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Abstract

In this paper we investigate the POD discretisation method for linear second order evolution equations. We present error estimates for two different choices of snapshot sets, one consisting of solution snapshots only and one consisting of solution snapshots and their derivatives up to second order. We show that the results of [7] for parabolic equations can be extended to linear second order evolution equations and that the derivative snapshot POD method behaves better than the classical one for small time steps. Numerical comparisons of the different approaches are presented underlining the theoretical results.

1 Introduction

Simulation of industrial problems like flow or heat transfer often requires the solution of large linear or nonlinear systems consisting of several ten thousands degrees of freedom [5]. Problems of such high dimensions can be handled by using powerful computers with large storage capabilities. Additionally, in some applications, these simulations need to be repeated several times with slightly different input, like in general controller design problems or in the durability simulation of wind turbines [9]. Often even real-time applicability is required, like in multibody dynamics with hardware-in-the-loop or human-in-the-loop systems. In these cases, simulation time becomes an important issue.
Over the years, various methods of model reduction for both linear and nonlinear systems have been developed [11]. These methods allow the construction of low-dimensional reduced models conserving the essential properties and features of the large model.
Whereas the most popular reduction methods such as balanced truncation, moment matching or analysis of eigenforms only seem to be suitable for linear problems, the method of *proper orthogonal decomposition (POD)* can also be applied to nonlinear systems. Its flexibility in application is based on analysing a given data set to provide the reduced model as described in Section 2 of this paper. Originating from fluid dynamic applications including turbulence and coherent structures [2], the method has also proved useful for certain problems in optimal control [6] and in circuit simulations [14].

To justify the method mathematically, Kunisch and Volkwein [7, 8] proved error bounds for POD-Galerkin approximations of linear and nonlinear parabolic equations, respectively. Lead by their numerical analysis they proposed to include also derivative information into the snapshot set and proved the superiority of this modified POD approach over the classical one both theoretically and numerically. In [6], the second author and Volkwein extended this analysis to optimal control problems using POD-surrogate models.
In Section 3 of this work we derive error estimates for POD Galerkin approximations to the linear second order evolution equations based on time discretization with Newmark's scheme. Similar to [7] and [8], we show that convergence can be guaranteed for the derivative approach and for the classical method if the time step size and the dimension of the POD subspace are coupled accordingly. Our numerical experiments for the wave equation described in Section 4 show that the error behaviour of both methods strongly depends on the eigenvalues of the correlation matrix.

2 POD for linear second order evolution equations

The linear wave equation is a simple example for a partial differential equation of second order. In this section, we want to outline the mathematical framework required to handle such problems. Furthermore, we describe the discretisation by Newmark's method and the POD scheme.

2.1 Problem description

Let V and H be real, separable Hilbert spaces for which we require [4, 7]

$$V \hookrightarrow H = H' \hookrightarrow V'$$

where V' denotes the dual of V. Each embedding is assumed to be dense and continuous. Further, let $a : V \times V \to \mathbb{R}$ be a continuous, coercive and symmetric bilinear form, i.e., there exist constants $\beta, \kappa \geq 0$ such that

$$\|a(\phi,\psi)\| \leq \beta \|\phi\|_V \|\psi\|_V , \tag{1}$$

$$\kappa \|\phi\|_V^2 \leq a(\phi,\phi), \tag{2}$$

for all $\phi, \psi \in V$.
As a simple example for a second order evolution equation, we chose the linear wave equation expressed in weak formulation:

$$\begin{aligned} \langle \ddot{x}(t), \phi\rangle_H + D\langle \dot{x}(t), \phi\rangle_H + a\,(x(t),\phi) &= \langle f(t),\phi\rangle_H & (3a)\\ &\quad \text{for all } \phi \in V \text{ and } t \in [0,T], & \\ \langle x(0),\psi\rangle &= \langle x_0,\psi\rangle_H \quad \text{for all } \psi \in H, & (3b)\\ \langle \dot{x}(0),\psi\rangle &= \langle \dot{x}_0,\psi\rangle_H \quad \text{for all } \psi \in H, & (3c) \end{aligned}$$

where $f(t) \in H$ is a given external force and $x(t) \in V$ denotes the sought deformation over time $t \in [0, T]$. Note that a damping term is incorporated which corresponds to a Rayleigh-type damping matrix C which is D times the mass matrix [3, 4].
Concerning the existence of a unique solution we have the following result available [3].

Proposition 1 *For $f \in L^2((0,T); H)$ and $x_0, \dot{x}_0 \in H$, problem 3 admits a unique weak solution.*

2.2 POD-Newmark scheme

For the time discretisation of (3) we divide the time interval $[0, T]$ into m subintervals of equal size $\Delta t = T/m$ and use Newmark's structure mechanics time integration scheme [4], i.e., we seek a sequence $(X_k) \subset V$, $k = 0, \ldots, m$, satisfying the following equations at each time level $t_k = k \cdot \Delta t$:

$$\begin{aligned} \langle \partial\partial X_k, \phi \rangle_H + D \langle \partial X_k, \phi \rangle_H + a(X_k, \phi) &= \langle f(t_k), \phi \rangle_H \qquad (4a) \\ &\qquad \text{for all } \phi \in V \text{ and } k = 1, \ldots m, \\ \langle X_0, \psi \rangle &= \langle x_0, \psi \rangle_H \quad \text{for all } \psi \in V, \qquad (4b) \\ \langle \partial X_0, \psi \rangle &= \langle \partial x_0, \psi \rangle_H \quad \text{for all } \psi \in V. \qquad (4c) \end{aligned}$$

Here we use the derivative approximations

$$\partial X_{k+1} = \frac{2}{\Delta t} X_{k+1} - \frac{2}{\Delta t} X_k - \partial X_k, \tag{5a}$$

$$\partial\partial X_{k+1} = \frac{4}{\Delta t^2} X_{k+1} - \frac{4}{\Delta t^2} X_k - \frac{4}{\Delta t} \partial X_k - \partial\partial X_k, \tag{5b}$$

for $k = 1, \ldots, m$.

Remark 1 *The case $k = 0$ is covered by the initial conditions x_0 and ∂x_0 for deformation X_0 and velocity ∂X_0, which yield the acceleration $\partial\partial X_0$ by solution of the equilibrium equation.*

Like any Galerkin-type method, *proper orthogonal decomposition* is a spatial discretisation scheme approximating the solution X_k by a linear combination of basis vectors $\varphi_i \in V$,

$$X_k = \sum_{i=1}^{l} \varphi_i \cdot p_i(t_k) \quad \text{for} \quad k = 1 \ldots m, \tag{6}$$

where p_i denotes the time-dependent participation factor of the basis vector i in the solution. Setting $V^l = span\{\varphi_1, \ldots, \varphi_l\} \subset V$, the POD-Newmark scheme for the wave equation consists in finding a sequence $\{X_k\}_{k=0,\ldots,m} \subset V^l$ which satisfies

$$\begin{aligned} \langle \partial\partial X_k, \phi \rangle_H + D \langle \partial X_k, \phi \rangle_H + a(X_k, \phi) &= \langle f(t_k), \phi \rangle_H \qquad (7a) \\ &\qquad \text{for all } \phi \in V^l \text{ and } k = 1, \ldots, m, \\ \langle X_0, \psi \rangle &= \langle x_0, \psi \rangle_H \quad \text{for all } \psi \in V^l, \qquad (7b) \\ \langle \partial X_0, \psi \rangle &= \langle \partial x_0, \psi \rangle_H \quad \text{for all } \psi \in V^l. \qquad (7c) \end{aligned}$$

The unique solvability of these equations follows from the following result [3].

