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Dedicated to Ivo Marek on the occasion of his 80th birthday

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THE ALGEBRAIC RICCATI EQUATION FOR QUATERNIONS

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Abstract. We will present the general solution of the algebraic Riccati equation for the quaternionic case, where also one additional variation is treated. For computational purpose a very simple form of the exact Jacobi matrix for Riccati polynomials is presented. There are several examples.

Key words. Quaternionic, algebraic Riccati equations, Jacobi matrix for quaternionic, algebraic Riccati equations.

AMS subject classifications. 11R52, 1204, 12E15, 12Y05, 1604, 65H04.

1. Introduction.

The algebraic Riccati equation, which is - roughly spoken - a matrix equation with a linear and a quadratic term, has attracted several authors. Recent examples include books by D. A. Bini, B. Iannazzo, B. Meini, [2, 2012], and by H. Abou-Kandil, G. Freiling, V. Ionescu, G. Jank, [1, 2003]. Another source is a book by P. Lancaster, L. Rodman, [13, 1995]. For a newer, numerical contribution see V. Simoncini, D. B. Szyld, M. Monsalve, [19, 2013]. In this paper we will present a complete solution for the algebraic, one dimensional quaternionic Riccati equation including one additional variation, which is not mentioned in the above literature of Riccati equations. It turns out, that in all cases we can use tools, that were only developed recently.

By $\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{C}, \mathbb{H}$ we will denote the set of integers, positive integers, real numbers, complex numbers and quaternions, respectively. By $\mathbb{F}^{m,n}$ we denote the set of matrices with entries from $\mathbb{F}$ distributed over m rows and n columns, where $\mathbb{F}$ denotes one of the mentioned number systems. The set of quaternions $\mathbb{H}$ should be regarded as the vector space $\mathbb{R}^4$ equipped with a special multiplication rule such that $\mathbb{H}$ becomes a skew field. Quaternions $a = (a_1, a_2, a_3, a_4)$ have an isomorphic representation as special matrices in $\mathbb{C}^{2\times 2}$ and as special matrices in $\mathbb{R}^{4\times 4}$. We may define these matrices by (see v. d. Waerden, [20, p. 55], Gürlebeck and Sprössig, [6, p. 5])

$$\text{(1.1)}\qquad \mathrm{i}_{\mathbb{C}}(a) := \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix}, \quad \alpha := a_1 + \mathbf{i}a_2, \beta := a_3 + \mathbf{i}a_4,$$

$$\text{(1.2)}\qquad \mathrm{i}_{\mathbb{R}}(a) := \begin{bmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & -a_4 & a_3 \\ a_3 & a_4 & a_1 & -a_2 \\ a_4 & -a_3 & a_2 & a_1 \end{bmatrix}.$$

It should be noted, that the representation (1.2) is not unique. There are other, equivalent forms. See Farebrother, Groß and Troschke, [4]. The first entry a_1 of a quaternion $a = (a_1, a_2, a_3, a_4)$ will be called *the real part of* a and denoted by $\Re a$. Let b be another quaternion. Then the real part $\Re$ has the property that $\Re(ab) = \Re(ba)$. By

$$|a| := \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2}$$

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we denote the *norm* or *length* of a and $\overline{a} = (a_1, -a_2, -a_3, -a_4)$ will be called the *conjugate of* a. We will need a third matrix derived from a quaternion $a = (a_1, a_2, a_3, a_4)$, namely

$$\mathrm{i}_{\mathbb{P}}(a) := \begin{bmatrix} a_1 & -a_2 & -a_3 & -a_4 \\ a_2 & a_1 & a_4 & -a_3 \\ a_3 & -a_4 & a_1 & a_2 \\ a_4 & a_3 & -a_2 & a_1 \end{bmatrix}. \tag{1.3}$$

The set of these matrices will be denoted by $\mathbb{H}_{\mathbb{P}}$, and we will name these matrices *pseudo quaternions*. See [8]. The column vector $(a_1, a_2, a_3, a_4)^{\mathrm{T}}$, where $^{\mathrm{T}}$ means transposition, will be abbreviated by $\mathrm{col}(a)$. There is the following essential property. Let a, b, c be three quaternions, then

$$\mathrm{col}(abc) = \mathrm{i}_{\mathbb{R}}(a)\mathrm{i}_{\mathbb{P}}(c)\mathrm{col}(b). \tag{1.4}$$

