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POLYNOMIAL INTERPOLATION IN NONDIVISION ALGEBRAS *

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Abstract. Algorithms for two types of interpolation polynomials in nondivision algebras are presented. One is based on the Vandermonde matrix, the other is close to the Newton interpolation scheme. Examples are taken from $\mathbb{R}^4$ algebras. In the Vandermonde case necessary and sufficient conditions for the existence of interpolation polynomials are given for commutative algebras. For noncommutative algebras there is a conjecture. This conjecture is true for equidistant nodes. It is shown that the Newton form of the interpolation polynom exists if and only if all node differences are invertible. There are several numerical examples.

Key words. Interpolation polynomials in nondivision algebras, Vandermonde type polynomials in nondivision algebras, Newton type polynomials in nondivision algebras, numerical examples of interpolation polynomials in nondivision algebras of $\mathbb{R}^4$.

AMS subject classifications. 15A66, 1604, 41A05, 65D05.

1. Introduction. The aim of this paper to provide an algorithm for solving the polynomial interpolation problem in nondivision algebras and studying the conditions under which such an algorithm works. As examples of nondivision algebras we will mainly use one of the eight $\mathbb{R}^4$ algebras. The letter $\mathbb{R}$ is used here for the field of real numbers. An algebra in general is the vector space $\mathbb{R}^N$ equipped with an additional, associative multiplication $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ which has a one, usually abbreviated by 1. More information can be found in a book by Garling, [3]. These algebras are also called *geometric algebras*, [5]. The names and the algebraic rules for the eight algebras in $\mathbb{R}^4$ can be found in a paper by Janovská, Opfer, [6]. The algebras are abbreviated by (the names are in parentheses. The names where given by Hamilton 1843 for $\mathbb{H}$, by Cockle, 1849 for $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm tes}, \mathbb{H}_{\rm cotes}$, [1, 2] and by Schmeikal, 2014 for the remaining four, [13])

$$\mathbb{H}(\text{quaternions}),\ \mathbb{H}_{\rm coq}(\text{coquaternions or split-quaternions}),\ \mathbb{H}_{\rm tes}(\text{tessarines}),$$
$$\mathbb{H}_{\rm cotes}(\text{cotessarines}),\ \mathbb{H}_{\rm nec}(\text{nectarines}),\ \mathbb{H}_{\rm con}(\text{conectarines}),$$
$$\mathbb{H}_{\rm tan}(\text{tangerines}),\ \mathbb{H}_{\rm cotan}(\text{cotangerines}).$$

Note that $\mathbb{H}_{\rm tes}$, $\mathbb{H}_{\rm cotes}$, $\mathbb{H}_{\rm tan}$, $\mathbb{H}_{\rm cotan}$ are commutative. We will in general use the notation $\mathcal{A}$ for one of these algebras. The problem to be considered will be called *interpolation problem* and we will consider two types of interpolation problems. One will be called *Vandermonde interpolation problem* and the other *Newton interpolation problem*. Both problem types rely on a set $(x_k, f_k) \in \mathcal{A} \times \mathcal{A}$ of *data*, where the x_k are referred to as *nodes*, and the f_k are referred to as *values*, $1 \le k \le n+1$. The minimum requirement for the nodes is that they are pairwise distinct.

We need the notion of similarity. For this purpose it is useful to introduce the simple notation

$$a = (a_1, a_2, a_3, a_4),\ a \in \mathcal{A}$$

for elements from $\mathbb{R}^4$ algebras. The first component, a_1 of a is called the *real part of* a and denoted by $a_1 = \Re(a)$. An element of the form $(a_1, 0, 0, 0)$ is called real, and the real elements of $\mathcal{A}$ can be identified by $\mathbb{R}$.

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DEFINITION 1.1. (1) Two elements $a, b \in \mathcal{A}$ will be called *similar*, denoted by $a \sim b$ if there is an invertible $h \in \mathcal{A}$ such that

$$b = h^{-1}ah.$$

The set of elements which are similar to a fixed $a \in \mathcal{A}$ is called *similarity class of* a, denoted by $[a]$ and formally defined by

$$[a] := \{b : b = h^{-1}ah \text{ for all invertible } h \in \mathcal{A}\}.$$

(2) Let $\mathcal{A}$ be one of the four noncommutative algebras. Define the *conjugate of* a by

$$\overline{a} = \mathrm{conj}(a) = (a_1, -a_2, -a_3, -a_4) \text{ and put } \mathrm{abs}_2(a) := a\overline{a}.$$

An early paper with the topic of similarity in connection with quaternions was already given 1936 by Wolf, [14]. It is clear, that similarity is an equivalence relation. Definition 1.1 part (1) applied to a commutative algebra $\mathcal{A}$ yields $[a] = \{a\}$, thus, in this case, the similarity class consists only of one element, which means, that two elements in a commutative algebra are similar if and only if they are identical. Now, by former results of the already quoted paper, [6], it is easily possible to identify similar elements, and to characterize invertible elements in one of the noncommutative algebras.

