

Hamburger Beiträge zur Angewandten Mathematik

The Jacobi matrix for functions in noncommutative algebras

Reiner Lauterbach and Gerhard Opfer

This paper has been submitted in April 2014 to Advances in Applied Clifford Algebras (AACAA). The final version may differ from this preprint.

Nr. 2014-11
May 2014

THE JACOBI MATRIX FOR FUNCTIONS IN NONCOMMUTATIVE ALGEBRAS

REINER LAUTERBACH* AND GERHARD OPFER†

Dedicated to Klaus G urlebeck, on the occasion of his 60th birthday

Abstract. We develop a general tool for constructing the exact Jacobi matrix for functions defined in noncommutative algebraic systems without using any partial derivative. The construction is applied to solving nonlinear problems of the form $f(x) = 0$ with the aid of Newton's method in algebras defined in \mathbb{R}^N . We apply this to commutative and noncommutative algebras in \mathbb{R}^4 . The Jacobi matrix is explicitly constructed for polynomials in $x - a$ and for polynomials in the reciprocals $(x - a)^{-1}$ such that Jacobi matrices for functions defined by Taylor and Laurent expansions can be constructed in general algebras over \mathbb{R}^N . The Jacobi matrix for Riccati's equation with matrix elements from an algebra in \mathbb{R}^N is presented, and one particular Riccati equation is numerically solved in 8 algebras. Another case treated was the exponential function with algebraic variables including a numerical example. For cases where the computation of the Jacobi matrix to the highest possible precision for finding solutions of $f(x) = 0$ is time consuming, a hybrid method is recommended, namely to start with the Jacobi matrix in low precision and only when $\|f(x)\|$ is sufficiently small, one switches to the Jacobi matrix with high precision.

Key words: Jacobi matrices in algebras over \mathbb{R}^N , Jacobi matrices for polynomials over noncommutative algebras, Jacobi's matrix for Riccati's equation, Jacobi matrices for functions defined via Taylor and Laurent expansions over noncommutative algebras, a hybrid Newton technique: low accuracy of the Jacobi matrix - high accuracy of the Jacobi matrix.

AMS Subject classification: 15A66, 1604, 26B10, 46B03, 46G05, 58C20, 65J15

1. Introduction. In this paper we want to solve nonlinear matrix equations $f(x) = 0$ using Newton's method, where $f : X \rightarrow X$ and X is a matrix space $X = \mathcal{A}^{m \times n}$ and \mathcal{A} is a finite dimensional algebra like quaternions. An important special case is $m = n = 1$. While it is straightforward to construct the linearization of a function, it not so obvious how to construct the Jacobi matrix representing this linearization, which we need for actual computations. So we describe how to construct the Jacobi matrix. In an example, the case $X = \mathcal{A}$ is presented in Section 9 by employing the exponential function defined on an algebra \mathcal{A} . The case $X = \mathcal{A}^{m \times n}$ is treated in Section 8, where we show how to construct the Jacobi matrix for Riccati's matrix equation where \mathcal{A} is an N -dimensional algebra. We solve $f(x) = 0$ numerically in this case, where 0 denotes the zero element in $\mathcal{A}^{m \times n}$ for eight \mathbb{R}^4 algebras.

The algebra of matrices with $m \in \mathbb{N}$ rows and $n \in \mathbb{N}$ columns with entries from \mathbb{K} will be denoted by $\mathbb{K}^{m \times n}$, where \mathbb{N} is the set of positive integers, \mathbb{K} is either \mathbb{R} or \mathbb{C} , the field of real, complex numbers, respectively. If $m = n$ we say that n is the *order* of the matrix. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{X}, \mathbf{0}$ be matrices over \mathbb{K} , all of the same order and $\mathbf{0}$ the corresponding zero matrix: all entries are zero. An example of the above f is the matrix valued function

$$(1.1) \quad f(\mathbf{X}) := \mathbf{AXBXC} + \mathbf{DXE} + \mathbf{F}.$$

A summary of results for quaternionic matrices is given in [24]. An example for a well investigated class of matrix equations is the algebraic Riccati equation (see [1, 2, 15,

* University of Hamburg, Faculty for Mathematics, Informatics, and Natural Sciences [MIN], Bundestra e 55, 20146 Hamburg, Germany, reiner.lauterbach@uni-hamburg.de

† University of Hamburg, Faculty for Mathematics, Informatics, and Natural Sciences [MIN], Bundestra e 55, 20146 Hamburg, Germany, gerhard.opfer@uni-hamburg.de

20, 22]) for $\mathbf{A}, \mathbf{X}, f(\mathbf{X}) \in \mathbb{K}^{m \times n}, \mathbf{B} \in \mathbb{K}^{m \times m}, \mathbf{C} \in \mathbb{K}^{n \times n}, \mathbf{D} \in \mathbb{K}^{n \times m}$, and

$$(1.2) \quad f(\mathbf{X}) := \mathbf{XDX} + \mathbf{XC} + \mathbf{BX} + \mathbf{A} = \mathbf{0}.$$

We will embed the stated matrix problem in a wider class of algebraic systems.

DEFINITION 1.1. Let \mathcal{A} be a finite dimensional vector space over \mathbb{K} . Assume, that in addition a multiplication $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ is declared, which is associative, and which contains an element *one*, denoted by 1 with the property $1a = a1 = a$ for all $a \in \mathcal{A}$. Then, we call \mathcal{A} a *finite dimensional algebra over \mathbb{K}* . If $\mathbb{K} = \mathbb{R}$ we call the algebra *real*, otherwise *complex*. For the *dimension* of \mathcal{A} we use the notation $\dim(\mathcal{A})$. Let $\dim(\mathcal{A}) = N$. An element $a \in \mathcal{A}$ will either be denoted by $a = (a_1, a_2, \dots, a_N)$, or by $a = \sum_{j=1}^N a_j \mathcal{N}_j$, $a_j \in \mathbb{K}$, $j = 1, 2, \dots, N$, where \mathcal{N}_j denotes the j th standard unit vector in \mathbb{K}^N , $j = 1, 2, \dots, N$ with the property $\mathcal{N}_1 = 1, \mathcal{N}_j^2 = \pm 1, j \geq 2$. Here, 1 is an abbreviation for $\mathcal{N}_1 = (1, 0, \dots, 0) \in \mathbb{K}^N$. The letter \mathcal{N} is pronounced “e”. By associating a norm with \mathbb{K}^n , the algebras we are considering are *Banach algebras*.

