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ZEROS AND SINGULAR POINTS FOR ONE-SIDED, COQUATERNIONIC POLYNOMIALS WITH AN EXTENSION TO OTHER $\mathbb{R}^4$ ALGEBRAS

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Abstract. For finding the zeros of a coquaternionic polynomial p of degree n, where p is given in the standard form $p(z) = \sum c_j z^j$, the concept of a (real) companion polynomial q of degree $2n$, as introduced for quaternionic polynomials, is applied. If z_0 is a root of q, then, based on z_0, there is a simple formula for an element z with the property that $\overline{p(z)}p(z) = 0$, thus, z is a singular point of p. Under certain conditions, the same z has the property that $p(z) = 0$, thus, z is a zero of p. There is an algorithm for finding zeros and singular points of p. This algorithm will find all zeros z with the property that in the equivalence class, to which z belongs, there are complex elements. For finding zeros which are not similar to complex numbers, Newton's method is applied and a simple technique for computing the exact Jacobi matrix is presented. We also show, that there is no "Fundamental Theorem of Algebra" for coquaternions, but there is a conjecture, that a "Weak Fundamental Theorem of Algebra" for coquaternions is valid. Several numerical examples are presented. It is also shown how to apply the given results to other algebras of $\mathbb{R}^4$ like tessarines, cotessarines, nectarines, conectarines, tangerines, cotangerines.

Key words: Zeros of coquaternionic polynomials, zeros of polynomials in split quaternions, companion polynomial for coquaternionic polynomials, singular points for coquaternionic polynomials, Newton method for coquaternionic polynomials, exact Jacobi matrix for coquaternionic polynomials, "Weak Fundamental Theorem of Algebra" for coquaternions, zeros of polynomials in other $\mathbb{R}^4$ algebras (tessarines, cotessarines, nectarines, conectarines, tangerines, cotangerines).

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1. Introduction. In this paper we will use the notations $\mathbb{Z}, \mathbb{R}, \mathbb{C}, \mathbb{H}$ for the integers, the real number system, the complex numbers and the quaternions, respectively. Coquaternions were introduced, 1849, by Sir James Cockle (1819–1895), [3], [4]. They may be regarded as elements of $\mathbb{R}^4$ of the form

$$a := a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}, \quad a_1, a_2, a_3, a_4 \in \mathbb{R} \tag{1.1}$$

which we also abbreviate by $a = (a_1, a_2, a_3, a_4)$ and which obey the multiplication rules given in Table 1.1.

TABLE 1.1. Multiplication table for coquaternions. Red entries differ in sign from the quaternionic case.

(1.2)

	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	1	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	1

The algebra of coquaternions will be abbreviated by $\mathbb{H}_{\rm coq}$. The explicit multiplication rule for the product ab of two coquaternions $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ derived from Table 1.1 is

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$$
\begin{aligned}
ab = \quad & a_1b_1 - a_2b_2 + a_3b_3 + a_4b_4 \\
&+(a_1b_2 + a_2b_1 - a_3b_4 + a_4b_3)\mathbf{i} \\
&+(a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2)\mathbf{j} \\
&+(a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1)\mathbf{k},
\end{aligned} \tag{1.3}
$$

which implies that

$$
a^2 = \quad a_1^2 - a_2^2 + a_3^2 + a_4^2 + 2a_1(a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k}). \tag{1.4}
$$

As elements of $\mathbb{R}^4$ we have $1 := (1,0,0,0)$, $\mathbf{i} := (0,1,0,0)$, $\mathbf{j} := (0,0,1,0), \mathbf{k} := (0,0,0,1)$. If a coquaternion has the form $a = (a_1,0,0,0)$ we will call it *real* and identify it with a_1. The real coquaternions and no others have the property that they commute with all coquaternions. In algebraic terms that means, that the *center* of $\mathbb{H}_{\rm coq}$ is $\mathbb{R}$. The first component, a_1, of a coquaternion $a = (a_1,a_2,a_3,a_4)$ will be called *real part* of a and abbreviated by $\Re(a)$. The second component, a_2, will be called the *imaginary part* of a and denoted by $\Im(a)$. Complex numbers $a_1+a_2\mathbf{i}$ will be identified with $a := (a_1,a_2,0,0) \in \mathbb{H}_{\rm coq}$ and vice versa, and the coquaternion a will be called *complex*. We will use the notations

$$
\mathrm{abs}_2(a) := a_1^2 + a_2^2 - a_3^2 - a_4^2, \quad \overline{a} := \mathrm{conj}(a) := (a_1,-a_2,-a_3,-a_4), \tag{1.5}
$$

where $\overline{a}$ and $\mathrm{conj}(a)$ as well are called the *conjugate of* a and it should be noted, that abs_2 is not the square of a norm since abs_2 may take on negative values. There are the following further properties:

$$
a\overline{a} = \overline{a}a = \mathrm{abs}_2(a),\ \mathrm{abs}_2(ab) = \mathrm{abs}_2(ba) = \mathrm{abs}_2(a)\mathrm{abs}_2(b),\ \Re(ab) = \Re(ba). \tag{1.6}
$$

A coquaternion a will be called *invertible* if $\mathrm{abs}_2(a) \neq 0$ and in this case

$$
a^{-1} = \frac{\overline{a}}{\mathrm{abs}_2(a)}, \quad \mathrm{abs}_2(a^{-1}) = \frac{1}{\mathrm{abs}_2(a)}. \tag{1.7}
$$

A noninvertible coquaternion is also called *singular*. An invertible coquaternion is called *nonsingular*. A product $a_1a_2\cdots a_k$, $k \geq 1$ is singular if and only if one of the factors is singular. This follows from the second formula in (1.6). A more detailed survey of properties of coquaterions is given in [12]. Applications to physical problems are treated in [2, 7]. Information regarding general algebraic systems can be found in [1, 5, 8, 18].

Let

$$
p(z) := \sum_{j=0}^{n} c_j z^j, \quad z,\ c_j \in, \mathbb{H}_{\rm coq},\ j = 0,1,\ldots,n, c_0 \neq 0,\ c_n \neq 0 \tag{1.8}
$$

be a coquaternionic polynomial. The two conditions in the end of (1.8) ensure that $p(0) \neq 0$ and that the degree of p is not smaller than n. Because of the noncommutativity of the elements in $\mathbb{H}_{\rm coq}$ we call this polynomial *one-sided*, since there are other forms of polynomials with terms of the form $c_j z^j d_j$, called *two-sided*, or even with terms of the form $c_0zc_1zc_2z\cdots c_{j-1}zc_j$, which will be called a *monomials of degree* j. An arbitrary finite sum of monomials of any degree would be called *general coquaternionic polynomial*. Since we are dealing here only with one-sided coquaternionic polynomials, we will delete the word one-sided. Solutions of $p(z) = 0$ will be called *zeros* of p. Before we start any general investigation we will treat the most simple quadratic case, since it sheds already some light on the general case.

EXAMPLE 1.2. Let

$$p(z) := z^2 - c, \quad z = (z_1, z_2, z_3, z_4),\ c = (c_1, c_2, c_3, c_4) \in \mathbb{H}_{\rm coq}, \tag{1.9}$$

and let us look for those z with $p(z) = 0$. We will call these z also *square roots of* c and use the notation $\sqrt{c}$. The equation $p(z) = 0$ splits into four real equations (see (1.4))

$$z_1^2 - z_2^2 + z_3^2 + z_4^2 = c_1, \tag{1.10}$$

$$2z_1 z_j = c_j,\ j = 2, 3, 4. \tag{1.11}$$

(a) Let $c \neq 0$ be nonreal, i. e. $c_j \neq 0$ for at least one $j \in \{2, 3, 4\}$. Then, (1.11) implies $z_1 \neq 0$ and

$$z_j = \frac{c_j}{2z_1},\ j = 2, 3, 4,$$

$$z_1^2 - \left(\frac{c_2}{2z_1}\right)^2 + \left(\frac{c_3}{2z_1}\right)^2 + \left(\frac{c_4}{2z_1}\right)^2 = c_1.$$

The last equation can also be written as

$$z_1^4 - c_1 z_1^2 + \frac{-c_2^2 + c_3^2 + c_4^2}{4} = 0. \tag{1.12}$$

The standard solution formula yields

$$z_1^2 = \frac{1}{2}\left(c_1 \pm \sqrt{\mathrm{abs}_2(c)}\right). \tag{1.13}$$

Since $z_1 \neq 0$ must be real, the solution depends on the validity of the conditions

$$(i)\ \mathrm{abs}_2(c) \geq 0, \quad (ii)\ c_1 - \sqrt{\mathrm{abs}_2(c)} > 0, \quad (iii)\ c_1 + \sqrt{\mathrm{abs}_2(c)} > 0, \tag{1.14}$$

and the result is summarized in Lemma 1.5.

(b) Let $c = \Re(c)$, i. e. $c = (c_1, 0, 0, 0)$. In this case, (1.11) has several solutions:

(b1) $z_1 = 0$ and $z_j, j = 2, 3, 4$ arbitrary, or

(b2) $z_j = 0, j = 2, 3, 4$ and z_1 arbitrary.

In case (b1), equations (1.10), (1.11) yield

$$z_1 = 0, \quad -z_2^2 + z_3^2 + z_4^2 = c_1. \tag{1.15}$$

In case (b2), equations (1.10), (1.11) yield

$$z_1^2 = c_1, \quad z_j = 0,\ j = 2, 3, 4. \tag{1.16}$$

Case (b2) will not have a solution if $c_1 < 0$, but case (b1) will have infinitely many solutions. For $c = 0$, e g., we obtain from (1.15) $\sqrt{c} = (0, z_2, z_3.z_4)$ with $-z_2^2 + z_3^2 + z_4^4 = 0$.

The just encountered phenomenon, that polynomials may have infinitely many zeros is a typical feature for all coquaternionic polynomials with real coefficients.

THEOREM 1.3. *Let p be a coquaternionic polynomial as defined in (1.8) where all coefficients $c_j, j = 0, 1, \ldots, n$ are real. Then,*

$$p(z) = 0 \Rightarrow p(h^{-1}zh) = 0 \text{ for all nonsingular } h \in \mathbb{H}_{\rm coq}. \tag{1.17}$$

In particular, if z is nonreal, p has infinitely many zeros.

Proof. From the fact, that all real elements (and no others) commute with all elements in $\mathbb{H}_{\rm coq}$ we deduce

$$h^{-1}p(z)h = h^{-1}\left(\sum_{j=0}^{n} c_j z^j\right) h = \sum_{j=0}^{n} c_j h^{-1} z^j h = \sum_{j=0}^{n} c_j (h^{-1}zh)^j = p(h^{-1}zh).$$

The set $\{u \in \mathbb{H}_{\rm coq} : u = h^{-1}zh \text{ for all nonsingular } h \in \mathbb{H}_{\rm coq}\}$ contains exactly one element if $z \in \mathbb{R}$, otherwise it contains infinitely many elements. ☐

Lemma 1.4. *Let p be given as in (1.9) with $c := (c_1, 0, 0, 0) \in \mathbb{H}_{\rm coq}$. Then, there exist infinitely many zeros $z = (0, z_2, z_3, z_4)$ of p defined in (1.15). If $c_1 \geq 0$ there are up to two additional zeros $\sqrt{c} = (\pm\sqrt{c_1}, 0, 0, 0)$. For $c_1 < 0$ there are no additional zeros.*

Proof. Follows from the solution formulas (1.15), (1.16). ☐

Lemma 1.5. *Let p be given as in (1.9) with $c \in \mathbb{H}_{\rm coq}\backslash\mathbb{R}$.* (I) *If (i) of (1.14) is not valid, p has no zeros.* (II) *If (i) is valid but (ii) and (iii) are not valid, p has no zeros.* (III) *If (i) and (ii) are valid, p has up to four zeros.* (IV) *If (i) and (iii) are valid, but (ii) is not valid, p has up to two zeros.*

Proof. In view of the solution formula (1.13) for z_1, which yields two solutions if the expression in parentheses is positive, the cases (I), (II), (IV) are clear. In (III) the validity of (ii) implies the validity of (iii). ☐

To summarize: Infinitely many square roots will always occur if c is real and in no other cases. Conditions that there is no square root at all are given in (I), (II) of Lemma 1.5.