Proposition 2 *Under the above assumptions there exists a unique solution $X_k \in V^l$ to problem (7) for each time level $k = 1, \ldots, m$.*

The essential step of the *snapshot POD method* [12] is the construction of the subspace V^l. Here, we use the snapshot POD method which consists in taking snapshots $X_k, k = 1, \ldots, m$, of the previously computed solution of problem (4). The subspace V^l is chosen as the best approximation of the snapshot set $\{X_k\}$ in a least squares sense [10]. In this paper we consider POD subspaces built from two different snapshot sets: set I consisting of deformation snapshots $\{x(t_k)\}$ at all time instances, and set II consisting of deformations and derivative approximations $\{x(t_k), \partial x(t_k), \partial\partial x(t_k)\}$. These sets yield the snapshot matrices Y_I and Y_{II} defined by:

$$Y_I = [x(t_0), \ldots, x(t_m)] \quad \text{and} \tag{8}$$

$$Y_{II} = [x(t_0), \ldots, x(t_m), \partial x(t_1), \ldots, \partial x(t_m), \partial\partial x(t_1), \ldots, \partial\partial x(t_{m-1})] \tag{9}$$

Note that the derivative approximations $\partial x(t_k)$ and $\partial\partial x(t_k)$ are elements of the space V. Furthermore, their inclusion does not change the dimension of the snapshot set, since they can be expressed as linear combinations of the deformation snapshots

$$\begin{aligned} \partial X_{k+1} + \partial X_k &= \frac{2}{\Delta t}\left(X_{k+1} - X_k\right), \\ \partial\partial X_{k+1} + 2\partial\partial X_k + \partial\partial X_{k-1} &= \frac{4}{\Delta t^2}\left(X_{k+1} - 2X_k + X_{k-1}\right). \end{aligned}$$

We write $Y_{I,II} = [y_0, \ldots, y_d]$ with either $d = m$ or $d = 3m - 1$. In both cases, we follow the regular POD recipe (see [13]) by constructing the correlation matrix C from scalar products of the snapshots y_i

$$\begin{aligned} C_{ij} &= \langle y_i, y_j \rangle_X, \quad i, j = 1, \ldots, d, \\ \text{solving the eigenvalue problem} \quad Cv^k &= \lambda_k v^k, \\ \text{and defining the POD basis vectors by} \quad \varphi_k &= Y \cdot v^k, \end{aligned}$$

with $X = V$ or $X = H$. Each eigenvector v^k of the correlation matrix defines a basis vector φ_k of the POD subspace. Depending on the number of basis vectors used for the subspace $V^l = span\{\varphi_1, \ldots, \varphi_l\}$, the projection error for

$$P^l y := \sum_{j=1}^{l} \langle y, \varphi_j \rangle_X \cdot \varphi_j$$

can be expressed as

$$\frac{1}{n} \sum_{k=1}^{n} \left\| y_k - \sum_{j=1}^{l} \langle y_k, \varphi_j \rangle_X \cdot \varphi_j \right\|_X^2 = \sum_{j=l+1}^{d} \lambda_j. \tag{10}$$

The integer $d < n$ shall denote the dimension of the snapshot set Y and $l < d$ is the number of POD basis vectors used for the projection.
The "optimal basis" consists of the eigenvectors corresponding to the l largest eigenvalues and spans the subspace V^l with the smallest projection error of all possible l-dimensional subspaces $\hat{V}^l \subset V$. This set of basis vectors is often called the *Karhunen-Loève basis* [11].

3 Error estimates

The error of the POD-Newmark scheme is defined as the difference between the numerical solution $X(t)$ of (7) and the analytical solution $x(t)$ of (3). Our goal consists in proving a bound for the H-Norm of the solution difference.

Theorem 1 *Let $x(t)$ be the regular solution of (3) and $X_k, k = 1, \ldots, m$, be the solution of (7) on each time level t_k. Let the POD subspace V^l be constructed from snapshot set Y_I or Y_{II},respectively. Then there exist constants C_I and C_{II} depending on T, D, $x^{(3)}$ and $x^{(4)}$, but not on Δt, m or l, such that it holds for $\Delta t \leq 1$ that:*

$$\begin{aligned} &\frac{1}{m}\sum_{k=1}^{m} \left\|X^k - x(t_k)\right\|_H^2 \leq \\ &\leq \; C_I \Big(\left\|X^0 - P^l x(t_0)\right\|_H^2 + \left\|X^1 - P^l x(t_1)\right\|_H^2 + \Delta t \left\|\partial X^0 - P^l \dot{x}(t_0)\right\|_H^2 \\ &\qquad + \Delta t \left\|\partial X^1 - P^l \dot{x}(t_1)\right\|_H^2 + \Delta t^4 + \left(\frac{1}{\Delta t^4} + \frac{1}{\Delta t} + 1\right) \sum_{j=l+1}^{d} \lambda_{Ij} \Big) \end{aligned} \tag{11}$$

for snapshots constructed via Y_I and

$$\begin{aligned} &\frac{1}{m}\sum_{k=1}^{m} \left\|X^k - x(t_k)\right\|_H^2 \leq \\ &\leq \; C_{II} \Big(\left\|X^0 - P^l x(t_0)\right\|_H^2 + \left\|X^1 - P^l x(t_1)\right\|_H^2 + \Delta t \left\|\partial X^0 - P^l \dot{x}(t_0)\right\|_H^2 \\ &\qquad + \Delta t \left\|\partial X^1 - P^l \dot{x}(t_1)\right\|_H^2 + \Delta t^4 + \sum_{j=l+1}^{d} \lambda_{IIj} \Big) \end{aligned} \tag{12}$$

for snapshots constructed via Y_{II}.

Remark 2 *These estimates are constructed in a similar way as the ones given in [7] and [8]. In analogy to [7, Lemma 2] we have: For all $x \in V$ it holds*

$$\|x\|_H \leq \sqrt{\|M\|_2 \cdot \|K^{-1}\|_2} \cdot \|x\|_V \text{ for all } x \in V^l, \tag{13}$$

$$\|x\|_V \leq \sqrt{\|K\|_2 \cdot \|M^{-1}\|_2} \cdot \|x\|_H \text{ for all } x \in V^l, \tag{14}$$

$$\text{with} \qquad M_{ij} = \langle \Phi_i, \Phi_j \rangle_H, \quad K_{ij} = \langle \Phi_i, \Phi_j \rangle_V, \tag{15}$$

where $\|\cdot\|_2$ denotes the spectral norm for symmetric matrices. M and K are called the system's mass and stiffness matrix, respectively.
These inequalities allow us to set up an error estimate in the H-norm and also to control the error in the V-norm as long as $\|M\|_2$, $\|M^{-1}\|_2$, $\|K\|_2$ and $\|K^{-1}\|_2$ are bounded. Hence, we restrict ourselves to the H-norm in the following.