The above definition does not include the claim that for all dimensions $N \in \mathbb{N}$, algebras with this dimension exist. The theory of Clifford algebras presents constructions for specific N . For more details, cf. Garling, [9, p. 16]. We will need another notion useful for noncommutative algebras.

DEFINITION 1.2. Let \mathcal{A} be a finite dimensional algebra over \mathbb{K} . The following set, $\mathcal{C}(\mathcal{A})$, will be called the *center* of \mathcal{A} :

$$(1.3) \quad \mathcal{C}(\mathcal{A}) := \{c \in \mathcal{A} : ca = ac \text{ for all } a \in \mathcal{A}\}.$$

It is well known that for $\mathcal{A} := \mathbb{K}^{n \times n}$ the center is the set of all multiples $\alpha \mathbf{I}, \alpha \in \mathbb{K}$, of the identity matrix $\mathbf{I} \in \mathbb{K}^{n \times n}$ and $\dim(\mathbb{K}^{n \times n}) = n^2$. Examples of algebras will be presented in a subsequent section.

2. The definition of a derivative. We recall the definition of a derivative for functions $f : X \rightarrow Y$ where X, Y are Banach spaces. We denote the set of linear, continuous mappings of $X \rightarrow Y$ by $\mathcal{L}(X, Y)$ and note, that $\mathcal{L}(X, Y)$ is again a Banach space. See [23, Theorem 1.1 on p. 189]. And also note, that all finite dimensional normed spaces are Banach spaces, [23, Theorem 3.2 on p. 63].

DEFINITION 2.1. The function f is called *differentiable at $x_0 \in X$* if there exists a (unique) mapping $L \in \mathcal{L}(X, Y)$, called the *derivative of f at x_0* such that

$$\lim_{h \rightarrow 0} \frac{\|f(x_0 + h) - f(x_0) - Lh\|}{\|h\|} = 0.$$

In order to accentuate the connection between the derivative L of f at x_0 , we denote the derivative L of f at x_0 by $f'(x_0) \in \mathcal{L}(X, Y)$ or by $Df(x_0) \in \mathcal{L}(X, Y)$. The evaluation of $f'(x_0)$ at some $h \in X$ is denoted by $f'(x_0)h \in Y$. The function f is called *differentiable* if it is differentiable at all points $x \in X$.

Higher order derivatives are defined inductively observing that $Df : X \rightarrow \mathcal{L}(X, Y)$, using, that the right hand side is a Banach space as well. With this definition most of the classical results on differentiable function can be established (see for example [6, Chapter 8]). A second derivative is denoted either by $f''(x_0) \in \mathcal{L}(X, \mathcal{L}(X, Y))$ or by $D^2f(x_0)$. The space $\mathcal{L}(X, \mathcal{L}(X, Y))$ will be identified with the set of bilinear maps $X \times X \rightarrow Y$, see [6, Chapter 8.12]. The evaluation is denoted by $f''(x_0)[h, h] \in Y$. The next theorem is one of the most practical theorems to determine the derivative of a differentiable function.

THEOREM 2.2. *Let X be a given Banach space and $f : X \rightarrow X$ be a twice continuously differentiable mapping with derivative $f'(x)$. Then, $f'(x)h$ is given by the collection of those terms of $f(x+h)$ which are linear in h .*

Proof. Apply Taylor's theorem with the second order remainder estimate to $f(x+h)$. \square

EXAMPLE 2.3. Let $f(x) := x^3$, where $x \in \mathcal{A}$. Assume that $\mathcal{C}(\mathcal{A})$ contains \mathbb{K} . Then $f(x+h) = (x+h)^3 = x^3 + x^2h + xhx + hx^2 + xh^2 + hxx + h^2x + h^3$. Thus,

$$(2.1) \quad f'(x)h = x^2h + xhx + hx^2 \text{ where } x \in \mathcal{A}.$$

In order to compute the second derivative we apply the same technique to $f'(x)h$:

$$\begin{aligned} f'(x+k)h &= (x+k)^2h + (x+k)h(x+k) + h(x+k)^2 \\ &= (x^2 + xk + kx + k^2)h + (x+k)h(x+k) + h(x^2 + xk + kx + k^2), \end{aligned}$$

and the bilinear part is

$$(2.2) \quad f''(x)[h, k] = x(kh + hk) + kxh + hxx + (kh + hk)x.$$

This example would apply to $\mathcal{A} := \mathbb{K}^{n \times n}$ the space of all square matrices of order n with entries from \mathbb{K} . In Section 8 we will solve Riccati's equation (1.2) in the more general setting $f : \mathcal{A}^{m \times n} \rightarrow \mathcal{A}^{m \times n}$.

With the help of Theorem 2.2 we can deduce the *product rule*:

LEMMA 2.4. *Assume Y is a Banach algebra and $f, g : X \rightarrow Y$ are twice differentiable then*

$$(2.3) \quad (f \cdot g)'(x_0)h = f(x_0) \cdot g'(x_0)h + f'(x_0)h \cdot g,$$

where the dot \cdot indicates multiplication in Y .

Proof. Observe: $f(x_0+h) = f(x_0) + f'(x_0)h + f''(x_0+th)[h, h]$ where $f''(x_0+th)$ is a bilinear mapping and of course similarly for g . Multiplication of these terms gives

$$\begin{aligned} f(x_0+h) \cdot g(x_0+h) &= \\ (f(x_0) + f'(x_0)h + f''(x_0+th)[h, h]) \cdot (g(x_0) + g'(x_0)h + g''(x_0+\tilde{t}h)[h, h]) \\ &= f(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)h + f'(x_0)h \cdot g(x_0) + O(\|h\|^2). \end{aligned}$$

Thus, the part linear in h is the desired formula (2.3). \square

It is a simple induction to generalize this to finitely many factors.

LEMMA 2.5. *Let Y is a Banach algebra and $f : X \rightarrow Y$ is a product*

$$f(x) = \prod_{j=1}^n g_j(x)$$

of twice differentiable functions, then

$$f'(x_0)h = \sum_{j=1}^n g_1(x_0) \cdot \dots \cdot g_{j-1}(x_0)(g'_j x_0)h \cdot g_{j+1}(x_0) \cdot \dots \cdot g_n(x_0).$$

Proof. This is a straightforward induction using Lemma 2.4. \square

For the sake of completeness we also look at compositions although the possible noncommutativity does not affect the results in this context.

