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NIVEN'S ALGORITHM APPLIED TO THE ROOTS OF THE COMPANION POLYNOMIAL OVER $\mathbb{R}^4$ ALGEBRAS

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Abstract. This paper will contain an extension of Niven's algorithm of 1941, which in its original form is designed for finding zeros of unilateral polynomials p over quaternions $\mathbb{H}$. The extensions will cover the algebra $\mathbb{H}_{\mathrm{coq}}$ of coquaternions, the algebra $\mathbb{H}_{\mathrm{nec}}$ of nectarines and the algebra $\mathbb{H}_{\mathrm{con}}$ of conectarines. These are nondivision algebras in $\mathbb{R}^4$. In addition, it is also shown that in all algebras the most difficult part of Niven's algorithm can easily be solved by inserting the roots of the companion polynomial c of p, with the result, that all zeros of all unilateral polynomials over all noncommutative $\mathbb{R}^4$ algebras can be found. In addition, for all four algebras the maximal number of zeros can be given. For the three nondivision algebras besides the known types of zeros: isolated, spherical, hyperbolic, a new type of zero will appear, which will be called *unexpected* zero of p.

Key words. Zeros of polynomials over noncommutative $\mathbb{R}^4$ algebras, Isolated, Spherical, Hyperbolic, Unexpected zeros.

AMS subject classifications. 12D10, 12E10, 15A66, 1604, 65J15.

1. Introduction. This paper will contain a revival and an extension of Niven's historical paper of 1941, [18]. Niven discussed in his paper the possibilities of finding the zeros of polynomials p with quaternionic coefficients by dividing p by a quadratic polynomial r with real coefficients. The result of this division can be put into the form

$$p(z) = q(z)r(z) + R_0 + R_1 z, \tag{1.1}$$

where q is the *quotient*, and $R_0 + R_1 z$ is the *remainder after division* (for short: *remainder*). If p is a polynomial of degree $n \geq 2$, then q is a polynomial of degree $n-2$. The extension mentioned refers to the possibility to apply Niven's algorithm also to polynomials with other coefficients than quaternionic coefficients. We will return to the details later in this paper.

We will use the following notations: $\mathbb{N}$ set of positive integers, $\mathbb{R}$ set of real numbers, $\mathbb{C}$ set of complex numbers, $\mathcal{A}$ one of the four algebras $\mathbb{H}$, $\mathbb{H}_{\mathrm{coq}}$, $\mathbb{H}_{\mathrm{nec}}$, $\mathbb{H}_{\mathrm{con}}$ defined as algebras in $\mathbb{R}^4$, where an algebra in $\mathbb{R}^4$ is the linear space $\mathbb{R}^4$ equipped with an associative multiplication $\mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^4$. $\mathbb{H}$ is the field of *quaternions*, discovered 1843 by Hamilton, which explains the letter $\mathbb{H}$. For Hamilton see [10]. $\mathbb{H}_{\mathrm{coq}}$ is the algebra of *coquaterions*, sometimes also called algebra of *split quaternions* introduced by Cockle, [2, 1849]. See also [3]. The algebras $\mathbb{H}_{\mathrm{nec}}, \mathbb{H}_{\mathrm{con}}$ are the algebras of *nectarines, conectarines*, respectively, introduced by Schmeikal, [24, 2014]. The new algebras $\mathbb{H}_{\mathrm{coq}}, \mathbb{H}_{\mathrm{nec}}, \mathbb{H}_{\mathrm{con}}$ belong to the class of *nondivision algebras*, which means that there exist noninvertible elements which differ from the zero element. All four algebras $\mathbb{H}$, $\mathbb{H}_{\mathrm{coq}}$, $\mathbb{H}_{\mathrm{nec}}$, $\mathbb{H}_{\mathrm{con}}$ are noncommutative. Algebras in $\mathbb{R}^N, N \in \mathbb{N}$ are often called *geometric algebras*. See [7]. For algebras in general see Garling, [5, 2011] and also Gürlebeck and Sprössig, [9, 1997]. More details about these algebras will be given later in this paper.