Remark 3 *Note that the eigenvalues λ_{Ij} and λ_{IIj} are not identical. The weighting of snapshots is changed by inclusion of the derivative approximations, which leads to different choices of basis vectors for the subspaces V_I^l and V_{II}^l. In both cases, the snapshot correlation matrix C is generally not invertible, so the sum of the eigenvalues remains finite.*

Proof of Theorem 1:
Let X^k be the solution of the POD system (7) for the time instances $t_k = k \cdot \Delta t, k = 0, \ldots, m$, and $x(t_k)$ be the corresponding solution of the original system (4). In order to estimate

$$\frac{1}{m} \sum_{k=1}^{m} \left\| X^k - x(t_k) \right\|_H^2 \tag{16}$$

we decompose the local error into a projection part ρ and a part ϑ arising from the numerical discretisation procedure:

$$X^k - x(t_k) = \underbrace{X^k - P^l x(t_k)}_{=:\vartheta_k} + \underbrace{P^l x(t_k) - x(t_k)}_{=:\rho_k}, \tag{17}$$

which yields

$$\frac{1}{m} \sum_{k=1}^{m} \left\| X^k - x(t_k) \right\|_H^2 \leq \frac{2}{m} \sum_{k=1}^{m} \|\vartheta_k\|_H^2 + \frac{2}{m} \sum_{k=1}^{m} \|\rho_k\|_H^2 . \tag{18}$$

For an estimate of $\|\rho_k\|_H^2$ we use the error bound (10). Case I is constructed in the "classical" way and simply yields the POD projection error ([13])

$$\frac{1}{m+1} \sum_{k=0}^{m} \left\| x_k - \sum_{j=1}^{l} \langle x_k, \phi_j \rangle_X \cdot \phi_j \right\|_X^2 = \sum_{j=l+1}^{d} \lambda_{Ij}. \tag{19}$$

Here, x_k denotes the snapshot $x(t_k)$.

For later use we derive

$$\begin{aligned}
&\sum_{k=1}^{m-1}\left\|\partial\left(x_{k+1}+2x_k+x_{k-1}\right)-\sum_{j=1}^{l}\langle\partial\left(x_{k+1}+2x_k+x_{k-1}\right),\phi_j\rangle_X\cdot\phi_j\right\|_X^2 = \\
&= \sum_{k=1}^{m-1}\frac{1}{\Delta t^4}\left\|x_{k+1}-2x_k+x_{k-1}-P^l x_{k+1}+2P^l x_k-P^l x_{k-1}\right\|_X^2 \\
&\le \frac{4}{\Delta t^2}\sum_{k=1}^{m-1}2\Big(\left\|x_{k+1}-P^l x_{k+1}\right\|_X^2+\left\|x_{k-1}-P^l x_{k-1}\right\|_X^2\Big) \\
&\le \frac{16}{\Delta t^2}\sum_{k=0}^{m}\left\|x_k-P^l x_k\right\|_X^2 \\
&\le \frac{16}{\Delta t^2}(m+1)\sum_{j=l+1}^{d}\lambda_{Ij}
\end{aligned} \tag{20}$$

and

$$\begin{aligned}
&\sum_{k=1}^{m-1}\left\|\partial\partial\left(x_{k+1}+2x_k+x_{k-1}\right)-\sum_{j=1}^{l}\langle\partial\partial\left(x_{k+1}+2x_k+x_{k-1}\right),\phi_j\rangle_X\cdot\phi_j\right\|_X^2 = \\
&= \sum_{k=1}^{m-1}\frac{16}{\Delta t^4}\left\|x_{k+1}-2x_k+x_{k-1}-P^l x_{k+1}+2P^l x_k-P^l x_{k-1}\right\|_X^2 \\
&\le \frac{16}{\Delta t^4}\sum_{k=1}^{m-1}4\Big(\left\|x_{k+1}-P^l x_{k+1}\right\|_X^2+\left\|2x_k-2P^l x_k\right\|_X^2+\left\|x_{k-1}-P^l x_{k-1}\right\|_X^2\Big) \\
&\le \frac{384}{\Delta t^4}\sum_{k=0}^{m}\left\|x_k-P^l x_k\right\|_X^2 \\
&\le \frac{384}{\Delta t^4}(m+1)\sum_{j=l+1}^{d}\lambda_{Ij}.
\end{aligned} \tag{21}$$

For the second case, the POD error is analogously defined for the sum over all snapshots (solution x and derivatives ∂x and $\partial\partial x$):

$$\begin{aligned}
&\frac{1}{3m}\sum_{k=0}^{m}\left\|x_k-\sum_{j=1}^{l}\langle x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2+\\
&+\frac{1}{3m}\sum_{k=1}^{m}\left\|\partial x_k-\sum_{j=1}^{l}\langle \partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2+\\
&+\frac{1}{3m}\sum_{k=1}^{m-1}\left\|\partial\partial x_k-\sum_{j=1}^{l}\langle \partial\partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2 \quad=\quad \sum_{j=l+1}^{d}\lambda_{IIj},
\end{aligned}\tag{22}$$

which yields

$$\begin{aligned}
\frac{1}{m}\sum_{k=1}^{m}\|\rho_k\|_X^2=\frac{1}{m}\sum_{k=1}^{m}\left\|x_k-\sum_{j=1}^{l}\langle x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2 &\leq 3\cdot\sum_{j=l+1}^{d}\lambda_{IIj},\\
\frac{1}{m}\sum_{k=1}^{m}\left\|\partial x_k-\sum_{j=1}^{l}\langle \partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2 &\leq 3\cdot\sum_{j=l+1}^{d}\lambda_{IIj},\\
\frac{1}{m}\sum_{k=1}^{m-1}\left\|\partial\partial x_k-\sum_{j=1}^{l}\langle \partial\partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2 &\leq 3\cdot\sum_{j=l+1}^{d}\lambda_{IIj}.
\end{aligned}$$

Hence, we get the estimate

$$\begin{aligned}
\sum_{k=1}^{m-1}\left\|\partial\left(x_{k+1}+2x_k+x_{k-1}\right)-\sum_{j=1}^{l}\langle\partial\left(x_{k+1}+2x_k+x_{k-1}\right),\phi_j\rangle_X\cdot\phi_j\right\|_X^2&\\
\leq 24\sum_{k=1}^{m-1}\left\|\partial x_k-\sum_{j=1}^{l}\langle \partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2&\\
\leq 72m\cdot\sum_{j=l+1}^{d}\lambda_{IIj},&
\end{aligned}\tag{23}$$

and an analogous estimate holds for the second derivatives

$$\sum_{k=1}^{m-1}\left\|\partial\partial\left(x_{k+1}+2x_k+x_{k-1}\right)-\sum_{j=1}^{l}\langle\partial\partial\left(x_{k+1}+2x_k+x_{k-1}\right),\phi_j\rangle_X\cdot\phi_j\right\|_X^2$$
$$\leq\ 24\sum_{k=1}^{m-1}\left\|\partial\partial x_k-\sum_{j=1}^{l}\langle\partial\partial x_k,\phi_j\rangle_X\cdot\phi_j\right\|_X^2$$
$$\leq\ 72m\cdot\sum_{j=l+1}^{d}\lambda_{IIj}. \tag{24}$$