THEOREM 2.6. *Let X, Y, Z be Banach space, $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be differentiable, then*

$$(2.4) \quad (g \circ f)'(x_0)h = g'(f(x_0))f'(x_0)h.$$

Proof. This can be shown by using the definition of differentiability. Taylor's theorem with remainder estimate proves it in spirit of the above statements:

$$\begin{aligned} g(f(x_0 + h)) &= g(f(x_0) + f'(x_0)h + f''(x_0 + th)[h, h]) \\ &= g(f(x_0) + k) \quad (k = f'(x_0)h + f''(x_0 + th)[h, h]) \\ &= g(f(x_0)) + g'(f(x_0))k + o(\|k\|) \\ &= g(f(x_0)) + g'(f(x_0))f'(x_0)h + g'(f(x_0))f''(x_0 + th)[h, h] + o(\|k\|) \\ &= g(f(x_0)) + g'(f(x_0))f'(x_0)h + o(\|h\|). \quad \square \end{aligned}$$

With the derivative $f'(x)$ we can solve the problem $f(x) = 0$ iteratively by Newton's method where x is known and h will be computed as solution of

$$(2.5) \quad f'(x)h + f(x) = 0, \quad x := x + h.$$

3. The definition of the Jacobi matrix. Let \mathcal{A} be a real algebra with $\dim(\mathcal{A}) = N$ and $a = (a_1, a_2, \dots, a_N) \in \mathcal{A}$, $a_j \in \mathbb{R}, j = 1, 2, \dots, N$. Let us start with the definition of a technical operator, col , pronounced *column* in the following form. Since \mathcal{A} has dimension N , we define for $a \in \mathcal{A}$

$$(3.1) \quad \text{col}(a) := \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix},$$

which is a real matrix consisting of one column. This applies also to the case where a is a matrix with entries from \mathcal{A} . In this case all columns of a , including the algebra elements are put into one column going from from the left to the right column of a . If a has the matrix size $m \times n$, then $\text{col}(a)$ has length mnN .

Let $f : \mathcal{A} \rightarrow \mathcal{A}$ be a mapping with derivative $f'(x)$ and let the center $\mathcal{C}(\mathcal{A})$ (see (1.3)) contain \mathbb{R} . According to the definition of a derivative, $f'(x)$ is a linear mapping if we fix x . Thus, there is a matrix $\mathbf{M}(x) \in \mathbb{R}^{N \times N}$ such that

$$(3.2) \quad \text{col}(f'(x)h) = \mathbf{M}(x)\text{col}(h).$$

See Horn and Johnson, [13, p. 5]. How to find this matrix? Let $\mathcal{N}_k, k = 1, 2, \dots, N$, be the k th standard unit vectors in the vector space \mathcal{A} . If we insert for h the basis element \mathcal{N}_k in (3.2), we obtain

$$(3.3) \quad \text{col}(f'(x)\mathcal{N}_k) = \mathbf{M}(x)\text{col}(\mathcal{N}_k), \quad k = 1, 2, \dots, N,$$

which defines the k th column of $\mathbf{M}(x)$. Since the left hand side $\text{col}(f'(x)\mathcal{N}_k)$ is known for all k , the matrix $\mathbf{M}(x)$ is known.

DEFINITION 3.1. Let $f : \mathcal{A} \rightarrow \mathcal{A}$ be a mapping with derivative $f'(x)$. Then, the matrix $\mathbf{M}(x)$ defined in (3.2) and in (3.3) is called the *Jacobi matrix of f* .

EXAMPLE 3.2. (a) Let \mathcal{A} have dimension N and let $f(x) = x^2$. Then, $f'(x)h = xh + hx$ and, applying (3.3), the Jacobi matrix of f is

$$\mathbf{M}(x) = [\text{col}(x\mathcal{N}_1 + \mathcal{N}_1x), \text{col}(x\mathcal{N}_2 + \mathcal{N}_2x), \dots, \text{col}(x\mathcal{N}_N + \mathcal{N}_Nx)] \in \mathbb{R}^{N \times N},$$

which means that each entry represents a column of length N .

(b) Let $x \in \mathcal{A} := \mathbb{R}^{n \times n}$. Then $N = n^2$. Rather than using $\mathcal{N}_\ell, \ell = 1, 2, \dots, N$ we use $e_{jk} \in \mathbb{R}^{n \times n}, j, k = 1, 2, \dots, n$ which is the zero matrix except at the position (j, k) , where there is a one. Now, xe_{jk} contains zeros with the exception of the k th column which is the j th column of x . And $e_{jk}x$ contains zeros with the exception of the j th row which is filled with the k th row of x . The matrix $xe_{jk} + e_{jk}x$ contains one column and one row of x with one overlapping point at position (j, k) with the value $x_{jj} + x_{kk}$. Thus, this matrix contains $n^2 - (n - 1)$ zeros. We can save further calculations, since for general matrices $\text{col}(\mathbf{A}\mathbf{X}\mathbf{B}) = (\mathbf{B}^T \otimes \mathbf{A})\text{col}\mathbf{X}$ (see Horn and Johnson, [12, p. 242]), where \otimes is the Kronecker product, the Jacobi matrix of x^2 is $\mathbf{M}(x) = x^T \otimes i + i \otimes x$, where i is the identity matrix in $\mathbb{R}^{n \times n}$.

Details for defining the Jacobi matrix depend on the special structure of the algebra \mathcal{A} and on the function class to be treated. In the following sections we will discuss some function classes for which the derivative and the Jacobi matrix can easily be computed. There is another idea by Falcão, [7], transferring the real form of Newton's method directly to quaternionic functions.

4. Polynomials. In the noncommutative case polynomials have a more complicated form than in the real or complex case. See [17, 18] for the quaternionic case and [14] for the coquaternionic case and other extensions to \mathbb{R}^4 algebras. Let the coefficients of the polynomials, to be defined, come from a given algebra \mathcal{A} .

DEFINITION 4.1. Let a_0, a_1, \dots, a_j be arbitrary elements in \mathcal{A} distinct from zero. A *monomial of degree $j \in \mathbb{N}$* is a function $m : \mathcal{A}^{j+1} \rightarrow \mathcal{A}$ defined by

$$(4.1) \quad m(x) := a_0 x a_1 x a_2 \cdots a_{j-1} x a_j, \quad x \in \mathcal{A}.$$

Here, the power \mathcal{A}^{j+1} is an abbreviation of the cartesian product $\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}$ with $j + 1$ factors. A *monomial of degree zero* is a constant. A *polynomial* is a finite sum of arbitrary monomials.