Niven's algorithm, [18, 1941] is tailored for finding zeros of unilateral polynomials with coefficients from the field $\mathbb{H}$ of quaternions. All polynomials to be treated in this paper will have the unilateral form

$$p(z) = \sum_{j=0}^{n} a_j z^j,\ z, a_j \in \mathcal{A},\ 0 \leq j \leq n,\ a_0, a_n \text{ invertible}, \tag{1.2}$$

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where $n \in \mathbb{N}$ is the *degree* of p. The existence of zeros of polynomials of a general type with coefficients from $\mathbb{H}$ is guaranteed under a very mild condition by a theorem of Eilenberg and Niven, [4, 1944]. Already in Niven's paper, [18, 1941], there is a theorem stating that unilateral, quaternionic polynomials have zeros, where results by Ore, [19, 20] are used. For polynomials with coefficients from the algebras $\mathbb{H}_{\rm coq}$, $\mathbb{H}_{\rm nec}$, $\mathbb{H}_{\rm con}$ the existence of zeros cannot be guaranteed. See [12, 13, 21]. E. g. the polynomial $p(z) = z^2 - (1, 2, 3, 4)$ has no zeros in $\mathbb{H}_{\rm coq}$. The form of the algebra element $(1, 2, 3, 4)$ is explained in (2.1).

The novelty of this paper will consist of two parts. 1. We will show that the roots of the real *companion polynomial* c (to be introduced in a later part of this paper as a companion polynomial of p) will serve to solve the most difficult part of Niven's algorithm and 2. we will show that Niven's algorithm can be extended to the algebras $\mathbb{H}_{\rm coq}$, $\mathbb{H}_{\rm nec}$, and $\mathbb{H}_{\rm con}$. In the course of this extension we will see, that a new type of zeros arises, which will be called *unexpected* zeros of p.

That the roots of the companion polynomial serve as a tool to find all zeros of a given polynomial with coefficients from $\mathbb{H}$ was shown in [15, 2010], and for coefficients from $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$ was shown in [11, 2016]. However, in the context of [11, 15] Niven's algorithm was not used.

2. Preliminaries. This part serves as an introduction into some technical parts which are needed for the understanding. In particular, we will present the corresponding multiplication tables for the new algebras. For simplicity we will denote the elements from $\mathcal{A}$ by

$$a = (a_1, a_2, a_3, a_4), a_j \in \mathbb{R}, 1 \le j \le 4. \tag{2.1}$$

The four units in $\mathcal{A}$ will be denoted by

$$1 := (1, 0, 0, 0), \quad \mathbf{i} := (0, 1, 0, 0), \quad \mathbf{j} = (0, 0, 1, 0), \quad \mathbf{k} = (0, 0, 0, 1).$$

With this notation it is also possible to write

$$a = a_1 + a_2\mathbf{i} + a_3\mathbf{j} + a_4\mathbf{k},$$

however, in most cases we will prefer the shorter notation given in (2.1). The first component a_1 of a will be called *real part* of a in the notation $a_1 = \Re(a)$. We will identify $(a_1, 0, 0, 0)$ with $\mathbb{R}$. Thus, a real element of $\mathcal{A}$ will have the form $(a_1, 0, 0, 0)$. The second component, a_2 will be called *imaginary part* with the notation $a_2 = \Im(a)$, the third component a_3 will be called **j**-*part* with the notation $a_3 = \Im_3(a)$, and the fourth component a_4 will be called **k**-*part* with the notation $a_4 = \Im_4(a)$.

LEMMA 2.1. *In all four algebras $\mathcal{A}$, the real elements are the only elements which commute with all algebra elements.*

Proof. [11]. ☐

In a general, noncommutative algebra G, the set of elements which commutes with all elements of G is called *center* of G and denoted by $\mathcal{C}_G$. Thus, Lemma 2.1 could be written as $\mathcal{C}_{\mathcal{A}} = \mathbb{R}$.