That means roughly, that the middle factor b can be pulled out of the product abc. See Gürlebeck and Sprössig, [6, p. 6], and Janovská, Opfer, [9, Lemma 3.5]. Because the product on the right hand side of (1.4) will appear several times later on, we will abbreviate it by

$$\mathrm{i}_3(a, b) := \mathrm{i}_{\mathbb{R}}(a)\mathrm{i}_{\mathbb{P}}(b), \tag{1.5}$$

such that (1.4) can be written as

$$\mathrm{col}(abc) = \mathrm{i}_3(a, c)\mathrm{col}(b). \tag{1.6}$$

2. The algebraic Riccati equation. The Riccati differential equation in its simplest, one dimensional, form reads

$$\dot{x}(t) = a(t) + b(t)x(t) + c(t)x^2(t). \tag{2.1}$$

It is named after Jacopo Francesco Riccati (1676 - 1754). See [22]. This equation can be reduced to a so-called Bernoulli differential equation if one could find a special solution $x(t) = u(t)$ of (2.1), and a Bernoulli equation can be reduced to a linear differential equation. This is all well known, and will not be pursued. See Reid, [18].

The *algebraic Riccati equation* reads (see [1, p. 22], [13, p. vii])

$$p(\mathbf{X}) := \mathbf{A} + \mathbf{B}\mathbf{X} + \mathbf{X}\mathbf{C} + \mathbf{X}\mathbf{D}\mathbf{X} = \mathbf{0}, \tag{2.2}$$

which apparently mimics the right hand side of (2.1) and the solutions of which represent the stationary solutions of (2.1). We assume that the matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are given matrices with $\mathbf{A} \in \mathbb{C}^{m\times n}, \mathbf{B} \in \mathbb{C}^{m\times m}, \mathbf{C} \in \mathbb{C}^{n\times n}, \mathbf{D} \in \mathbb{C}^{n\times m}$, and that $\mathbf{X} \in \mathbb{C}^{m\times n}$ is an unknown matrix such that $p(\mathbf{X}) \in \mathbb{C}^{m\times n}$. The most important class of algebraic Riccati equations is the *Hermite, algebraic Riccati equation* where $m = n$ and $\mathbf{C} = \mathbf{B}^*$. See also [1, p. 21/22]. By $\mathbf{B}^*$ we understand the transposed, complex conjugate matrix of $\mathbf{B}$. One is interested to find all $\mathbf{X}$ which solve (2.2). This is the topic of the mentioned literature, [1, 2, 13, 19].

3. The algebraic Riccati equation for quaternions. If we reduce (2.2) to the one dimensional, quaternionic case we obtain

$$p(x) := a + bx + xc + xdx = 0, \quad a, b, c, d, x \in \mathbb{H}. \tag{3.1}$$

The solutions of $p(x) = 0$ will also be called *zeros* of p. This polynomial p is a quaternionic polynomial of degree two and we will show that this polynomial can be reduced to a polynomial of the so-called *one-sided type* (sometimes also called *simple*, or *unilateral*) quaternionic polynomial which is a polynomial of the form

$$p(x) := \sum_{j=0}^{n} a_j x^j, \quad x, a_j \in \mathbb{H},\ j = 0, 1, \ldots, n. \tag{3.2}$$

The theory for quaternionic polynomials of the one-sided type was developed to a final state by Janovská, Opfer, [10]. In that paper it is shown, that the zeros may have two different types, called *isolated* and *spherical* (see Definition 3.2) and it is also shown how to compute all zeros. In addition, a result by Gordon and Motzkin, [5, 1965], that the *essential number of zeros* is bounded by n is also included. For a definition of this notion see a later part of this paper, Definition 4.4, p. 8. Quaternionic polynomials of the *two-sided type* have the form

$$p(x) := \sum_{j=0}^{n} a_j x^j b_j, \quad x, a_j, b_j \in \mathbb{H},\ j = 0, 1, \ldots, n, \tag{3.3}$$

and they will also play a role in this paper. The classification of the zeros of quaternionic polynomials of the two-sided type was introduced by Janovská, Opfer, [9]. It also allows multiple terms of the same degree and it was shown, that there are three more types of zeros than in the one-sided case, and that the essential number of zeros is not bounded by n.