THEOREM 4.2. Let the monomial m of (4.1) be given. Define

$$(4.2) \quad s_k(x)h := a_0 x a_1 x a_2 x \cdots a_{k-1} h a_k x \cdots x a_{j-1} x a_j, \quad 1 \leq k \leq j.$$

Assume that s_k is linear in h over \mathbb{K} . Then, the derivative of m is

$$m'(x)h = \sum_{k=1}^j s_k(x)h.$$

Proof. Apply Theorem 2.2. □

The quantities s_k defined in (4.2) have all the same form

$$(4.3) \quad s_k(x)h = A_k(x)hB_k(x), \quad k = 1, 2, \dots, j, \quad \text{where} \\ A_k := a_0 x a_1 x \cdots a_{k-1}, \quad (A_1 = a_0), \quad B_k := a_k x a_{k+1} x \cdots a_{j-1} x a_j, \quad (B_j = a_j).$$

Assume that $\dim(\mathcal{A}) = N$ and that \mathcal{N}_k , $k = 1, 2, \dots, N$ are the standard unit vectors in \mathcal{A} . Then

$$(4.4) \quad \mathbf{M}_k(x) := [\text{col}(A_k \mathcal{N}_1 B_k), \text{col}(A_k \mathcal{N}_2 B_k), \dots, \text{col}(A_k \mathcal{N}_N B_k)] \in \mathbb{K}^{N \times N}, \\ k = 1, 2, \dots, j.$$

This is the matrix (see [16]) which represents the linear mapping defined by $s_k(x)h$ in (4.3), thus,

$$(4.5) \quad \text{col}(s_k(x)h) = \mathbf{M}_k(x)\text{col}(h).$$

Therefore,

$$(4.6) \quad \mathbf{M}(x) := \sum_{k=1}^j \mathbf{M}_k(x)$$

defines the Jacobi matrix of the monomial $m(x)$, given in (4.1) at the point x .

COROLLARY 4.3. *Let m be a monomial of the simple form*

$$(4.7) \quad m(x) := x^j, \quad j \in \mathbb{N}.$$

Then, the derivative of $m(x)$ is

$$(4.8) \quad m'(x)h = \sum_{k=1}^j x^{k-1} h x^{j-k}.$$

The derivative of a constant is zero (multiplied by h).

Proof. The proof follows from Theorem 4.2 by putting $a_k = 1, k = 0, 1, \dots, j$. \square

It is clear that in the above corollary, we can replace $m(x)$ by $m(x) = (x - a)^j$ where in formula (4.8) x has to be replaced on the right hand side by $x - a$ in order to find the derivative of the newly defined $m(x)$. The computation of the Jacobi matrix \mathbf{M} depends on \mathcal{A} . In the polynomial case it is sufficient to compute \mathbf{M}_k which defines the linear mapping defined by $s_k(s)h$, see (4.3), to (4.5). For the matrix case mentioned in the end of the introduction we have

$$(4.9) \quad \mathbf{M}_k = B_k^T \otimes A_k, \quad \mathbb{K} = \mathbb{C},$$

where \otimes denotes the Kronecker product and A_k, B_k are defined in (4.3), $k = 1, 2, \dots, j$. See [12, p. 242].

5. Inverse powers. Here we are interested in computing the derivative and the Jacobi matrix of the negative powers x^{-k} , $k \in \mathbb{N}$ for $x \in \mathcal{A}$.

LEMMA 5.1. *Let $f : \mathcal{A} \rightarrow \mathcal{A}$ be defined by $f(x) = x^{-1}$ for nonsingular $x \in \mathcal{A}$. Then,*

$$(5.1) \quad f'(x)h = -x^{-1}hx^{-1}.$$

Proof. We apply the product rule (2.3) to $G(x) := f(x) \cdot g(x) := x^{-1} \cdot x = 1$, where 1 is the element one of \mathcal{A} . Then, $0h = G'(x)h = f(x) \cdot g'(x)h + f'(x)h \cdot g(x) = x^{-1} \cdot h + f'(x)h \cdot x \Rightarrow$ (5.1), where the dot \cdot denotes algebra multiplication. \square

THEOREM 5.2. *Let $f_k : \mathcal{A} \rightarrow \mathcal{A}$ be defined by $f_k(x) = x^{-k}$, $k \in \mathbb{N}$ for nonsingular $x \in \mathcal{A}$. Then,*

$$(5.2) \quad f'_k(x)h = - \sum_{\substack{r+s=k+1 \\ r, s \geq 1}} x^{-r} h x^{-s}.$$

Proof: The proof will be by induction with respect to k . For $k = 1$, Lemma 5.1 contains the proof. We apply the product rule to $f_{k+1}(x) = x^{-k}x^{-1}$ and employ Lemma 5.1 again and obtain $f'_{k+1}(x)h = f'_k(x)hx^{-1} + x^{-k}f'_1(x)h =$

$$= - \left(\sum_{\substack{r+s=k+1 \\ r,s \geq 1}} x^{-r}hx^{-s} \right) x^{-1} - x^{-k}x^{-1}hx^{-1} = - \sum_{\substack{r+s=k+2 \\ r,s \geq 1}} x^{-r}hx^{-s}. \quad \square$$

THEOREM 5.3. *The Jacobi matrix of $f_k(x) := x^{-k}$, $k \in \mathbb{N}$ in an algebra \mathcal{A} of dimension N is*

$$(5.3) \quad \mathbf{M}(x) = - \sum_{\substack{r+s=k+1 \\ r,s \geq 1}} [\text{col}(x^{-r}\mathcal{N}_1x^{-s}), \text{col}(x^{-r}\mathcal{N}_2x^{-s}), \dots, \text{col}(x^{-r}\mathcal{N}_Nx^{-s})].$$

COROLLARY 5.4. *Let $f_k(x) := (x - a)^{-k}$ for nonsingular $x - a$. Then, for the derivative of $f_k(x)$ we can apply formula (5.2) and obtain*

$$f'_k(x)h = - \sum_{\substack{r+s=k+1 \\ r,s \geq 1}} (x - a)^{-r}h(x - a)^{-s}.$$

This formula allows to evaluate Laurent series.