The process of conjugation plays an important role. For $a = (a_1, a_2, a_3, a_4) \in \mathcal{A}$ we define the *conjugate* of a in the notation $\overline{a}$ or $\operatorname{conj}(a)$ by

$$\overline{a} = \operatorname{conj}(a) = (a_1, -a_2, -a_3, -a_4). \tag{2.2}$$

An important new function is

$$\operatorname{abs}_2(a) = a\overline{a} \tag{2.3}$$

implying $\mathrm{abs}_2(1) = 1 \cdot \overline{1} = 1$. In the following lemma several properties of $\mathrm{conj}(a)$ and of $\mathrm{abs}_2(a)$ are collected.

LEMMA 2.2. *Let $a, b \in \mathcal{A}$. Then*

1. $\Re(ab) = \Re(ba)$,
2. $\overline{ab} = \overline{b}\,\overline{a}, \quad a + \overline{a} = 2\Re(a)$,
3. $\mathrm{abs}_2(a) = a\overline{a} = \overline{a}a \in \mathbb{R}, \quad \Re(\overline{a}) = \Re(a),\ \mathrm{abs}_2(\overline{a}) = \mathrm{abs}_2(a)$,
4. a *is invertible if and only if* $\mathrm{abs}_2(a) \neq 0$.
5. *Let* $\mathrm{abs}_2(a) \neq 0$. *Then*

$$a^{-1} = \frac{\overline{a}}{\mathrm{abs}_2(a)}. \tag{2.4}$$

6. *The function* $\mathrm{abs}_2 : \mathcal{A} \to \mathbb{R}$ *defined in (2.3) is multiplicative, which means*

$$\mathrm{abs}_2(ab) = \mathrm{abs}_2(ba) = \mathrm{abs}_2(a)\mathrm{abs}_2(b). \tag{2.5}$$

For invertible a (2.5) implies

$$1 = \mathrm{abs}_2(aa^{-1}) = \mathrm{abs}_2(a)\mathrm{abs}_2(a^{-1}). \tag{2.6}$$

7. *For* abs_2 *there are the following formulas.*

$$\mathrm{abs}_2(a) = \begin{cases} a_1^2 + a_2^2 + a_3^2 + a_4^2 & \text{for } a \in \mathbb{H}, \\ a_1^2 + a_2^2 - a_3^2 - a_4^2 & \text{for } a \in \mathbb{H}_{\mathrm{coq}}, \\ a_1^2 - a_2^2 + a_3^2 - a_4^2 & \text{for } a \in \mathbb{H}_{\mathrm{nec}}, \\ a_1^2 - a_2^2 - a_3^2 + a_4^2 & \text{for } a \in \mathbb{H}_{\mathrm{con}}. \end{cases} \tag{2.7}$$

Proof. See [13]. □

Thus, in $\mathbb{H}$ we have $\mathrm{abs}_2(a) = ||a||^2$ where $||\cdot||$ is the euclidean norm in $\mathbb{R}^4$.

For completeness we present the multiplication rules for $\mathbb{H}, \mathbb{H}_{\mathrm{coq}}, \mathbb{H}_{\mathrm{nec}}, \mathbb{H}_{\mathrm{con}}$ in Table 2.3.

TABLE 2.3. The multiplication tables for $\mathbb{H}, \mathbb{H}_{\mathrm{coq}}, \mathbb{H}_{\mathrm{nec}}, \mathbb{H}_{\mathrm{con}}$.

(2.8)

$\mathbb{H}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	-1

$\mathbb{H}_{\mathrm{coq}}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	-1	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	1	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$	1

(2.9)

$\mathbb{H}_{\mathrm{nec}}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	1	$\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	-1	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$	1

$\mathbb{H}_{\mathrm{con}}$	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
1	1	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	1	$\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	1	$-\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$-\mathbf{j}$	$\mathbf{i}$	-1

For all algebras $\mathcal{A}$ the reals $\mathbb{R}$ are a subalgebra of $\mathcal{A}$. However, $\mathbb{C}$ is a subalgebra only of $\mathbb{H}$ and of $\mathbb{H}_{\mathrm{coq}}$.