We see, that the algebraic Riccati equation (3.1) is neither of the form (3.2) nor of the form (3.3) because the quadratic term in (3.1) has the form xdx where in comparison with (3.2) the quadratic term has the form a_2x^2 and with (3.3) it has the form $a_2x^2b_2$.

If $d = 0$, the algebraic Riccati equation (3.1) reduces to a quaternionic, linear equation of Sylvester's type. The solution of this linear equation is given in [11, Theorem 2.3].

Theorem 3.1. *Let $d \neq 0$. Then, (a) the algebraic, quaternionic Riccati equation (3.1) always has a solution. (b) The equation (3.1) can be simplified to*

$$r(z) := \alpha + \beta z + z^2 = 0, \quad \alpha, \beta \in \mathbb{H},\ \Re(\beta) = 0, \ \textit{where} \tag{3.4}$$

$$x = d^{-1}(z - c - e), \tag{3.5}$$

$$\alpha := da - dbd^{-1}c + e(e - dbd^{-1} + c), \tag{3.6}$$

$$\beta := dbd^{-1} - c - 2e, \tag{3.7}$$

$$e := \frac{\Re(dbd^{-1} - c)}{2}. \tag{3.8}$$

Proof. Part (a) follows from [3, Eilenberg, Niven, 1944]. (b) Since $d \neq 0$ we can introduce $y = dx \Leftrightarrow x = d^{-1}y$ into (3.1), and we obtain

$$dp(x) = \tilde{p}(y) := \tilde{a} + \tilde{b}y + y\tilde{c} + y^2 = 0, \quad \tilde{a} = da,\ \tilde{b} = dbd^{-1},\ \tilde{c} = c.$$

A further simplification (see [9, formula (5.12)] and [14, p. 311]) by introducing $y := u - \tilde{c}$ yields $\tilde{p}(y) = \hat{p}(u) := \hat{a} + \hat{b}u + u^2$, $\hat{a} := \tilde{a} - \tilde{b}\tilde{c}$, $\hat{b} := \tilde{b} - \tilde{c}$. Eventually, by putting $u = z - \frac{\Re\hat{b}}{2}$ we obtain the final form $r(z) := \hat{p}(u) = \alpha + \beta z + z^2, \alpha = \hat{a} - \hat{b}\frac{\Re\hat{b}}{2} + \left(\frac{\Re\hat{b}}{2}\right)^2, \beta = \hat{b} - \Re\hat{b}$ and the real part of β is zero. ∎

We have to introduce some terminology in order to explain the solution structure of the Riccati equation (3.4).

Definition 3.2. Let $x, y \in \mathbb{H}$. We call x, y *equivalent* if there is an $h \in \mathbb{H}\backslash\{0\}$ such that $y = h^{-1}xh$ and denote this by $x \sim y$. Let $x \in \mathbb{H}$. By

$$[x] := \{y : y \sim x\} = \{y : y = h^{-1}xh, \text{ for all } h \in \mathbb{H}\backslash\{0\}\} \tag{3.9}$$

we denote the set of all elements equivalent to x. Let p be a one-sided or two-sided quaternionic polynomial and x be a nonreal zero of p. If $p(y) = 0$ for all $y \in [x]$, the zero x and the equivalence class $[x]$, as well, will be called a *spherical zero* of p. If x is the only zero of p in $[x]$, which includes real zeros, the zero will be called *isolated zero* of p.

Remark 3.3. Two quaternions x, y are equivalent if and only if

$$|x| = |y|, \quad \Re x = \Re y. \tag{3.10}$$

See [12, Lemma 2.2]. Thus, it is easy to recognize two equivalent quaternions. In particular, two conjugate quaternions $a, \overline{a}$ are equivalent.