THEOREM 1.2. *Let $a, b \in \mathcal{A}\backslash\mathbb{R}$ and let $\mathcal{A}$ be one of the four noncommutative $\mathbb{R}^4$ algebras. Then $a \sim b$ if and only if*

$$\Re(a) = \Re(b), \quad \mathrm{abs}_2(a) = \mathrm{abs}_2(b), \tag{1.1}$$

where $\mathrm{abs}_2(a)$ is a real quantity with $\mathrm{abs}_2(a) \neq 0$ if and only if a is invertible and

$$a^{-1} = \frac{\overline{a}}{\mathrm{abs}_2(a)} \text{ if } \mathrm{abs}_2(a) \neq 0. \tag{1.2}$$

For $\mathrm{abs}_2(a)$ there is the formula

$$\mathrm{abs}_2(a) = \begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 & \text{for } a \in \mathbb{H}, \\ a_1^2 + a_2^2 - a_3^2 - a_4^2 & \text{for } a \in \mathbb{H}_{\mathrm{coq}}, \\ a_1^2 - a_2^2 + a_3^2 - a_4^2 & \text{for } a \in \mathbb{H}_{\mathrm{nec}}, \\ a_1^2 - a_2^2 - a_3^2 + a_4^2 & \text{for } a \in \mathbb{H}_{\mathrm{con}}, \end{cases} \tag{1.3}$$

and the property

$$\mathrm{abs}_2(ab) = \mathrm{abs}_2(ba) = \mathrm{abs}_2(a)\mathrm{abs}_2(b). \tag{1.4}$$

Proof. See [6]. □

The side condition for (1.1) that $a, b \notin \mathbb{R}$ can be omitted for $\mathbb{H}$ but not for the other three noncommutative algebras. This can be easily seen by the example $a = (1, 0, 0, 0)$, $b = (1, 3, 3, 0)$ for $\mathbb{H}_{\mathrm{coq}}$ and $\mathbb{H}_{\mathrm{nec}}$ and by $a = (1, 0, 0, 0)$, $b = (1, 3, 0, 3)$ for $\mathbb{H}_{\mathrm{con}}$. The condition (1.1) is valid for a, b, but $a = (a_1, 0, 0, 0)$ is not similar to any nonreal element. In a corresponding theorem for coquaternions by Pogoruy and Rodríguez-Dagnino, [11], this condition is missing.

2. The Vandermonde approach. Given a *polynomial* p of degree $n \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of positive integers) in the form

$$p(x) := \sum_{j=1}^{n+1} a_j x^{j-1}, \quad x, a_j \in \mathcal{A},\ 1 \le j \le n+1, \tag{2.1}$$

and pairwise distinct *nodes*

$$x_k \in \mathcal{A},\ 1 \le k \le n+1, \tag{2.2}$$

$$\text{such that } x_k - x_{k+1} \text{ are invertible for all } 1 \le k \le n, \tag{2.3}$$

and *values*

$$f_k \in \mathcal{A},\ 1 \le k \le n+1 \tag{2.4}$$

with the requirement that

$$p(x_k) = f_k,\ 1 \le k \le n+1. \tag{2.5}$$

We use this notation because it is convenient in some programming languages, like MATLAB. The cases with $n = 1,\ n = 2,\ n = 3$ will be called *linear, quadratic, cubic*, respectively. The given condition (2.3) for the nodes augments the standard condition, that the nodes are pairwise distinct. In a nondivision algebra two distinct nodes do not necessarily have the property that the difference is invertible. This will be seen in the following. The requirement given in (2.5) leads to a linear system in the algebra $\mathcal{A}$ with $n+1$ unknowns and equations, defined by the Vandermonde matrix. Therefore we call this approach the Vandermonde approach.

LEMMA 2.1. *The interpolation problem, defined by (2.1) to (2.5) does not necessarily have a solution and if there is a solution it may not be unique.*

Proof. Let there be two solutions p, q to the interpolation problem. Then, $(p-q)(x_k) = 0,\ 1 \le k \le 1+n$, which implies that the difference polynomial $p - q$ of degree n has $n+1$ zeros. However, the polynomials considered here do not necessarily obey the Haar condition, which means that a polynomial of degree n with more than n zeros vanishes identically. See [6] and [10], also for the case that the Vandermonde matrix is singular. ◻

The algebra $\mathbb{H}$ of quaternions is a division algebra (i. e. the only noninvertible element is the zero element) and the problem to be considered has been solved for $\mathbb{H}$ by Lam, [8] with extensions by Lam and Leroy, [9]. The result is the following:

THEOREM 2.2. *The interpolation problem in $\mathbb{H}$ as stated above has a unique solution if and only if the nodes obey the following rule: Not three of them belong to the same similarity class.*

Proof. By Lam, [8]. ◻

DEFINITION 2.3. Let there be $n+1$ pairwise distinct nodes $x_k \in \mathcal{A},\ 1 \le k \le n+1$. If there does not exist a subset of three nodes which belong to the same similarity class, we say that the nodes satisfy the *Lam condition*.

