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THE LENGTH OF THE PRIMAL-DUAL PATH IN MOREAU-YOSIDA-BASED PATH-FOLLOWING FOR STATE CONSTRAINED OPTIMAL CONTROL

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Abstract. A priori estimates on the length of the primal-dual path that results from a Moreau-Yosida approximation of the feasible set for state constrained optimal control problems are derived. These bounds depend on the regularity of the state and the dimension of the problem. Numerical results indicate that the bounds are indeed sharp and are typically attained in cases where the active set consists of isolated active points. Further conditions on the multiplier approximation are identified which guarantee higher convergence rates for the feasibility violation due to the Moreau-Yosida approximation process. Numerical experiments show again that the results are sharp and accurately predict the convergence behavior.

Key words. Moreau-Yosida regularization, PDE constrained optimization, path-following, pointwise state constraints, regularization error

AMS subject classifications. 49M30, 49J52, 49J20

1. Introduction. In recent years, path-following methods based on the Moreau-Yosida regularization of state constrained problems have received considerable attention. While general results on the convergence of this method can be derived under fairly mild assumptions, deriving estimates on the length of corresponding homotopy path, the "primal-dual path", and its asymptotic behavior is more delicate. In particular, numerical experience shows that the asymptotic behavior varies from problem to problem. This has lead to the consideration of a posteriori error estimators for the regularization error; see, e.g., [11, 14].

The purpose of this note is twofold. First, we present a priori error estimates on the order of convergence of the primal-dual path that depend on the dimension of the problem and the smoothness of the solution. In comparison to the estimates that were derived in [13] we obtain an improvement in the rate, compared to [6] we can also improve upon the obtained convergence rate. We note that the techniques of [6] when augmented by a bootstrapping argument also have the potential to yield the same rate as ours; see Section 2.5 below. However, we point out that our results are based on considerably weaker assumptions than those in the aforementioned work. Finally, we note that in some cases one may derive our worst case estimates also from the work contained in [4].

Secondly, we develop an understanding of the principles that govern the rate of convergence of the primal-dual path. This will be accomplished by a comparison of numerical and theoretical results. In fact, it turns out that the topology of the active set plays a decisive role for the rate of convergence. In the "worst case" scenario, which is the case of the active set being a single touch point (or a set of isolated touch points), our theoretical estimates coincide with numerical observations. Moreover, we are able to predict the convergence behavior in several prototypical cases depending on the shape of the active set.

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In order to render our discussion concrete, we consider the primal-dual path-following method for a state constrained model problem in optimal control. We emphasize, however, that the techniques presented here are applicable in a much broader context; see, e.g., [15] for constraints on the gradient of the state. Thus, for our specific discussion we throughout consider the model problem (1.1) below. Applying the Moreau-Yosida approximation to the indicator function of the set related to the pointwise inequality constraint in

$$
\begin{aligned}
&\text{minimize} \quad J(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} \quad \text{over } y\in H^1(\Omega), u\in L^2(\Omega)\\
&\text{subject to} \quad -\Delta y - u = 0 \text{ in } \Omega, \quad y=0 \text{ on } \partial\Omega,\\
&\qquad\qquad y \le \psi \text{ in } \Omega,
\end{aligned}
\tag{1.1}
$$

results in the family of problems

$$
\begin{aligned}
&\text{minimize} \quad J^\gamma(y,u) = \frac{1}{2}\|y-y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|\max(y-\psi,0)\|^2_{L^2(\Omega)}\\
&\qquad\qquad \text{over } y\in H^1(\Omega), u\in L^2(\Omega)\\
&\text{subject to} \quad -\Delta y - u = 0 \text{ in } \Omega, \ y=0 \text{ on } \partial\Omega
\end{aligned}
\tag{1.2}
$$

for $\gamma > 0$. Above, we assume for simplicity that Ω is a smoothly bounded domain in $\mathbb{R}^d$ for $d = 1,2,3$, $y_d \in L_2(\Omega)$, and ψ is a smooth, strictly positive function on $\overline{\Omega}$ and $\alpha > 0$ some given constant. Let $x_\gamma := (y_\gamma, u_\gamma)$ denote the unique solution of (1.2). It was shown in [7] that the sequence (x_γ) converges to the original solution $x_* := (y_*, u_*)$ of (1.1) as γ tends to infinity. Given the structure of our model problem, we again remark that the following analysis does not depend on the special form of the (partial differential) equation or the inequality constraint in (1.1), but rather on the fact that the feasibility violation, i.e., $\max(y-\psi,0)$ in our case, is not only in $L^2(\Omega)$ but in fact it is in some space $C^\beta(\overline{\Omega})$ with $0 < \beta \le 2$.

Using slightly non-standard notation we define the space C^β as follows. Let $m \in \mathbb{N}_0$. For $m < \beta \le m+1$ denote by $C^\beta(\overline{\Omega})$ the subspace of $C^m(\overline{\Omega})$ of functions which have Hölder continuous derivatives of order up to m. These spaces are equipped with the usual norms

$$
\|v\|_{C^\beta} := \|v\|_{C^m} + \|\nabla^{(m)} v\|_{C^{\beta-m}}.
$$

Practical algorithms use a semi-smooth Newton method to solve discretizations of the problems (1.2) approximately or exactly; see, e.g., [7, 8]. For this purpose, the first order necessary conditions are derived for (1.2) which assert the existence of an adjoint state $p_\gamma \in H^2(\Omega)$ such that

$$
y_\gamma - y_d + \gamma\max(y_\gamma-\psi,0) - \Delta p_\gamma = 0 \text{ in } \Omega, \quad p_\gamma = 0 \text{ on } \partial\Omega, \tag{1.3}
$$
$$
\alpha u_\gamma - p_\gamma = 0 \text{ in } \Omega, \tag{1.4}
$$
$$
-\Delta y_\gamma - u_\gamma = 0 \text{ in } \Omega, \quad y_\gamma = 0 \text{ on } \partial\Omega. \tag{1.5}
$$

The system (1.3)–(1.5) approximates the first order necessary (and in our case also sufficient) conditions for the original problem (1.1), which yield the existence of a measure-valued Lagrange multiplier $m_* \in \mathcal{M}(\overline{\Omega})$ and an adjoint state $p_* \in W^{1,q'}(\Omega)$ ($q' < d/(d-1)$) such that

$$
\begin{aligned}
y_* - y_d + m_* - \Delta p_* &= 0 \text{ in } \Omega, \quad p_* = 0 \text{ on } \partial\Omega,\\
-\Delta y_* - u_* &= 0 \text{ in } \Omega, \quad y_* = 0 \text{ on } \partial\Omega,\\
\alpha u_* - p_* &= 0 \text{ in } \Omega,\\
m_* \geq 0, \quad y_* \leq \psi, \quad \langle m_*, y_* - \psi\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})} &= 0 \text{ in } \overline{\Omega},
\end{aligned}
$$

where $\langle\cdot,\cdot\rangle_{\mathcal{M}(\overline{\Omega}),C(\overline{\Omega})}$ denotes the duality pairing between $C(\overline{\Omega})$ and its dual space $\mathcal{M}(\overline{\Omega})$, the regular Borel measures. We observe that, in the regularized setting, the function $\gamma\max(y_\gamma - \psi, 0)$ plays the role of m_*.