In order to see the possibility of infinitely many zeros, we consider the polynomial

$$p(x) := x^2 + 1, \tag{3.11}$$

regarded as a polynomial over $\mathbb{H}$. There are apparently two complex zeros $\pm\mathbf{i}$. For all $h \in \mathbb{H}\backslash\{0\}$ we have $h^{-1}p(x)h = h^{-1}(x^2+1)h = (h^{-1}xh)^2 + 1 = p(h^{-1}xh)$. Thus, $p(x) = 0$ implies $p(h^{-1}xh) = 0$ for all $h \in \mathbb{H}\backslash\{0\}$, and the infinite equivalence class $[\mathbf{i}]$ is the set of all zeros of p. Thus, there are infinitely many zeros. According to Eilenberg and Niven, [3], all quaternionic polynomials (regardless of their form) have at least one zero, provided the term with the highest degree appears only once. A counterexample is $p(x) := ax^2 - x^2a + 1$ for $a \in \mathbb{H}$ where there are two terms with the highest degree 2. In this case $\Re(p(x)) = \Re(ax^2 - x^2a + 1) = \Re(ax^2) - \Re(x^2a) + 1 = 1$ for all $x \in \mathbb{H}$, because the real part of a product commutes. Thus, p has no zero. The theory developed by Janovská, Opfer, [9, 10] implies, that one-sided quaternionic polynomials only have either isolated or spherical zeros, whereas two-sided quaternionic polynomials may have three more types of zeros than isolated or spherical zeros.

Let us consider the polynomial r, defined in (3.4) which also represents the general Riccati equation (3.1).

Lemma 3.4. *Let p be the quaternionic Riccati equation defined in (3.1) with given coefficients $a, b, c, d \in \mathbb{H}$, $d \neq 0$, and let r be defined as in (3.4) with coefficients α, β defined in (3.6), (3.7), where we assume that $\alpha \neq 0$. Then $p(x) = 0 \Leftrightarrow r(z) = 0$ where the relation between z and x is given in (3.5).*

Proof. The proof of Theorem 3.1 shows that the transformations applied to p in order to arrive at r have the property that the variables of the newly developing polynomials remain in a one to one relation to their predecessors. Thus, the variable x of the original polynomial p is in a one to one relation to the final variable z in r. □

The problem of finding the zeros of p is thus, reduced to finding the zeros of r. For this problem we already have the following result.

Theorem 3.5. *Let r be given as in (3.4): $r(z) := \alpha + \beta z + z^2$, $\alpha \neq 0$, $\Re\beta = 0$.*

1. *If both α, β are real (hence, $\beta = 0$), then r has either two different isolated real zeros in $\mathbb{H}$ (if $\alpha < 0$), or one spherical zero in $\mathbb{H}$ (if $\alpha > 0$). The zeros in the first case are $\pm\sqrt{-\alpha}$, the spherical zero is $[r_0]$, where $r_0 := \sqrt{\alpha}\,\mathbf{i}$.*

2. *If at least one of the coefficients α, β is not real, then r has either one or two isolated zeros in $\mathbb{H}$. It has one isolated zero if*

$$2\Re(\alpha\overline{\beta}) = (2\Re\alpha + |\beta|^2)^2 - 4|\alpha|^2 = 0. \tag{3.12}$$

It has two isolated zeros, otherwise.

Proof. Janovská, Opfer, [10, Theorem 6.1], and Niven, [15, Theorem 2].□

Thus, the polynomial r may have spherical zeros only in the case that both coefficients α, β are real, and in this case the zeros are known. How do we find the zeros in case 2 (which means that at least one of the coefficients α, β is not real) of the above theorem?

Theorem 3.6. *Let $r(z) := \alpha + \beta z + z^2, \alpha \neq 0, \Re\beta = 0$, and let at least one of the coefficients α, β be nonreal. Define the so-called companion polynomial*

$$q(u) := |\alpha|^2 + 2\Re(\alpha\overline{\beta})u + (2\Re\alpha + |\beta|^2)u^2 + u^4, \tag{3.13}$$

which has real coefficients and has degree four. Let there be a pair of complex conjugate zeros $\gamma := \gamma_1 \pm \gamma_2\mathbf{i}$ of q. Define the auxiliary quantity

$$h := (\overline{\beta} + 2\gamma_1)(\alpha - |\gamma|^2) =: (h_1, h_2, h_3, h_4). \tag{3.14}$$

Then

$$r_0 := \left(\gamma_1, -\frac{|\gamma_2|}{\hat{|h|}}h_2, -\frac{|\gamma_2|}{\hat{|h|}}h_3, -\frac{|\gamma_2|}{\hat{|h|}}h_4\right), \quad \hat{|h|} := \sqrt{h_2^2 + h_3^2 + h_4^2} \tag{3.15}$$

is a zero of r.