The Lam condition is satisfied if the underlying algebra $\mathcal{A}$ is commutative or if the number of nodes is at most two. In the algebra of quaternions $\mathbb{H}$, two distinct nodes have the property that the difference is invertible. This is not true for the other $\mathbb{R}^4$ algebras. And therefore, the Lam condition is not good enough to guarantee the solvability of the interpolation problem for the other algebras.

2.1. The linear and the quadratic case. We will treat the linear and quadratic case, individually, in order to obtain some information for the general case. The linear interpolation problem can be written as

$$a_1 + a_2 x_1 = f_1,$$
$$a_1 + a_2 x_2 = f_2.$$

By subtracting the second equation from the first equation we obtain

$$a_2(x_1 - x_2) = f_1 - f_2,$$

and the solution is

$$a_2 = (f_1 - f_2)(x_1 - x_2)^{-1}, \qquad a_1 = f_1 - a_2 x_1.$$

COROLLARY 2.4. *The linear interpolation problem in any algebra $\mathcal{A}$ has a unique solution if and only if the difference of the two nodes, $x_1 - x_2$, is invertible.*

Let $n = 2$. Then, (2.5) implies

$$a_1 + a_2 x_1 + a_3 x_1^2 = f_1, \tag{2.6}$$
$$a_1 + a_2 x_2 + a_3 x_2^2 = f_2, \tag{2.7}$$
$$a_1 + a_2 x_3 + a_3 x_3^2 = f_3. \tag{2.8}$$

We will introduce some notation, which will also be used later:

$$g_1(j,k) := x_k^{j-1}, \quad \varphi_1(k) := f_k, \quad 1 \le j,k \le 3, \tag{2.9}$$
$$g_2(3,k) := \big(g_1(3,k) - g_1(3,k+1)\big)(x_k - x_{k+1})^{-1}, \tag{2.10}$$
$$\varphi_2(k) := \big(\varphi_1(k) - \varphi_1(k+1)\big)(x_k - x_{k+1})^{-1},\ 1 \le k \le 2, \tag{2.11}$$
$$\varphi_3(1) := \big(\varphi_2(1) - \varphi_2(2)\big)\big(g_2(3,1) - g_2(3,2)\big)^{-1}. \tag{2.12}$$

Subtracting equation (2.7) from (2.6) and (2.8) from (2.7) yields

$$\sum_{j=2}^{3} a_j(x_k^{j-1} - x_{k+1}^{j-1}) = f_k - f_{k+1},\ 1 \le k \le 2, \tag{2.13}$$

Multiplying each of the two equations by $(x_k - x_{k+1})^{-1},\ 1 \le k \le 2$ from the right, yields

$$a_2 + a_3 g_2(3,k) = \varphi_2(k),\ 1 \le k \le 2. \tag{2.14}$$

By subtracting and multiplying again, we obtain the final solution

$$a_3 = \varphi_3(1). \tag{2.15}$$

If a_3 is known, a_2 can be computed from (2.14), and a_1 from (2.6).

There are two critical steps. In order that the quadratic interpolation problem has a solution, it is necessary and sufficient that

$$(i) : (x_1 - x_2)^{-1},\ (x_2 - x_3)^{-1},\ (ii) : \big(g_2(3,1) - g_2(3,2)\big)^{-1}$$

exist. For the second part we define

$$\begin{aligned} f(x_1,x_2,x_3) &:= g_2(3,1) - g_2(3,2) \\ &= (x_1^2 - x_2^2)(x_1 - x_2)^{-1} - (x_2^2 - x_3^2)(x_2 - x_3)^{-1}. \end{aligned} \tag{2.16}$$

The central question is, whether f is invertible. Here we have to distinguish between the commutative and the noncommutative algebras.

THEOREM 2.5. *Let $\mathcal{A}$ be one of the commutative algebras. Then the quadratic interpolation problem has a unique solution if and only if the three differences $x_1 - x_2, x_2 - x_3, x_1 - x_3$ are invertible.*

Proof. Because of the commutativity we have $f(x_1, x_2, x_3) = x_1 - x_3$. ◻

We will use the following lemma.