EXAMPLE 2.4. Let

$$p(z) = z^2 + 1, \quad z \in \mathbb{H} \text{ or } z \in \mathbb{H}_{\mathrm{coq}}.$$

Apparently, $p(\pm\mathbf{i}) = 0$ in $\mathbb{H}$ and in $\mathbb{H}_{\rm coq}$. See Table 2.3. Let $p(z_0) = 0$ for nonreal z_0 and let h be an invertible element in $\mathbb{H}$ or in $\mathbb{H}_{\rm coq}$. We multiply equation $p(z_0) = 0$ by h^{-1} from the left and by h from the right. This yields

$$h^{-1}p(z_0)h = h^{-1}z_0^2h + 1 = (h^{-1}z_0h)^2 + 1 = p(h^{-1}z_0h) = 0.$$

The equality $1 = h^{-1}1h$ follows from Lemma 2.1. Thus, not only z_0 but also $h^{-1}z_0h$ for all invertible h are zeros of p, and, therefore, the number of zeros of p is infinite.

DEFINITION 2.5. Two elements $a, b \in \mathcal{A}$ will be called *similar*, denoted by $a \sim b$, if there is an invertible $h \in \mathcal{A}$ such that

$$a = h^{-1}bh. \tag{2.10}$$

The set of elements which are similar to a fixed element $a \in \mathcal{A}$ is called the *similarity class* of a and denoted by $[a]$.

Similarity is an equivalence relation. The number of elements of a similarity class $[a]$ is either one or infinite. It is one if and only if $a \in \mathbb{R}$. If $a \sim b$ in any of the four algebras $\mathcal{A}$, the transformation $h \in \mathcal{A}$ which defines the similarity can be computed by a method given in [11]. In Example 2.4 $[z_0]$ is a similarity class which consists only of zeros of p.

LEMMA 2.6. *Let $a, b \in \mathcal{A}$ and let $a \sim b$. Then,*

$$\Re(a) = \Re(b), \quad \mathrm{abs}_2(a) = \mathrm{abs}_2(b). \tag{2.11}$$

Proof. Apply Lemma 2.2 part 1 and formula (2.5) to $a = h^{-1}bh$. □

THEOREM 2.7. *Let $a, b \in \mathbb{H}$. Then (2.11) is a necessary and sufficient condition for $a \sim b$. Let $\mathcal{A} \neq \mathbb{H}$, then (2.11) is a necessary and sufficient condition for $a \sim b$ under the restriction $a, b \notin \mathbb{R}$.*

Proof. See [15]. Compare also with [23]. □

DEFINITION 2.8. In $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$ two elements $a, b \in \mathcal{A}$ will be called *quasi similar* denoted by $a \overset{q}{\sim} b$, if (2.11) is valid. The set of elements which are quasi similar to a fixed a is called the *quasi similarity class* of a and denoted by $[a]_q$.

Quasi similarity is an equivalence relation. In $\mathbb{H}$ similarity and quasi similarity is the same. Thus, if we, occasionally, use the term quasi similarity also for $\mathbb{H}$, similarity is meant. In general, similarity implies quasi similarity and

$$[a] \subset [a]_q \text{ for all } a \in \mathcal{A}, \quad [a] = [a]_q \text{ if } a \notin \mathbb{R} \text{ and } \mathcal{A} \neq \mathbb{H}.$$

In $\mathbb{H}_{\rm coq}$ the two elements $a = (a_1, 0, 0, 0), b = (a_1, a_2, a_3, a_4)$ with $\mathrm{abs}_2(b) = a_1^2$ are quasi similar, but not similar. Two real elements are similar if and only if they are identical. Or, in other words, a real and a nonreal element can never be similar.

DEFINITION 2.9. Let z_0 be a zero of the polynomial p. If in the quasi similarity class $[z_0]_q$ there is no other zero, then the zero z_0 is called an *isolated zero* of p. If all elements of $[z_0]_q$ are zeros of p then for $\mathcal{A} = \mathbb{H}$ the zero z_0 is called *spherical*. For $\mathcal{A} \neq \mathbb{H}$ the zero is called *hyperbolic*.

As we have seen, it may happen that all elements of a similarity class contain zeros of a given polynomial, therefore, we cannot count zeros piece by piece.

DEFINITION 2.10. Let p be a polynomial over one of the algebras $\mathcal{A}$. Let there be κ quasi similarity classes, which contain at least one zero of p. Then we say that the *number of zeros of* p is κ.