Proof. See Janovská, Opfer, [10]. □

If in the above theorem, q has two distinct pairs of complex conjugate zeros, then each pair defines a zero of r. In this case r has two (isolated) zeros. If q has a pair of complex zeros $\gamma = \gamma_1 \pm \gamma_2\mathbf{i}$ which is a double zero, then $q(u) = (u-\gamma)(u-\overline{\gamma})^2 = (u^2 + |\alpha|)^2$. And this happens if and only if (3.12) is valid, which implies that q is a complete square and $\gamma = \sqrt{|\alpha|}\,\mathbf{i}$. See [10, Lemma 6.2].

Let the Riccati equation, defined by p, have real coefficients $a, b, c, d, d \neq 0$. Then, it is clear that the coefficients of the polynomial r are

$$\alpha = ad - \frac{(b+c)^2}{4}, \ \beta = 0$$

which are also real and the first part of Theorem 3.5 applies.

Example 3.7. We study the Riccati equation $p(x) := a + bx + xc + xdx$ with the coefficients

$$a = (-9, 21, -19, 3), \ b = \mathbf{i}, \ c = \mathbf{j}, \ d = \mathbf{k}.$$

The quaternionic polynomial $r(z) = \alpha + \beta z + z^2$ has the coefficients (apply (3.6), (3.7))

$$\alpha = (-3, 19, 21, -8), \quad \beta = (0, -1, -1, 0).$$

The second part of Theorem 3.5 applies, where condition (3.12) *is not satisfied*. Thus, there are two isolated zeros. The companion polynomial q reads here

$$\begin{aligned} q(u) &= 875 - 80u - 4u^2 + u^4 \\ &= (u-\gamma)(u-\overline{\gamma})(u-\delta)(u-\overline{\delta}), \ \gamma = -4 + \sqrt{19}\,\mathbf{i}, \ \delta = 4 - 3\mathbf{i}. \end{aligned}$$

Formula (3.15) for the zero γ of q yields the isolated zero $r_0 = (-4, 3, 3, -1)$ of r and the zero δ of q yields the second isolated zero $r_1 = (132, -58, -74, 31)/33$ of r. The two corresponding isolated zeros z_0, z_1 of the original Riccati equation p are computed from r_0, r_1, respectively, by applying (3.5) which yields

$$z_0 = (-1, 2, -3, 4), \quad z_1 = (31, -107, 58, -132)/33.$$

Example 3.8. We study the Riccati equation $p(x) := a + bx + xc + xdx$ with the coefficients

$$a = (1, 2, -1, 1),\ b = (-2, -2, -1, 0),\ c = \mathbf{j},\ d = (1, 0, -1, -1).$$

The quaternionic polynomial $r(z) := \alpha + \beta z + z^2$ has by (3.6) and (3.7) the coefficients

$$\alpha = (1, -2, -2, 0), \quad \beta = (0, 0, 0, 1)$$

with $|\alpha| = 3$. The second part of Theorem 3.5 applies, where in this example condition (3.12) *is satisfied*. As a consequence, the companion polynomial q of r is a complete square

$$q(u) = 9 + 6u^2 + u^4 = (|\alpha| + u^2)^2$$

with zeros $\pm\sqrt{|\alpha|}\,\mathbf{i}$. The polynomial r has, by using (3.14), (3.15), only the isolated zero at $r_0 = (0, -1, 1, 1)$ and the given polynomial p has also only one isolated zero at

$$z_0 = \mathbf{k},$$

which can be computed from (3.5).

We note, that examples where the Riccati equation has exactly one (isolated) zero are not so easy to find.

We see, that apart from finding the zeros of the companion polynomial q, which is of degree four, no numerical tools are necessary in order to find the zeros of the given Riccati polynomial p defined in (3.1).