LEMMA 2.6. *Let $z, h \in \mathcal{A}$ and let $\mathcal{A}$ be one of the noncommutative algebras. Then*

$$hz^k - z^k h = c_k(hz - zh),\ c_k \in \mathbb{R} \text{ for all } k \in \mathbb{N}.^* \tag{2.17}$$

Proof. In all four noncommutative algebras there is the formula

$$z^2 = -\mathrm{abs}_2(z) + 2\Re(z)z =: b_2 + c_2 z,$$

which implies (multiply by z and use the formula for z^2 again)

$$z^k = b_k + c_k z,\ b_k, c_k \in \mathbb{R},\ \text{for all } k \in \mathbb{N}. \tag{2.18}$$

For more details and formulas for computing the constants b_k, c_k see [6, p. 138] or [7, p. 247]. Thus, $z^k - c_k z = b_k \in \mathbb{R}$ and therefore, $h(z^k - c_k z) = (z^k - c_k z)h$ for all $h \in \mathcal{A}$. Rearranging yields (2.17). ◻

In the next lemma we gather some information of the consequences of the violation of the Lam condition.

LEMMA 2.7. *Let $\mathcal{A}$ be one of the noncommutative algebras and let $x, y \in \mathcal{A}$ such that $x - y$ is invertible and $x \sim y$. Then*

$$(x^k - y^k)(x - y)^{-1} = c_k(x) \in \mathbb{R} \text{ for all } k \in \mathbb{N} \text{ and all } y \in [x]. \tag{2.19}$$

Proof. Assume that the similarity is defined by $y = h^{-1}xh$. We multiply equation (2.17) form the left by h^{-1} and use that the real number c_k commutes with all algebra elements and obtain $x^k - y^k = c_k(x - y)$ from where (2.19) follows. ◻

This lemma says, that in all four noncommutative algebras $\mathcal{A}$ there are real numbers $c_k, k \in \mathbb{N}$ associated to all equivalence classes $[a]$. This also implies, that under the conditions of Lemma 2.7 we have $f(x, y, z) = 0$ where f is defined in (2.16). Therefore, the Lam condition excludes the case $f(x, y, z) = 0$, but not necessarily the case that $f(x, y, z)$ is not invertible.

THEOREM 2.8. *Let $\mathcal{A}$ be one of the noncommutative algebras. Then, the quadratic interpolation problem has a unique solution if in addition to $x_1 - x_2, x_2 - x_3$ being invertible, the quantity $f(x_1, x_2, x_3)$, defined in (2.16) is invertible.*

Proof. Follows from the above derivation of the highest coefficient a_3. The definition of $\varphi_3(1)$ depends on the invertibility of $f(x_1, x_2, x_3)$. ◻

EXAMPLE 2.9. Let $x_1 = (2, 8, 4, 9),\ x_2 = (8, 5, 5, 1),\ x_3 = (4, 0, 2, 1)$, $f_1 = (1, 2, 1, 1),\ f_2 = (8, 6, 3, 5), f_3 = (1, 2, 4, 0)$. Note that $x_1 - x_3 = (-2, 8, 2, 8)$ is not invertible in $\mathbb{H}_{\mathrm{coq}}$. Nevertheless, this problem has a solution in $\mathbb{H}_{\mathrm{coq}}$, which shortened to four digits is

$$\begin{aligned}
a_1 &= (357.1411,\ 479.8347,\ 185.6411,\ 567.8347),\\
a_2 &= (-86.1452,\ -141.9758,\ -65.7823,\ -152.5806),\\
a_3 &= (5.2460,\ 10.0121,\ 5.1411,\ 10.0202).
\end{aligned}$$

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*This is according to some experiments possibly also true for other algebras $\mathcal{A}$ than $\mathbb{R}^4$ algebras, only $\mathbb{R}$ has to be replaced by the center $\mathcal{C}_{\mathcal{A}}$ of $\mathcal{A}$. The center of an algebra is the set whose elements commute with all algebra elements.

This example shows, that the Vandermonde interpolation problem is not invariant under permutation of the nodes and values. In the above example an exchange of x_1 and x_2 and of f_1 and f_2 will render the problem unsolvable.

2.2. The general case. The technique to find the solution of a polynomial interpolation problem for degree n has been sketched already for the quadratic case. It consists essentially of a triangulation of the underlying Vandermonde matrix. The general form of this procedure will be summarized in the following theorem.