The elimination of u_γ from (1.3)–(1.5) yields the system

$$
F(x;\gamma) := \left\{ \begin{array}{rcll} y - y_d + \gamma\max(y-\psi,0) - \Delta p &=& 0 \text{ in } \Omega, & p = 0 \text{ on } \partial\Omega,\\ -\Delta y - \alpha^{-1} p &=& 0 \text{ in } \Omega, & y = 0 \text{ on } \partial\Omega, \end{array}\right. \tag{1.6}
$$

which can be tackled by a semi-smooth Newton method as shown in [7].

2. Analysis of the length of the primal-dual path. Our analysis proceeds in three main steps. First, we derive uniform L^1-bounds on the constraint violation. In a second step we conclude that, given these L^1-bounds, the length of the primal-dual path depends on L^∞-bounds on the constraint violation. Then, finally, these latter bounds are derived by exploiting the Hölder continuity of the states.

2.1. A priori bounds for the constraint violation in L^1. Abusing notation, in what follows we write y_γ^+ for $\max(y_\gamma - \psi, 0)$. Our first aim is to show that $\gamma\left\|y_\gamma^+\right\|_{L^1}$ is bounded uniformly as $\gamma\to\infty$. The following technique is well established by now and has been used in various contexts (see, e.g., [5, 8, 12]). Below, c denotes a generic constant which may take different values at different occasions.

LEMMA 2.1. *The expression $\gamma\left\|y_\gamma^+\right\|_{L^1}$ is uniformly bounded for $\gamma\to\infty$.*

Proof. Let S be the solution operator, i.e., the control-to-state mapping, of the state equation which is the partial differential equation constraint in (1.2). We test (1.3) and (1.4) by a feasible direction $(Sv, v) \in H^1(\Omega)\times L^2(\Omega)$ and add the resulting two equations up (taking into account that $\langle -\Delta p_\gamma, Sv\rangle = (p_\gamma, v)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^1(\Omega)$ and its dual $H^1(\Omega)^*$ and $(\cdot,\cdot)_{L^2}$ is the usual inner product in $L^2(\Omega)$). As a consequence, we obtain

$$
\alpha(u_\gamma, v)_{L^2} + (y_\gamma - y_d, Sv)_{L^2} + \gamma(y_\gamma^+, Sv)_{L^2} = 0 \qquad \forall v\in L^2(\Omega).
$$

Inserting $v := u_\gamma, Su_\gamma = y_\gamma$ we obtain

$$
\alpha(u_\gamma, u_\gamma)_{L^2} + (y_\gamma - y_d, y_\gamma)_{L^2} - \gamma(y_\gamma^+, y_\gamma)_{L^2} = 0.
$$

From (1.2) one readily finds that u_γ, y_γ, y_d are bounded in L^2 independently of γ. Thus, we conclude that

$$
\gamma(y_\gamma^+, y_\gamma)_{L^2} = \gamma\int_{\overline{\Omega}} y_\gamma y_\gamma^+\, d\omega \leq c
$$

for some constant $c > 0$ independent of γ. Hence, by the non-negativity of y_γ^+ and $0 < \underline{\psi} \leq \psi \leq y_\gamma$, for some $\underline{\psi} > 0$, on $\{y_\gamma^+ \neq 0\}$ we get

$$
\gamma\left\|y_\gamma^+\right\|_{L^1} \leq \gamma\underline{\psi}(y_\gamma^+, y_\gamma)_{L^2} \leq c.
$$

This concludes the proof. □

One may wonder which exponents $s > 0$ one might expected for estimates of the form $\gamma^s \left\|y_\gamma^+\right\|_{L^q} \leq c$ in generic situations. Now assume that such an estimate would hold true. If $s \geq 1$, then we find that

$$\gamma \left\|y_\gamma^+\right\|_{L^q} \leq c.$$

Hence, γy_γ^+ is at least bounded in $L^q(\Omega)$ and converges to 0 in $L^q(\Omega)$. If $q > 1$, then one infers for the original problem that a Lagrange multiplier m_* exists in $L^q(\Omega)$. This is the contents of the following proposition.

PROPOSITION 2.2. *Assume that* $\left\|y_\gamma^+\right\|_{L^q} \leq c\gamma^{-1}$ *holds true for some* $1 < q < \infty$. *Then there exists a Lagrange multiplier for the pointwise inequality constraints on the state in problem* (1.1) *which is an element of* $L^q(\Omega)$.

Proof. Since $\gamma \left\|y_\gamma^+\right\|_{L^q} \leq c$, the function γy_γ^+ has a weak accumulation point $m_* \in L^q(\Omega)$, which is positive due to weak closedness of the positive cone in $L^q(\Omega)$. Moreover,

$$(m_*, y_*)_{L^2} = \lim_{\gamma\to\infty} (\gamma y_\gamma^+, y_*)_{L^2} = 0$$

by strong convergence of $y_\gamma \to y_* \leq \psi$ in $H^1(\Omega)$; see [8] for the latter. From the adjoint equation (1.3) one readily infers uniform boundedness of (p_γ) in $H^1(\Omega)$. Then, by compactness, p_γ converges strongly in $L^2(\Omega)$ to some p_* as $\gamma \to \infty$. Consequently, $u_\gamma \to u_*$ in $L^2(\Omega)$, as well, and (1.4) is satisfied in the limit. Hence, m_* is the Lagrange multiplier for the pointwise inequality constraints on the state. □

In general, one observes that the multipliers m_* are measures only. Thus, the case $s \geq 1$ for $q > 1$ only appears in exceptionally regular situations with respect to Lagrange multipliers (or completely inactive state inequality constraints). Hence, we generically expect $s < 1$ for $q > 1$. Consequently, the result of Lemma 2.1 appears to be optimal.

2.2. Estimates depending on the constraint violation in L^∞. We approach our aim via the value functional

$$V(\gamma) := J^\gamma(x_\gamma) = J(x_\gamma) + \frac{\gamma}{2}\left\|y_\gamma^+\right\|_{L^2}^2.$$

It was shown in [7] that $\lim_{\gamma\to\infty} V(\gamma) = J(x_*)$. Here we show that the rate of convergence depends on $\|y_\gamma^+\|_\infty$.