4. A variation of the algebraic Riccati equation for quaternions. If we have a look at the transition from the Riccati differential equation (2.1) to the algebraic Riccati equation (2.2), the occurrence of the quadratic term in the form $\mathbf{XDX}$ is not implied by logic. A quadratic term of the form $\mathbf{DX}^2\mathbf{E}$ could be justified as well. However, because of the term $\mathbf{X}^2$, the corresponding algebraic Riccati equation must consist of square matrices of the same size, and it would read

$$p(\mathbf{X}) := \mathbf{A} + \mathbf{BX} + \mathbf{XC} + \mathbf{DX}^2\mathbf{E}, \quad \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{X} \in \mathbb{C}^{n\times n}. \tag{4.1}$$

If we reduce this equation to the quaternionic case, then the quaternions corresponding to $\mathbf{D}, \mathbf{E}$ can be assumed to be nonzero, because otherwise, the remaining part is linear. Thus, the corresponding quaternionic Riccati equation can be written in the form

$$p(x) := a + bxc + dxe + x^2, \quad a, b, c, d, e, x \in \mathbb{H}. \tag{4.2}$$

The quadratic polynomial p is now two-sided where the linear part appears twice. The corresponding equation with one and two linear terms has been treated in [9], and it was shown that besides isolated and spherical zeros, there are three more types of zeros. We assume for the remaining part of the paper that p, given in (4.2), cannot be reduced to the one-sided case. Otherwise, the theory just developed would apply. This would happen if both c, d or both b, e are real. In order to be able to treat this case we replace the quaternions a, b, c, d, e, x by the isomorphic, real (4×4) matrices as defined in (1.2). We will denote the corresponding matrix space by $\mathbb{H}_{\mathbb{R}}$. We use the fact, that arbitrary powers $\mathbf{A}^j, j \geq 0$ of (real or complex) square matrices $\mathbf{A}$ of order n which have a minimal polynomial of degree $\nu \leq n$, can be represented in the form

$$\mathbf{A}^j \in \langle \mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^{\nu-1} \rangle, \quad j \geq 0,$$

where $\langle \cdots \rangle$ indicates the span of $\cdots$ See Horn and Johnson, [7, p. 87]. All matrices in $\mathbb{H}_{\mathbb{R}}$ which do not correspond to real quaternions have a minimal polynomial of degree two. Let $\mathbf{A} := \mathrm{i}_{\mathbb{R}}(a)$, $a = (a_1, a_2, a_3, a_4)$ be defined as in (1.2). The explicit form of the minimal polynomial of $\mathbf{A}$ is given in [8, formula (2.16)]. It reads

$$\mu(z) := z^2 - 2a_1 z + |a|^2. \tag{4.3}$$

Thus,

$$\begin{aligned} \mathbf{A}^j &= u_j \mathbf{I} + v_j \mathbf{A}, \quad \mathbf{A} \in \mathbb{H}_{\mathbb{R}},\ j \geq 0 \text{ and in particular} \\ \mathbf{I} = \mathbf{A}^0 &= 1 \cdot \mathbf{I} + 0 \cdot \mathbf{A} \Rightarrow u_0 = 1,\ v_0 = 0, \\ \mathbf{A} = \mathbf{A}^1 &= 0 \cdot \mathbf{I} + 1 \cdot \mathbf{A} \Rightarrow u_1 = 0\,, v_0 = 1, \\ \mathbf{A}^2 &= -|a|^2\, \mathbf{I} + 2a_1 \mathbf{A} \Rightarrow u_2 = -|a|^2,\ v_2 = 2a_1, \end{aligned} \tag{4.4}$$

where the last formula follows via Hamilton's theorem from (4.3). From here it is easy to derive a recursion formula for all $u_j, v_j, j \geq 0$, but we don't need it here. See [9, Lemma 3.1]. Since $\mathbb{H}$ and $\mathbb{H}_{\mathbb{R}}$ are isomorphic, we can rewrite (4.4) as

$$z^j = u_j + v_j z, \quad z \in \mathbb{H},\ j \geq 0, \tag{4.5}$$

where $u_j, v_j,\ j \geq 0$ are the same factors as in (4.4). See also [17]. If we apply (4.5) for $j = 2$ and (1.4) to the polynomial p defined in (4.2) we obtain