Example 1.6. For each case mentioned in Lemma 1.5 there is one example:

$$\begin{aligned}
\text{(I)}: &\ c := (1,2,3,4) \text{ has no square root since } \mathrm{abs}_2(c) = -20 < 0,\\
\text{(II)}: &\ c := (-2,1,2,0) \text{ has no square root since } \mathrm{abs}_2(c) = 1 > 0 \text{ but,}\\
&\ c_1 - \sqrt{\mathrm{abs}_2(c)} = -3 < 0, c_1 + \sqrt{\mathrm{abs}_2(c)} = -1 < 0,\\
\text{(III)}: &\ c := (2,1,2,0) \text{ has four square roots since } \mathrm{abs}_2(c) = 1 > 0,\\
&\ c_1 - \sqrt{\mathrm{abs}_2(c)} = 1 > 0, c_1 + \sqrt{\mathrm{abs}_2(c)} = 3 > 0,\\
&\ \sqrt{c} = \pm\sqrt{2}\,(1/2, 1/2, 1, 0),\ \sqrt{c} = \pm\sqrt{6}\,(1/2, 1/6, 1/3, 0),\\
\text{(IV)}: &\ c := (4,6,5,1) \text{ has two square roots since } \mathrm{abs}_2(c) = 26 > 0,\\
&\ c_1 - \sqrt{\mathrm{abs}_2(c)} = 4 - \sqrt{26} < 0,\ a := c_1 + \sqrt{\mathrm{abs}_2(c)} = 4 + \sqrt{26} > 0,\\
&\ \sqrt{c} = \pm\sqrt{2a}\left(\frac{1}{2}, \frac{3}{a}, \frac{5}{2a}, \frac{1}{2a}\right).
\end{aligned}$$

Corollary 1.7. *There is no "Fundamental Theorem of Algebra" for coquaternions.*

Proof. There is a well known "Fundamental Theorem of Algebra" for quaternions by Eilenberg and Niven, 1944, [6] ensuring, that all general quaternionic polynomials of degree $n \geq 1$ have at least one zero, provided that there is only one monomial term of the highest degree n. However, in the coquaternionic case, the first two parts of Lemma 1.5 show, that there are coquaternionic, quadratic polynomials without any zeros. ☐

2. Similarity and quasi similarity of coquaternions. In Lam, p. 52, [15] and in Janovská and Opfer, [12] we find that $\mathbb{H}_{\mathrm{coq}}$ is isomorphic to $\mathbb{R}^{2\times 2}$, where this isomorphism is defined by

$$\hat{\imath}(a) = \hat{\imath}(a_1, a_2, a_3, a_4) := \begin{bmatrix} a_1 + a_4 & a_2 + a_3 \\ -a_2 + a_3 & a_1 - a_4 \end{bmatrix}. \tag{2.1}$$

The mapping $\hat{\imath} : \mathbb{H}_{\mathrm{coq}} \to \mathbb{R}^{2\times 2}$ is invertible. Let $\mathbf{B} := \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$ be arbitrary. Then,

$$\hat{\imath}^{-1}(\mathbf{B}) = \frac{1}{2}\big(b_{11} + b_{22}, b_{12} - b_{21}, b_{12} + b_{21}, b_{11} - b_{22}\big).$$

Note that

$$\mathrm{abs}_2(a) = \det(\hat{\imath}(a)), \quad 2\Re(a) = \mathrm{tr}(\hat{\imath}(a)), \tag{2.2}$$

where det, tr stand for *determinant, trace*, respectively. We will transfer the similarity concept of matrices to coquaternions.

Definition 2.1. Two coquaternions a, b will be called *similar*, denoted by $a \sim b$ if the corresponding matrices $\hat{\imath}(a), \hat{\imath}(b)$ are similar, or in other words if there is a nonsingular (= invertible) coquaternion h such that $a = h^{-1}bh$. By

$$[a] := \{b : b = h^{-1}ah, \text{ for all invertible } h \in \mathbb{H}_{\mathrm{coq}}\}$$

we denote the equivalence class of all coquaternions which are similar to a.

That similarity is an *equivalence relation* is shown by Horn and Johnson, p. 45, [10]. We have the following property:

$$a = \Re(a) \Leftrightarrow [a] = \{a\}. \tag{2.3}$$

Which means that the equivalence class $[a]$ consists of one single element if and only if a is real.

Lemma 2.2. *Let $a \sim b$. Then*

$$\Re(a) = \Re(b), \quad \mathrm{abs}_2(a) = \mathrm{abs}_2(b). \tag{2.4}$$

Proof. Both parts follow easily from conditions given in (1.6). ∎

In contrast to the quaternionic case, the conditions (2.4) are not sufficient for similarity. Take $a = (\alpha, 0, 0, 0)$, $b = (\alpha, 5, 4, 3)$ for an arbitrary $\alpha \in \mathbb{R}$. Since $[a]$ consists of one single element only (see (2.3)), a, b are not similar. However, (2.4) is valid.

Similarity is a very useful tool when investigating properties of matrices, like the determination of the rank of a matrix. However, in this investigation we are mainly interested in a consequence of similarity, namely in the properties mentioned in (2.4) which in some cases are also valid for nonsimilar matrices or nonsimilar coquaternions.

Definition 2.3. Two coquaternions a, b are called *quasi similar*, in signs $a \overset{q}{\sim} b$, if (2.4) is valid.

Lemma 2.4. *The relation $\overset{q}{\sim}$ is an equivalence relation.*

Proof. The three properties $a \overset{q}{\sim} a$ (reflexive), $a \overset{q}{\sim} b \Leftrightarrow b \overset{q}{\sim} a$ (symmetric), $a \overset{q}{\sim} b$, $b \overset{q}{\sim} c \Rightarrow a \overset{q}{\sim} c$ (transitive) are easily verified. ∎

The corresponding equivalence classes are denoted by

$$[a]_q := \{b : b \overset{q}{\sim} a\}.$$

There are the following simple properties:

$$a \sim b \Rightarrow a \overset{q}{\sim} b, \quad [a] \subset [a]_q.$$

Because of the first condition of (2.4), distinct real numbers are in different equivalence classes with respect to $\sim$ and to $\overset{q}{\sim}$. In a later section (Section 4, p. 8) we will see that there is only a small difference between similarity and quasi similarity.

3. Reformulation of coquaternionic polynomials via matrix equivalents. Let $\mathbf{A} \in \mathbb{R}^{2\times 2}$ be arbitrary. Then, (see Horn and Johnson, p. 87, [10])

$$\mathbf{A}^j \in \langle \mathbf{I}, \mathbf{A}\rangle, \quad j \in \mathbb{N} \cup \{0\}, \tag{3.1}$$

where $\langle \cdots \rangle$ is the linear span of what is between the brackets. The characteristic polynomial of $\mathbf{A}$ is (see (2.2))

$$\chi_{\mathbf{A}}(z) := z^2 - \operatorname{tr}(\mathbf{A})z + \det(\mathbf{A}) = z^2 - 2a_1 z + \operatorname{abs}_2(a),$$

The Cayley-Hamilton Theorem (see Horn and Johnson, p. 86, [10]) implies $\mathbf{A}^2 = -\operatorname{abs}_2(a)\mathbf{I} + 2\Re(a)\mathbf{A}$. Because of the isomorphism between $\mathbb{R}^{2\times 2}$ and $\mathbb{H}_{\rm coq}$ and the relation (3.1) we have

$$z^j = \alpha_j + \beta_j z, \quad j = 0, 1, \ldots, \tag{3.2}$$

where the coefficients can be determined by the recursion

$$\alpha_0 = 1, \quad \beta_0 = 0, \tag{3.3}$$

$$\alpha_{j+1} = -\operatorname{abs}_2(z)\beta_j, \quad \beta_{j+1} = \alpha_j + 2\Re(z)\beta_j,\ j = 0, 1, \ldots \tag{3.4}$$

We observe, that all coefficients α_j, β_j, $j \geq 0$ are real and, more important, they depend only on $\Re(z)$ and $\operatorname{abs}_2(z)$ and not fully on z.

LEMMA 3.1. *Let $a, b \in \mathbb{H}_{\rm coq}$ and $a \overset{q}{\sim} b$. Then, a^j and b^j have the same recursion coefficients α_j, β_j, $j \geq 0$ as defined in (3.2).*

Proof. The quasi similarity implies, that the coefficients $\Re, \operatorname{abs}_2$ occurring in (3.4), are the same for a and b. Thus, the recursion coefficients α_j, β_j are the same for a and b. $\square$

There is another difference between coquaternions and quaternions. In the corresponding quaternionic equivalence classes $[a]$ there are always complex elements, provided $a \notin \mathbb{R}$. See [14].

LEMMA 3.2. *Let $a = (a_1, a_2, a_3, a_4) \in \mathbb{H}_{\rm coq}\backslash\mathbb{C}$. Then, $\Delta := [a]_q \cap \mathbb{C} \neq \emptyset$ if and only if*

$$a_2^2 - a_3^2 - a_4^2 \geq 0.$$

And in this case $\Delta = \{b^+, b^-\}$, where $b^\pm = a_1 \pm \sqrt{a_2^2 - a_3^2 - a_4^2}\,\mathbf{i}$.

Proof. Let $b \in \Delta$. Then, $b := b_1 + b_2\mathbf{i}$ and $a_1 = \Re(a) = \Re(b) = b_1$, $a_1^2 + a_2^2 - a_3^2 - a_4^2 = \operatorname{abs}_2(a) = \operatorname{abs}_2(b) = b_1^2 + b_2^2 \Leftrightarrow b_2^2 = a_2^2 - a_3^2 - a_4^2 \geq 0$. By similar arguments, $a_2^2 - a_3^2 - a_4^2 < 0$ implies $\Delta = \emptyset$. $\square$

Thus, in $[a := (1, 2, 3, 4)]_q$ there is no complex number.

We replace z^j in the definition of the polynomial p, defined in (1.8), by the recursion formula for z^j and obtain

$$p(z) := \sum_{j=0}^{n} c_j z^j = \sum_{j=0}^{n} c_j(\alpha_j + \beta_j z) = \sum_{j=0}^{n} \alpha_j c_j + \left(\sum_{j=0}^{n} \beta_j c_j\right) z \tag{3.5}$$
$$=: A(\Re(z), \mathrm{abs}_2(z)) + B(\Re(z), \mathrm{abs}_2(z))z.$$

The two newly defined quantities, A, B, depend only on $\Re(z)$ and $\mathrm{abs}_2(z)$. Therefore, we have put this dependence in parentheses.

Lemma 3.3. *The quantities $A(\Re(z), \mathrm{abs}_2(z))$, $B(\Re(z), \mathrm{abs}_2(z))$ defined in (3.5) are constant on the quasi equivalence class $[z]_q$.*

Proof. The properties (2.4) are valid for all elements in the same quasi equivalence class $[z]_q$. □

Let $a = (a_1, a_2, a_3, a_4)$. In contrast to (2.3) we have

$$a \in [a_1]_q \Leftrightarrow a_2^2 - a_3^2 - a_4^2 = 0. \tag{3.6}$$

Thus, there is no quasi equivalence class with only one element. Let $a = (a_1, 0, 0, 0)$, then, for all $b \in [a]_q$ we have according to (3.6), $\mathrm{abs}_2(a) = \mathrm{abs}_2(b) = a_1^2$. Condition (3.6) also implies that there is no complex element with nonvanishing imaginary part in $[a_1]_q$.