THEOREM 2.3. *The value functional* V *is differentiable and the following estimate for the derivative holds true:*

$$0 \leq \frac{d}{d\gamma} V(\gamma) \leq \frac{c}{\gamma}\left\|y_\gamma^+\right\|_{L^\infty}. \tag{2.1}$$

If $\left\|y_\gamma^+\right\|_{L^\infty} \leq c\gamma^{-s}$ *for some* $s > 0$, *then one has*

$$0 \leq J(x_*) - V(\gamma) \leq c\gamma^{-s}, \tag{2.2}$$

and further

$$\sqrt{\alpha}\left\|u_* - u_\gamma\right\|_{L^2} \leq c\gamma^{-s/2}. \tag{2.3}$$

Proof. In [7, Proposition 4.1] differentiability of V was shown and the expression

$$\frac{d}{d\gamma}V(\gamma) = \frac{1}{2}\left\|y_\gamma^+\right\|_{L^2}^2$$

was derived. Hence V is monotonically increasing, and from the estimate $\|v\|_{L^2}^2 \leq \|v\|_{L^1}\|v\|_{L^\infty}$, it follows that

$$0 \leq \frac{d}{d\gamma}V(\gamma) \leq \left\|y_\gamma^+\right\|_{L^1}\left\|y_\gamma^+\right\|_{L^\infty}.$$

Due to Lemma 2.1 we have that $\gamma\left\|y_\gamma^+\right\|_{L^1} \leq c$. Hence, (2.1) follows.

Now assume that $\left\|y_\gamma^+\right\|_{L^\infty} \leq c\gamma^{-s}$ holds true. Then we obtain

$$0 \leq \frac{d}{d\gamma}V(\gamma) \leq c\gamma^{-1-s}$$

Let $\gamma_1 > \gamma_2$ be given. Then, from the fundamental theorem of calculus we infer

$$V(\gamma_1) - V(\gamma_2) = \int_{\gamma_2}^{\gamma_1}\frac{d}{d\gamma}V(\gamma)\,d\gamma \leq \int_{\gamma_2}^{\gamma_1}c\gamma^{-1-s}\,d\gamma = c(\gamma_2^{-s} - \gamma_1^{-s}).$$

Since $\lim_{\gamma\to\infty}V(\gamma) = J(x_*)$ this estimate yields

$$J(x_*) - V(\gamma_2) = \lim_{\gamma_1\to\infty}V(\gamma_1) - V(\gamma_2) \leq c\gamma_2^{-s},$$

which implies (2.2). Finally, (2.3) readily follows from the uniform convexity of J^γ with respect to u in $L^2(\Omega)$ and (2.2). Indeed, we have

$$\begin{aligned}\frac{\alpha}{2}\|u_* - u_\gamma\|_{L^2}^2 &\leq J^\gamma(x_*) + J^\gamma(x_\gamma) - 2J^\gamma\left(\frac{1}{2}x_* + \frac{1}{2}x_\gamma\right)\\ &\leq J^\gamma(x_*) + J^\gamma(x_\gamma) - 2J^\gamma(x_\gamma) = J^\gamma(x_*) - J^\gamma(x_\gamma)\\ &= J(x_*) - V(\gamma) \leq c\gamma^{-s},\end{aligned}$$

which concludes the proof. ∎

2.3. A worst case estimate for the constraint violation in L^∞. The bottom line of the previous section is that we have to find an estimate for $\|y_\gamma^+\|_\infty$, which is as sharp as possible. This can be achieved by exploiting the smoothness of y.

The essence of our technique is a geometric idea, which we explain by means of a simple example for illustration purposes; cf. Figure 2.1. Let $f(\omega) = a(-\omega^2 + \varepsilon^2)$ be a concave parabola, and $f^+(\omega) := \max(f(\omega), 0)$ its positive part. Then $f(\omega) \geq 0$ for $\omega \in [-\varepsilon, +\varepsilon]$ with a maximum $\|f^+\|_\infty = f(0) = a\varepsilon^2$, and $\|f^+\|_{L^1} = \int_{-\varepsilon}^{\varepsilon} f(\omega)\,d\omega = \frac{4}{3}a\varepsilon^3$. Thus, we have

$$\left\|f^+\right\|_\infty = a\varepsilon^2 = a^{1/3}\left(\frac{3}{4}\right)^{2/3}\left(\frac{4}{3}a\varepsilon^3\right)^{2/3} \leq c\,\|f\|_{C^2}^{1/3}\left\|f^+\right\|_{L^1}^{2/3}.$$

Hence, from the boundedness of the second derivatives (by $2a$) one can conclude a relation between the L^1-norm and the L^∞-norm of a function with zero boundary values. The following proposition generalizes this observation.

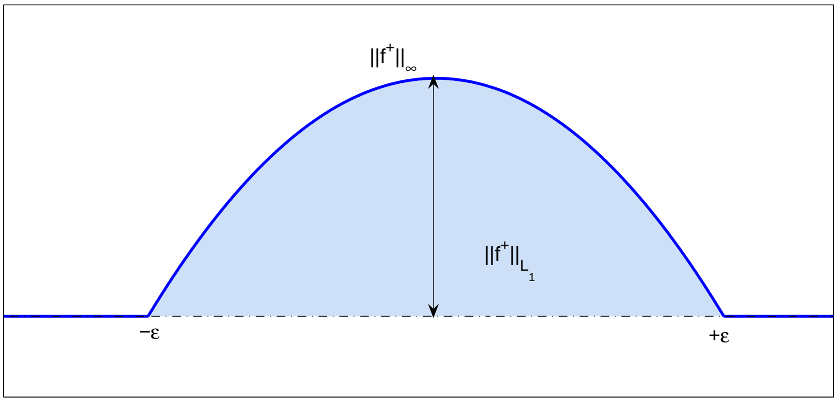

FIG. 2.1. *Geometrical idea of our considerations.*

PROPOSITION 2.4. *Let $\Omega \subset \mathbb{R}^d$ be bounded and open, $0 \le y \in C^\beta(\overline{\Omega})$, with $0 < \beta \le 2$, and $y \in L^1(\Omega)$. Moreover, assume that $y = 0$ on $\partial\Omega$. Then*

$$\|y\|_{L^\infty} \le c \, \|y\|_{C^\beta}^{1-\Theta} \, \|y\|_{L^1}^{\Theta} \tag{2.4}$$

with $\Theta = \frac{\beta}{\beta+d}$. The constant c is independent of Ω.

Proof. Without loss of generality assume that $0 \in \Omega$ and $y(0) = \|y\|_{L^\infty}$., and denote by $B_r(0)$ the ball of radius r and with center 0.