$$\begin{aligned} \mathrm{col}(p(x)) :&= \mathrm{col}(a) + \mathrm{i}_3(b,c)\mathrm{col}(x) + \mathrm{i}_3(d,e)\mathrm{col}(x) + 2x_1 \mathrm{col}(x) - |x|^2 \mathrm{col}(1) \\ &= \mathrm{col}(a) - |x|^2 \mathrm{col}(1) + \left(\mathrm{i}_3(b,c) + \mathrm{i}_3(d,e) + 2x_1 \mathbf{I}\right) \mathrm{col}(x) \\ &=: B(x) + \mathbf{A}(x)\mathrm{col}(x), \end{aligned} \tag{4.6}$$

where i_3 was defined at (1.5), and where $B(x) \in \mathbb{R}^{4\times 1}$ and $\mathbf{A}(x) \in \mathbb{R}^{4\times 4}$.

Conditions (3.10) characterize the equivalence of two quaternions. Therefore we have:

Corollary 4.1. *Let x_0 be a nonreal zero of p, defined in (4.2). Then, the vector $B(x)$ and the matrix $\mathbf{A}(x)$ are constant on $x \in [x_0]$. This implies that all zeros of p in $[x_0]$ can be found by solving the linear system $B(x_0) + \mathbf{A}(x_0)\mathrm{col}(x) = 0$.*

The restriction that x_0 should be nonreal excludes the trivial case that $[x_0] = \{x_0\}$, which means that the equivalence class $[x_0]$ consists only of one point. This corollary gives rise to the classification of the zeros of p.

Definition 4.2. Let x_0 be a zero of p. We call x_0 a *zero of rank k* if $\mathrm{rank}(\mathbf{A}(x_0)) = k,\ k = 0, 1, 2, 3, 4$ where $\mathbf{A}(x_0)$ is defined in (4.6).

If x_0 is a zero of rank 4, then it is clear, that the linear system $B(x_0) + \mathbf{A}(x_0)\mathrm{col}(x) = 0$ has a unique solution in $[x_0]$, namely x_0, and the zero x_0 is isolated. If it happens that x_0 is a zero of rank zero, which implies $\mathbf{A}(x_0) = \mathbf{0}, B(z_0) = \mathbf{0}$ then all $x \in [x_0]$ will be zeros of p. Zeros of this type are spherical. The examples 5.1, (a), (b), (c) in [9] show, that all ranks 2, 3, 4 can appear. Whether rank 1 is also possible is unknown to the authors.

The task to find one zero is still remaining. The application of Newton's method is in many cases very successful. In order to employ Newton's method the Jacobi matrix of the mapping defined by the polynomial p is needed.

THEOREM 4.3. *Let p be given as in (4.2). Then the Jacobi matrix, $\mathbf{J}$, of p is*

$$\mathbf{J}(x) = \mathrm{i}_3(b, c) + \mathrm{i}_3(d, e) + \mathrm{i}_3(x, 1) + \mathrm{i}_3(1, x), \tag{4.7}$$

where i_3 is defined in (1.5).

Proof. The Jacobi matrix is the matrix which defines the linear mapping given by $p(x + h)$ with respect to h. Now

$$\begin{aligned} p(x + h) &= a + b(x + h)c + d(x + h)e + (x + h)(x + h) \\ &= a + bxc + bhc + dxe + dhe + x^2 + xh + hx + h^2. \end{aligned}$$

The part which is linear in h is

$$L(h) := bhc + dhe + xh1 + 1hx,$$

where 1 stands for the quaternion $(1, 0, 0, 0)$. An application of (1.6) to L yields

$$\mathrm{col}(L(h)) = (\mathrm{i}_3(b, c) + \mathrm{i}_3(d, e) + \mathrm{i}_3(x, 1) + \mathrm{i}_3(1, x))\,\mathrm{col}(h),$$

whiche proves (4.7). $\square$

Let $x = (x_1, x_2, x_3, x_4)$. The last two terms of (4.7) can be combined to

$$\mathrm{i}_3(x, 1) + \mathrm{i}_3(1, x) = 2 \begin{bmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & 0 & 0 \\ x_3 & 0 & x_1 & 0 \\ x_4 & 0 & 0 & x_1 \end{bmatrix}.$$

This matrix is a so-called *arrow matrix*. See [21]. Since this matrix is occurring in a sum with matrices of other types, this arrow structure is, however, not so important here.