Lemma 3.4. *Let $p(z) = 0$. Then*

$$p(z) = A(\Re(z_0), \mathrm{abs}_2(z_0)) + B(\Re(z_0), \mathrm{abs}_2(z_0))z = 0 \textit{ for all } z_0 \in [z]_q, \tag{3.7}$$
$$Bz = 0 \Leftrightarrow A = 0,\ Bz = 0 \Rightarrow B \textit{ is singular}, \tag{3.8}$$
$$B = 0 \Rightarrow A = 0. \tag{3.9}$$

If $B = 0$ or $Bz = 0$, then all $z_0 \in [z]_q$ are zeros of p where we have deleted the arguments of A, B in (3.8), (3.9).

Proof. (3.7) follows from (3.5) and the second part of Lemma (2.2). In (3.8) $Bz = 0$ implies that B is singular, because $z \neq 0$. In (3.9) $B = 0 \Rightarrow A = 0$ is obvious. The cases $B = 0$ and $Bz = 0$ imply $p(z_0) = 0$ for all z_0 in $[z]_q$. □

Definition 3.5. Let $z \in \mathbb{H}_{\mathrm{coq}} \backslash \mathbb{R}$. If $p(z_0) = 0$ for all $z_0 \in [z]_q$, we say that z *generates a class of hyperbolic zeros* or *z is a hyperbolic zero*. If there is exactly one zero in $[z]_q$, we call that zero *isolated*.

There is one example of a hyperbolic zero in Example 1.2, formula (1.15). The conditions given there (second condition of (1.15)), abbreviated to the hyperbola equation $x^2 - y^2 = c$, are the reason for using the word *hyperbolic*.

Example 3.6. Let p be a coquaternionic polynomial with only real coefficients. Then, all nonreal zeros of p are hyperbolic. See Theorem 1.3.

Lemma 3.7. *Let u, v be two distinct zeros of a coquaternionic polynomial p in the same quasi equivalence class $[u]_q = [v]_q$. Let $z \in [u]_q$. Then, B in the representation $p(z) = A + Bz$, where the arguments of A, B are deleted, is singular. And $B = 0$ if $u - v$ is nonsingular. If $B = 0$ all elements in $[u]_q = [v]_q$ are zeros of p, and all zeros are hyperbolic.*

Proof. It follows that $p(u) = A + Bu = 0, p(v) = A + Bv = 0$, with the consequence that $p(u) - p(v) = B(u - v) = 0$, because A, B are constant on $[u]_q$. Thus, B is singular. If $u - v$ is nonsingular, it follows that $B = 0$. If $B = 0$, the statement follows from Lemma 3.4. Two distinct zeros in the same equivalence class can never be real. Thus, $[u]_q$ consists of (infinitely many) hyperbolic zeros of p. □

We can refine the representation (3.5) of the coquaternionic polynomial p, by applying the *column operator* $\mathrm{col}: \mathbb{H}_{\mathrm{coq}} \to \mathbb{R}^{4\times 1}$ which is defined by

$$\mathrm{col}(h) = \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}, \tag{3.10}$$

where $h = (h_1, h_2, h_3, h_4) \in \mathbb{H}_{\mathrm{coq}}$. The following lemma is very useful in this context.

LEMMA 3.8. *Let B, z be two coquaternions. (α) There is a real 4×4 matrix $\mathbf{C}$, such that*

$$\mathrm{col}(Bz) = \mathbf{C}\,\mathrm{col}(z) \ \textit{and} \tag{3.11}$$

$$\mathbf{C} = [\mathrm{col}(B), \mathrm{col}(B\mathbf{i}), \mathrm{col}(B\mathbf{j}), \mathrm{col}(B\mathbf{k})]. \tag{3.12}$$

(β) The matrix $\mathbf{C}$ is singular if and only if B is singular. (γ) If $B \neq 0$ is singular, $\mathrm{rank}(\mathbf{C}) = 2$.

Proof. (α) and (β) follow from [12], Formula(5.2) and Theorem 5.1. (γ) Let $B \neq 0$ be singular. One can explictly show, that the last two columns of $\mathbf{C}$ can be linearly expressed by the first two columns. ◻

The application of the col operator to (3.5) yields now

$$\mathrm{col}(p(z)) = \mathrm{col}(A) + \mathrm{col}(Bz) = \mathrm{col}(A) + \mathbf{C}\mathrm{col}(z), \tag{3.13}$$

where $\mathbf{C}$ is defined in Lemma 3.8.

THEOREM 3.9. *Let $[z_0]_q$ contain a zero of the coquaternionic polynomial p. Then, all zeros of p in $[z_0]_q$ can be found by solving the linear, inhomogeneous system*

$$\mathbf{C}\mathrm{col}(z) = -\mathrm{col}(A), \ \textit{subject to } z \in [z_0]_q. \tag{3.14}$$

(i) Let B be nonsingular.Then, there is exactly one (isolated) zero $z \in [z_0]_p$ which can be computed by

$$z = -B^{-1}A = -\frac{\mathrm{conj}(B)A}{\mathrm{abs}_2(B)} \Leftrightarrow \mathrm{col}(z) = -\mathbf{C}^{-1}\mathrm{col}(A), \tag{3.15}$$

where A, B are short forms for $A(\Re(z_0), \mathrm{abs}_2(z_0)), B(\Re(z_0), \mathrm{abs}_2(z_0))$, respectively, and where $\mathbf{C}$ is defined in (3.12). (ii) Let B be singular. Then the linear system given in (3.14) has a solution if the extended matrix $[\mathbf{C}, col(A)]$ has the same rank as $\mathbf{C}$.

Proof. Follows from Lemma 3.8 and from (3.7) in Lemma 3.4 and some simple facts from linear algebra. ◻

4. Symmetric and normal coquaternions. A comparison of similarity and quasi similarity. We will make a very short excursion to symmetric and normal coquaternions and show that in most cases, similarity and quasi similarity are identical notions.

DEFINITION 4.1. A coquaternion a is called *symmetric, normal* if the matrix $\hat{\imath}(a)$, defined in (2.1) has these properties, respectively. Thus, a is symmetric, normal if

$$\hat{\imath}(a) = \hat{\imath}(a)^{\mathrm{T}}, \quad \hat{\imath}(a)\hat{\imath}(a)^{\mathrm{T}} - \hat{\imath}(a)^{\mathrm{T}}\hat{\imath}(a) = \mathbf{0},$$

respectively, where $^{\mathrm{T}}$ indicates transposition.

Lemma 4.2. *A coquaternion a is symmetric, if and only if $\Im(a) = 0$. A coquaternion a is normal if and only if $\Im(a) = 0$ or $a \in \mathbb{C}$.*

Proof. The condition for symmetry follows immediately from (2.1). Let $a \in \mathbb{H}_{\text{coq}}$ be arbitrary and $a = (a_1, a_2, a_3.a_4)$. Then

$$\hat{\imath}(a)\hat{\imath}(a)^{\mathrm{T}} - \hat{\imath}(a)^{\mathrm{T}}\hat{\imath}(a) = -4a_2 \begin{bmatrix} -a_3 & a_4 \\ a_4 & a_3 \end{bmatrix}.$$

□

Thus, a coquaternion is normal if and only if it is symmetric or complex. Let $a, b \in \mathbb{H}_{\text{coq}}$ be normal. Then, the conditions in (2.4) are sufficient for similarity of a, b. See Horn and Johnson, p. 109, Problem 15, [10]. Thus, $a \sim b \Leftrightarrow a \overset{q}{\sim} b$ if a and b are normal. Two normal coquaternions $a = (a_1, 0, a_3, a_4), b = (b_1, b_2, 0, 0)$ are similar if and only if $a_1 = b_1$ and $b_2 = a_3 = a_4 = 0$, which means $a = b \in \mathbb{R}$. Two distinct complex numbers are similar (=quasi similar) if and only if they are conjugate complex to each other. Two normal coquaternions $a = (a_1, 0, a_3, a_4), b = (b_1, 0, b_3, b_4)$ are similar (=quasi similar) if and only if

$$a_1 = b_1, \quad a_3^2 + a_4^2 = b_3^2 + b_4^2.$$

Lemma 4.3. *Let $a, b \in \mathbb{H}_{\text{coq}} \backslash \mathbb{R}$ and let (2.4) be valid. Then $a \sim b$.*

Proof. The assumption (2.4) implies that the two characteristic polynomials for $\hat{\imath}(a)$ and $\hat{\imath}(b)$ (see 2.1) are identical. The assumption $a \notin \mathbb{R}$ is in matrix terms equivalent to $\hat{\imath}(a) \neq \alpha\mathbf{I}$, $\alpha \in \mathbb{R}$, where $\mathbf{I}$ is the identity matrix in $\mathbb{R}^{2\times 2}$. In other words $\hat{\imath}(a)$ does not belong to the center of $\mathbb{R}^{2\times 2}$. The two possible canonical Jordan normal forms of $\hat{\imath}(a)$ and $\hat{\imath}(b)$ are

$$(i): \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}, \quad (ii): \begin{bmatrix} z & 1 \\ 0 & z \end{bmatrix}.$$

The assumptions imply $x \neq y$ and thus, all eigenvalues (in both forms) have geometric multiplicity one. See Horn and Johnson, p. 135, [10]. That means, similarity $a \sim b$ is equivalent to the two conditions (2.4), and thus, equivalent to quasi similarity, provided, both, $a, b \notin \mathbb{R}$. □

5. Singular points of coquaternionic polynomials. The zero point in $\mathbb{R}, \mathbb{C}, \mathbb{H}$ is in all these three cases the only singular point, in the sense that it has no inverse. This is one of the reasons why polynomials in these three spaces are investigated for its zeros. This applies also to eigenvalues in matrix spaces over $\mathbb{R}, \mathbb{C}$ because they are zeros of the corresponding characteristic polynomials. In the space $\mathbb{H}_{\text{coq}}$, however, there are infinitely many singular points. Therefore, it is not surprising that we will encounter points z such that $p(z)$ is singular, i. e. $\mathrm{abs}_2(p(z)) = 0$, but $p(z) \neq 0$, where p is defined in (1.8). We will conjecture in Section 11, that all coquaternionic polynomials of degree $n \geq 1$ have at least one singular point.

Definition 5.1. Let p be given as in (1.8). We will say that $z \in \mathbb{H}_{\text{coq}}$ is a *singular point for* p if

$$\mathrm{abs}_2(p(z)) := p(z)\overline{p(z)} = 0. \tag{5.1}$$

It is clear, that a zero of p is also a singular point for p. In contrast to the polynomial equation $p(z) = 0$, which may be regarded as four real equations in four real unknowns, equation (5.1) is only one equation in four real unknowns. Thus, it is not a polynomial equation.