For an estimate of $\|\vartheta_k\|_H^2=\left\|X^k-P^lx(t_k)\right\|_H^2$ we state the following idenity:

$$\begin{aligned}
&\langle\partial\partial\vartheta_k,\psi\rangle_H+D\langle\partial\vartheta_k,\psi\rangle_H+a\left(\vartheta_k,\psi\right)=\\
&=\ \langle\partial\partial X_k,\psi\rangle_H-\langle\partial\partial P^lx(t_k),\psi\rangle_H+D\cdot\langle\partial X_k,\psi\rangle_H-D\cdot\langle\partial P^lx(t_k),\psi\rangle_H+\\
&\qquad+a\left(X_k,\psi\right)-a\left(P^lx(t_k),\psi\right)\\
&=\ \langle f(t_k),\psi\rangle_H-a\left(P^lx(t_k),\psi\right)-\langle\partial\partial P^lx(t_k),\psi\rangle_H-D\langle\partial P^lx(t_k),\psi\rangle_H\\
&=\ \langle f(t_k),\psi\rangle_H-a\left(x(t_k),\psi\right)-\langle\partial\partial P^lx(t_k),\psi\rangle_H-D\langle\partial P^lx(t_k),\psi\rangle_H\\
&=\ \langle\ddot{x}(t_k),\psi\rangle_H+D\langle\dot{x}(t_k),\psi\rangle_H-\langle\partial\partial P^lx(t_k),\psi\rangle_H\\
&=\ \langle\left(\ddot{x}(t_k)-\partial\partial P^lx(t_k)\right)+D\left(\dot{x}(t_k)-\partial P^lx(t_k)\right),\psi\rangle_H\\
&=:\ \langle v_k,\psi\rangle_H,
\end{aligned}\tag{25}$$

which holds for all $\psi\in V^l$.
Hence, that the sequence ϑ_k can be regarded as the solution of a linear, damped wave equation with the "force term" v_k. In analogy to the centered scheme described in [4], the Newmark scheme for this equation can be written as:

$$\frac{1}{\Delta t^2}\langle\vartheta_{k+1}-2\vartheta_k+\vartheta_{k-1},\psi\rangle_H+\frac{2D}{\Delta t}\langle\vartheta_{k+1}-\vartheta_{k-1},\psi\rangle_H+$$
$$+\frac{1}{4}a\left(\vartheta_{k+1}+2\vartheta_k+\vartheta_{k-1},\psi\right)=\frac{1}{4}\langle v_{k+1}+2v_k+v_{k-1},\psi\rangle_H$$

For notational convenience we define $\gamma_k=v_{k+1}+2v_k+v_{k-1}$. Choosing $\psi=\vartheta_{k+1}-\vartheta_{k-1}\in V^l$ as a test function in (25) we get

$$\underbrace{\frac{1}{\Delta t^2}\langle\vartheta_{k+1}-2\vartheta_k+\vartheta_{k-1},\vartheta_{k+1}-\vartheta_{k-1}\rangle_H}_{=:T_1}+\underbrace{\frac{2D}{\Delta t}\left\|\vartheta_{k+1}-\vartheta_{k-1}\right\|_H^2}_{=:d^k\leq 0}+$$
$$+\underbrace{\frac{1}{4}a\left(\vartheta_{k+1}+2\vartheta_k+\vartheta_{k-1},\vartheta_{k+1}-\vartheta_{k-1}\right)}_{=:T_2}\ =\ \underbrace{\frac{1}{4}\langle\gamma_k,\vartheta_{k+1}-\vartheta_{k-1}\rangle_H}_{=:W^k}.$$

Further, it holds

$$\begin{aligned} T_1 = \frac{1}{\Delta t^2} \langle \vartheta_{k+1} - 2\vartheta_k + \vartheta_{k-1}, \vartheta_{k+1} - \vartheta_{k-1} \rangle_H = \\ = \frac{1}{\Delta t^2} \langle (\vartheta_{k+1} - \vartheta_k) - (\vartheta_k - \vartheta_{k-1}), (\vartheta_{k+1} - \vartheta_k) + (\vartheta_k - \vartheta_{k-1}) \rangle_H = \\ = \frac{1}{\Delta t^2} \left(\|\vartheta_{k+1} - \vartheta_k\|_H^2 - \|\vartheta_k - \vartheta_{k-1}\|_H^2 \right) \end{aligned}$$

and

$$\begin{aligned} T_2 = \frac{1}{4} a \left(\vartheta_{k+1} + 2\vartheta_k + \vartheta_{k-1}, \vartheta_{k+1} - \vartheta_{k-1} \right) = \\ = \frac{1}{4} a \left((\vartheta_{k+1} + \vartheta_k) + (\vartheta_k + \vartheta_{k-1}), (\vartheta_{k+1} + \vartheta_k) - (\vartheta_k + \vartheta_{k-1}) \right) \\ = \frac{1}{4} \left[a \left(\vartheta_{k+1} + \vartheta_k, \vartheta_{k+1} + \vartheta_k \right) - a \left(\vartheta_k + \vartheta_{k-1}, \vartheta_k + \vartheta_{k-1} \right) \right]. \end{aligned}$$

This yields

$$\begin{aligned} E^{k+1} + d^k &= E^k + W^k \\ \text{with} \quad E^{k+1} &:= \left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|_H^2 + \frac{1}{4} a \left(\vartheta_{k+1} + \vartheta_k, \vartheta_{k+1} + \vartheta_k \right). \end{aligned}$$

Due to the coercivity of the bilinear form a we have

$$\left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|_H^2 \leq E^{k+1} \tag{26}$$

and

$$\begin{aligned}
E^{k+1}+d^k &= E^1+\sum_{i=1}^{k}W^i = E^1+\frac{1}{4}\sum_{i=1}^{k}\langle\gamma_i,\vartheta_{i+1}-\vartheta_{i-1}\rangle_H \\
&= E^1+\frac{1}{4}\sum_{i=1}^{k}\langle\gamma_i,(\vartheta_{i+1}-\vartheta_i)+(\vartheta_i-\vartheta_{i-1})\rangle_H \\
&= E^1+\frac{1}{4}\sum_{i=1}^{k}\langle\gamma_i,\vartheta_{i+1}-\vartheta_i\rangle_H+\frac{1}{4}\sum_{i=1}^{k}\langle\gamma_i,\vartheta_i-\vartheta_{i-1}\rangle_H \\
&= E^1+\frac{1}{4}\left(\sum_{i=1}^{k-1}\langle\gamma_i,\vartheta_{i+1}-\vartheta_i\rangle_H+\langle\gamma_k,\vartheta_{k+1}-\vartheta_k\rangle_H\right)+ \\
&\qquad +\frac{1}{4}\left(\sum_{p=1}^{k-1}\langle\gamma_{p+1},\vartheta_{p+1}-\vartheta_p\rangle_H+\langle\gamma_1,\vartheta_1-\vartheta_0\rangle_H\right) \\
&= E^1+\frac{1}{4}\langle\gamma_k,\vartheta_{k+1}-\vartheta_k\rangle_H+\frac{1}{4}\langle\gamma_1,\vartheta_1-\vartheta_0\rangle_H+ \\
&\quad +\frac{1}{4}\sum_{i=1}^{k-1}\langle\gamma_{i+1}+\gamma_i,\vartheta_{i+1}-\vartheta_i\rangle_H.
\end{aligned}$$