6. The exponential function. Let \mathcal{A} be a real, finite dimensional algebra with $\mathbb{R} \subset \mathcal{C}(\mathcal{A})$ and $\dim(\mathcal{A}) = N$. We use in all cases of such algebras \mathcal{A} the standard power series representation of the *exponential function*

$$(6.1) \quad \exp(x) := \sum_{j=0}^{\infty} \frac{x^j}{j!}, \quad x \in \mathcal{A}.$$

THEOREM 6.1. *The derivative of the exponential function \exp is*

$$(6.2) \quad \exp'(x)h = \sum_{j=1}^{\infty} \frac{\sum_{k=1}^j x^{k-1}hx^{j-k}}{j!},$$

which implies that the Jacobi matrix of the exponential function is

$$(6.3) \quad \mathbf{M}(x) = \sum_{j=1}^{\infty} \frac{\sum_{k=1}^j \mathbf{M}_k(x)}{j!}, \quad \text{where}$$

$$(6.4) \quad \mathbf{M}_k(x) = [\text{col}(x^{k-1}\mathcal{N}_1x^{j-k}), \text{col}(x^{k-1}\mathcal{N}_2x^{j-k}), \dots, \text{col}(x^{k-1}\mathcal{N}_Nx^{j-k})].$$

The first column of $\mathbf{M}(x)$ is $\text{col}(\exp(x))$. If \mathcal{A} happens to be commutative, the j th column of $\mathbf{M}(x)$ is $\text{col}(\exp(x)\mathcal{N}_j)$, $j = 1, 2, \dots, N$.

Proof. The matrix $\mathbf{M}_k(x)$ is a special case of (4.4). See also (4.8). Since $\mathcal{N}_1 = 1$, the first column of $\mathbf{M}_k(x)$ is $\text{col}(x^{j-1})$, $j \geq 1$ for all k . In a commutative case, the j th column of $\mathbf{M}_k(x)$ is $\text{col}(x^{j-1}\mathcal{N}_j)$, $j \geq 1$ for all k . \square

The computation of the Jacobi matrix (6.3) to the highest possible precision is very time consuming. Therefore, it is reasonable to use a hybrid technique. In the beginning of the Newton iteration where $f(x)$ is still large (in euclidean norm) one computes the Jacobi matrix $\mathbf{M}(x)$ only to a low precision, and only if $f(x)$ is small,

say $\|f(x)\| < 10^{-4}$, one switches to a higher precision. If this does not happen in a reasonable number of steps, one changes the initial guess.

Let \mathcal{A} be an algebra with $\dim(\mathcal{A}) = N \geq 2$ and with $\mathcal{N}_k^2 = -1$ for a fixed k with $2 \leq k \leq N$. Then, an element $a = (a_1, a_2, \dots, a_N) \in \mathcal{A}$ will be called *complex* if $a_j = 0$ for $j \neq k$ and $j \neq 1$.

LEMMA 6.2. *Let \mathcal{A} be an algebra with $\dim(\mathcal{A}) = N \geq 2$ and with the property that $\mathcal{N}_k^2 = -1$, for a fixed $k \geq 2$. Then, the subset $\{a \in \mathcal{A} : a = (a_1, 0, \dots, 0, a_k, 0, \dots, 0)\}$ is a subalgebra isomorphic to \mathbb{C} .*

Proof. Let $a = \sum_{j=1}^N a_j \mathcal{N}_j$, $b = \sum_{j=1}^N b_j \mathcal{N}_k$ be two complex elements of \mathcal{A} . Then, $ab = \sum_{j,\ell=1}^2 a_j b_\ell \mathcal{N}_j \mathcal{N}_\ell = a_1 b_1 - a_k b_k + (a_1 b_k + a_k b_1) \mathcal{N}_k = ba$. \square

THEOREM 6.3. *Let $x = u + v \mathcal{N}_j \in \mathcal{A}$ for an arbitrary $2 \leq j \leq N$. Then*

$$(6.5) \quad \exp(x) = \exp(u) \exp(v \mathcal{N}_j) = \begin{cases} \exp(u) (\cos(v) + \sin(v) \mathcal{N}_j) & \text{if } \mathcal{N}_j^2 = -1, \\ \exp(u) (\cosh(v) + \sinh(v) \mathcal{N}_j) & \text{if } \mathcal{N}_j^2 = +1. \end{cases}$$

Proof. The proof of Lemma 6.2 implies that all powers in the definition (6.1) of the exponential function remain in the first and in the j th component of \mathcal{A} . Thus, standard techniques from real analysis can be applied to derive (6.5). \square

COROLLARY 6.4. *Let $x = \pi \mathcal{N}_j \in \mathcal{A}$, $2 \leq j \leq N$. Then*

$$(6.6) \quad \exp(x) = \begin{cases} -1 & \text{if } \mathcal{N}_j^2 = -1, \\ \cosh(\pi) + \sinh(\pi) \mathcal{N}_j \approx 11.5920 + 11.5484 \mathcal{N}_j & \text{if } \mathcal{N}_j^2 = +1. \end{cases}$$

It is clear, that a similar derivation can be made for all functions whose power series expansion is real, like the trigonometric functions.

In the next section we will present some examples of algebras in \mathbb{R}^4 and we will solve a numerical problem related to the exponential function in all of these algebras by the means developed.

7. Examples of algebras in \mathbb{R}^4 . As examples we will present eight \mathbb{R}^4 algebras, where in all cases the four basis elements are the standard unit vectors in \mathbb{R}^4 which are denoted by

$$(7.1) \quad \mathbf{1} := (1, 0, 0, 0), \mathbf{i} := (0, 1, 0, 0), \mathbf{j} := (0, 0, 1, 0), \mathbf{k} := (0, 0, 0, 1),$$

and where for the three squares $\mathbf{i}^2, \mathbf{j}^2, \mathbf{k}^2$ all eight possibilities to assign ± 1 to them are used. The essential multiplication rules are given in Table 7.1. The first component a_1 of an element $a := (a_1, a_2, a_3, a_4) \in \mathbb{R}^4$ will always be called *real part* and denoted by $\Re(a)$, and a real number, a_1 , will be identified with $(a_1, 0, 0, 0)$.

TABLE 7.1. Essential multiplication rules for eight algebras in \mathbb{R}^4 .