If $\beta \le 1$ then by the definition of Hölder continuity, it follows that $y(\omega) > y(0) - \|y\|_{C^\beta} \, r^\beta$ for all $\omega \in B_r(0)$. If $\beta > 1$ then y is once continuously differentiable and attains a maximum at 0. Hence, we conclude that $\nabla y(0) = 0$ for $\beta > 1$. Moreover we can compute the value at any point $\omega \in B_r(0)$ using the fundamental theorem of calculus along the line $[0, \omega]$. Using the Hölder continuity of the first derivative one then obtains $y(\omega) > y(0) - \|y\|_{C^\beta} \, r^\beta$ for all $\omega \in B_r(0)$ also in this case. In particular, $y(\omega)$ is positive for $\omega \in B_R(0)$ with

$$R = \left(\frac{y(0)}{\|y\|_{C^\beta}} \right)^{1/\beta} = \left(\frac{\|y\|_{L^\infty}}{\|y\|_{C^\beta}} \right)^{1/\beta} .$$

From the assumption $y = 0$ on $\partial\Omega$ it follows that $B_R(0) \subset \overline{\Omega}$. Hence, we can compute

$$\begin{aligned} \|y\|_{L^1} = \int_{\overline{\Omega}} |y(\omega)| \, d\omega &\ge c \int_{[0,R]} \left| y(0) - \|y\|_{C^\beta} \, r^\beta \right| r^{d-1} \, dr \\ &= c \, \|y\|_{C^\beta} \int_{[0,R]} \left(\frac{y(0)}{\|y\|_{C^\beta}} - r^\beta \right) r^{d-1} \, dr \\ &= c \, \|y\|_{C^\beta} \int_{[0,R]} \left(R^\beta - r^\beta \right) r^{d-1} \, dr \\ &\ge c \, \|y\|_{C^\beta} \, R^{\beta+d} = c \, \|y\|_{C^\beta}^{1-\frac{\beta+d}{\beta}} \, \|y\|_{L^\infty}^{\frac{\beta+d}{\beta}} . \end{aligned} \tag{2.5}$$

Solving for $\|y\|_{L^\infty}$, we obtain

$$\|y\|_{L^\infty} \le c \, \|y\|_{L^1}^{\frac{\beta}{\beta+d}} \, \|y\|_{C^\beta}^{1-\frac{\beta}{\beta+d}}$$

as stated in the assertion. ∎

REMARK 2.5. *Observe that* (2.4) *is only true for* $\beta \leq 2$*, which means that we can only use smoothness up to order* 2*. This corresponds to the fact that maximizers yield vanishing derivatives of first, but not of higher order.*

For $1 \leq \beta \leq 2$ *our argumentation is only valid due to our assumption* $y|_{\partial\Omega} = 0$*. This allows to exploit that the maximizer of* y *lies in the interior of* $\overline{\Omega}$*. If this assumption is dropped, then our result still holds for* $\beta \leq 1$ *under a cone condition on* Ω*. This is of interest when boundary conditions different from Dirichlet conditions are considered.*

A similar technique was previously used in the context of interior point methods to show positivity of the distance of the central path to the bounds [12]. Also, in [9, Lemma 4.7] similar techniques seem to be employed, at least for the case $\beta \leq 1$ for the virtual control approach considered in that work.

COROLLARY 2.6. *If* y_γ *is uniformly bounded in* $C^\beta(\overline{\Omega})$ *for* $\gamma \to \infty$ *and some* $0 < \beta \leq 2$*, we have the following estimate on the constraint violation:*

$$\|y_\gamma^+\|_\infty \leq c\gamma^{-s}, \ \text{where } s = \frac{\beta}{\beta + d}. \tag{2.6}$$

In particular, for the example problem (1.1) *we have for every* $\varepsilon > 0$*:*

$$\|y_\gamma^+\|_\infty \leq \begin{cases} c\gamma^{-2/3} & : \ d = 1, \\ c\gamma^{-1/2+\varepsilon} & : \ d = 2, \\ c\gamma^{-1/4+\varepsilon} & : \ d = 3. \end{cases} \tag{2.7}$$

Proof. The result (2.6) follows readily from Lemma 2.1 and Proposition 2.4.

For deriving (2.7), we have to invoke standard regularity results for partial differential equations with measures in the right hand side; see, e.g., [3, Theorem 4]. Let $q' = d/(d-1)$ for $d > 1$ and q' arbitrarily large if $d = 1$. Then, since $\gamma \left\|y_\gamma^+\right\|$ is uniformly bounded in $L^1(\Omega)$, it follows that u_γ is uniformly bounded in $W^{1,q'}(\Omega)$. This implies that y_γ is uniformly bounded in $W^{3,q'}(\Omega)$. It follows from Sobolev embedding theorems that y_γ is uniformly bounded in C^β for $\beta = 3 - \varepsilon, 2 - \varepsilon, 1 - \varepsilon$ in the cases $d = 1, 2, 3$, respectively, and for every $\varepsilon > 0$. ∎

In our numerical experiments below, we shall see that our technique yields sharp estimates in geometric situations where the graph of the state has a shape similar to an elliptic paraboloid, as modeled in our proof above. Such a configuration occurs when the active set is indeed a single point, only. In more regular situations, however, larger values of s are observed. We provide some further insight into this in section 2.6.

REMARK 2.7. *Under the assumption that* y_γ *is uniformly bounded in* C^2*, we get the heuristic bound* $O(\gamma^{-2/5})$ *for the constraint violation in the case* $d = 3$*.*

Finally, we establish a generic upper bound on s.

PROPOSITION 2.8. *If* $s \geq 1$*, then problem* (1.1) *has a Lagrangian multiplier in* $L^q(\Omega)$ *for each* $1 \leq q < \infty$*.*

Proof. This is a direct consequence of Proposition 2.2. ∎

2.4. The length of the primal-dual path. When we combine our estimates above, then this yields the following convergence estimate.

THEOREM 2.9. *If* y_γ *is uniformly bounded in* $C^\beta(\Omega)$ *for* $\gamma \to \infty$ *and some* $0 < \beta \leq 2$*, then the primal-dual path satisfies the following convergence estimate:*

$$\sqrt{\alpha}\,\|u_* - u_\gamma\|_{L^2} \leq c\gamma^{-\frac{\beta}{2(\beta+d)}}. \tag{2.8}$$

In particular, for every $\varepsilon > 0$ it holds that

$$\sqrt{\alpha}\,\|u_* - u_\gamma\|_U \leq \begin{cases} c\gamma^{-1/3} & : \quad d = 1, \\ c\gamma^{-1/4+\varepsilon} & : \quad d = 2, \\ c\gamma^{-1/8+\varepsilon} & : \quad d = 3. \end{cases} \tag{2.9}$$

In both situations, the constant $c > 0$ is independent of γ.

Proof. The result (2.8) follows from Theorem 2.3 and Corollary 2.6. In the same way, (2.9) follows from (2.7). ∎

2.5. A bootstrapping argument. We now come back to the possible improvement of the results obtained in [6] using bootstrapping arguments. For this purpose, we derive a bound on the L^∞-norm based on a boundedness result for the L^2-norm of the feasibility violation.

PROPOSITION 2.10. *Let $\Omega \subset \mathbb{R}^d$ be bounded and open, $0 \leq y \in C^\beta(\overline{\Omega}), y \in L^2(\Omega)$, and $0 < \beta \leq 2$. Moreover, assume that $y = 0$ on $\partial\Omega$. Then*

$$\|y\|_{L^\infty} \leq c\,\|y\|_{C^\beta}^{1-\Theta}\,\|y\|_{L^2}^{\Theta} \tag{2.10}$$

with $\Theta = \frac{2\beta}{2\beta+d}$. The constant c is independent of y.