EXAMPLE 5.2. Let $p(z) := z^2 - c$ where c is a given coquaternion. Then,

$$\mathrm{abs}_2(p(z)) = (z^2 - c)\overline{(z^2 - c)} = \mathrm{abs}_2(z^2) - 2\Re(c\overline{z}^2) + \mathrm{abs}_2(c).$$

Now, $\mathrm{abs}_2(z^2) = (\mathrm{abs}_2(z))^2$, which follows from (1.6). The middle term is

$$\Re(c\,\overline{z}^2) = \Re(z^2\,\overline{c}) = c_1(z_1^2 - z_2^2 + z_3^2 + z_4^2) + 2z_1(c_2z_2 - c_3z_3 - c_4z_4).$$

Altogether,

$$\mathrm{abs}_2(p(z)) = (z_1^2+z_2^2-z_3^2-z_4^2)^2-2c_1(z_1^2-z_2^2+z_3^2+z_4^2)-4z_1(c_2z_2-c_3z_3-c_4z_4)+\mathrm{abs}_2(c).$$

If we choose $c = (1, 2, 3, 4)$ as in Example 1.6, part I, p. 4 and $z = (z_1, z_2, z_3, z_4)$ with

$$z_1 = z_2 = z_3 = z_4 = \pm\frac{1}{2}\sqrt{5},$$

then $z^2 = 2.5(1, 1, 1, 1)$, $p(z) = z^2 - c = (1.5, 0.5, -0.5, -1.5)$ and $\mathrm{abs}_2(p(z)) = 0$, thus, z is a singular point for $p(z) := z^2 - c$, though p is a polynomial without any zero. The same applies for $c = (-2, 1, 2, 0)$ as in Example 1.6, part II, p. 4. In this case $z = 0.5(1, 1, -1, 1)$ yields $p(z) = z^2 - c = 0.5(5, -1, -5, 1)$ and $\mathrm{abs}_2(p(z)) = 0$.

THEOREM 5.3. *Let $z_0 \in \mathbb{R}$ be a zero of a coquaternionic polynomial p and let $v \in [z_0]_q$. Then $\mathrm{abs}_2(p(v)) = 0$ hence, v is a singular point for p.*

Proof. An element $v \in [z_0]_q$ has the form $v = z_0 + u$, where $u = (0, u_2, u_3, u_4)$ with $\mathrm{abs}_2(u) = u_2^2 - u_3^2 - u_4^2 = 0$. As a consequence $\Re(v) = \Re(z_0) = z_0$, $\mathrm{abs}_2(v) = \mathrm{abs}_2(z_0)$. If $p(z_0) = A(\Re(z_0), \mathrm{abs}_2(z_0)) + B(\Re(z_0), \mathrm{abs}_2(z_0))z_0 = 0$, then (using Lemma 3.3) $p(v) = A + B(z_0 + u) = A + Bz_0 + Bu = Bu$ and $\mathrm{abs}_2(Bu) = \mathrm{abs}_2(B)\mathrm{abs}_2(u) = 0$. □

COROLLARY 5.4. *Let z_0 be a zero of a coquaternionic polynomial p and let $[z_0]_q$ contain a real element. Then all elements $v \in [z_0]_q$ are singular points for p.*

THEOREM 5.5. *Let there be a $z \in \mathbb{H}_{\mathrm{coq}}$ such that $A(\Re(z), \mathrm{abs}_2(z)) = 0$, and $B(\Re(z), \mathrm{abs}_2(z))$ singular, where A, B are defined in (3.5). Then z is singular for p.*

Proof. In the above case we have $\mathrm{abs}_2(p(z)) = p(z)\overline{p(z)} = (A + Bz)(\overline{A + Bz}) = Bz\overline{Bz} = Bz\overline{z}\overline{B} = \mathrm{abs}_2(B)\mathrm{abs}_2(z) = 0$. □

6. The companion polynomial. In Theorem 3.9 there is a simple formula for computing a zero of a coquaternionic polynomial p, provided one knows an equivalence class $[z_0]_q \subset \mathbb{H}_{\mathrm{coq}}$ which contains a zero of p. This knowledge will be provided by the so-called companion polynomial of p which will be introduced in this section and will be denoted by q. The concept of a companion polynomial was very successful in treating one-sided quaternionic polynomials. See [14]. It was originally already introduced by Niven, 1941, [17] and later used by Pogorui and Shapiro, 2004, [19] to find an alternative proof for the number of zeros of a one-sided quaternionic polynomials as given by Gordon and Motzkin, 1965, [9]. The companion polynomial q is a polynomial with real coefficients, the roots of which determine - under certain conditions - equivalence classes which contain zeros of p.

DEFINITION 6.1. Let p be the polynomial of degree n, given in (1.8). The polynomial q of degree $\le 2n$ defined by

$$q(z) := \sum_{j,k=0}^{n} \overline{c_j}\,c_k z^{j+k}, \quad z \in \mathbb{C} \tag{6.1}$$

will be called the *companion polynomial of* p. The zeros of q will be called *roots* of q.

Lemma 6.2. *The coefficients of q defined in (6.1) are all real, and q can be written as*

$$q(z) := \sum_{k=0}^{2n} b_k z^k, \quad b_k := \sum_{j=\max(0,k-n)}^{\min(k,n)} \overline{c_j}\, c_{k-j} \in \mathbb{R},\ k = 0,1,\ldots,2n. \tag{6.2}$$

Proof. Formula (6.2) is obtained from (6.1) by putting $\kappa := j + k$ and observing the restrictions $0 \le \kappa \le 2n, 0 \le j,k \le n$ and in the end renaming κ in k. For a fixed $0 \le k \le 2n$ there are $\min(k+1, 2n+1-k)$ terms in the representation of b_k given in (6.2). If k is odd, the number of terms is even, and in this case, if $\overline{c_j} c_{k-j}$ is one of the terms, then there is another, distinct term $\overline{c_{k-j}} c_j$, and the sum is real. If k is even, the number of terms is odd. We use the same argument as before and note that in this case there is an additional, single real term $\overline{c_{\frac{k}{2}}} c_{\frac{k}{2}}$. □

The companion polynomial q should only be regarded as a polynomial over $\mathbb{C}$, not over $\mathbb{H}_{\mathrm{coq}}$. The highest coefficient of the companion polynomial q is $b_{2n} = \overline{c_n} c_n = \mathrm{abs}_2(c_n)$. Thus, if c_n is singular, the degree of the companion polynomial is less than $2n$. If c_0 is singular, then, the constant term of q is $b_0 = \mathrm{abs}_2(c_0) = 0$ and $q(0) = 0$. It is even possible, that the companion polynomial q vanishes identically. The companion polynomial q can be computed by $q(z) := p(z)\overline{p(z)}$, assuming that $z \in \mathbb{R}$, using, that in this case, $z = \overline{z}$ commutes with all coefficients. This implies that a real zero of p appears as a double root of q.

Example 6.3. Let $p(z) = bz^2 - c$. Then, $q(z) = \mathrm{abs}_2(b)z^4 - 2\Re(b\overline{c})z^2 + \mathrm{abs}_2(c)$, and q vanishes if $\mathrm{abs}_2(b) = \mathrm{abs}_2(c) = \Re(b\overline{c}) = 0$. In this case

$$\mathrm{abs}_2(p(z)) = \mathrm{abs}_2(b)\mathrm{abs}_2(z^2) - 2\Re(bz^2\overline{c}) + \mathrm{abs}_2(c) = -2\Re(bz^2\overline{c}),$$

and $\mathrm{abs}_2(p(z))$ vanishes for all $z \in \mathbb{R}$. Now, let $b = 1$. Then, for the roots z of q we have

$$z^2 = \Re(c) \pm \sqrt{-c_2^2 + c_3^2 + c_4^2}.$$

Put $\sigma := \pm\sqrt{-c_2^2 + c_3^2 + c_4^2}$ and assume that $\sigma \in \mathbb{R}$. Then, $z^2 = c_1 + \sigma \in \mathbb{R}$, and

$$\underbrace{c_1^2 - c_2^2 + c_3^2 + c_4^2 + 2c_1\sigma}_{\mathrm{abs}_2(z^2)} \underbrace{-2c_1^2 - 2c_1\sigma}_{-2\Re(z^2\overline{c})} + \underbrace{c_1^2 + c_2^2 - c_3^2 - c_4^2}_{\mathrm{abs}_2(c)} = 0.$$

Thus, the real roots z of q are singular points for p.

Lemma 6.4. *Let $q(z) = 0$ for some $z \in \mathbb{R}$. Then z is a singular point for p. If q vanishes identically, then, all $z \in \mathbb{R}$ are singular points for p.*

Proof. $q(z) = p(z)\overline{p(z)} = \mathrm{abs}_2(p(z)) = 0$ since $z \in \mathbb{R}$. □

Lemma 6.5. *Let p have the form $p(z) = A\big(\Re(z), \mathrm{abs}_2(z)\big) + B\big(\Re(z), \mathrm{abs}_2(z)\big)z$, see (3.5). Then, the companion polynomial q can be written as (deleting the arguments of A and of B)*

$$q(z) = \mathrm{abs}_2(A) + 2\Re(\overline{B}A)z + \mathrm{abs}_2(B)z^2. \tag{6.3}$$

Proof. Let $z^j = \alpha_j + \beta_j z$. See (3.2) to (3.4). Then,

$$q(z) = \sum_{j,k=0}^{n} \overline{c_j}\, c_k z^{j+k} = \sum_{j=0}^{n} \overline{c_j} \left(\sum_{k=0}^{n} c_k z^k \right) z^j = \sum_{j=0}^{n} \overline{c_j}\,(A + Bz) z^j \tag{6.4}$$

$$= \sum_{j=0}^{n} \overline{c_j}\,(A+Bz)(\alpha_j+\beta_j z) \quad [\alpha_j,\beta_j \in \mathbb{R}] \tag{6.5}$$

$$= \sum_{j=0}^{n} \alpha_j\overline{c_j}\,A + \sum_{j=0}^{n} \beta_j\overline{c_j}\,Az + \sum_{j=0}^{n} \alpha_j\overline{c_j}\,Bz + \sum_{j=0}^{n} \beta_j\overline{c_j}\,Bz^2 \tag{6.6}$$

$$= \overline{A}A + \overline{B}Az + \overline{A}Bz + \overline{B}Bz^2. \tag{6.7}$$

□

We will show that $z \overset{q}{\sim} z_0$ if $[z_0]_p$ contains a zero, which is computed by the formula given in Theorem 3.9.

Lemma 6.6. *Let $z_0 = x + \mathbf{i}y$ be a root of q with $y \neq 0$ and $B(\Re(z_0), \mathrm{abs}_2(z_0))$ be nonsingular. Then, for z defined in (3.15) we have $z \in [z_0]_q$, or more explicitly*

$$\Re(z_0) = \Re(z), \quad \mathrm{abs}_2(z_0) = \mathrm{abs}_2(z).$$

Proof. Put $A := A(\Re(z_0), \mathrm{abs}_2(z_0))$, $B := B(\Re(z_0), \mathrm{abs}_2(z_0))$. Lemma 6.5 implies

$$q(z_0) = \mathrm{abs}_2(A) + 2\Re(\overline{B}A)z_0 + \mathrm{abs}_2(B)z_0^2 = 0.$$

Denote $\quad v := \overline{B}A =: (v_1, v_2, v_3, v_4), \quad z_0 =: (x, y, 0, 0),$ where $y \neq 0$.