THEOREM 2.10. *In order to solve the polynomial interpolation problem as stated in (2.1) to (2.5) one has to do the following. Define*

$$g_1(j,k) := x_k^{j-1}, \quad \varphi_1(k) := f_k,\ 1 \le j,k \le n+1, \tag{2.20}$$

$$g_\ell(j,k) := \big(g_{\ell-1}(j,k) - g_{\ell-1}(j,k+1)\big)\big(g_{\ell-1}(\ell,k) - g_{\ell-1}(\ell,k+1)\big)^{-1}, \tag{2.21}$$

$$\varphi_\ell(k) := \big(\varphi_{\ell-1}(k) - \varphi_{\ell-1}(k+1)\big)\big(g_{\ell-1}(\ell,k) - g_{\ell-1}(\ell,k+1)\big)^{-1}, \quad \ell \le 2 \le n+1,\ 1 \le k \le n-\ell+2,\ \ell+1 \le j \le n+1, \tag{2.22}$$

where we assume that all inverses of $g_{\ell-1}(\ell,k) - g_{\ell-1}(\ell,k+1)$ occurring in (2.21) and (2.22) exist. Then

$$a_\ell + \sum_{j=\ell+1}^{n+1} a_j g_\ell(j,k) = \varphi_\ell(k), \quad 1 \le \ell \le n+1,\ 1 \le k \le n-\ell+2, \text{ and} \tag{2.23}$$

$$a_{n+1} = \varphi_{n+1}(1). \tag{2.24}$$

If a_{n+1} is known by (2.24), we compute the coefficients $a_n, a_{n-1}, \ldots, a_1$ backwards by formula (2.23) inserting at all places $k = 1$.

Proof. We start with the representation

$$a_1 + \sum_{j=2}^{n+1} a_j x_k^{j-1} = f_k,\ 1 \le k \le n+1,$$

which is the same as

$$a_1 + \sum_{j=2}^{n+1} a_j g_1(j,k) = \varphi_1(k),\ 1 \le k \le n+1. \tag{2.25}$$

Subtracting equation $k+1$ from equation k in (2.25) and multiplicating the difference by $(x_k - x_{k+1})^{-1}$, assuming that this is possible, and applying (2.21) and (2.22) yields

$$a_2 + \sum_{j=3}^{n+1} a_j g_2(j,k) = \varphi_2(k). \tag{2.26}$$

Having arrived at (2.23), we can use induction, to show that

$$a_{\ell+1} + \sum_{j=\ell+2}^{n+1} a_j g_{\ell+1}(j,k) = \varphi_{\ell+1}(k). \qquad \square$$

Let us assume we have solved an interpolation problem successfully. How to judge the quality of the computed coefficients $a_1, a_1, \ldots, a_{n+1}$. One possibility is to compute $\tilde{f}_k := p(x_k)$ and compare these values with the given values f_k for all $1 \le k \le n+1$. For test purposes it is a good idea to choose all components of f_k as integers. Then the errors are the deviation of $\tilde{f}_k$ from being integer. This can be measured in the maximum norm by putting both $\tilde{f}_k$ and f_k in one real $4(n+1)$ column vector and then form $e := \max|\mathrm{col}(\tilde{f}_k)-\mathrm{col}(f_k)|$ as measure for the *error* where col indicates the forming of the column vector.

EXAMPLE 2.11. Let $n = 3$ and $x_1 = (2,8,4,9),\ x_2 = (8,5,5,1),\ x_3 = (4,0,2,1),\ x_4 = (9,9,4,4),\ f_1 = (1,2,1,1),\ f_2 = (8,6,3,5),\ f_3 = (1,2,4,0), f_4 = (3,9,3,1)$. These data were chosen randomly. There are no solutions in $\mathbb{H}_{\mathrm{cotes}}$ because $x_1 - x_2$ is not invertible in $\mathbb{H}_{\mathrm{cotes}}$ and in $\mathbb{H}_{\mathrm{nec}}$ because $x_2 - x_3$ is not invertible in $\mathbb{H}_{\mathrm{nec}}$. For all other cases the interpolation problem has a solution, given in the following tables, shortened to 4 digits.
Solution for $\mathbb{H}$ with error $= 2.5757e \cdot 10^{-14}$:

$$\begin{aligned}
a_1 &= (-6.4416, & -15.2697, && 8.2518, && 2.6443),\\
a_2 &= (1.0192, & 4.8057, && 0.9450, && -3.7386),\\
a_3 &= (-0.0542, & -0.0930, && -0.4554, && 0.4117),\\
a_4 &= (-0.0063, & -0.0076, && 0.0215, && 0.0002).
\end{aligned}$$

Solution for $\mathbb{H}_{\mathrm{coq}}$ with error $= 5.4001 \cdot 10^{-13}$: ($x_1 - x_3$ is not invertible)

$$\begin{aligned}
a_1 &= (176.1447, & 257.9025, && 114.5842, && 290.1670),\\
a_2 &= (-69.1053, & -115.1597, && -55.6326, && -122.4325),\\
a_3 &= (10.2252, & 14.3371, && 6.0045, && 16.4766),\\
a_4 &= (-0.5421, & -0.4238, && -0.0404, && -0.6820).
\end{aligned}$$