Proof. The proof is analog to the one of Proposition 2.10 with the following replacement for (2.5):

$$\begin{aligned} \|y\|_{L^2}^2 = \int_{\overline{\Omega}} |y(\omega)|^2\,d\omega &\geq c\int_{[0,R]} \left|y(0) - \|y\|_{C^\beta} r^\beta\right|^2 r^{d-1}\,dr \\ &= c\,\|y\|_{C^\beta}^2 \int_{[0,R]} \left(\frac{y(0)}{\|y\|_{C^\beta}} - r^\beta\right)^2 r^{d-1}\,dr \\ &= c\,\|y\|_{C^\beta}^2 \int_{[0,R]} \left(R^\beta - r^\beta\right)^2 r^{d-1}\,dr \\ &\geq c\,\|y\|_{C^\beta}^2 R^{2\beta+d} = c\,\|y\|_{C^\beta}^{2-\frac{2\beta+d}{\beta}}\,\|y\|_{L^\infty}^{\frac{2\beta+d}{\beta}}. \end{aligned} \tag{2.11}$$

Solving for $\|y\|_{L^\infty}$, we conclude that

$$\|y\|_{L^\infty} \leq c\,\|y\|_{L^2}^{\frac{2\beta}{2\beta+d}}\,\|y\|_{C^\beta}^{1-\frac{2\beta}{2\beta+d}}$$

holds with a constant independent of y. ∎

To proceed we next show that, for the L^∞-norm of the feasibility violation, one obtains the same results as in Corollary 2.6 from the feasibility violation in $L^2(\Omega)$ when using a bootstrapping argument.

THEOREM 2.11. *Let y_γ be uniformly bounded in $C^\beta(\overline{\Omega})$ for $\gamma \to \infty$ and some $0 < \beta \leq 2$. Then the following estimate on the constraint violation holds true:*

$$\|y_\gamma^+\|_\infty \leq c\gamma^{-s}, \text{ where } s = \frac{\beta}{\beta+d}.$$

Proof. We begin by noting that by Theorem 2.3 and in particular (2.2) we have that

$$\gamma\|y_\gamma^+\|_{L^2}^2 \leq c\|y_\gamma^+\|_{L^\infty} \leq c.$$

In particular, for a constant independent of $t \geq 1$ it holds that

$$\|y_\gamma^+\|_{L^2}^2 \leq c\gamma^{-t}. \tag{2.12}$$

In order to obtain the best possible exponent t we start with $t_1 = 1$ and get from Proposition 2.10 that

$$\|y_\gamma^+\|_{L^\infty} \leq c\gamma^{-\left(\frac{1}{2}\frac{2\beta}{2\beta+d}\right)}.$$

Combining this estimate with our initial bound $\gamma\|y_\gamma^+\|^2 \leq c\|y_\gamma^+\|_{L^\infty}$ yields the improved rate

$$\|y_\gamma^+\|_{L^2}^2 \leq c\gamma^{-1-\theta}$$

with $\theta = \frac{1}{2}\frac{2\beta}{2\beta+d}$ and a constant independent of θ. We, thus, arrive at $t_2 = 1 + \theta$.

By induction it follows that for any $n \geq 1$ it holds

$$\|y_\gamma^+\|_{L^2}^2 \leq c\gamma^{-\sum_{k=0}^{n}\theta^k}$$

with a constant independent of n and θ. Thus, in the limit for $n \to \infty$ we obtain

$$\|y_\gamma^+\|_{L^2}^2 \leq c\gamma^{-\sum_{k=0}^{\infty}\theta^k} = c\gamma^{-1-\frac{\beta}{\beta+d}},$$

as well as

$$\|y_\gamma^+\|_{L^\infty} \leq c\gamma^{-\sum_{k=1}^{\infty}\theta^k} = c\gamma^{-\frac{\beta}{\beta+d}}.$$

This ends the proof. □

2.6. Structural assumptions and refined results. In the numerical section below we shall see that the rate of convergence of the primal-dual path depends on the structure of the Lagrangian multiplier at the optimal solution. For example, it can be observed that rates differ for line measures and for point measures. In order to clarify this issue, in this section we provide an in-depth discussion.

The following structural assumptions are motivated by our model situation where a lower dimensional problem is lifted to a higher space dimension through a parallel translation along an additional degree of freedom. A similar situation can be observed locally in many practically relevant cases.

We also note that the following regularity Assumption 2.12, applied to a measure m, is widely used in the theory of Sobolev spaces. It is the simplest instance of a regularity condition for measures, used in refined theories of Sobolev spaces, as for examples those treated in the monograph [10]. This assumption holds, for instance, when m is a measure with support on a δ-dimensional sub-manifold in Ω and represented by a bounded function. In order to emphasize the role of γy_γ^+ as a measure approximation we abbreviate $m_\gamma := \gamma y_\gamma^+$.

Assumption 2.12. *For $\omega \in \Omega$ denote by $B_R(\omega)$ the ball of radius R and with center ω. Let $\delta \geq 0$ be given. Assume that there exists $K < \infty$, independent of x, R, δ, and γ, such that the following inequality holds:*

$$\int_{B_R(x)} dm_\gamma \leq KR^\delta \tag{2.13}$$

for all sufficiently small $R > 0$.

Let us motivation our requirement (2.13). The exponent δ measures the actual amount of singularity of the measure. For instance, if m is induced by a bounded function on Ω, then $\delta = d$. On the other hand, if m is a line measure, induced by a bounded function on this line, then $\delta = 1$. Non-integer values of δ may arise in cases where the measures are induced by L^q-functions on submanifolds of $\overline{\Omega}$.

THEOREM 2.13. *Suppose that Assumption 2.12 holds true for some $\delta \geq 0$. Then the constraint violation can be computed as*

$$\|y_\gamma^+\|_\infty \leq c\gamma^{-s}\|y_\gamma\|_{C^\beta}^{1-s} \quad \textit{with } s = \frac{\beta}{\beta + d - \delta}. \tag{2.14}$$

Proof. Assume that y_γ attains its maximum at x_γ. As in Proposition 2.4 we choose $R = (\|y_\gamma^+\|_\infty / \|y_\gamma\|_{C^\beta})^{1/\beta}$, and compute

$$K\gamma^{-1}R^\delta \geq \int_{B_R(x_\gamma)} y_\gamma^+ \, dx \geq c\|y_\gamma\|_{C^\beta} R^{\beta+d} = c\|y_\gamma\|_{C^\beta}^{1-\frac{\beta+d}{\beta}} \|y_\gamma^+\|_\infty^{\frac{\beta+d}{\beta}}.$$

We solve the above inequality for $\|y_\gamma^+\|_\infty$ and set $s = \frac{\beta}{\beta+d-\delta}$ to obtain

$$\|y_\gamma^+\|_\infty \leq c\gamma^{-s}\|y_\gamma\|_{C^\beta}^{1-s},$$

which concludes the proof. ◻

Our assumption yields additional regularity. In particular we can obtain higher regularity of the adjoint state (and thus of the constraint violation) if the exponent δ in (2.13) is large.

