and by the present authors [7, 8].

If $\mathbf{A}$ is a real or a complex matrix of order n with a minimal polynomial of degree $\nu \leq n$, then (2.17) is a special case of

$$\mathbf{A}^j \in \langle \mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^{\nu-1} \rangle \text{ for all } j = 0, 1, \ldots.$$

where $\mathbf{I}$ is the identity matrix of the same size as $\mathbf{A}$ and $\langle \cdots \rangle$ denotes the linear hull of what is between the parentheses. This is a consequence of the Cayley-Hamilton theorem. For more details see Horn and Johnson, p. 87, [4].

Now, the question is, whether we can find a mapping $\mathbb{H} \to \mathbb{C}^{2\times 2}$ which has the same properties as $\imath_2$. This will be the topic of the next section.

3. Quaternions and pseudoquaternions in the matrix space $\mathbb{C}^{2\times 2}$. For all $a := (a_1, a_2, a_3, a_4) \in \mathbb{H}$, we define the mapping $\jmath_1 : \mathbb{H} \to \mathbb{C}^{2\times 2}$ by

$$\jmath_1(a) := \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad \alpha := a_1 + a_2\mathbf{i}, \quad \beta := a_3 + a_4\mathbf{i}. \tag{3.1}$$

This mapping is again an isomorphism between $\mathbb{H}$ and

$$\mathbb{H}_{\mathbb{C}} := \jmath_1(\mathbb{H}) \subset \mathbb{C}^{2\times 2}. \tag{3.2}$$

See van der Waerden, p. 55, [10]. We will keep parts of the notation of the last section, however, defined in $\mathbb{C}^{2\times 2}$. The basis representation of $\jmath_1(a)$ is

$$\jmath_1(a) = a_1\mathbf{I}_1 + a_2\mathbf{I}_2 + a_3\mathbf{I}_3 + a_4\mathbf{I}_4,$$

where $\mathbf{I}_1$ is the identity matrix in $\mathbb{R}^{2\times 2}$ and

$$\mathbf{I}_2 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \quad \mathbf{I}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{I}_4 = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}.$$

They obey the already mentioned rules (2.6) of the quaternionic multiplication:

$$\mathbf{I}_2^2 = \mathbf{I}_3^2 = \mathbf{I}_4^2 = -\mathbf{I}_1, \quad \mathbf{I}_2\mathbf{I}_3 = \mathbf{I}_4,\ \mathbf{I}_3\mathbf{I}_4 = \mathbf{I}_2,\ \mathbf{I}_4\mathbf{I}_2 = \mathbf{I}_3.$$

A change of the basis elements in the form

$$\mathbf{I}_2 \to -\mathbf{I}_2, \quad \mathbf{I}_3 \to -\mathbf{I}_3$$

would not change the above multiplication rules. That means, that also other representations of the isomorphism $\jmath_1$ are possible.

Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}$. We define a new *column operator* by

$$\operatorname{col}(a) := \begin{pmatrix} a_1 + a_2\mathbf{i} \\ -a_3 + a_4\mathbf{i} \end{pmatrix} = \begin{pmatrix} \alpha \\ -\overline{\beta} \end{pmatrix}, \tag{3.3}$$

where α, β are defined in (3.1). Note also here, that $\operatorname{col}(a)$ is the first column of $\jmath_1(a)$. This definition implies

$$\operatorname{col}(ab) = \begin{pmatrix} a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)\mathbf{i} \\ -a_1b_3 + a_2b_4 - a_3b_1 - a_4b_2 + (a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)\mathbf{i} \end{pmatrix}.$$

DEFINITION 3.1. Let $a, b \in \mathbb{H}$. A matrix $\mathbf{M} \in \mathbb{C}^{2\times 2}$ depending only on b will be called a *pseudoquaternion* in $\mathbb{C}^{2\times 2}$ if it has the property

$$\mathbf{M}\operatorname{col}(a) = \operatorname{col}(ab) \text{ for all } a, b \in \mathbb{H}. \tag{3.4}$$

In the next theorem we show that (2.11) has a unique equivalent in $\mathbb{C}^{2\times 2}$ but (2.12) (which is the analogue of (3.4)) has no equivalent. Which means that pseudoquaternions do not exist in $\mathbb{C}^{2\times 2}$.