Using Young's inequality and $\Delta t\le 1$ we get

$$\begin{aligned}
\left\|\frac{\vartheta_{k+1}-\vartheta_k}{\Delta t}\right\|_H^2 &\le E^1+\frac{\Delta t}{32}\|\gamma_1\|_H^2+\frac{\Delta t}{2}\left\|\frac{\vartheta_1-\vartheta_0}{\Delta t}\right\|_H^2+\frac{\Delta t}{32}\|\gamma_k\|_H^2+\frac{\Delta t}{2}\left\|\frac{\vartheta_{k+1}-\vartheta_k}{\Delta t}\right\|_H^2+ \\
&\quad +\sum_{i=1}^{k-1}\frac{\Delta t}{32}\|\gamma_{i+1}+\gamma_i\|_H^2+\sum_{i=1}^{k-1}\frac{\Delta t}{2}\left\|\frac{\vartheta_{i+1}-\vartheta_i}{\Delta t}\right\|_H^2 \\
&\le E^1+\frac{\Delta t}{32}\|\gamma_1\|_H^2+\frac{\Delta t}{2}\left\|\frac{\vartheta_1-\vartheta_0}{\Delta t}\right\|_H^2+\frac{\Delta t}{32}\|\gamma_k\|_H^2+\frac{1}{2}\left\|\frac{\vartheta_{k+1}-\vartheta_k}{\Delta t}\right\|_H^2+ \\
&\quad +\sum_{i=1}^{k-1}\frac{\Delta t}{32}\|\gamma_{i+1}+\gamma_i\|_H^2+\sum_{i=1}^{k-1}\frac{\Delta t}{2}\left\|\frac{\vartheta_{i+1}-\vartheta_i}{\Delta t}\right\|_H^2.
\end{aligned}$$

This yields

$$\begin{aligned}
\left\|\frac{\vartheta_{k+1}-\vartheta_k}{\Delta t}\right\|_H^2 &\le 2\cdot E^1+\frac{\Delta t}{16}\|\gamma_1\|_H^2+\Delta t\left\|\frac{\vartheta_1-\vartheta_0}{\Delta t}\right\|_H^2+\frac{\Delta t}{16}\|\gamma_k\|_H^2+ \\
&\quad +\sum_{i=1}^{k-1}\frac{\Delta t}{16}\|\gamma_{i+1}+\gamma_i\|_H^2+\sum_{i=1}^{k-1}\Delta t\left\|\frac{\vartheta_{i+1}-\vartheta_i}{\Delta t}\right\|_H^2 \\
&\le 2\cdot E^1+\Delta t\left\|\frac{\vartheta_1-\vartheta_0}{\Delta t}\right\|_H^2+\sum_{i=1}^{k}\frac{\Delta t}{4}\|\gamma_i\|_H^2+\sum_{i=1}^{k-1}\Delta t\left\|\frac{\vartheta_{i+1}-\vartheta_i}{\Delta t}\right\|_H^2
\end{aligned}$$

We use the discrete Gronwall lemma [1], which yields

$$\begin{aligned}
&\sum_{i=1}^{k} \Delta t \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 \leq \\
&\leq \; (1+\Delta t)^k \sum_{i=1}^{k} (1+\Delta t)^{-i} \left(2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \sum_{j=1}^{i-1} \frac{\Delta t}{4} \left\| \gamma_j \right\|_H^2 \right) \\
&\leq \; e^T \cdot \left(2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 \right) + \sum_{i=1}^{k} \frac{\Delta t}{4} \left\| \gamma_i \right\|_H^2 + \sum_{i=2}^{k} \sum_{j=1}^{i-1} \frac{\Delta t}{4} \left\| \gamma_j \right\|_H^2 \\
&\leq \; e^T \cdot \left(2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 \right) + \sum_{i=1}^{k} \frac{\Delta t}{4} \left\| \gamma_i \right\|_H^2 + \frac{\Delta t}{4} \sum_{i=2}^{k} (k-i) \left\| \gamma_i \right\|_H^2 \\
&\leq \; e^T \cdot \left(2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{4} \sum_{i=1}^{k} \left\| \gamma_i \right\|_H^2 \right).
\end{aligned}$$

Therefore,

$$\Delta t \sum_{i=1}^{m} \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 \leq e^T \cdot \left(2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{4} \sum_{i=1}^{m} \left\| \gamma_i \right\|_H^2 \right),$$

which depends only on the initial conditions ϑ_0 and $\dot{\vartheta}_0$ and on the sequence (γ_k). Further, we have

$$\begin{aligned}
\left\| \vartheta_{k+1} \right\|_H^2 \; &\leq \; \left\| \vartheta_1 + \sum_{i=1}^{k-1} (\vartheta_{i+1} - \vartheta_i) \right\|_H^2 \\
&\leq \; 2 \left\| \vartheta_1 \right\|_H^2 + 2k \sum_{i=1}^{k} \left\| \vartheta_{i+1} - \vartheta_i \right\|_H^2 \\
&\leq \; 2 \left\| \vartheta_1 \right\|_H^2 + 2m\Delta t \sum_{i=1}^{m} \left\| \frac{\vartheta_{i+1} - \vartheta_i}{\Delta t} \right\|_H^2 \\
&\leq \; 2 \left\| \vartheta_1 \right\|_H^2 + 2Te^T \left(2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{4} \sum_{i=1}^{m} \left\| \gamma_i \right\|_H^2 \right),
\end{aligned}$$

which yields for the averaged sum

$$\frac{1}{m} \sum_{k=0}^{m} \left\| \vartheta_k \right\|_H^2 \; \leq \; 2 \left\| \vartheta_1 \right\|_H^2 + 2Te^T \left(2 \cdot E^1 + \Delta t \left\| \frac{\vartheta_1 - \vartheta_0}{\Delta t} \right\|_H^2 + \frac{\Delta t}{4} \sum_{i=1}^{m} \left\| \gamma_i \right\|_H^2 \right).$$

In the following we construct a bound for the right hand side terms which are dominated by the sum over $\|\gamma_k\|_H^2$. Again, this sequence $\|\gamma_k\|_H^2 = \|v_{k+1} + 2v_k + v_{k-1}\|_H^2$ is separated into two terms, a "projection" and a "discretisation" part:

$$
\begin{aligned}
v_k &= \ddot{x}(t_k) - \partial\partial P^l x(t_k) + D\left(\dot{x}(t_k) - \partial P^l x(t_k)\right) \\
&= \underbrace{\ddot{x}(t_k) - \partial\partial x(t_k)}_{=:w_k} + \underbrace{\partial\partial x(t_k) - \partial\partial P^l x(t_k)}_{=:z_k} + D\left(\underbrace{\dot{x}(t_k) - \partial x(t_k)}_{=:\widetilde{w}_k} + \underbrace{\partial x(t_k) - \partial P^l x(t_k)}_{=:\widetilde{z}_k}\right),
\end{aligned}
$$

yielding finally

$$
\begin{aligned}
\|\gamma_k\|_H^2 \;\le\; & 4\,\|w_{k+1} + 2w_k + w_{k-1}\|_H^2 + 4\,\|z_{k+1} + 2z_k + z_{k-1}\|_H^2 + \\
& + 4\,\|\widetilde{w}_{k+1} + 2\widetilde{w}_k + \widetilde{w}_{k-1}\|_H^2 + 4\,\|\widetilde{z}_{k+1} + 2\widetilde{z}_k + \widetilde{z}_{k-1}\|_H^2 .
\end{aligned}
$$