PROPOSITION 2.14. *Suppose that Assumption 2.12 holds true for some $0 \leq \delta \leq d$. Then* (2.13) *also holds for the Lagrangian multiplier m_*, i.e., $\int_{B_R(x)} dm_* \leq KR^\delta$. Moreover, for any*

$$l \in \{0, 1\} \textit{ and } q' \in (\max(1, (d-\delta)/(2-l)), d),$$

one has $p_\gamma \in W^{l,q}(\Omega)$ uniformly, where $1/q + 1/q' = 1$. The same holds true for p_*.

Proof. Due to weak*-convergence $\gamma y_\gamma^+ \rightharpoonup^* m_*$ we have

$$\int_{B_R(x)} dm_\gamma \to \int_{B_R(x)} dm_*,$$

which implies that (2.13) also holds for m.

According to [10, Section 1.4.5] we have the embedding $W^{2-l,q'}(\Omega) \hookrightarrow L^1(\Omega, m)$ for all l and q' that satisfy our assumptions. Here $L1(\Omega, m)$ denotes the space of all functions that are integrable with respect to the measure m. Hence, γy_γ^+ and m yield continuous linear functionals on $W^{2-l,q'}(\Omega)$ via the mapping

$$v \to \int_\Omega v \, dm_\gamma \text{ and } v \to \int_\Omega v \, dm_*,$$

respectively. Now the differential operator $A = -\Delta : W^{2-l,q'}(\Omega) \to (W^{l,q}(\Omega))^*$ is an isomorphism due to our regularity assumptions. Then $A^* : W^{l,q}(\Omega) \to (W^{2-l,q'}(\Omega))^*$ is an isomorphism as well, which implies our result. ◻

COROLLARY 2.15. *Suppose that Assumption 2.12 holds true in the following cases. Then we can conclude:*

(i) (Line measure in 2d) If $d = 2, \delta = 1$, then $s = 2/3$.
(ii) (Surface measure in 3d) If $d = 3, \delta = 2$, then $s = 2/3$.
(iii) (Line measure in 3d) If $d = 3, \delta = 1$, then $s = 1/2 - \varepsilon$ for each $\varepsilon > 0$.

Proof. From our results of Proposition 2.14 we obtain p_γ uniformly bounded in $W^{1,q}(\Omega)$ for arbitrary $q < \infty$ in the cases (i) and (ii), and thus $y \in W^{3,q}(\Omega) \hookrightarrow C^2(\overline{\Omega})$ uniformly. Hence, we can apply Theorem 2.13 with $\beta = 2$.

In case (iii) we obtain $p_\gamma \in L^q(\Omega)$ for arbitrary $q < \infty$, only, and thus $y \in W^{2,q}(\Omega) \hookrightarrow C^{2-\varepsilon}(\overline{\Omega})$ for all $\varepsilon > 0$. Theorem 2.13 with $\beta = 2 - \varepsilon$ then yields the result. ∎

Comparing this corollary with Corollary 2.6 we see that we obtain the same rates for d and δ as for $d - \delta$ in Corollary 2.6. This confirms that problems with higher regularity of the Lagrangian multipliers essentially behave like lower dimensional problems.

2.6.1. Active sets with non-empty interior. Finally, we show that further improvement of the convergence rate is possible, when the active set has a non-empty interior. Here, we confine our discussion to the 1d-case, but augment our findings by comments concerning the general case.

As a motivation for the following proof, consider the typical setting for $d = 1$ with an active set which has a non-empty interior; for numerical details we refer to Section 3 below. Since y_* is constant on the active interval the derivatives $y_*^{(m)}$ satisfy $y_*^{(m)} = 0$ for all $m \in \mathbb{N}_0$ on this set. Consider the functions $f(\omega) = a(-\omega^3 + \varepsilon\omega^2)$ and $f^+ = \max(f, 0)$ on $[0, \infty[$; see the left plot of Figure 2.2 for a graphical illustration.

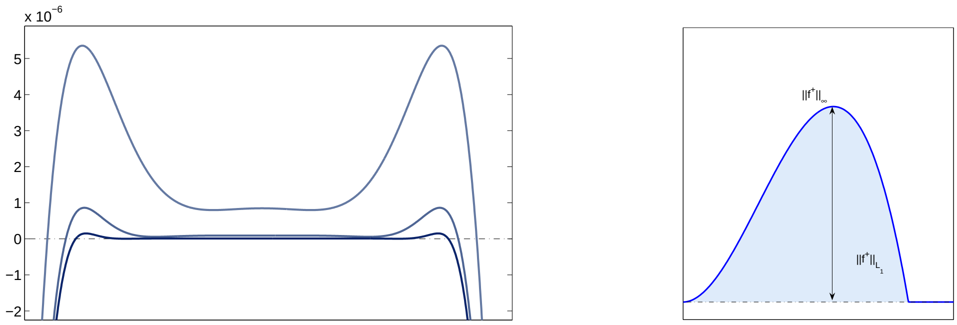

FIG. 2.2. *Geometric situation for y_γ near an active interval. Left: observed in numerical experiments for $\gamma = 10^7, 10^8$, and 10^9. Right: idealized model.*

Obviously, $f^{(m)}(0) = 0$ for $m = 1, 2$. Clearly, f is positive on $[0, \varepsilon]$ and has a maximum at $\omega = 2/3\varepsilon$ with $f(2/3\varepsilon) = 10/27a\varepsilon^3$, thus, $\|f^+\|_\infty = ca\varepsilon^3$. Further, $\|f^+\|_{L^1} = \int_0^\varepsilon f(\omega)\, d\omega = a\varepsilon^4/12$. Hence, similarly as above and using $6a = \|f\|_{C^3}$, we conclude that

$$\left\|f^+\right\|_\infty \leq c \left\|f\right\|_{C^3}^{1/4} \left\|f^+\right\|_{L^1}^{3/4}.$$

THEOREM 2.16. *Let $d = 1$, Ω a bounded interval, and assume that for all sufficiently large γ the support of y_γ^+ consists of finitely many intervals of positive length. Assume that the minimal length of these intervals is bounded from below by $a_0 > 0$, which is independent of γ. Then, for each $\varepsilon > 0$ one has*

$$\|y_\gamma^+\|_\infty \leq c\gamma^{-s} \text{ with } s = \frac{3}{4} - \varepsilon.$$

Proof. Since our discussion is limited to the one-dimensional case, we obtain that y_γ is bounded in $W^{3,\infty}(\Omega)$ by a constant M as $\gamma \to \infty$. Moreover, we have shown that $\|y_\gamma^+\|_\infty \leq c\gamma^{-s}$ for $s = 2/3$. A bootstrapping technique with respect to s allows us to increase s to any value lower than $3/4$.