Separating $q(z_0)$ in real and imaginary part yields

$$\Re(q(z_0)) = \mathrm{abs}_2(A) + 2v_1x + \mathrm{abs}_2(B)(x^2 - y^2) = 0. \tag{6.8}$$

$$\Im(q(z_0)) = 2\big(\Re(\overline{B}A) + \mathrm{abs}_2(B)x\big)y = 0 \Rightarrow v_1 = -\mathrm{abs}_2(B)x. \tag{6.9}$$

From the definition of z it follows that

$$\Re(z) = -\frac{\Re(\overline{B}A)}{\mathrm{abs}_2(B)} = -\frac{v_1}{\mathrm{abs}_2(B)} = x = \Re(z_0),$$

where the last part follows from (6.9). Thus, the first part is shown. Now, from (3.15)

$$\mathrm{abs}_2(z) = \mathrm{abs}_2(-B^{-1}A) = \frac{\mathrm{abs}_2(A)}{\mathrm{abs}_2(B)}. \tag{6.10}$$

If we insert $v_1 = -\mathrm{abs}_2(B)x$ from (6.9) into (6.8) we obtain

$$\mathrm{abs}_2(A) - 2\mathrm{abs}_2(B)x^2 + \mathrm{abs}_2(B)(x^2 - y^2) = \mathrm{abs}_2(A) - \mathrm{abs}_2(B)\mathrm{abs}_2(z_0) = 0,$$

and together with (6.10) the second part, $\mathrm{abs}_2(z_0) = \mathrm{abs}_2(z)$, follows. □

Lemma 6.7. *Let the companion polynomial q have a pair of complex conjugate roots $z_0^{\pm}$. Then, the formula (3.15) yields the same value for z for both roots $z_0^{\pm}$.*

Proof. The formula for z depends only on $\Re(z_0)$ and on $\mathrm{abs}_2(z_0)$ which are the same for both $z_0^{\pm}$. □

Lemma 6.8. *Let p be a given coquaternionic polynomial, see (1.8), and let q be the companion polynomial of p. Let $z_0 \in \mathbb{C}$ be a root of q and let $B(\Re(z_0), \mathrm{abs}_2(z_0))$ be nonsingular, and let z be determined by formula (3.15). Then*

$$\mathrm{abs}_2(A) = \mathrm{abs}_2(Bz), \quad \Re(\overline{A}Bz) = -\mathrm{abs}_2(A). \tag{6.11}$$

Proof. The definition (3.15) of z implies $Bz = -A$ which proves both parts. □

THEOREM 6.9. *Let p be a given coquaternionic polynomial, see (1.8), and let q be the companion polynomial of p, see 6.1, 6.2. Let $z_0 \in \mathbb{C}$ be a root of q and let $B(\Re(z_0), \mathrm{abs}_2(z_0))$ be nonsingular, and let z be determined by formula (3.15). Then,*

$$\overline{p(z)}\,p(z) = \mathrm{abs}_2(p(z)) = 0. \tag{6.12}$$

If $z_0 \notin \mathbb{R}$, then,

$$p(z) = 0. \tag{6.13}$$

Proof. We have $\overline{p(z)}\,p(z) = \overline{(A+Bz)}(A+Bz) = \mathrm{abs}_2(A) + \overline{A}Bz + \overline{Bz}A + \mathrm{abs}_2(Bz) = \mathrm{abs}_2(A) + 2\Re(\overline{A}Bz) + \mathrm{abs}_2(Bz)$. Conditions (6.11) yield $\overline{p(z)}\,p(z) = 2\mathrm{abs}_2(A) - 2\mathrm{abs}_2(A) = 0$, which proves (6.12). In the second case, Lemma 6.6 is valid, and following (3.5)

$$\begin{aligned}
p(z) &= A(\Re(z), \mathrm{abs}_2(z)) + B(\Re(z), \mathrm{abs}_2(z))z \\
&= A(\Re(z), \mathrm{abs}_2(z)) + B(\Re(z), \mathrm{abs}_2(z))\left(-B(\Re(z_0), \mathrm{abs}_2(z_0)))^{-1}A(\Re(z_0), \mathrm{abs}_2(z_0)\right) \\
&= A(\Re(z_0), \mathrm{abs}_2(z_0)) - B(\Re(z_0), \mathrm{abs}_2(z_0))(B(\Re(z_0), \mathrm{abs}_2(z_0)))^{-1}A(\Re(z_0), \mathrm{abs}_2(z_0)) \\
&= 0,
\end{aligned}$$

which proves (6.13). □

The assumption in Lemma 6.6 that z_0 is not real, is essential. If $z_0 \in \mathbb{R}$ we have (6.12). However, this case does not exclude the case $p(z) = 0$.

THEOREM 6.10. *Let $p(z) = 0$, where p is defined in (1.8). Assume that*

$$\Delta := [z]_q \cap \mathbb{C} \neq \emptyset. \tag{6.14}$$

Then, there is $z_0 \in \Delta$ with $q(z_0) = 0$, where q is defined in Definition 6.1.

Proof. (a) Let $z \in \mathbb{R}$, then, $\Delta = \{z\}$ and $q(z) = \overline{p(z)}p(z) = 0$. (b) Let $z \notin \mathbb{R}$ and $B(\Re(z), \mathrm{abs}_2(z)) = 0$ or $B(\Re(z), \mathrm{abs}_2(z))z = 0$, then $A(\Re(z), \mathrm{abs}_2(z)) = 0$ and $\mathrm{abs}_2(B(\Re(z), \mathrm{abs}_2(z))) = 0$ according to Lemma 3.4. Let $z_0 \in \Delta$. The companion polynomial in the representation (6.3) yields $q(z_0) = 0$. (c) Let $z \notin \mathbb{R}$ and B be nonsingular. In this case $p(z) = A(\Re(z), \mathrm{abs}_2(z)) + B(\Re(z), \mathrm{abs}_2(z))z = 0$ and $z = -B^{-1}A$, deleting the arguments. The quadratic polynomial for q in the form (6.3) has real coefficients and can be solved in standard technique with the zero

$$z_0 = -\frac{\Re(\overline{B}A)}{\mathrm{abs}_2(B)} + \frac{\mathbf{i}}{|\mathrm{abs}_2(B)|}\sqrt{\mathrm{abs}_2(AB) - (\Re(\overline{B}A))^2}. \tag{6.15}$$

The radicand is positive if $z \notin \mathbb{R}$ and the formula (6.15) yields

$$\Re(z_0) = -\frac{\Re(\overline{B}A)}{\mathrm{abs}_2(B)} = \Re(z).$$

We use formula (6.15) again, and since $z_0 \in \mathbb{C}$ we have

$$\mathrm{abs}_2(z_0) = |z_0|^2 = \left(\frac{\Re(\overline{B}A)}{\mathrm{abs}_2(B)}\right)^2 + \frac{\mathrm{abs}_2(AB) - (\Re(\overline{B}A))^2}{(\mathrm{abs}_2(B))^2} = \frac{\mathrm{abs}_2(A)}{\mathrm{abs}_2(B)} = \mathrm{abs}_2(z).$$

Property (6.14) is governed by Lemma 3.2. □

The previous theorem tells us, that we will find all zeros of p by employing the companion polynomial, provided the zero has a complex number in its equivalence class. Or in other words, Theorem 6.10 says, that all zeros z of p with the property $z_2^2 - z_3^2 - z_4^2 \geq 0$ can be found by applying the companion polynomial, but all others cannot be found. More precisely, the assumption $\Delta \neq \emptyset$ (see (6.14)) is equivalent to $c_2^2 = z_2^2 - z_3^2 - z_4^2 \geq 0$, which implies that $c_1 + c_2\mathbf{i} \in \mathbb{C}$, where $c_2 := \pm\sqrt{z_2^2 - z_3^2 - z_4^2}$. Thus, the complex zeros of q can be recovered from the zeros of p. If we have a look at Example 1.6, part (III), we see that all four square roots $\sqrt{c}$ given, do not satisfy condition (6.14), whereas the two roots $\sqrt{c}$ of part (IV) do satisfy (6.14).

Example 6.11. Let us treat the most trivial case

$$p(z) := z - c, \quad c = (c_1, c_2, c_3, c_4).$$

In terms of (3.5) we have $p(z) = A + Bz$ with $A = -c$, $B = 1$. Both, A and B do not depend on $z \in \mathbb{H}_{\text{coq}}$. The assumptions of Theorem 3.9 are in this case met since p always has a zero. Formula (3.15) yields the correct answer $z = c$. Let us now apply the companion polynomial $q(z) = z^2 - 2c_1 z + \text{abs}_2(c)$, the roots of which are $z_{1,2} = c_1 \pm \sqrt{-c_2^2 + c_3^2 + c_4^2}$. If we apply Theorem 6.9 we also obtain the correct answer, however, independent of the roots $z_{1,2}$.

Example 6.12. Let a quadratic, coquaternionic polynomial p be defined by the coefficients $c_0 = 1, c_1 = (-15, 6, -6, -25)/25, c_2 = 1$. Then, the companion polynomial

$$q(z) = z^4 - (6/5)z^3 + (43/25)z^2 - (6/5)z + 1$$

has two pairs $r_1 = a_1 \pm b_1\mathbf{i}, r_2 = a_2 \pm b_2\mathbf{i}$ of complex conjugate roots, and for the first pair we have $A_1 = (0, 0, 0, 0), B_1 = (4/5, 6/25, -6/25, -4/5)$ and for the second pair we have $A_2 = (0, 0, 0, 0), B_2 = (-4/5, 6/25, -6/25, -4/5)$. Both, B_1, B_2 are singular and Theorem 5.5 applies.

Experiments with random integer coefficients for the coquaternionic polynomial show, that cases where B is singular are very rare.

7. Application to the quadratic case. Let us treat the quadratic case

$$p(z) := c_2 z^2 + c_1 z + c_0, \quad c_0, c_2 \neq 0. \tag{7.1}$$

Equation (3.5) applied to (7.1) yields

$$p(z) = c_0 - \text{abs}_2(z)c_2 + (c_1 + 2\Re(z)c_2)z := A(\text{abs}_2(z)) + B(\Re(z))z.$$

The companion polynomial q of p is (deleting the arguments of A and B)

$$q(z) = \text{abs}_2(c_2)z^4 + 2\Re(c_1\overline{c_2})z^3 + \big(2\Re(c_0\overline{c_2}) + \text{abs}_2(c_1)\big)z^2 + 2\Re(c_0\overline{c_1})z + \text{abs}_2(c_0) \tag{7.2}$$

$$= \text{abs}_2(A) + 2\Re(\overline{B}A)z + \text{abs}_2(B)z^2, \tag{7.3}$$

where the second equation is derived from (6.3). The existence of the factor c_2 at z^2 in (7.1) is only justified if $\text{abs}_2(c_2) = 0$. In this case, q has degree ≤ 3 and the formalism just described has to be applied.

Let $\text{abs}_2(c_2) \neq 0$. We may assume that $c_2 = 1$ and $\Re(c_1) = 0$. The latter condition can always be achieved by introducing the transformation $z = u - \frac{\Re(c_1)}{2}$. The transformed quadratic polynomial in u has the property that the real part of the coefficient at the linear term is zero. See also Niven, [17]. Thus, if $\operatorname{abs}(c_2) \neq 0$, we can put (7.1) into the simpler form

$$p(z) := z^2 + c_1 z + c_0, \quad c_0 \neq 0, \Re(c_1) = 0, \tag{7.4}$$

and if c_1 is real, we have $c_1 = 0$ and in this case the complete solution is described in the two Lemmas 1.4, 1.5. Let c_1 be nonreal. Then the above q specializes to

$$q(z) = z^4 + \big(2\Re(c_0) + \operatorname{abs}_2(c_1)\big)z^2 + 2\Re(c_0\overline{c_1})z + \operatorname{abs}_2(c_0). \tag{7.5}$$

If $c_0 \in \mathbb{R}$ the linear term cancels and the roots z of q, defined in (7.5), obey

$$z^2 = \frac{1}{2}\left(-2c_0 - \operatorname{abs}_2(c_1) \pm \sqrt{\operatorname{abs}_2(c_1)\big(\operatorname{abs}_2(c_1) + 4c_0\big)}\right).$$

For singular c_1 it follows, that $z^2 = -c_0$. If c_0, c_1 are both nonreal, equation (7.5) has to be further investigated by the given means.