Due to Taylor's theorem we have

$$
\begin{aligned}
& \|w_{k+1} + 2w_k + w_{k-1}\|_H^2 \\
&= \|\ddot{x}(t_{k+1}) + 2\ddot{x}(t_k) + \ddot{x}(t_{k-1}) - (\partial\partial x(t_{k+1}) + 2\partial\partial x(t_k) + \partial\partial x(t_{k-1}))\|_H^2 \\
&= \left\|\ddot{x}(t_{k+1}) + 2\ddot{x}(t_k) + \ddot{x}(t_{k-1}) - \frac{4}{\Delta t^2}\left(x(t_{k+1}) - 2x(t_k) + x(t_{k-1})\right)\right\|_H^2 \\
&\le K\Delta t^4,
\end{aligned}
$$

where K is independent of Δt, m and l, which leads to:

$$
\sum_{k=1}^{m-1} \|w_{k+1} + 2w_k + w_{k-1}\|_H^2 \;\le\; K\Delta t^3.
$$

Accordingly, we find for $\widetilde{w}$

$$
\sum_{k=1}^{m-1} \|\widetilde{w}_{k+1} + 2\widetilde{w}_k + \widetilde{w}_{k-1}\|_H^2 \;\le\; K\Delta t^3,
$$

where $K > 0$ is independent of Δt, m and l. The estimates for $z_k = \partial\partial x(t_k) - \partial\partial P^l x(t_k)$ and $\widetilde{z}_k = \partial x(t_k) - \partial P^l x(t_k)$ depend on the particular choice of the POD subspace, see equations (20), (21), resp. (23) and (24) : For case I we have

$$
\sum_{k=1}^{m-1} \|z_{k+1} + 2z_k + z_{k-1}\|_X^2 \;\le\; \frac{24}{\Delta t^4}(m+1) \sum_{j=l+1}^{d} \lambda_{Ij}
$$

$$
\text{and} \qquad \sum_{k=1}^{m-1} \|\widetilde{z}_{k+1} + 2\widetilde{z}_k + \widetilde{z}_{k-1}\|_X^2 \;\le\; \frac{16}{\Delta t^2}(m+1) \sum_{j=l+1}^{d} \lambda_{Ij}.
$$

For case II we get

$$\sum_{k=1}^{m-1} \|z_{k+1} + 2z_k + z_{k-1}\|_X^2 \;\leq\; 72m \cdot \sum_{j=l+1}^{d} \lambda_{IIj}$$

$$\text{and} \qquad \sum_{k=1}^{m-1} \|\widetilde{z}_{k+1} + 2\widetilde{z}_k + \widetilde{z}_{k-1}\|_X^2 \;\leq\; 72m \cdot \sum_{j=l+1}^{d} \lambda_{IIj}.$$

Combining the estimates for z_k , $\widetilde{z}_k$, w_k and $\widetilde{w}_k$ we have

$$\begin{aligned} \|\gamma_k\|_H^2 &= \|v_{k+1} + 2v_k + v_{k-1}\|_H^2 \\ &\leq 4\|w_{k+1} + 2w_k + w_{k-1}\|_H^2 + 4\|z_{k+1} + 2z_k + z_{k-1}\|_H^2 + \\ &\quad + 4D\|\widetilde{w}_{k+1} + 2\widetilde{w}_k + \widetilde{w}_{k-1}\|_H^2 + 4D\|\widetilde{z}_{k+1} + 2\widetilde{z}_k + \widetilde{z}_{k-1}\|_H^2 . \end{aligned}$$

We get for case I:

$$\sum_{i=1}^{m} \|\gamma_i\|_H^2 \leq K_1 \Delta t^2 + K_2 \frac{m+1}{\Delta t^4} \sum_{j=l+1}^{d} \lambda_{Ij}$$

and for case II:

$$\sum_{i=1}^{m} \|\gamma_i\|_H^2 \leq K_3 \Delta t^3 + K_4\, m \sum_{j=l+1}^{d} \lambda_{IIj}.$$

In conclusion, the error estimate for case I can be written as:

$$\begin{aligned} &\frac{1}{m}\sum_{k=1}^{m} \left\|X^k - x(t_k)\right\|_H^2 \leq \\ &\leq\; 4\sum_{j=l+1}^{d} \lambda_{Ij} + 4\|\vartheta_1\|_H^2 + \\ &\quad + 4Te^T\left(2\cdot E^1 + \Delta t\left\|\frac{\vartheta_1 - \vartheta_0}{\Delta t}\right\|_H^2 + K\Delta t^4 + \left(\frac{12}{\Delta t^3} + \frac{4D}{\Delta t}\right)(m+1)\sum_{j=l+1}^{d} \lambda_{Ij}\right). \end{aligned}$$

The term E^1 contains the expression $(\vartheta_1 - \vartheta_0)/\Delta t$ which can be regarded as an extended initial condition for the velocities $(\partial\vartheta_1 + \partial\vartheta_0)/2$ due to Newmark's scheme (5a). Hence, we get

$$\begin{aligned} &\frac{1}{m}\sum_{k=1}^{m} \left\|X^k - x(t_k)\right\|_H^2 \leq \\ &\leq\; C_I\Big(\left\|X^0 - P^l x(t_0)\right\|_H^2 + \left\|X^1 - P^l x(t_1)\right\|_H^2 + \Delta t\left\|\partial X^0 - P^l \dot{x}(t_0)\right\|_H^2 \\ &\quad + \Delta t\left\|\partial X^1 - P^l \dot{x}(t_1)\right\|_H^2 + \Delta t^4 + \left(\frac{1}{\Delta t^4} + \frac{1}{\Delta t} + 1\right)\sum_{j=l+1}^{d} \lambda_{Ij}\Big), \end{aligned}$$

with C_I independent of Δt and m.
Case II yields the following estimate:

$$\begin{aligned}
&\frac{1}{m}\sum_{k=1}^{m}\left\|X^k - x(t_k)\right\|_H^2 \leq \\
&\leq \; 6\sum_{j=l+1}^{d}\lambda_{IIj} + 4\left\|\vartheta_1\right\|_H^2 + \\
&\qquad + 4Te^T\left(2\cdot E^1 + \Delta t\left\|\frac{\vartheta_1-\vartheta_0}{\Delta t}\right\|_H^2 + K\Delta t^4 + 36T(1+D)\cdot\sum_{j=l+1}^{d}\lambda_{IIj}\right),
\end{aligned}$$

which can similarly be interpreted as

$$\begin{aligned}
&\frac{1}{m}\sum_{k=1}^{m}\left\|X^k - x(t_k)\right\|_H^2 \leq \\
&\leq \; C_{II}\left(\left\|X^0 - P^l x(t_0)\right\|_H^2 + \left\|X^1 - P^l x(t_1)\right\|_H^2 + \Delta t\left\|\partial X^0 - P^l\dot{x}(t_0)\right\|_H^2\right. \\
&\qquad \left. + \Delta t\left\|\partial X^1 - P^l\dot{x}(t_1)\right\|_H^2 + \Delta t^4 + \sum_{j=l+1}^{d}\lambda_{IIj}\right), \qquad (27)
\end{aligned}$$

where C_{II} is independent of Δt and m.