Newton's method consists of repeatedly solving the linear system

$$p(x_{\mathrm{old}}) + \mathbf{J}(x_{\mathrm{old}})h = 0, \quad x_{\mathrm{new}} := x_{\mathrm{old}} + h,\ x_{\mathrm{old}} \leftarrow x_{\mathrm{new}}$$

for h, where the first x_{old} has to be guessed.

The simple technique, shown in the proof of Theorem 4.3, for finding the (exact) Jabobi matrix applies to many other cases, as well, in particular, for p being a general polynomial in quaternions or in matrices or in other noncommutative algebras. It is much superior to what we have shown in [9, Section 7].

We will mention another topic. How many solutions can be expected from quaternionic Riccati equations? Since it is possible that a whole equivalence class consists of zeros (see the example on p. 4), it does not make sense to count the individual zeros.

DEFINITION 4.4. Let p be a quaternionic Riccati polynomial, either of the form (3.1) or of the form (4.2). The number of equivalence classes which contain zeros of p will be called the *essential number of zeros of p*.

The already mentioned result by Gordon and Motzkin, [5] is, that n is the maximal essential number of zeros for one-sided polynomials of degree n. Thus, the essential number of zeros of a Riccati polynomials of type (3.1) is two. For two-sided quaternionic polynomials a bound of the essential number of zeros is not known. We will discuss an example from [9].

Example 4.5. Let

$$p(x) := 1 + \mathbf{i}x\mathbf{j} + x^2. \tag{4.8}$$

In [9, Lemma 4.1] it was shown, that the zeros of all quadratic polynomials of the type $p(x) = a + bxc + x^2$ can have only even ranks. This applies to p of (4.8) as well. In order to find the zeros we employ Newton with the exact Jacobi matrix, (4.7), and obtain the following zeros:

Table 4.6. Zeros of Riccati polynomial p of (4.8).

zero	rank	equivalent zero	rank
$z_1 = 0.5(1,1,1,-1)$	2	$z_2 = 0.5(1,-1-1,-1)$	2
$z_3 = 0.5(1-\sqrt{5})\mathbf{k}$	4		
$z_4 = 0.5(1+\sqrt{5})\mathbf{k}$	4		
$z_5 = -0.5(1,1,-1,1)$	2	$z_6 = 0.5(-1,1,-1,-1)$	2

Thus, the essential number of zeros of p of (4.8) is at least four. We cannot prove, that the above list of zeros is exhaustive. But there is strong evidence from many numerical tests. This example shows that the essential number of zeros of two-sided quaternionic polynomials is not bounded by the degree n.

Rather than counting the zeros, it is possible (see [16, Section 4]) to find a region Z in $\mathbb{R}^4$ which must contain all zeros of p, where

$$Z := \{z \in \mathbb{R}^4 : r \le |z| \le R\}, \tag{4.9}$$

and the two quantities r, R are determined in the next theorem.

Theorem 4.7. *Let*

$$p(x) := a + bxc + dxe + x^2, \quad a, b, c, d \in \mathbb{H},\ a \neq 0.$$

Put $A_0 = |a|$, $A_1 = |bc| + |de|$. Denote the only positive zero of $-A_0 + A_1 x + x^2$ by r, and define $R := \max\{1, A_0 + A_1\}$. Then, all zeros of p are located in Z, defined in (4.9). For r we have $r = 0.5\left(\sqrt{A_1^2 + 4A_0} - A_1\right)$.

Proof. [16, Corollary 4.4 and Corollary 4.5]. □

If we apply this theorem to the polynomial $p(x) := 1 + \mathbf{i}x\mathbf{j} + x^2$ of Example 4.5, we obtain $r = 0.5(\sqrt{5} - 1) \approx 0.6180$, $R = 2$. And for the 6 zeros of p we have $1 = |z_1| = |z_2| = |z_5| = |z_6|$, $|z_3| = r$, $|z_4| = 0.5(1 + \sqrt{5}) \approx 1.6180$. Thus, the lower bound r is attained.

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