8. Newton's method for finding coquaternionic zeros and the determination of the exact Jacobi matrix. Let p be the coquaternionic polynomial (1.8). Since the method of using the companion polynomial q to find zeros of p is restricted to those zeros which share a complex number in their equivalence class, we turn to the computation of the zeros of p by Newton's method in order to find the remaining zeros, if any. Let $p'(z,h)$ be the (Fréchet-) derivative of p. We will see immediately, how to find it. In abbreviation, Newton's method consists of solving the real, linear (4×4) system

$$\operatorname{col}(p(z_k)) + \operatorname{col}(p'(z_k,h)) = 0, \quad z_{k+1} := z_k + h, \ k = 0, 1, \ldots, \tag{8.1}$$

for h, where z_0 is the *initial guess*, and where col is defined in (3.10), p. 8. If $\mathbf{J}(z)$ is the *Jacobi matrix* of the mapping $p : \mathbb{R}^4 \to \mathbb{R}^4$, then

$$\operatorname{col}(p'(z,h)) = \mathbf{J}(z)\operatorname{col}(h), \tag{8.2}$$

where $\mathbf{J}(z)$ is a real (4×4) matrix. How do we find it? We refer to a paper by Lauterbach and Opfer, [16].

THEOREM 8.1. *Let p be a given coquaternionic polynomial with coefficients $c_0, c_1, \ldots, c_n, n \geq 1$. Define*

$$\lambda_j(z,h) := \sum_{\substack{k+\ell=j-1\\ k,\ell\geq 0}} z^k h z^\ell, \ j \geq 1, \quad L(z,h) := \sum_{j=1}^n c_j \lambda_j(z,h), \tag{8.3}$$

where $L : \mathbb{H}_{\mathrm{coq}} \to \mathbb{H}_{\mathrm{coq}}$ is a linear mapping over $\mathbb{R}$ with respect to h. Its matrix representation is the Jacobi matrix of p,

$$\mathbf{J}(z) := \sum_{j=1}^n \mathbf{M}_j(z) \in \mathbb{R}^{4\times 4}, \ \textit{where} \tag{8.4}$$

$$\mathbf{M}_j(z) := \sum_{\substack{k+\ell=j-1\\ k,\ell\geq 0}} \big[\operatorname{col}(z^k z^\ell), \operatorname{col}(z^k \mathbf{i} z^\ell), \operatorname{col}(z^k \mathbf{j} z^\ell), \operatorname{col}(z^k \mathbf{k} z^\ell)\big]. \tag{8.5}$$

Proof. The quantity $\lambda_j(z,h)$ is the derivative of z^j, $j \geq 1$. It is a linear mapping over $\mathbb{R}$ with respect to h since real numbers (and no others) commute with coquaternions. Therefore the derivative of p is $p'(z,h) = L(z,h)$. The formula (8.5) for $\mathbf{M}_j(z)$ is formula (5.4) in [12]. ∎

The right hand side of $\lambda_j(z,h)$ is obtained by computing $(z+h)^j$ and deleting all terms which are not linear in h. The result is the derivative of z^j. To mention two examples, $\lambda_2(z,h) = zh + hz$, $\lambda_3(z,h) = z^2h + zhz + hz^2$. An application of the method just described to the quaternionic algebraic Riccati equation is treated in [11].

Lemma 8.2. *Let $z, c_j \in \mathbb{Z}^4 \subset \mathbb{H}_{\text{coq}}$, $j = 1,2,\ldots,n$. Then, $\mathbf{J}(z) \in \mathbb{Z}^{4\times 4}$.*

Proof. The formula for computing the Jacobi matrix $\mathbf{J}(z)$ involves only additions and multiplications of the data $z, c_j, j = 1,2,\ldots,n$. See (8.3) to (8.5). ∎

The property given in Lemma 8.2 is not shared by the numerical version of the Jacobi matrix which is columnwise computed by the formula

$$\frac{p(z+\alpha\mathscr{N}_j) - p(z)}{\alpha}, \quad j = 1,2,3,4,\ \alpha \approx 10^{-6},$$

where $\mathscr{N}_j \in \mathbb{H}_{\text{coq}}$, $j = 1,2,3,4$ represent the four units of $\mathbb{H}_{\text{coq}}$ and α is roughly the square root of the machine precision.

Example 8.3. We take a cubic polynomial p with data from Example 10.5, p. 18. Then, for the start value $z_0 = (0,0,0,0)$ the Jacobi matrix is

$$\mathbf{J}(z_0) = \begin{bmatrix} 2 & -3 & 5 & 7 \\ 3 & 2 & 7 & -5 \\ 5 & 7 & 2 & -3 \\ 7 & -5 & 3 & 2 \end{bmatrix}.$$

After six Newton steps we arrive at the zero z which is listed in third position of Table 10.6, and $||p(z)|| = 1.8609 \cdot 10^{-14}$, where $||\quad||$ is the euklidean norm in $\mathbb{R}^4$.

9. The Algorithms for finding zeros and singular points of coquaternic polynomials. The techniques to find zeros and singular points for the coquaternionic polynomial p with the methods described in the foregoing sections can be summarized in the following algorithms.

Algorithm 9.1. Algorithm I for finding zeros and singular points for coquaternionic polynomials by means of the companion polynomial.

1. Let $c_0, c_1, \ldots, c_n$ be the coquaternionic coefficients of the polynomial p.
2. Compute the real coefficients $b_0, b_1, \ldots, b_{2n}$ of the companion polynomial q by formula (6.2).
3. If the companion polynomial vanishes, all points $z \in \mathbb{R}$ are singular points for p, stop here, otherwise continue.
4. Compute all (real and complex) roots of q. There are at most $2n$ roots.
5. Delete all complex roots with negative imaginary part. The remaining roots will be denoted by $r_1, r_2, \ldots, r_m$, $m \leq 2n$.
6. Define an integer vector *ind* (indicator) of length m and put all entries to zero.
   `for` $j = 1$ `to` m `do`
7. Compute A_j, B_j at the root r_j by using formulas (3.3), (3.4), and (3.5).
8. Compute $\rho_j = -A_j B_j^{-1}$ if B_j is nonsingular ($\text{abs}_2(B_j) \neq 0$), otherwise put $\rho_j = r_j$ and identify ρ_j with a coquaternion.

9. Check whether $p(\rho_j) = 0$, in this case put $ind(j) = 2$ (ρ_j is a zero of p). Otherwise, check whether ρ_j is a singular point for p ($\text{abs}_2(p(\rho_j)) = 0$). In this case put $ind(j) = 1$.
   `end for` j

Comments: The result of Algorithm I is a vector of coquaternions ρ_j and an integer vector $ind_j,\ j = 1, 2, \ldots, m$ with

$$ind_j = \begin{cases} 0: & \rho_j \text{ is neither a zero of } p \text{ nor a singular point for } p, \\ 1: & \rho_j \text{ is a singular point for } p \text{ but not a zero of } p, \\ 2: & \rho_j \text{ is a zero of } p. \end{cases}$$

In the above algorithm we have not paid special attention to the case, where one of the zeros of p is real. In this case a real double root appears in the set of roots of q. In order that the above algorithm works smoothly, one should use a so-called *overloading technique* which is available in several programming systems. Overloading means that the elementary arithmetic operations and functions can be extended to coquaternions keeping the standard notations like $+, -, *, /$. In particular, one needs a subprogram with which one can evaluate coquaternionic polynomials at coquaternions. If the overloading technique has been implemented, then polynomial evaluation can be done simply by the application of Horner's scheme. For finding the roots of a real polynomial (like the companion polynomial) standard programs are ordinarily available. However, the standard program in MATLAB is called `roots` and it suffers from a severe loss in accuracy if there are multiple roots. For a remedy, there are hints in [14]. If one uses the overloading technique, typically, the definition for a vector of coquaternions c reads $c(j)$ =coquaternion($[c_{j1}, c_{j2}, c_{j3}, c_{j4}]$), $j = 1, 2, \ldots$

ALGORITHM 9.2. Algorithm II for finding zeros of coquaternionic polynomials by means of Newton's method.

1. Let $c_0, c_1, \ldots, c_n$ be the coquaternionic coefficients of the polynomial p.
2. Define a list L of already known zeros of p, possibly empty in the beginning or filled by zeros obtained from Algorithm I.
3. Define a number of trials: no_trials, a maximal number of Newton iterations: max_Newton, and a stopping criterion: $crit\_stop$.
   `for` $j = 1$ `to` no_trials `do`
4. Define an initial guess $z = z_0$, compute $y = p(z)$, and $ynorm = ||y||$ (euklidean norm). Put $\ell = 0$.
   `while` $ynorm \geq crit\_stop$ `and` $\ell \leq max\_Newton$ `do`
5. $\ell = \ell + 1$; execute one Newton step: $z = Newton(z)$.
   Compute $y = p(z), ynorm = ||y||$.
   `end while`
6. If $\ell < max\_Newton$ or $ynorm < crit\_stop$, check whether the last computed z is already contained in the list L. If not, add it to L.
   `end for` j

Comments: The result of this algorithm will be a list L of zeros of p, which in general is not exhaustive. The start value is - according to experience of the authors - best selected by a random, integer choice, where the integers should be restricted to a small interval, say to $[-5, 5]$. It is possible (in rare cases) that the Jacobi matrix is singular. This can be easily overcome by choosing a new initial guess.

10. Numerical Examples. We present some examples of cubic coquaternionic polynomials, to show that the number of zeros and singlar points, computed by the companion polynomial and by Newton's methods is rather unpredictable. Nevertheless, there is some pattern we would like to show. The examples are ordered with respect to the number of zeros one finds by using the companion polynomial. The corresponding zeros are printed in red.

EXAMPLE 10.1. The cubic coquaternionic polynomial p with coefficients $c_0 = (2, -2, 2, 3)$, $c_1 = (-4, -5, 1, 1)$, $c_2 = (-1, 0, -5, -1)$, $c_3 = (2, 2, -1, 0)$ has **no zeros** similar to complex numbers and six singular points. Applying Newton's method with exact Jacobi matrix yields the following eight zeros listed in Table 10.2. They are all not similar to a complex number.

TABLE 10.2. Eight zeros of a cubic coquaternionic polynomial p defined in Example 10.1, all found by Newton's method.

0.920792194877860,	−0.477350655832754,	2.428458796390070,	−1.654108298624764;
2.450727144208431,	0.977395660928656,	0.317301767845470,	−1.652159929881599;
0.038499359300800,	−0.459455816622210,	0.517633403455030,	0.178975644511005;
0.040708445821269,	−0.839407205920705,	−0.328922433104592,	1.295950832229326;
0.709019332932411,	−0.294621664264792,	−0.112477811982268,	0.461335678870540;
−0.448518057687961,	1.536978387850394,	2.034978412068015,	0.749577465189058;
0.410896918015976,	−0.063043960237222,	0.222327740640928,	0.635721599205228;
−1.489226503509231,	−0.051244615268034,	0.422127971968205,	−0.252540209891112.

EXAMPLE 10.3. The cubic coquaternionic polynomial p with coefficients $c_0 = (1, -5, -2, 0)$, $c_1 = (3, 3, -2, 4)$, $c_2 = (-4, -3, -5, 2)$, $c_3 = (-3, -4, 1, -2)$ has **one zero** which is similar to a complex number, and four singular points, and six zeros computed by Newton's method, not similar to a complex number which together are listed in Table 10.4.

TABLE 10.4. Seven zeros of a cubic coquaternionic polynomial p, defined in Example 10.3, the last six found by Newton's method.

−0.084025738354299,	1.111175126311441,	−0.574783886624048,	0.584853095346396;
−1.280365616247547,	0.020877114875100,	0.503907316675033,	2.157051290547817;
−0.285608645398092,	1.407387895553819,	1.602481962888596,	−0.292825912129321;
0.734696869093826,	−0.802514241229524,	−0.739507355478451,	0.370330803674946;
−1.480332927529147,	−0.481980935905488,	0.945158732810532,	1.843761755812835;
−2.300671130739401,	0.360373154160493,	−0.067402575042700,	1.246070549138632;
−0.085641334116581,	3.501590113862619,	3.639869657498098,	0.231312656003601.