No	Name of algebra	Short name	\mathbf{i}^2	\mathbf{j}^2	\mathbf{k}^2	\mathbf{ij}	\mathbf{jk}	\mathbf{ki}
1	Quaternions	\mathbb{H}	-1	-1	-1	\mathbf{k}	\mathbf{i}	\mathbf{j}
2	Coquaternions	\mathbb{H}_{coq}	-1	1	1	\mathbf{k}	$-\mathbf{i}$	\mathbf{j}
3	Tessarines	\mathbb{H}_{tes}	-1	1	-1	\mathbf{k}	\mathbf{i}	$-\mathbf{j}$
4	Cotessarines	$\mathbb{H}_{\text{cotes}}$	1	1	1	\mathbf{k}	\mathbf{i}	\mathbf{j}
5	Nectarines	\mathbb{H}_{nec}	1	-1	1	\mathbf{k}	\mathbf{i}	$-\mathbf{j}$
6	Conectarines	\mathbb{H}_{con}	1	1	-1	\mathbf{k}	$-\mathbf{i}$	$-\mathbf{j}$
7	Tangerines	\mathbb{H}_{tan}	1	-1	-1	\mathbf{k}	$-\mathbf{i}$	\mathbf{j}
8	Cotangerines	$\mathbb{H}_{\text{cotan}}$	-1	-1	1	\mathbf{k}	$-\mathbf{i}$	$-\mathbf{j}$

The complete multiplication tables for these eight algebras can be found by multiplying the last three columns from the right by $\mathbf{j}, \mathbf{k}, \mathbf{i}$, respectively. It turns out, that the algebras with the numbers 3,4,7,8 are commutative.

Quaternions were invented by Hamilton, 1843, [10] and three algebras called *coquaternions* (also called *split quaternions*), *tessarines*, *cotassarines* were introduced in a paper by Cockle, 1849, [4, 5]. The names *nectarines*, *conectarines*, *tangerines*, *cotangerines* were introduced in a recent (Jan. 18, 2014, not yet published) paper by Bernd Schmeikal (Vienna), [21]. The coquaternions have recently gained some interest in quantum mechanics. See [3, 8]. However, all other algebras, numbered 3 to 8 in Table 7.1 have not left traces in the mathematical literature. The explicit form of the matrix representations of the 8 algebras including the rules for computing the inverses is given in [14].

8. The Jacobi matrix for the algebraic Riccati equation with matrix entries from an algebra \mathcal{A} . As a nontrivial application of the section on polynomials we will derive the Jacobi matrix for the Riccati equation (1.2) under the assumption that the matrix elements in the Riccati equation stem from an algebra \mathcal{A} , where \mathcal{A} is supposed to be an algebra in \mathbb{R}^N with center $\mathcal{C}(\mathcal{A})$ that contains \mathbb{R} . We keep the notation of (1.2), p. 2, where \mathbb{K} is replaced with \mathcal{A} .

THEOREM 8.1. *The Jacobi matrix $\mathbf{M}(\mathbf{X})$ for the Riccati equation f , defined in (1.2), with matrix elements from an N -dimensional algebra in \mathbb{R}^N is*

$$(8.1) \quad \mathbf{M}(\mathbf{X}) = \mathbf{M}_1(\mathbf{X}) + \mathbf{M}_2(\mathbf{X}), \quad \mathbf{M}_1(\mathbf{X}), \mathbf{M}_2(\mathbf{X}) \in \mathbb{R}^{mnN \times mnN}.$$

In order to find $\mathbf{M}_1, \mathbf{M}_2$, we enumerate the elements of a matrix $\mathbf{H} \in \mathcal{A}^{m \times n}$ column-wise, counting also the N elements of the entries of \mathbf{H} and denote the j th element in this enumeration by \mathbf{H}_j , $j = 1, 2, \dots, mnN$. For $j = 1, 2, \dots, mnN$ we define the matrix $\mathbf{H}^{(j)} \in \mathcal{A}^{m \times n}$ by putting $\mathbf{H}_j = 1$ and $\mathbf{H}_k = 0$ for all $k \neq j$. Then, the j th columns of $\mathbf{M}_1, \mathbf{M}_2$ are, respectively,

$$\text{col} \left((\mathbf{B} + \mathbf{XD})\mathbf{H}^{(j)} \right), \quad \text{col} \left(\mathbf{H}^{(j)}(\mathbf{C} + \mathbf{DX}) \right), \quad j = 1, 2, \dots, mnN.$$

Proof. The derivative of the algebraic Riccati equation f defined in (1.2) is the derivative of the sum of the three terms $\mathbf{BX}, \mathbf{XC}, \mathbf{XDX}$. These terms were already considered in the Section Polynomials. And a previous result implies

$$f'(\mathbf{X})\mathbf{H} = (\mathbf{B} + \mathbf{XD})\mathbf{H} + \mathbf{H}(\mathbf{C} + \mathbf{DX}) \in \mathcal{A}^{m \times n}, \quad \mathbf{H}, \mathbf{X} \in \mathcal{A}^{m \times n}.$$

This is a linear mapping $\mathcal{A}^{m \times n} \rightarrow \mathcal{A}^{m \times n}$ over \mathbb{R} and thus,

$$(8.2) \quad \text{col}(f'(\mathbf{X})\mathbf{H}) = \mathbf{M}_1(\mathbf{X})\text{col}(\mathbf{H}) + \mathbf{M}_2(\mathbf{X})\text{col}(\mathbf{H}) = \mathbf{M}(\mathbf{X})\text{col}(\mathbf{H}), \quad \mathbf{H} \in \mathcal{A}^{m \times n}.$$

For the remaining part we proceed as in Example 3.2, part (b). \square

EXAMPLE 8.2. We choose $m = 2, n = 3$ and define the Riccati equation by $\mathbf{A} \in \mathcal{A}^{2 \times 3}, \mathbf{B} \in \mathcal{A}^{2 \times 2}, \mathbf{C} \in \mathcal{A}^{3 \times 3}, \mathbf{D} \in \mathcal{A}^{3 \times 2}$, and define the matrix elements as members of one of the mentioned \mathbb{R}^4 algebra as follows, where $\mathbf{A} = (a_{jk})$, etc.