■

In both cases, we find terms that are independent of the time step Δt. Both cases also contain terms that depend on Δt in the numerator. These terms vanish as Δt goes to zero. In case I, which only uses the deformation snapshots, the error estimate additionally contains a term that carries Δt in the denominator. For this particular choice of the POD subspace the error bound tends to infinity with $\Delta t \to 0$. This means that convergence cannot be assured formally, if a snapshot set consisting of deformations only is used. If velocities and accelerations are added into the set, convergence can be deduced from (27).

4 Numerical results

For a numerical comparison of the different POD techniques discussed above, a simple test model was set up in MATLAB. The example shows a one-dimensional linear wave equation on the interval $\Omega = (0, L)$ with homogeneous Dirichlet boundary conditions, which can be regarded as a vibrating string fixed at both ends.
Mathematically, our model problem is described by the following initial-boundary value

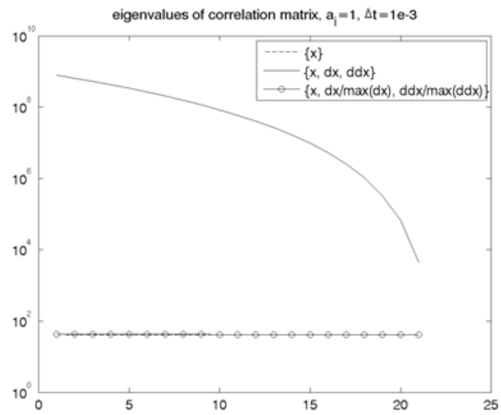


Figure 1: Decay of eigenvalues of the snapshot correlation matrix, for $a_i = 1$

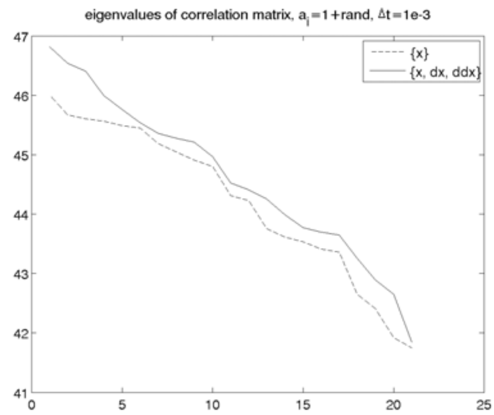


Figure 2: Decay of eigenvalues of the snapshot correlation matrix, for $a_i = 1 + rand$

problem:

$$\begin{aligned}
\mu \cdot \ddot{x}(s,t) - S \cdot x''(s,t) &= f(s,t) \quad \text{in } (0,L) \times (0,T), && \text{(28a)}\\
x(s,0) &= x_0 \quad \text{in } (0,L), && \text{(28b)}\\
\dot{x}(s,0) &= \dot{x}_0 \quad \text{in } (0,L), && \text{(28c)}\\
x(s,t) &= 0 \quad \text{on } \partial\Omega = \{0, L\} \quad \text{for all } t \in (0,T). && \text{(28d)}
\end{aligned}$$

We chose $L = 1$, $S = 1$, $\mu = 1$ and $T = 2$ and the initial deformation x_0 is a weighted sum of sinus shapes

$$x_0 = x(t_0) = \sum_{i=1}^{n} a_i \cdot sin\left(i \cdot \pi \frac{s}{L}\right),$$

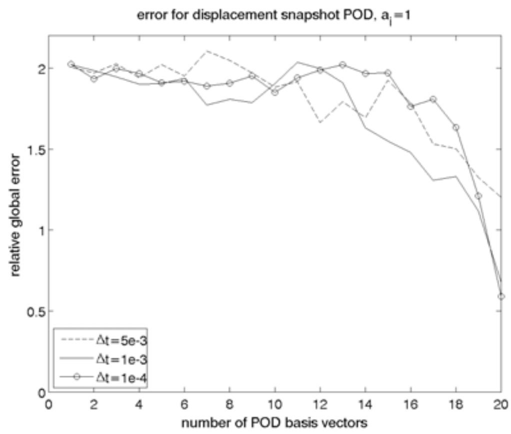
with weights $a_i \in \mathbb{R}$.
Furthermore, we set the external force $f(t)$ to zero, yielding the analytical solution and its derivatives:

$$x(s,t) = \sum_{i=1}^{n} a_i \cdot sin\left(i\pi \frac{s}{L}\right) \cdot cos\left(i\pi \frac{c}{L} t\right), \tag{29}$$

$$\dot{x}(s,t) = \sum_{i=1}^{n} -a_i \cdot sin\left(i\pi \frac{s}{L}\right) \cdot sin\left(i\pi \frac{c}{L} t\right) \cdot i\pi \frac{c}{L}, \tag{30}$$

$$\ddot{x}(s,t) = \sum_{i=1}^{n} -a_i \cdot sin\left(i\pi \frac{s}{L}\right) \cdot cos\left(i\pi \frac{c}{L} t\right) \cdot i^2 \pi^2 \frac{c^2}{L^2}, \quad \text{with} \quad c = \sqrt{\frac{S}{\mu}}. \tag{31}$$

The POD method was realized using snapshots at $m + 1$ uniformly distributed points in time. To observe the error behaviour with decreasing time step, we investigate three



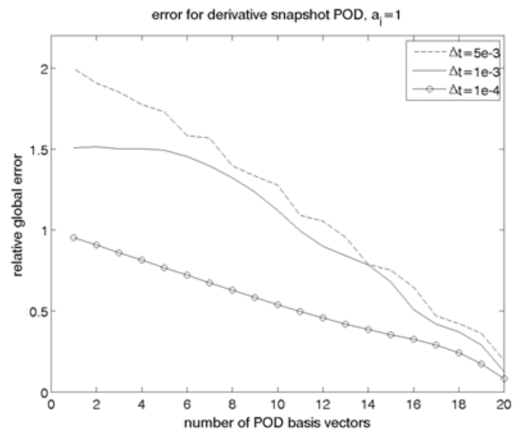


Figure 3: Error norms for deformation snapshot set, $a_i = 1$

Figure 4: Error norms for derivative snapshot set, $a_i = 1$

different step sizes dividing the interval into $m = 400$, 2000 and 20000 sub intervals. With this setup, we use Newmark's method for the time integration. The spatial discretisation is done by a linear Finite Element approach consisting of 500 elements.
For the case of only including the displacements into the POD set, the snapshots are simply $\{x(t_k)\}_{k=0,\dots,m}$. In the second case, where deformations, velocities and accelerations were taken into account, the snapshot set was built using the resulting quantities from the analytical solution: $x(t_0), \dots, x(t_m), \dot{x}(t_0), \dots, \dot{x}(t_m), \ddot{x}(t_0), \dots, \ddot{x}(t_m)$.