Solution for $\mathbb{H}_{\mathrm{tes}}$ (commutative) with error $= 3.7303 \cdot 10^{-14}$:

$$\begin{aligned}
a_1 &= (-5.1033, & 9.7931, && -5.4347, && 5.3327),\\
a_2 &= (1.0124, & -2.5193, && 0.8486, && -1.9091),\\
a_3 &= (-0.1439, & -0.0969, && 0.2835, && 0.4170),\\
a_4 &= (0.0535, & 0.0014, && -0.0606, && -0.0041).
\end{aligned}$$

Solution for $\mathbb{H}_{\mathrm{con}}$ with error $= 6.3594 \cdot 10^{-13}$ ($x_1 - x_3$ is not invertible):

$$\begin{aligned}
a_1 &= (-1.4124, & -6.9135, && 11.4218, && -16.4555),\\
a_2 &= (9.9449, & 0.5466, && -3.5760, && 10.3660),\\
a_3 &= (-2.9882, & 0.6470, && 1.0686, && -1.9385),\\
a_4 &= (0.1220, & -0.0208, && -0.0261, && 0.1024).
\end{aligned}$$

Solution for $\mathbb{H}_{\mathrm{tan}}$ (commutative) with error $= 3.4195 \cdot 10^{-14}$:

$$\begin{aligned}
a_1 &= (-23.9102, & -17.9102, && 3.6414, && 1.6414),\\
a_2 &= (6.6223, & 4.1439, && -4.5867, && -3.8300),\\
a_3 &= (-0.2334, & -0.0737, && 0.528, && 0.4998),\\
a_4 &= (-0.0036, & -0.0038, && 0.0198, && -0.0454).
\end{aligned}$$

Solution for $\mathbb{H}_{\mathrm{cotan}}$ (commutative) with error $= 5.4179 \cdot 10^{-14}$:

$$\begin{aligned}
a_1 &= (2.7916, & 46.4053, && -41.7540, && 4.0301),\\
a_2 &= (0.4118, & -14.6794, && 14.4364, && -0.4932),\\
a_3 &= (0.1728, & 1.5803, && -1.4896, && 0.1744),\\
a_4 &= (-0.0166, & -0.0503, && 0.0482, && -0.0125).
\end{aligned}$$

There is one general result for interpolation polynomials in commutative algebras.

THEOREM 2.12. *Let $\mathcal{A}$ be one of the commutative algebras. Then, the Vandermonde interpolation polynomial of degree n exists if and only if for the underlying nodes $x_1, x_2, \ldots, x_{n+1}$ the node differences*

$$x_k - x_{\ell+k-1},\ 2 \leq \ell \leq n+1,\ 1 \leq k \leq n-\ell+2 \tag{2.27}$$

are invertible.

Proof. The formulas (2.20) to (2.22) for computing the interpolation polynomial contain one part which requires computing the inverse and this is

$$g_{\ell-1}(\ell, k) - g_{\ell-1}(\ell, k+1) = x_k - x_{\ell+k-1}. \tag{2.28}$$

For $\ell = 2$ this follows directly, because $g_1(2,k) - g_1(2,k+1) = x_k - x_{k+1},\ 1 \leq k \leq n$. Now, for $\ell = 3$

$$\begin{aligned}
&g_2(3,k) - g_2(3,k+1) = \\
&= \big(g_1(3,k) - g_1(3,k+1)\big)\big(g_1(2,k) - g_1(2,k+1)\big)^{-1} - \\
&\quad -\big(g_1(3,k+1) - g_1(3,k+2)\big)\big(g_1(2,k+1) - g_1(2,k+2)\big)^{-1} = \\
&= (x_k^2 - x_{k+1}^2)(x_k - x_{k+1})^{-1} - (x_{k+1}^2 - x_{k+2}^2)(x_{k+1} - x_{k+2})^{-1} = \\
&= (x_k + x_{k+1}) - (x_{k+1} + x_{k+2}) = x_k - x_{k+2},\ 1 \leq k \leq n-1.
\end{aligned}$$

For $\ell = 4$ we compute

$$g_3(4,k) - g_3(4,k+1).$$

The first part is

$$\begin{aligned}
&g_3(4,k) = \big(g_2(4,k) - g_2(4,k+1)\big)\big(g_2(3,k) - g_2(3,k+1)\big)^{-1} = \\
&= \big(g_2(4,k) - g_2(4,k+1)\big)\big(x_k - x_{k+2}\big)^{-1}.
\end{aligned}$$