TABLE 8.3. Definition of the coefficients of Riccati's equation.

$$\begin{array}{l} a_{11} = (1, -4, -2, 0) \\ a_{22} = (2, -3, -4, 0) \\ b_{21} = (-1, 3, 1, 0) \\ c_{13} = (1, 0, -5, -2) \\ c_{31} = (-3, 4, -3, 3) \\ d_{12} = (-3, 0, -1, 1) \\ d_{32} = (-4, -2, 3, -2) \end{array} \left| \begin{array}{l} a_{12} = (-1, 4, 3, 5) \\ a_{23} = (4, 5, 0, -4) \\ b_{22} = (4, -2, 3, 3) \\ c_{21} = (-3, 3, -2, 0) \\ c_{32} = (0, 5, -4, -1) \\ d_{21} = (2, -5, 3, 4) \end{array} \right| \begin{array}{l} a_{13} = (2, -3, 2, -5) \\ b_{11} = (3, -3, 4, -2) \\ c_{11} = (-1, 1, -4, -4) \\ c_{22} = (-3, 1, -2, 2) \\ c_{33} = (-4, 5, -5, 3) \\ d_{22} = (5, -2, 1, -3) \end{array} \left| \begin{array}{l} a_{21} = (-1, -1, 3, 3) \\ b_{12} = (-3, -2, 1, 0) \\ c_{12} = (0, 3, 4, -4) \\ c_{23} = (2, 2, 0, -4) \\ d_{11} = (5, 5, -3, 5) \\ d_{31} = (-2, -5, -4, 3) \end{array} \right.$$

The data of Table 8.3 are randomly chosen integers in $[-5, 5]$. We choose $\mathbf{X} = \mathbf{0}$ as initial guess with the exception of Algebra 6, where another guess is used. Newton's method converges then in all 8 cases. The solutions are given in Table 8.4. These solutions are not necessarily the only solutions.

9. An example involving the exponential function. Let us treat the problem

$$(9.1) \quad f(x) := \exp(x) - \mathbf{j}x^{-1} = 0$$

in the algebras of Table 7.1. This example is chosen because it employs the Jacobi matrix of both the exponential and the inverse function. For the function f defined in (9.1) we have the following information:

LEMMA 9.1. *Let f be defined as in (9.1) and let $x_\ell = a + b\mathcal{V}_\ell$ for $\ell = 2, 3, 4$ and arbitrary $a, b \in \mathbb{R}$. Then, (i) For x_2 the function values $f(x_2)$ are the same in algebra 1,2, in algebra 3,8, in algebra 4,7, and in algebra 5,6. (ii) For x_3 the function values $f(x_3)$ have the form $A + B\mathbf{j}$ in all 8 algebras with real A, B and $f(x_3)$ are the same in algebra 1,5,7,8 and in algebra 2,3,4,6. (iii) For x_4 the function values $f(x_4)$ are the same in algebra 1,3, in algebra 2,8, in algebra 4,5, and in algebra 6,7.*

Proof. (i) The algebras 1,2,3,8 have $\mathbf{i}^2 = -1$ and for the algebras 4,5,6,7 we have $\mathbf{i}^2 = +1$. The algebras 1,2,5,6 are noncommutative and 3,4,7,8 are commutative. Together with Theorem 6.3 the result follows for x_2 . The other two cases are similar, with the additional property that $x_3^{-1} = A + B\mathbf{j}$. Explicit formulas for x^{-1} are given in [14]. \square

Problem 9.1 has one special type of solutions, namely solutions of the form $x = a + b\mathbf{j}$, $a, b \in \mathbb{R}$. Solutions of this form are given in Table 9.2. They appear in the algebras numbered 1,5,7,8 (see Lemma 9.1 (ii)). Solutions $x = (x_1, x_2, x_3, x_4)$ where all four components do not vanish were found only in algebra 3 and 7. They are presented in Table 9.3. The norm $\|\cdot\|$ used is the euclidean norm in \mathbb{R}^4 . All numerical results were obtained by using MATLAB, using Newton's method.

TABLE 9.2. Solutions $x = a + b\mathbf{j}$ of $\exp(x) = \mathbf{j}x^{-1}$ with respect to algebra \mathcal{A} .

Algebras No	a	b	$\ f(x)\ $
1, 5, 7, 8	-1.834 271 700 407 948	5.985 834 988 966 563	1.206e - 14
	-2.246 613 246 955 871	-9.184 889 165 789 079	8.997e - 14
	0.374 699 020 737 117	0.576 412 723 031 436	2.221e - 16
	-1.089 648 913 877 818	-2.766 362 603 273 831	1.786e - 14

We did not find solutions in algebras 2, 4, 6. For computational details compare the remarks following the proof of Theorem 6.1.

Table 8.4 Solutions $\mathbf{X} \in \mathcal{A}^{2 \times 3}$ of Riccati's equation in all 8 algebras of Table 7.1.

1	(-0.1045, -0.1495, 0.2660, 0.0712)	(0.2731, -0.1913, 0.1003, 0.6167)	(0.9956, 0.1306, 0.3851, 0.6174)
	(0.3071, 0.3399, -0.6350, 0.1765)	(0.5842, 0.7970, -0.0900, -0.4788)	(-0.5317, -0.3326, -0.5753, -0.3978)
2	(0.1544, 0.6186, -0.0883, 0.3512)	(0.1612, 0.0891, -0.6692, -0.2445)	(-0.1351, 0.2386, -0.6664, 0.8003)
	(0.2732, 0.3379, -0.5506, 0.2582)	(0.5916, 1.3237, 0.1198, 0.6950)	(-1.2738, 0.5379, 0.6102, 1.0119)
3	(-0.8748, -0.1261, 0.0210, 0.0205)	(0.1632, 0.0329, -0.1188, -0.4736)	(-0.6189, 0.5054, -0.8030, -0.4800)
	(-0.1044, -0.4106, 1.0738, -0.9254)	(0.0282, 0.4299, 0.3035, 0.4502)	(0.8554, -0.0146, 0.2011, -1.1011)
4	(0.0749, 0.5097, 0.8189, -0.7321)	(-1.7670, 0.0836, 0.3282, -1.2972)	(1.0919, -1.0135, -0.5820, 1.0441)
	(0.1177, -0.0515, -0.7539, 0.0017)	(0.3680, 0.1663, 0.7493, 1.4654)	(-0.5413, -0.2660, 0.0956, -0.2739)
5	(0.1490, -0.1557, 0.6250, 0.3262)	(0.5285, 0.1447, -0.4479, 0.0074)	(-0.2660, -0.3224, 0.3163, 0.3754)
	(0.1598, -0.5805, -0.9796, -1.2966)	(-0.2607, -0.4621, -0.8051, -0.4240)	(-0.7931, -0.9394, 0.4466, 0.6899)
6	(1.3763, -0.5053, 0.8822, -1.0762)	(1.5484, -4.3525, -1.4404, 5.4397)	(1.7588, -0.5349, -0.3498, 0.8295)
	(0.4270, 0.0322, 0.5237, 0.8558)	(-0.5729, 1.4386, 1.0001, -1.7677)	(-0.3289, 0.4151, -0.0512, -0.3035)
7	(0.0238, 0.1553, -0.6713, 0.3590)	(-0.1408, -0.0282, 0.2520, 0.8164)	(0.1099, -0.4139, 0.0090, -0.1696)
	(0.3589, -0.1427, -0.2777, -0.6520)	(-0.1875, -0.0557, 0.5904, 0.4577)	(-0.1393, -0.2064, 0.0938, 0.0396)
8	(-0.4374, -0.8677, -0.4367, 0.8265)	(0.2330, -0.6134, -0.4203, 0.2160)	(0.2155, -0.2357, -0.4028, 0.0678)
	(-0.4075, 0.6499, 0.5927, -0.0469)	(-0.4898, -0.1886, 0.2013, 0.4882)	(0.2917, 0.1488, 0.3577, -0.2748)