As a special case of the Gagliardo-Nirenberg inequalities (cf., e.g., [16, Appendix]) it follows for a *fixed* domain (say, the unit interval I) that

$$\|\nabla^j f\|_{L^\infty(I)} \leq K \|f\|^{\theta}_{W^{m,\infty}(I)} \|f\|^{1-\theta}_{L^\infty(I)},$$

where $\theta = j/m$. Transforming I to an interval of arbitrary positive length a through $x \mapsto ax + x_0$ shows that the constant K depends on the length a, only. The constant degenerates only if $a \to 0$ or $a \to \infty$.

Let I_γ denote the subinterval where y_γ^+ attains its maximum. By assumption, the length of I_γ is bounded from below by a_0 and from above by the length of Ω. Using our convention that $C^\beta(\Omega)$ for integral $\beta > 1$ contains elements whose derivatives up to the order $(\beta - 1)$ are Lipschitz continuous, we hence, get

$$\|y_\gamma^+\|_{C^2(I_\gamma)} \leq K(a_0) \|y_\gamma^+\|^{2/3}_{W^{3,\infty}(I_\gamma)} \|y_\gamma^+\|^{1/3}_{L^\infty(I_\gamma)} \leq K(a_0) M^{2/3} \gamma^{-s/3}.$$

Employing Theorem 2.13 we obtain with $s_0 = \beta/(1+\beta) = 2/3$ that

$$\|y_\gamma^+\|_\infty \leq \|y_\gamma^+\|^{1-s_0}_{C^2(I_\gamma)} \gamma^{-s_0} \leq c\gamma^{-(\frac{s}{3}\frac{1}{3})-\frac{2}{3}}.$$

Thus, we conclude that

$$\|y_\gamma^+\|_\infty \leq c\gamma^{-s} \Longrightarrow \|y_\gamma^+\|_\infty \leq c\gamma^{-\frac{s}{9}-\frac{2}{3}}.$$

Consequently, starting with $s = s_0 = 2/3$ we obtain a sequence (s_n) whose elements are defined recursively by

$$s_{n-1} \mapsto s_n := \frac{s_{n-1}}{9} + \frac{2}{3}. \tag{2.15}$$

A short computation yields that this sequence converges monotonically to the fixed point $s = 3/4$ of (2.15), which shows our result. □

REMARK 2.17. *When one tries to extend this result to higher dimensions, one has to find an analogue to the stated minimal length assumption, in the sense that the maximum of y_γ^+ lies in the interior of a subset of* $\operatorname{supp} y_\gamma^+$*, the support of y_γ^+, that can be transformed to a fixed domain, say the unit square, such that the transformations do not degenerate for $\gamma \to \infty$. This will lead to uniform cone conditions on the boundaries of the active sets, independent of γ. However while the idea is quite clear, the mathematical realization is technical and, in our opinion, yields no further insight. Hence we refrain from pursuing this direction here.*

3. Comparison with experimental results. We conclude our paper with numerical experiments in 1d and 2d, which illustrate the close relationship between our theoretical estimates and the convergence behavior in practice.

In order to measure the exponent s we perform a numerical path-following method for the system (1.6), implemented in MATLAB for the case $d = 1$. For this purpose, we use a prescribed sequence of parameters γ_j. In our test problems we choose a constant desired state $y_d \equiv 10$, the control cost $\alpha = 1$, and obtain different types of

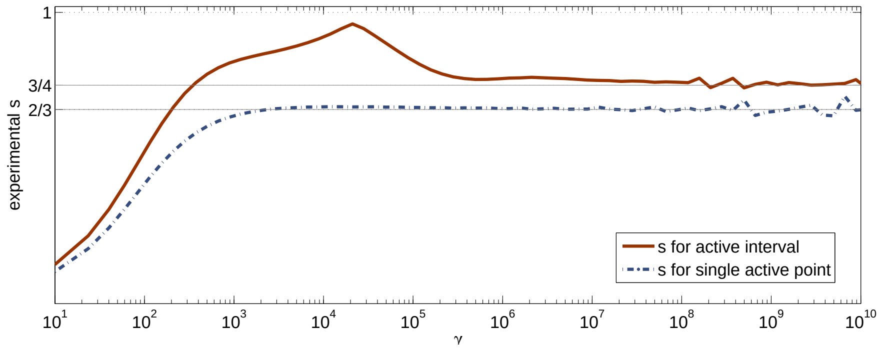

FIG. 3.1. *Experimental results on the exponent s of the relation* $\left\|y_\gamma^+\right\|_{L^\infty} = O(\gamma^{-s})$ *in a one-dimensional setting.*

active sets by varying the constant upper bound ψ. The discretization of the state y and the adjoint state p is based on classical finite differences, i.e, the 3-point stencil in the one-dimensional setting. In the 2d-case we use the optimization toolbox `DOpE` based on the finite element library `deal.ii` [1, 2]. For obtaining a high resolution of the quantity $\|y_\gamma^+\|_\infty$ we use anisotropic mesh refinement and locally refined grids with piecewise bilinear finite elements.

An estimate for s is then computed by

$$s_j := \frac{\ln(\|y_{\gamma_j}^+\|_\infty) - \ln(\|y_{\gamma_{j+1}}^+\|_\infty)}{\ln \gamma_{j+1} - \ln \gamma_j}.$$

Observe that this formula is quite sensitive to perturbations of y, which partially explains the slightly oscillatory behavior observed in the plots.

3.1. Experimental results in 1d. In the one-dimensional case our computational domain is the unit interval, discretized uniformly by 10000 nodes. In our first setting (with $\psi \equiv 0.06$), the active set consists of a single point, and the optimal state y_* has a parabolic shape, i.e., a second derivative, which is bounded away from zero. Here, according to Figure 3.1, $s \approx 2/3$ as predicted by Corollary 2.6.

In our second setting (with $\psi \equiv 0.01$), the active set is a proper interval. Here we observe experimentally $s \approx 3/4$, which is a higher rate than predicted by Corollary 2.6. However, this rate can be explained by the results of Theorem 2.16.

Another interesting aspect is the large value of s in the range $\gamma \in [10^4, 10^5]$. A close look at the corresponding intermediate solutions shows that for these values y_γ has a very flat maximum, which explains this behavior. In terms of Assumption 2.12 this means $\delta \approx d$ for an intermediate range of γ. For larger values of γ this single maximum splits into two maxima, which can be nicely observed in Figure 3.2. For larger values of γ the qualitative behavior of y_γ^+ can be observed in Figure 2.2.

3.2. Experimental results in 2d. Compared to the 1d-case, the geometric situation for $d = 2$ (and even more for $d = 3$) is much more complex, but still our analysis and numerical experiments suggest the conjecture that the rate of convergence of the primal-dual path will depend largely on the geometry of the active set, or equivalently on the structure of the Lagrangian multipliers.