THEOREM 3.2. *(1) With the column operator defined in (3.3) we have for all $a, b \in \mathbb{H}$*

$$\jmath_1(a)\operatorname{col}(b) = \operatorname{col}(ab). \tag{3.5}$$

There is no other matrix than $\jmath_1(a)$ with this property.
(2) There is no matrix $\mathbf{M} \in \mathbb{C}^{2\times 2}$ depending only on b such that

$$\mathbf{M}\operatorname{col}(a) = \operatorname{col}(ab) \textit{ for all } a, b \in \mathbb{H}. \tag{3.6}$$

Proof: (1) We have

$$\jmath_1(a)\operatorname{col}(b) = \begin{pmatrix} a_1 + a_2\mathbf{i} & a_3 + a_4\mathbf{i} \\ -a_3 + a_4\mathbf{i} & a_1 - a_2\mathbf{i} \end{pmatrix} \begin{pmatrix} b_1 + b_2\mathbf{i} \\ -b_3 + b_4\mathbf{i} \end{pmatrix}$$

and this coincides with the above given $\operatorname{col}(ab)$. Let

$$\mathbf{M} := \begin{pmatrix} u_1 + u_2\mathbf{i} & v_1 + v_2\mathbf{i} \\ x_1 + x_2\mathbf{i} & y_1 + y_2\mathbf{i} \end{pmatrix} \tag{3.7}$$

for $u_1, u_2, v_1, v_2, x_1, x_2, y_1, y_2 \in \mathbb{R}$. Now,

$$\mathbf{M}\operatorname{col}(b) = \begin{pmatrix} u_1b_1 - u_2b_2 - v_1b_3 - v_2b_4 + (u_1b_2 + u_2b_1 + v_1b_4 - v_2b_3)\mathbf{i} \\ -y_1b_3 - y_2b_4 + x_1b_1 - x_2b_2 + (y_1b_4 - y_2b_3 + x_1b_2 + x_2b_1)\mathbf{i} \end{pmatrix}.$$

A comparison with $\operatorname{col}(ab)$ shows, that the only solution, valid for all $a, b \in \mathbb{H}$, is $\mathbf{M} = \jmath_1(a)$.
(2) Let $\mathbf{M}$ be defined as in (3.7). The first row of $\mathbf{M}\operatorname{col}(a)$ reads:

$$a_1u_1 - a_2u_2 - a_3v_1 - a_4v_2 + (a_1u_2 + a_2u_1 - a_3v_2 + a_4v_1)\mathbf{i}.$$

A comparison with the real and imaginary part of the first row of $\operatorname{col}(ab)$ yields, respectively,

$$v_1 = b_3, v_2 = b_4; \quad v_1 = -b_3, v_2 = -b_4.$$

In other words, there is no solution for all b. $\square$

In particular, this implies that we cannot find an analogue of property (2.14).

COROLLARY 3.3. *There is no matrix* $\mathbf{M} \in \mathbb{C}^{2\times 2}$ *such that*

$$\mathbf{M}\mathrm{col}(b) = \mathrm{col}(abc) \text{ for all } a, b, c \in \mathbb{H}, \tag{3.8}$$

where $\mathbf{M}$ *depends only on* a, c.

Proof: Assume, there is such a matrix. Put $a = 1$. Then, $\mathbf{M}\mathrm{col}(b) = \mathrm{col}(bc)$. However, this contradicts Theorem 3.2, part (2). □

The fact that the minimal and characteristic polynomials coincide for $\jmath_1(a)$ for all $a \in \mathbb{H}$ has the consequence that the centralizer $\mathcal{C}(\jmath_1(a))$ consists exactly of the polynomials $\mathcal{P}(\jmath_1(a))$. Thus, all solutions $\mathbf{M}$ of the equation

$$\jmath_1(a)\mathbf{M} = \mathbf{M}\,\jmath_1(a)$$

are located in $\mathcal{P}(\jmath_1(a))$, which means that an equation of the form (2.9) is impossible in $\mathbb{C}^{2\times 2}$. See Horn and Johnson, Corollary 4.4.18, [5].

The characteristic polynomial of $\jmath_1(a)$, identical with the minimal polynomial of both $\imath_k(a),\ k = 1, 2$, is given in (2.16). Therefore, we also have the analogue of (2.17), namely

$$(\jmath_1(a))^j = \alpha_j\,\jmath_1(a) + \beta_j,\ j = 0, 1, \ldots; \tag{3.9}$$

where the coefficients $\alpha_j, \beta_j, j = 0, 1, \ldots$ are the same as in (2.17).

Define

$$\mathbf{M}(a) := \jmath_1(a)^{\mathrm{T}}. \tag{3.10}$$

This matrix has trivially the property

$$\mathbf{M}(ab) = \mathbf{M}(b)\mathbf{M}(a).$$

Matrix $\mathbf{M}(a)$, defined in (3.10), has the following basis representation:

$$\mathbf{M} = a_1\mathbf{I}_1 + a_2\mathbf{J}_2 + a_3\mathbf{J}_3 + a_4\mathbf{J}_4,$$

where

$$\mathbf{J}_2 := \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \quad \mathbf{J}_3 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{J}_4 := \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix},$$

and these basis elements follow (2.7). However, this is not enough for the matrix $\mathbf{M}$ to qualify for a pseudoquaternion, as we have seen.