EXAMPLE 10.5. The cubic coquaternionic polynomial p with coefficients $c_0 = (7, 6, 5, 1)$, $c_1 = (2, 3, 5, 7)$, $c_2 = (4, -3, 2, 1)$, $c_3 = (1, 3, 2, 4)$ has **two zeros** which are similar to a complex number, two singular points, and one zero computed by Newton's method, listed in Table 10.6.

TABLE 10.6. Three zeros of a cubic coquaternionic polynomial p of Example 10.5, the last one found by Newton's method.

−1.618852521797113,	6.463899263531390,	2.829324921055154,	5.651970856832540;
0.418326476405790,	−1.691555573954496,	0.998887526357887,	0.395365114055260;
−0.099473954608707,	−1.081012068817781,	−0.782231163978552,	−1.127180514797187.

EXAMPLE 10.7. The cubic coquaternionic polynomial p with coefficients $c_0 = (0, 2, 0, 5)$, $c_1 = (0, 1, 0, 1)$, $c_2 = (-2, -4, 4, 1)$, $c_3 = (1, 0, 4, -2)$ has **three zeros** which are similar to a complex number, no singular points. The three zeros

are listed in Table 10.8. No zeros were found by applying Newton's method (several thousand trials).

TABLE 10.8. Three zeros of a cubic coquaternionic polynomial p, defined in Example 10.7 all found by using the companion polynomial.

−1.466507448592167,	1.324915491617470,	1.123223813460332,	−0.564677198394439;
0.781247091809576,	0.634161128551769,	−0.200695566535362,	0.065867128807512;
−0.156844906375301,	−2.299180524759707,	1.304072974458774,	−1.766122605663109.

We made many more tests with coquaternionic polynomials of degree $n \geq 3$ which attained the maximal number n of zeros found by the companion polynomial. In all these cases we did not find additional zeros computed by Newton's method. This applies also to Example 1.6, part (IV), p. 4.

CONJECTURE 10.9. *Let p be a coquaternionic polynomial of degree n which has n zeros which are similar to complex numbers. Then, there are no additional zeros not similar to complex numbers.*

Another experience is the following: If a coquaternionic polynomial p has many zeros which are not similar to complex numbers, then there are only few zeros which are similar to complex numbers. This applies, in particular, to Example 1.6, part (III), p. 4. However, this is a very vague statement, which needs more attention.

11. On the number of zeros and singular points. We have already seen, that coquaternionic polynomials may have no zeros. See Corollary 1.7. On the other hand there is a maximum number of zeros which contain complex numbers in the corresponding equivalence class.

THEOREM 11.1. *A coquaternionic polynomial p of degree $n \geq 1$ has at most n zeros in equivalence classes which contain complex numbers. The number n will be attained, if all roots of the companion polynomial q are either pairs of complex conjugate numbers or pairs of real numbers and the corresponding quantities B are nonsingular.*

Proof. Let the companion polynomial q may have the degree $d \leq 2n$. It may have $2m_1 \leq d$ real double roots, $2m_2 \leq d$ complex roots, and $m_3 \leq d$ real single roots. The total number of roots is $2m_1 + 2m_2 + m_3 = d$. Only the real double roots and the complex roots may lead to a zero of p. Thus, the maximum number of zeros of p is $m_1 + m_2 = (d - m_3)/2 \leq n$. The maximum number is attained if $d = 2n$ and $m_3 = 0$. □

THEOREM 11.2. *Let p be a coquaternionic polynomial of degree n and let the companion polynomial q have degree $d \leq 2n$. If n_z is the number of zeros of p and n_s the number of singular points for p, different from zeros, both computed from the roots of q, then $2n_z + n_s \leq d$. In particular, $n_s \leq 2n$, which means that there are at most $2n$ singular points for p, that are different from zeros of p and that are computed by means of the companion polynomial q.*

Proof. Singular points for p are derived from single, real roots of q. See Theorem 6.9. □

Our experiments implied the following conjecture.

CONJECTURE 11.3. *All coquaternionic polynomials p (defined in (1.8)) which do not reduce to a constant have singular points.*

Since $p(0) = c_0$ the conjecture is true if c_0 is singular. Let c_0 be nonsingular. In this case a proof could consist of showing that there is a $z \in \mathbb{H}_{\mathrm{coq}}$ such that

$$\mathrm{abs}_2(p(0))\mathrm{abs}_2(p(z)) = \mathrm{abs}_2(c_0)\mathrm{abs}_2(p(z)) \leq 0. \tag{11.1}$$

In our tests we always found a very simple z such that (11.1) was valid. It was sufficient to choose either one of the four unit vectors (possibly multiplied by 2) for z, or one of the square roots of $z = 0$. See p. 3.

This conjecture (if true) could be called the "Weak Fundamental Theorem of Algebra" for coquaternions.

12. Extension to algebras in $\mathbb{R}^4$. If we go from the algebra of coquaternions to other algebras in $\mathbb{R}^4$, we observe many similarities with the coquaternionic case. The algebras to be considered are given in Table 12.1.

TABLE 12.1. Eight $\mathbb{R}^4$ algebras with multiplication rules.

No	Name of algebra	Short name	$\mathbf{i}^2$	$\mathbf{j}^2$	$\mathbf{k}^2$	$\mathbf{ij}$	$\mathbf{jk}$	$\mathbf{ki}$
1	Quaternions	$\mathbb{H}$	-1	-1	-1	$\mathbf{k}$	$\mathbf{i}$	$\mathbf{j}$
2	Coquaternions	$\mathbb{H}_{\text{coq}}$	-1	1	1	$\mathbf{k}$	$-\mathbf{i}$	$\mathbf{j}$
3	Tessarines	$\mathbb{H}_{\text{tes}}$	-1	1	-1	$\mathbf{k}$	$\mathbf{i}$	$-\mathbf{j}$
4	Cotessarines	$\mathbb{H}_{\text{cotes}}$	1	1	1	$\mathbf{k}$	$\mathbf{i}$	$\mathbf{j}$
5	Nectarines	$\mathbb{H}_{\text{nec}}$	1	-1	1	$\mathbf{k}$	$\mathbf{i}$	$-\mathbf{j}$
6	Conectarines	$\mathbb{H}_{\text{con}}$	1	1	-1	$\mathbf{k}$	$-\mathbf{i}$	$-\mathbf{j}$
7	Tangerines	$\mathbb{H}_{\text{tan}}$	1	-1	-1	$\mathbf{k}$	$-\mathbf{i}$	$\mathbf{j}$
8	Cotangerines	$\mathbb{H}_{\text{cotan}}$	-1	-1	1	$\mathbf{k}$	$-\mathbf{i}$	$-\mathbf{j}$

The full multiplication table of the eight listed algebras can be obtained by multiplying the last three columns in Table 12.1 by $\mathbf{j}, \mathbf{k}, \mathbf{i}$, respectively. The table is obtained by allowing all eight combinations of signs ± 1 for the squares $\mathbf{i}^2, \mathbf{j}^2, \mathbf{k}^2$ and keeping the product $\mathbf{ij} = \mathbf{k}$ the same for all algebras. The names given for the algebras with numbers 2 to 4 are from Cockle, 1849, [3], the names for the algebras 5 to 8 are from Schmeikal, 2014, [20], who also establishes the connection to Clifford algebras in his paper. In a first draft, these algebras were called New Algebra 1 to New Algebra 4 by the present authors.

The first mentioned algebra, the algebra of quaternions, goes back to Hamilton, 1843. The problem of finding zeros of unilateral and bilateral polynomials in quaternionic variables has been treated by the current authors already in [13, 14].

The 8 algebras separate into 4 noncommutative ones, namely those with numbers 1,2,5,6, and into four commutative ones, those with numbers 3,4,7,8. We note that the center of all 8 algebras is or contains $\mathbb{R}$, which means that the real numbers commute with all members of all algebras. In all eight cases we define the conjugate of an algebraic element a as in (1.5).

LEMMA 12.2. *Let $\mathcal{A}$ be one of the four noncommutative algebras of Table 12.1 (No 1, 2, 5, 6) and $a \in \mathcal{A}$. Then, the product* $a\operatorname{conj}(a) = \operatorname{conj}(a)a$ *is real and*

$$a^{-1} = \frac{\operatorname{conj}(a)}{\operatorname{conj}(a)a} \text{ if } \operatorname{conj}(a)a \neq 0. \tag{12.1}$$

Proof. As in (1.5) we put $\operatorname{abs}_2(a) := \operatorname{conj}(a)a$, and by applying the multiplication rules we obtain

$$\operatorname{abs}_2(a) = \begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 & \text{for quaternions,} \\ a_1^2 + a_2^2 - a_3^2 - a_4^2 & \text{for coquaternions,} \\ a_1^2 - a_2^2 + a_3^2 - a_4^2 & \text{for nectarines,} \\ a_1^2 - a_2^2 - a_3^2 + a_4^2 & \text{for conectarines.} \end{cases} \tag{12.2}$$

Formula (12.1) follows by multiplying from the left or right by a and from the fact that $\operatorname{conj}(a)a$ is real. $\square$

It is clear that all 8 algebras contain $\mathbb{R}$ as a subalgebra by defining the set of elements of the form $a = (a_1, 0, 0, 0), a_1 \in \mathbb{R}$. However, this is not in general true for the field $\mathbb{C}$.

LEMMA 12.3. *The cotessarines $\mathbb{H}_{\text{cotes}}$ (algebra No 4) do not contain the field of complex numbers $\mathbb{C}$ as a subalgebra. Let $z = x + \mathbf{i}y \in \mathbb{C}$. Then, $\mathbb{C}$ is a subalgebra of one of the remaining seven algebras $\mathcal{A}$ if $\mathcal{A}$ is reduced to the following form:*

$$z = x + \mathbf{i}y \to \begin{cases} (x, y, 0, 0) & \text{for } \mathcal{A} = \text{quaternions, or} \\ (x, 0, y, 0) & \text{for } \mathcal{A} = \text{quaternions, or} \\ (x, 0, 0, y) & \text{for } \mathcal{A} = \text{quaternions,} \\ (x, y, 0, 0) & \text{for } \mathcal{A} = \text{coquaternions,} \\ (x, y, 0, 0) & \text{for } \mathcal{A} = \text{tessarines, or,} \\ (x, 0, 0, y) & \text{for } \mathcal{A} = \text{tessarines,} \\ (x, 0, y, 0) & \text{for } \mathcal{A} = \text{nectarines,} \\ (x, 0, 0, y) & \text{for } \mathcal{A} = \text{conectarines,} \\ (x, 0, y, 0) & \text{for } \mathcal{A} = \text{tangerines or,} \\ (x, 0, 0, y) & \text{for } \mathcal{A} = \text{tangerines,} \\ (x, y, 0, 0) & \text{for } \mathcal{A} = \text{cotangerines or,} \\ (x, 0, y, 0) & \text{for } \mathcal{A} = \text{cotangerines.} \end{cases} \tag{12.3}$$

Proof. We put the imaginary part y of z at the positions where the squares $\mathbf{i}^2, \mathbf{j}^2, \mathbf{k}^2$ in Table 12.1 are equal to -1. However, there is not such a position in the case of cotessarines. □

The task of finding zeros (or singular points) of polynomials p defined with coefficients from one of the noncommutative algebras (No 1,2,5,6) can now be achieved as described for coquaternions. The complex solutions of the companion polynomial must be inserted in the corresponding algebra according to the rules given in (12.3).

The fact that (quasi) similarity classes may not contain complex numbers is virulent in all noncommutative algebras apart from quaternions. Compare Theorem 6.10 and Lemma 3.2.