If the active set is a single point (for $\psi \equiv 0.04$), we observe in Figure 3.3 that our estimates of Corollary 2.6 coincide with the numerical rate of convergence, namely

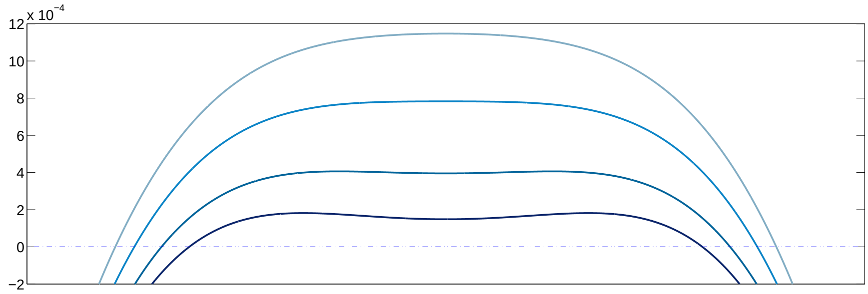

FIG. 3.2. *Transition from single maximum to double maximum of y_γ for $\gamma = 7.5 \cdot 10^{-5}, 5 \cdot 10^{-5}, 2.5 \cdot 10^{-5}$, and 10^{-5}.*

$s \approx 0.5$. We observe the predicted rate of $s \approx 0.75$ in the case of the active set with nonempty interior (for $\psi \equiv 0.01$).

The computations were done using the optimization toolkit `DOpE` based upon `deal.ii`. Using adaptive mesh refinement to accurately capture the behavior of $\|y_\gamma^+\|_\infty$. On the final mesh, the smallest elements where of diameter 2^{-12} with a total of approximately 1.5 million unknowns.

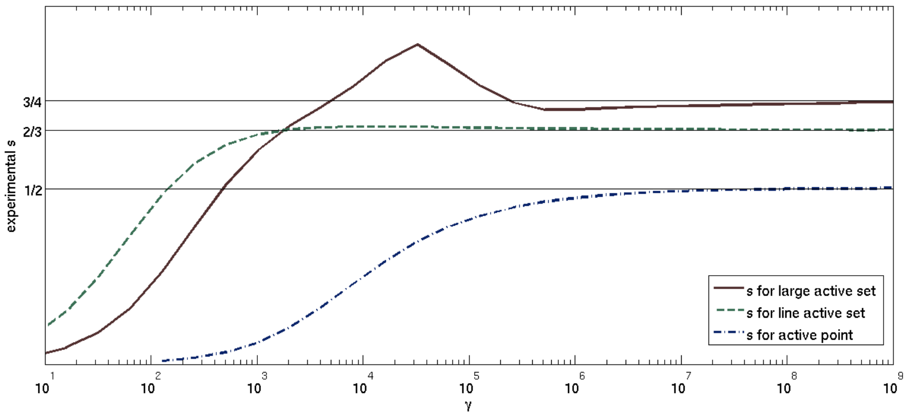

FIG. 3.3. *Experimental results on the exponent s of the relation $\left\|y_\gamma^+\right\|_{L^\infty} = O(\gamma^{-s})$ in a two-dimensional setting.*

In order to obtain numerical results for a line measure we use a slightly different numerical setting. In fact, we chose homogeneous Neumann boundary conditions on two opposing boundary faces (i.e., $\{\omega_2 \in \{0,1\}\}$) of the unit square and homogeneous Dirichlet conditions on the other two faces (i.e., $\{\omega_1 \in \{0,1\}\}$). As mentioned before, we use the optimization toolkit `DOpE` based on `deal.ii`. For obtaining a high resolution for the line measure case an anisotropic refinement is employed such that the mesh is refined more in the ω_1-direction yielding an aspect ratio of 1:128 with a total number of 1065090 degrees of freedom.

3.3. Effect of a fixed discretization on the rate. On a fixed mesh, when y_γ is computed up to very large values of γ, then it can be observed (see Figure 3.4) that a rate of $s = 1$ finally occurs. This can simply be explained by the fact that in finite dimensional spaces all norms are equivalent, so that for uniformly bounded discrete (indicated by subscript h) $\gamma\|(y_\gamma^+)_h\|_{L^1}$ we eventually observe the upper bound

$$\|(y_\gamma^+)_h\|_{L^\infty} \le c(h)\|(y_\gamma^+)_h\|_{L^1} \le c(h)\gamma^{-1} \quad \Longleftrightarrow \quad s = 1.$$

If such a behavior is observed in practice, it is a clear indication that the problem has been "over-solved" numerically, or that the multiplier m_* is in $L^q(\Omega)$; see Proposition 2.8.

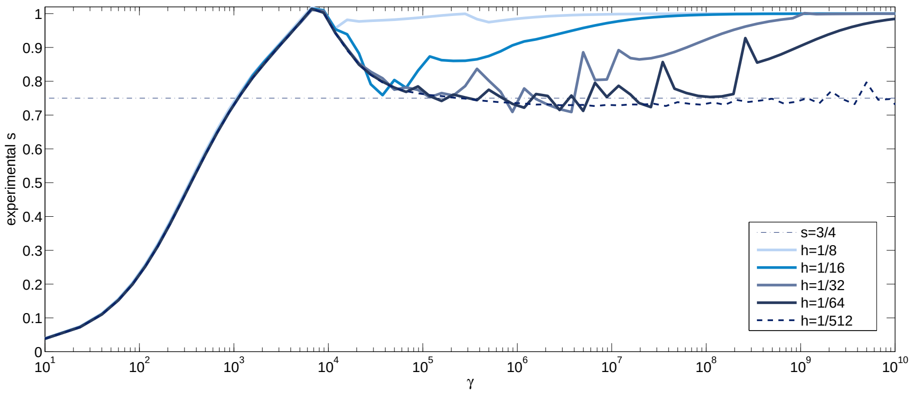

FIG. 3.4. *Experimental results on the exponent in a two-dimensional setting with large active set for various coarse grids. For comparison the asymptotics for a fine grid $h = 1/512$ is added as a dashed line.*

4. Conclusion. Numerical experiments indicate that the asymptotics on the constraint violation derived in this note are sharp, if the active set is a singleton, at least in one- and two-dimensional problems. Moreover, we gave some arguments concerning the implication of the regularity of the multiplier onto the observed convergence of the constraint violation. Numerical experiments indicate that the derived bounds are sharp in all these cases.

From a practical point of view, the observation that one has

$$\|y_\gamma^+\|_{L^\infty} \le c\gamma^{-s}$$

where s generically varies in an interval $s \in [s_{\text{point}}, 1[$ in turn implies a rate of convergence for the error in the control which varies between $[s_{\text{point}}/2, 1/2[$. This may help in the construction of adaptive algorithms, which try to balance algebraic errors and discretization errors and clearly shows that an a posteriori estimation of this error is necessary as the rates depend on the a priori unknown behavior of the feasibility violation.

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