Table 9.3 Solutions $x = (x_1, x_2, x_3, x_4)$ of $\exp(x) = \mathbf{j}x^{-1}$ with respect to algebra \mathcal{A} .

No	x_1	x_2	x_3	x_4	$\ f(x)\ $
3	-0.926022412499154	2.856210427246304	-0.607890907294389	1.518974725815617	3.362e - 15
	-1.798095524696101	1.606723012705319	0.264182204902493	-5.981908165767104	6.048e - 14
7	-1.812521021860694	0.722872107972611	4.798875608829636	-7.565238212111327	4.614e - 12
	-0.357474946570336	0.732173967307453	-1.094974940121210	1.671387663152645	1.495e - 15
	-1.461960307142823	-0.372311393265054	1.609736192846364	4.376098796120243	6.517e - 15

Acknowledgment. The research of the second mentioned author was supported by the German Science Foundation, DFG, GZ: OP 33/19-1. The numerical results were obtained by using MATLAB, Version: 7.14.0.739 (R2012a), Operating System: Mac OS X Version: 10.6.8.

REFERENCES

- [1] H. Abou-Kandill, G. Freiling, V. Ionescu, G. Jank, *Matrix Riccati equations in control and systems theory*, Birkhäuser, Basel, Boston, Berlin, 2003, 572 pp.
- [2] D. A. Bini, B. Iannazzo, B. Meini, *Numerical solution of algebraic Riccati equations*, siam, Philadelphia, 2012, 250 pp.
- [3] D. C. Brody and E.-M. Graefe, *On complexified mechanics and coquaternions*, J. Phys. A: Math. Theory, **44** (2011) 072001, 9 pp., also: arXiv:1012.0757v2 [math-ph] 7 Jan 2011, 11 pp.
- [4] J. Cockle, *On systems of algebra involving more than one imaginary; and on equations of the fifth degree*, Phil. Mag. (3) 35, 1849, pp. 434–437.
- [5] J. Cockle, <http://www.oocities.org/cocklebio/>
- [6] J. Dieudonné, *Foundation of modern analysis*, Academic Press, New York, London, 1960, 361 pp.
- [7] M. I. Falcão, *Newton method in the context of quaternionic analysis*, Appl. Math. Comput., **236** (2014), pp. 458–470.
- [8] I. Frenkel, M. Libine, *Split quaternionic analysis and separation of the series for $SL(2, \mathbb{R})$ and $SL(2, \mathbb{C})/SL(2, \mathbb{R})$* , Advances in Mathematics 228, (2011), 678–763 pp., also: arXiv:1009.2532v2 [math.RT] 22 Jul 2011, 70 pp.
- [9] D. J. H. Garling, *Clifford algebras: an introduction*, Cambridge University Press, Cambridge, 2011, 200 pp.
- [10] W. R. Hamilton, http://en.wikipedia.org/wiki/William_Rowan_Hamilton
- [11] K. Gürlebeck, W. Sprössig, *Quaternionic and Clifford calculus for physicists and engineers*, Wiley, Chichester, 1997, 371 pp.
- [12] R. A. Horn and C. R. Johnson, *Topics in matrix analysis*, Cambridge University Press, Cambridge, 1991, 607 pp.
- [13] R. A. Horn and C. R. Johnson, *Matrix analysis*, Cambridge University Press, Cambridge, 1992, 561 pp.
- [14] D. Janovská, G. Opfer, *Zeros and singular points for one-sided, coquaternionic polynomials with an extension to other \mathbb{R}^4 algebras*, accepted for publication in ETNA, March 2014.
- [15] D. Janovská, G. Opfer, *The algebraic Riccati equation for quaternions*, dedicated to Ivo Marek on the occasion of his 80th birthday, Adv. Appl. Clifford Algebras, **23** (2013), pp. 907–918.
- [16] D. Janovská, G. Opfer, *Linear equations and the Kronecker product in coquaternions*, Mitt. Math. Ges. Hamburg, **33** (2013), pp. 181–196.
- [17] D. Janovská, G. Opfer, *The classification and the computation of the zeros of quaternionic, two-sided polynomials*, Numer. Math., **115** (2010), pp. 81–100.
- [18] D. Janovská, G. Opfer, *A note on the computation of all zeros of simple quaternionic polynomials*, SIAM J. Numer. Anal., **48** (2010), pp. 244–256.
- [19] D. Janovská, G. Opfer, *Computing quaternionic roots by Newton's method*, Electron. Trans. Numer. Anal., **26** (2007), pp. 82–102.
- [20] P. Lancaster, L. Rodman, *Algebraic Riccati equations*, Clarendon Press, 1995, 498p.
- [21] B. Schmeikal, *Tessarinen, Nektarinen und andere Vierheiten. Beweis einer Vermutung von Gerhard Opfer*, submitted to Mitt. Math. Ges. Hamburg, **34** (2014).
- [22] V. Simoncini, D. B. Szyld, M. Monsalve, *On two numerical methods for the solution of large-scale algebraic Riccati equations*, Report 12-11-26, November 2012, revised February 2013, to appear in the IMA Journal on Numerical Analysis.
- [23] A. E. Taylor, D. C. Lay, *Introduction to functional analysis*, 2nd ed., Wiley, New York, 1980, 467 pp.
- [24] F. Zhang, *Quaternions and matrices of quaternions*, Linear Algebra Appl., **251** (1997), pp. 21–57.