We continue with the first factor

$$\begin{aligned}
&g_2(4,k) - g_2(4,k+1) = \\
&= \big(g_1(4,k) - g_1(4,k+1)\big)\big(g_1(2,k) - g_1(2,k+1)\big)^{-1} - \\
&\quad -\big(g_1(4,k+1) - g_1(4,k+2)\big)\big(g_1(2,k+1) - g_1(2,k+2)\big)^{-1} = \\
&= (x_k^3 - x_{k+1}^3)(x_k - x_{k+1})^{-1} - (x_{k+1}^3 - x_{k+2}^3)(x_{k+1} - x_{k+2})^{-1} = \\
&= (x_k^2 + x_k x_{k+1}) - (x_{k+1}x_{k+2} + x_{k+2}^2) = x_k^2 - x_{k+2}^2 + x_{k+1}(x_k - x_{k+2}) = \\
&= (x_k - x_{k+2})(x_k + x_{k+1} + x_{k+2}).
\end{aligned}$$

Thus, the first and second part are

$$g_3(4,k) = x_k + x_{k+1} + x_{k+2}, \quad g_3(4,k+1) = x_{k+1} + x_{k+2} + x_{k+3},$$

and the final result, the difference is $x_k - x_{k+3},\ 1 \leq k \leq n-2$, as desired. For general ℓ induction should be used with respect to ℓ to prove (2.28). ∎

A simple count reveals that the list of node differences in (2.27) contains all possible differences $x_{k_1} - x_{k_2}$ with $1 \leq k_1 < k_2 \leq n+1$.

COROLLARY 2.13. *Let $\mathcal{A}$ be one of the commutative algebras. Then, the Vandermonde interpolation polynomial exists if and only if all node differences are invertible.*

In $\mathbb{H}$, the field of quaternions, the Lam condition is equivalent to the condition that all quadratic interpolation problems on three arbitrarily selected, pairwise distinct nodes have a solution.

DEFINITION 2.14. Let there be $n+1 \geq 3$ pairwise distinct nodes in one of the algebras $\mathcal{A}$. If for all subsets of three nodes the quadratic interpolation problem has a solution, we say that the nodes obey the *extended Lam condition*.

CONJECTURE 2.15. *Let the extended Lam condition be valid. Then, the Vandermonde interpolation polynomial exists.*

The conjecture is apparently true for commutative algebras. It is also true for equidistant nodes, which have the form $x_{k+1} := (\ell k + k_0)\xi, 0 \leq k \leq n$, where ξ is a fixed, invertible algebra element, and ℓ, k_0 are fixed integers with $\ell \neq 0$. The standard case is $\ell = k_0 = 1$. It is not difficult to show, that equidistant nodes always lead to an invertible f, where f is defined in (2.16). Thus, the extended Lam condition is valid, and the decisive quantity $g_{\ell-1}(\ell, k) - g_{\ell-1}(\ell, k+1)$ needed in (2.21), (2.22) is always an integer multiple of ξ, thus, it is invertible and the Vandermonde interpolation polynomial exists in this case.

3. The Newton approach. The Vandermonde approach has the advantage of working with a polynomial in standard form (2.1), but it has also several disadvantages. Just by looking at the nodes, it is difficult to judge whether an interpolation polynomial exists for data from nondivision algebras. And if it exists, then the Vandermonde approach may lead to numerically bad results, since it is known already for a long time, (see Gautschi, [4]) that the Vandermonde matrix in the standard form has a very bad condition number. There are also several papers to prevent this difficulty by various measures. One example is a paper by Reichel, Opfer, [12]. Though the present author does not know about investigations of the condition number for Vandermonde matrices with entries from nondivision algebras, it cannot be expected, that the condition number is smaller than for the standard case.

The interpolation problem to be treated here has the following setting. Given data $(x_k, f_k), 1 \leq k \leq n+1$ representing the nodes, and values, respectively, which are members of an algebra $\mathcal{A}$. The minimal requirement is, that the nodes are pairwise distinct. Wanted is a polynomial p of degree n in the form

$$p(x) := \sum_{j=1}^{n+1} a_j p_{j-1}(x), \ x, a_j \in \mathcal{A}, \ 1 \leq j \leq n+1, \text{ where} \tag{3.1}$$

$$p_0(x) := 1, \text{ for all } x \in \mathcal{A}, \tag{3.2}$$

$$p_j(x) := \prod_{k=1}^{j} (x - x_k), \ 1 \leq j \leq n \text{ and the requirement that} \tag{3.3}$$

$$p(x_k) \ = f_k, \ 1 \leq k \leq n+1. \tag{3.4}$$

We will call this problem the *Newton interpolation problem*.