Remark 4 *Note that when using unweighted snapshots, the eigenvalues of the derivative set differ from the ones of the deformation set by a factor of* 10^8 *(Figure 1). This observation originates from the fact that the velocities and accelerations are about* 2*, respectively* 3*, orders of magnitude larger than the deformations. This difference leads not only to large eigenvalues but also to an overrating of the derivatives in the correlation matrix. For this reason, the derivative snapshots were divided by the respective maximum over space and time.*

Furthermore, to investigate the influence of the eigenvalues of the correlation matrix, we compare two different initial conditions: one consisting of uniformly weighted sinus shapes ($a_i = 1$), and one additionally containing small random numbers ($a_i = 1 + rand$, $max(rand) = 0.05$). The former yields a nearly constant distribution of eigenvalues up to the dimension of a, whereas the eigenvalues for the latter set decay linearly (Figures 1 and 2). Note that for problems including damping, the eigenvalues usually decay exponentially.
Figures 3 and 4 compare the norms of the relative global errors for the case $a_i = 1$. In this setup, the classical snapshot POD method shows no improvement with decreasing time step size whereas the one which uses derivative snapshots performs significantly better.
In the case of a random distribution of sinus weights, both methods show a diminishing

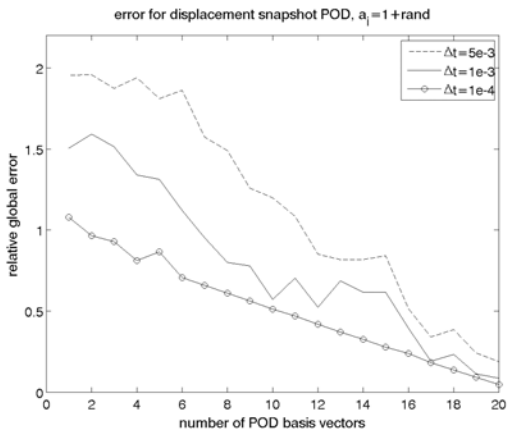


Figure 5: Error norms for deformation snapshot set, $a_i = 1 + rand$

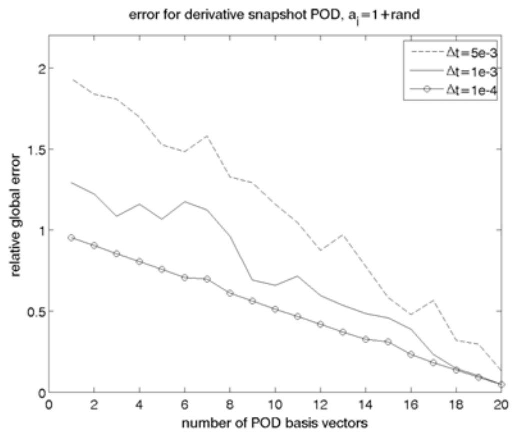


Figure 6: Error norms for derivative snapshot set, $a_i = 1 + rand$

error for smaller time steps (Figures 5 and 6). One possible reason is the influence of the eigenvalue decay on the error norm which dominates the error in this case.

Remark 5 *Note that in all cases mentioned above the absolute values of the error norms are still high (> 10%). The dimension of the model corresponds to the dimension of a (here: $dim(a) = 21$). As soon as a larger number of POD vectors is used, the error drops instantly. This behaviour is also seen in the eigenvalue distribution, yielding $\lambda_i = 0$ for $i > dim(a)$. Therefore, a setup with such weighting of modes actually forces the user to work with all occurring eigenvalues as every neglected basis vector still has a considerable influence on the solution. In this case, dimension reduction is risky and the example shall only be seen as a constructed model to demonstrate the error behaviour.*

As a second example, we use a non-smooth initial condition u_0 (Figure 7) on the same setup as above and compare the POD methods with the classical eigenmode method frequently used for linear systems. Furthermore, we set the damping factor $d = 10$.
In the case of high damping, we get an exponential decay of eigenvalues (Figure 8). A fast eigenvalue decay leads to a small error in subspace approximation of the snapshot set (see (10)). This yields a better condition for the POD method than in the example above.
Figure 9 shows a comparison of the relative global errors of both POD and the eigenmode methods. The errors are computed in the H-norm $\frac{1}{m}\sum_{k=1}^{m}\left\|X^k - x(t_k)\right\|_H^2$. In this case, the derivative POD method performs slightly worse than the classical one. The errors for the eigenmode method range between the ones of both POD methods.
If we measure the error in the V-norm, we find that both POD methods perform better than the eigenmodes (Figure 10).

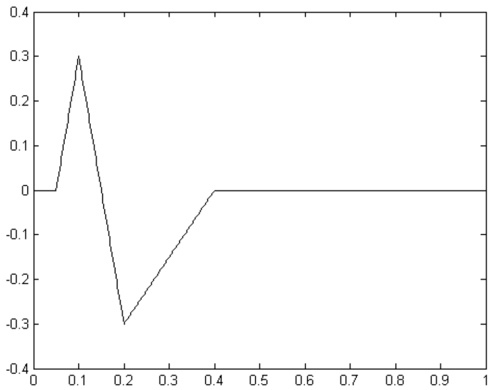


Figure 7: Initial condition for the linear wave equation

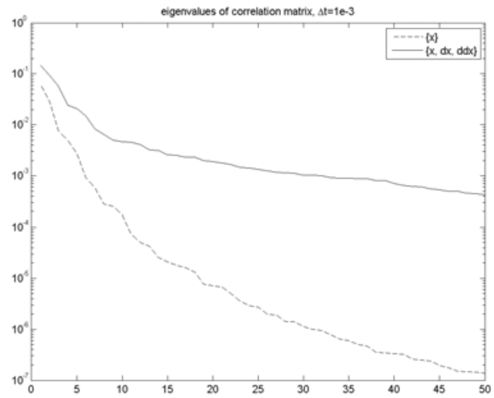


Figure 8: Decay of eigenvalues for the POD snapshot sets, $\Delta t = 10^{-3}$

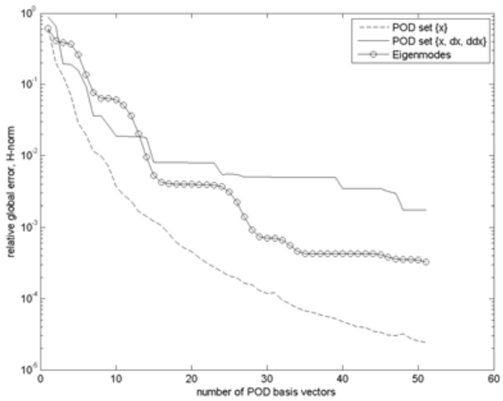


Figure 9: Error norms for POD and eigenmode analysis, H-norm, $\Delta t = 10^{-3}$

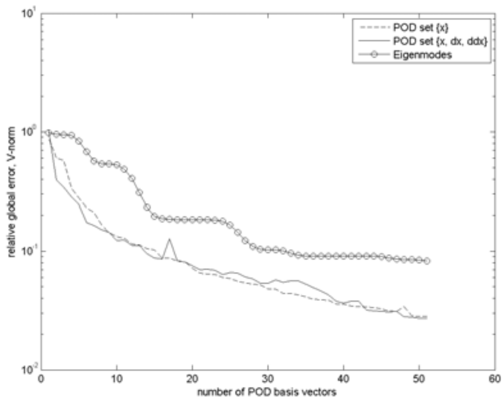


Figure 10: Error norms for POD and eigenmode analysis, V-norm, $\Delta t = 10^{-3}$

5 Conclusion

We study the POD method for the linear wave equation comparing two different choices of snapshot sets. Set I consists of deformation snapshots, and set II additionally contains velocities and accelerations. As for parabolic problems, there is no convergence guarantee for simple deformation snapshots. Only the incorporation of additional derivative snapshots yields an error bound which is diminishing for small time steps.

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