LEMMA 12.4. *Let $\mathcal{A}$ be one of the noncommutative algebras and $a = a_1 + a_2\mathbf{i} \in \mathbb{C}$. In order that $b = (b_1, b_2, b_3, b_4) \in \mathcal{A}$ is quasi similar to a it is necessary and sufficient that $a_1 = b_1$ and*

$$a_2^2 = \left\{ \begin{array}{ll} +b_2^2 + b_3^2 + b_4^2 & \text{for quaternions} \\ +b_2^2 - b_3^2 - b_4^2 & \text{for coquaternions} \\ -b_2^2 + b_3^2 - b_4^2 & \text{for nectarines} \\ -b_2^2 - b_3^2 + b_4^2 & \text{for conectarines} \end{array} \right\} \geq 0. \tag{12.4}$$

Proof. Follows from (12.2). □

For the commutative cases, the companion polynomial is not well defined since in these cases $\text{conj}(a)a$ is not real. In addition, in commutative algebras all similarity classes shrink to one point. However, in this case another idea is applicable. Let $a = (a_1, a_2, a_3, a_4) \in \mathcal{A}$, where $\mathcal{A}$ is one of the eight algebras. Then $a = a_1 + a_2\mathbf{i} + (a_3 + a_4\mathbf{i})\mathbf{j}$ since $\mathbf{ij} = \mathbf{k}$ in all eight cases. Thus, for all $z \in \mathcal{A}$ and all eight $\mathcal{A}$ we can write

$$z = x + y\,\mathbf{j}, \quad x, y \in \mathbb{C}. \tag{12.5}$$

Let $\mathcal{A}$ be commutative. Then z^ℓ, where z is in (12.5), can be evaluated by the binomial formula and we obtain $z^\ell = u_\ell + v_\ell\,\mathbf{j}, u_\ell, v_\ell \in \mathbb{C}$. Let $c_\ell = a_\ell + b_\ell\,\mathbf{j}, a_\ell, b_\ell \in \mathbb{C}$ be the corresponding polynomial coefficients. Then

$$c_\ell z^\ell = a_\ell u_\ell + b_\ell v_\ell \mathbf{j}^2 + (a_\ell v_\ell + b_\ell u_\ell)\mathbf{j}, \text{ where } \mathbf{j}^2 = \pm 1.$$

Thus, finding solutions of $p(z) = \sum_{\ell=0}^n c_\ell z^\ell = 0$ splits into two complex equations

$$\text{(12.6)} \quad p_1(u,v) := \sum_{\ell=0}^n (a_\ell u_\ell + b_\ell v_\ell \mathbf{j}^2) = 0, \quad p_2(u,v) := \sum_{\ell=0}^n (a_\ell v_\ell + b_\ell u_\ell) = 0.$$

Having a look at Table 12.1, reveals that $\mathbf{j}^2 = +1$ for tessarines and cotessarines and $\mathbf{j}^2 = -1$ for tangerines and cotangerines.

If we write

$$z^{\ell+1} = u_{\ell+1} + v_{\ell+1}\mathbf{j} = zz^\ell = (u_1 + v_1\mathbf{j})(u_\ell + v_\ell\mathbf{j}) = u_1 u_\ell + v_1 v_\ell \mathbf{j}^2 + (u_1 v_\ell + v_1 u_\ell)\mathbf{j},$$

we obtain the following recursion only in complex numbers:

$$u_{\ell+1} = u_1 u_\ell + v_1 v_\ell \mathbf{j}^2,\ v_{\ell+1} = u_1 v_\ell + v_1 u_\ell, \quad u_0 = 1, v_0 = 0,\ 0 \le \ell \le n-1,$$

which allows the evaluation of (12.6) in an easy manner. However, the use of these formulas has to be postponed to another occasion.

Let $\alpha \in \mathcal{A}$, and $\mathcal{A}$ one of the eight algebras, where $\alpha := (a,b,c,d)$. Since the mapping $l : \mathcal{A} \to \mathcal{A}$ defined by

$$l(x) := \alpha x, x \in \mathcal{A}$$

is linear in x (over $\mathbb{R}$), there must be a matrix $\mathbf{M} \in \mathbb{R}^{4\times 4}$, depending on $\mathcal{A}$, such that

$$\text{(12.7)} \qquad \mathrm{col}(l(x)) = \mathbf{M}\mathrm{col}(x).$$

We will denote the four unit standard vectors also as

$$\mathscr{e}_1 := 1 := (1,0,0,0), \mathscr{e}_2 := \mathbf{i} := (0,1,0,0), \mathscr{e}_3 := \mathbf{j} := (0,0,1,0), \mathscr{e}_4 := \mathbf{k} := (0,0,0,1)$$

where $\mathscr{e}$ is pronounced "e". By putting $x = \mathscr{e}_j, j = 1,2,3,4$ in (12.7) we obtain the jth column of $\mathbf{M}$, such that

$$\mathbf{M} := [\mathrm{col}(\alpha\mathscr{e}_1), \mathrm{col}(\alpha\mathscr{e}_2), \mathrm{col}(\alpha\mathscr{e}_3), \mathrm{col}(\alpha\mathscr{e}_4)],$$

and these eight matrices are given in Table 12.5 using the notation $\imath : \mathcal{A} \to \mathbb{R}^{4\times 4}$.

TABLE 12.5. Representations of the above eight algebras in $\mathbb{R}^{4\times 4}$.

$$\begin{aligned}
\imath(\mathbb{H}) &= \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}, & \imath(\mathbb{H}_{\mathrm{coq}}) &= \begin{bmatrix} a & -b & c & d \\ b & a & d & -c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}, \\
\imath(\mathbb{H}_{\mathrm{tes}}) &= \begin{bmatrix} a & -b & c & -d \\ b & a & d & c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}, & \imath(\mathbb{H}_{\mathrm{cotes}}) &= \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, \\
\imath(\mathbb{H}_{\mathrm{nec}}) &= \begin{bmatrix} a & b & -c & d \\ b & a & -d & c \\ c & -d & a & b \\ d & -c & b & a \end{bmatrix}, & \imath(\mathbb{H}_{\mathrm{con}}) &= \begin{bmatrix} a & b & c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & -c & b & a \end{bmatrix}, \\
\imath(\mathbb{H}_{\mathrm{tan}}) &= \begin{bmatrix} a & b & -c & -d \\ b & a & -d & -c \\ c & d & a & b \\ d & c & b & a \end{bmatrix}, & \imath(\mathbb{H}_{\mathrm{cotan}}) &= \begin{bmatrix} a & -b & -c & d \\ b & a & -d & -c \\ c & -d & a & -b \\ d & c & b & a \end{bmatrix}.
\end{aligned}$$

The inverses of elements of the noncommutative algebras can simply be computed by formula (12.1). The inverses of elements in one of the four commutative algebras can be computed by rules given in Table 12.6. The computations were facilitated by using `maple`.

TABLE 12.6. Rules for computing the inverses $a^{-1} = b/\det = (b_1, b_2, b_3, b_4)/\det$ of $a = (a_1, a_2, a_3, a_4) \in \mathcal{A}$ in the four commutative algebras $\mathcal{A}$.

No	Algebra	$a^{-1} = (b_1, b_2, b_3, b_4)/\det$
3	$\mathbb{H}_{\text{tes}}$	$\det = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^2 - 4(a_1a_3 + a_2a_4)^2$
		$b_1 = a_1(a_1^2 + a_2^2 - a_3^2 + a_4^2) - 2a_2a_3a_4$
		$b_2 = -a_2(a_1^2 + a_2^2 + a_3^2 - a_4^2) + 2a_1a_3a_4$
		$b_3 = a_3(-a_1^2 + a_2^2 + a_3^2 + a_4^2) - 2a_1a_2a_4$
		$b_4 = -a_4(a_1^2 - a_2^2 + a_3^2 + a_4^2) + 2a_1a_2a_3$
4	$\mathbb{H}_{\text{cotes}}$	$\det = (a_1^2 + a_2^2 - a_3^2 - a_4^2)^2 - 4(a_1a_2 - a_3a_4)^2$
		$b_1 = a_1(a_1^2 - a_2^2 - a_3^2 - a_4^2) + 2a_2a_3a_4$
		$b_2 = a_2(-a_1^2 + a_2^2 - a_3^2 - a_4^2) + 2a_1a_3a_4$
		$b_3 = a_3(-a_1^2 - a_2^2 + a_3^2 - a_4^2) + 2a_1a_2a_4$
		$b_4 = a_4(-a_1^2 - a_2^2 - a_3^2 + a_4^2) + 2a_1a_2a_3$
7	$\mathbb{H}_{\text{tan}}$	$\det = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^2 - 4(a_1a_2 + a_3a_4)^2$
		$b_1 = a_1(a_1^2 - a_2^2 + a_3^2 + a_4^2) - 2a_2a_3a_4$
		$b_2 = a_2(-a_1^2 + a_2^2 + a_3^2 + a_4^2) - 2a_1a_3a_4$
		$b_3 = -a_3(a_1^2 + a_2^2 + a_3^2 - a_4^2) + 2a_1a_2a_4$
		$b_4 = -a_4(a_1^2 + a_2^2 - a_3^2 + a_4^2) + 2a_1a_2a_3$
8	$\mathbb{H}_{\text{cotan}}$	$\det = (a_1^2 + a_2^2 + a_3^2 + a_4^2)^2 - 4(a_1a_4 - a_2a_3)^2$
		$b_1 = a_1(a_1^2 + a_2^2 + a_3^2 - a_4^2) + 2a_2a_3a_4$
		$b_2 = -a_2(a_1^2 + a_2^2 - a_3^2 + a_4^2) - 2a_1a_3a_4$
		$b_3 = -a_3(a_1^2 - a_2^2 + a_3^2 + a_4^2) - 2a_1a_2a_4$
		$b_4 = a_4(-a_1^2 + a_2^2 + a_3^2 + a_4^2) + 2a_1a_2a_3$

The determinant of $\imath(\mathbb{H}_{\text{cotes}})$ factors in the form
$\det(\imath(\mathbb{H}_{\text{cotes}})) = (a_1 + a_2 - a_3 - a_4)(a_1 - a_2 - a_3 + a_4)(a_1 + a_2 + a_3 + a_4)(a_1 - a_2 + a_3 - a_4)$, such that $\det(\imath(\mathbb{H}_{\text{cotes}})) = 0$ if and only if $|a_1 + a_2| = |a_3 + a_4|$ or $|a_1 - a_2| = |a_3 - a_4|$. Since all other determinants have the form $\det = x^2 - 4y^2$, they also factor in the form $\det = (x - 2y)(x + 2y)$, where the meaning of x, y has to read from Table 12.6.

In (2.1) we have already seen, that the algebra of coquaternions is isomorph to the algebra of all real 2×2 matrices. This is also true for the algebra of nectarines and conectarines, though the representation differs as follows (using $\hat{\imath} : \mathcal{A} \to \mathbb{R}^{2\times 2}$):

$$\hat{\imath}(\mathbb{H}_{\text{nec}}) = \begin{bmatrix} a-d & b+c \\ b-c & a+d \end{bmatrix}, \quad \hat{\imath}(\mathbb{H}_{\text{con}}) = \begin{bmatrix} a-c & b+d \\ b-d & a+c \end{bmatrix}. \tag{12.8}$$

This can be verified by putting $\alpha = (a, b, c, d) = \mathscr{N}_j$, $j = 1, 2, 3, 4$ and checking the multiplication rules of Table 12.1.

The noncommutative algebras No 2,5,6 (coquaternions, nectarines, conectarines) are all isomorphic to the set of real 2×2 matrices $\mathbb{R}^{2\times 2}$ with representations given in (2.1), (12.8). Thus it is sufficient to study one of these algebras in order to study $\mathbb{R}^{2\times 2}$.

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