In summary, we have shown that in $\mathbb{C}^{2\times 2}$ it is not possible to define a subspace over $\mathbb{R}$ with dimension four with the same properties as the corresponding subspace $\mathbb{H}_{\mathbb{P}}$ of pseudoquaternions in $\mathbb{R}^{4\times 4}$.

4. The application of Kronecker's product. Let $\mathbf{A}, \mathbf{B}, \mathbf{X}$ be real or complex matrices, not necessarily square such that

$$f(\mathbf{X}) := \mathbf{AXB}$$

can be defined. Then, there is a matrix $\mathbf{P}(\mathbf{A}, \mathbf{B})$ such that

$$\mathrm{col}(f(\mathbf{X})) = \mathbf{P}(\mathbf{A}, \mathbf{B})\mathrm{col}(\mathbf{X}).$$

This matrix $\mathbf{P}(\mathbf{A},\mathbf{B})$ is called the *Kronecker product* of $\mathbf{A}$ and $\mathbf{B}$. It can be applied to matrices of all sizes. The col operator applied to a matrix puts all columns of this matrix into one column, starting with the left column. The details can be found in Horn and Johnson, Chapter 4, [5]. For quaternionic matrices the product $\mathbf{P}(\mathbf{A},\mathbf{B})$ can also be defined. This was shown by Janovská and Opfer, [6].

Let us return to the topic of this paper and assume that a, b, x are quaternions. In the previous section, we have shown in Corollary 3.3 that there is no matrix $\mathbf{M} \in \mathbb{C}^{2\times 2}$ with the property

$$\mathrm{col}(axb) = \mathbf{M}\mathrm{col}(x),$$

where the col operator is defined in equation (3.3). Define

$$f(\mathrm{J}_1(x)) := \mathrm{J}_1(a)\mathrm{J}_1(x)\mathrm{J}_1(b).$$

Since all occurring matrices are complex 2×2 matrices, the general theory for Kronecker products applies and yields

$$\mathrm{col}(f(\mathrm{J}_1(x)) := \mathbf{P}(\mathrm{J}_1(a), \mathrm{J}_1(b))\mathrm{col}(\mathrm{J}_1(x)), \quad \mathbf{P}(\mathrm{J}_1(a), \mathrm{J}_1(b)) \in \mathbb{C}^{4\times 4}. \tag{4.1}$$

In order to find out how $\mathbf{P}$ looks, we have to introduce some notation. Let $a := (a_1, a_2, a_3, a_4)$, $b := (b_1, b_2, b_3, b_4), x := (x_1, x_2, x_3, x_4)$ and

$$\mathrm{J}_1(a) = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\overline{\alpha_2} & \overline{\alpha_1} \end{pmatrix}, \quad \mathrm{J}_1(b) = \begin{pmatrix} \beta_1 & \beta_2 \\ -\overline{\beta_2} & \overline{\beta_1} \end{pmatrix}, \quad \mathrm{J}_1(x) = \begin{pmatrix} \xi_1 & \xi_2 \\ -\overline{\xi_2} & \overline{\xi_1} \end{pmatrix};$$

$\alpha_1 := a_1 + a_2\mathbf{i}$, $\alpha_2 := a_3 + a_4\mathbf{i}$; $\beta_1 := b_1 + b_2\mathbf{i}$, $\beta_2 := b_3 + b_4\mathbf{i}$; $\xi_1 := x_1 + x_2\mathbf{i}$, $\xi_2 := x_3 + x_4\mathbf{i}$.

Then (see Horn and Johnson, p. 243 and p. 255, [5])

$$\mathbf{P}(\mathrm{J}_1(a), \mathrm{J}_1(b)) = \begin{pmatrix} \beta_1 \mathrm{J}_1(a) & -\overline{\beta_2}\mathrm{J}_1(a) \\ \beta_2 \mathrm{J}_1(a) & \overline{\beta_1}\mathrm{J}_1(a) \end{pmatrix}, \quad \mathrm{col}(\mathrm{J}_1(x)) = \begin{pmatrix} \xi_1 \\ -\overline{\xi_2} \\ \xi_2 \\ \overline{\xi_1} \end{pmatrix}. \tag{4.2}$$

This complex (4×4) block matrix may be regarded as a replacement for the missing complex (2×2) pseudoquaternionic matrix.

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