LEMMA 3.1. *Let the data $(x_k, 0)$ be given with the property that the differences $x_{k_1} - x_{k_2}$ of the nodes are invertible for all $1 \leq k_1 < k_2 \leq n+1$. Then the solution of the Newton interpolation problem is $p(x) = 0$ for all $x \in \mathcal{A}$.*

Proof. The requirement for the nodes implies that all differences $x_k - x_\ell, k \neq \ell$ are invertible. We show that $a_j = 0, 1 \leq j \leq n+1$. For this purpose, we insert $x_1, x_2, \ldots, x_{n+1}$ in p, in this order, and obtain $a_1 = p(x_1) = 0$, $p(x_2) = a_1 + a_2(x_2 - x_1) = 0$ which implies $a_2 = 0$ since $x_2 - x_1$ is invertible. Then, $p(x_3) = a_1 + a_2(x_3 - x_1) + a_3(x_3 - x_1)(x_3 - x_2) = 0$ implies $a_3 = 0$, since $(x_3 - x_1)(x_3 - x_2)$ is invertible. Thus, all $a_j, 1 \leq j \leq n+1$ vanish. $\square$

This implies that solutions are unique.

COROLLARY 3.2. *Assume that all node differences $x_k - x_\ell$ for $k \neq \ell$ are invertible. Let there be two solutions, p and q to the Newton interpolation problem. Then, $p = q$.*

For the general problem we can state the following theorem.

THEOREM 3.3. *Let all node differences $x_k - x_\ell$ for $k \neq \ell$ be invertible. Then, there is a unique Newton interpolation problem for the data $(x_k, f_k), 1 \leq k \leq n+1$. If one of the node differences $x_k - x_\ell$ for $k \neq \ell$ is not invertible, then no Newton interpolation polynomial exists.*

Proof. The assumption that all node differences are invertible, implies that

$$p_j(x_\ell) = \begin{cases} 0 & \text{for } \ell \leq j, \\ \text{invertible} & \text{for } \ell > j. \end{cases}$$

Now,

$$p(x_1) = a_1 = f_1. \tag{3.5}$$

Assume that a_j are known for all $1 \leq j \leq \ell$. Then,

$$p(x_{\ell+1}) = \sum_{j=1}^{\ell} a_j p_{j-1}(x_{\ell+1}) + a_{\ell+1} p_\ell(x_{\ell+1}) = f_{\ell+1}.$$

This implies

$$a_{\ell+1} = \big(f_{\ell+1} - \sum_{j=1}^{\ell} a_j p_{j-1}(x_{\ell+1})\big)\big(p_\ell(x_{\ell+1})\big)^{-1},\ 1 \leq \ell \leq n, \tag{3.6}$$

and all coefficients are known. If one of the differences has no inverse, then there will be one ℓ such that $p_\ell(x_{\ell+1})$ has no inverse and formula (3.6) cannot be applied. □

EXAMPLE 3.4. We use the data from Example 2.11 and choose the commutative tessarine case $\mathcal{A} = \mathbb{H}_{\rm tes}$. For commutative algebras the two types of polynomials must coincide in the sense that they have the same values at all $x \in \mathcal{A}$, which does not imply that the coefficients are the same with the exception of the highest coefficient.

Solution for $\mathbb{H}_{\rm tes}$ (commutative) with error $= 1.7764 \cdot 10^{-15}$:

$$\begin{array}{rrrrrl} a_1 = & (1.0000, & 2.0000, & 1.0000, & 1.0000) & = f_1, \\ a_2 = & (0.1765, & 0.2059, & -0.3235, & 0.7059), & \\ a_3 = & (-0.0335, & -0.0933, & 0.0626, & 0.1760), & \\ a_4 = & (0.0535, & 0.0014, & -0.0606, & -0.0041). & \end{array}$$

Note, that the error here is by a factor 21 smaller than the corresponding error for $\mathbb{H}_{\rm tes}$ in Example 2.11. The polynomial value at $x := (1, 2, 3, 4)$ is

$$(6.458660398875651, 4.787370206864643, 1.650198860414113, 4.589677899172335).$$

It differs from the corresponding value for the Vandermonde polynomial with the same data by at most two digits in the last two places. The coefficient a_4 coincides in all computed places with the corresponding coefficient a_4 from the Vandermonde polynomial.

It would be of interest to see an error development with growing degree n for both types of interpolation. However, this may be a topic for another paper.

4. Concluding remarks. Though we have chosen examples from $\mathbb{R}^4$ algebras, the two algorithms given in (2.20) to (2.24) and in (3.5), (3.6) are valid for all types of algebras. An easy way of implementing them is to use the "overloading technique" offered by MATLAB.

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