LEMMA 2.11. *Let $a \in \mathcal{A}$, where $\mathcal{A}$ is any of the four algebras $\mathbb{H}, \mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$. Then*

$$a \sim \overline{a}. \tag{2.12}$$

Proof. If $a \in \mathbb{R}$, then $a = \overline{a}$ and (2.12) is valid. Let $a \notin \mathbb{R}$. Then, $\overline{a} \notin \mathbb{R}$. The remaining part follows from Lemma 2.2, part 3 and Theorem 2.7. $\square$

From Lemma 2.11 we learn that $\mathbf{i}$ and $-\mathbf{i}$ are similar in all algebras. This implies that according to Definition 2.10, p of Example 2.4 has one zero. Similarity for quaternionic matrices was already considered by Wolf, [26, 1936] and early contributions to quaternionic matrices can be found in Brenner, [1, 1951]. and Lee, [17, 1949].

For more details related to this section see [13].

3. The companion polynomial. Let p be a polynomial of degree n given as in (1.2), namely

$$p(z) = \sum_{j=0}^{n} a_j z^j, \ z, a_j \in \mathcal{A}, \ 0 \le j \le n, \ a_0, a_n \text{ invertible.} \tag{3.1}$$

The introduction of the *companion polynomial* of p serves as a means to find all similarity classes or quasi similarity classes which contain zeros of the polynomial p. The companion polynomial of p, denoted by c, will be a real polynomial of degree $2n$. It is defined by

$$c(z) = \sum_{j,k=0}^{n} \overline{a_j} a_k z^{j+k} = \sum_{\ell=0}^{2n} b_\ell z^\ell, \quad b_\ell = \sum_{j=\max(0,\ell-n)}^{\min(\ell,n)} \overline{a_j} a_{\ell-j} \in \mathbb{R}, \ 0 \le \ell \le 2n. \tag{3.2}$$

For the quadratic case $n = 2$, the coefficients of the companion polynomial c in all noncommutative $\mathbb{R}^4$ algebras are

$$\begin{aligned}
b_0 &= \mathrm{abs}_2(a_0), \\
b_1 &= \overline{a_0} a_1 + \overline{a_1} a_0 = 2\Re(\overline{a_0} a_1), \\
b_2 &= \overline{a_0} a_2 + \mathrm{abs}_2(a_1) + \overline{a_2} a_0 = 2\Re(\overline{a_0} a_2) + \mathrm{abs}_2(a_1), \\
b_3 &= \overline{a_1} a_2 + \overline{a_2} a_1 = 2\Re(\overline{a_1} a_2), \\
b_4 &= \mathrm{abs}_2(a_2).
\end{aligned} \tag{3.3}$$

We will call the solutions of $c(z) = 0$ *roots* of c and will keep the name *zeros* for the solutions of $p(z) = 0$. The companion polynomial c has an even number (including zero) of real and an even number (including zero) of complex roots. The name *companion polynomial* has been introduced in a paper by Janovská and Opfer, [15, 2010]. It was introduced already by Niven, 1941, [18] without a specific name and also used by Pogorui and Shapiro, 2004, [22] under the name *basic polynomial*.

The essential property of the companion polynomial is that it defines by its roots similarity classes or quasi similarity classes which may contain zeros of p. And outside these similarity classes there are no zeros of p. See [15, 13].

Let c be the companion polynomial for p, where c has degree $2n$. Let the roots of c be

$$\rho_1, \rho_2, \ldots, \rho_{2\kappa}, \quad \zeta_1, \zeta_2, \ldots, \zeta_{n-\kappa}, \quad \overline{\zeta_1}, \overline{\zeta_2}, \ldots, \overline{\zeta_{n-\kappa}}, \tag{3.4}$$

where $\rho_j, \ j = 1, 2, \ldots, 2\kappa$ are the real roots, and $\zeta_j, \ j = 1, 2, \ldots, n - \kappa$ are the complex roots with positive imaginary part of the companion polynomial c. If $\kappa = n$, all roots are real, and if $\kappa = 0$, all roots are complex and the number of complex roots with positive real part is n. The existence of one real root implies the existence of another real root.

4. Niven's algorithm. We will give a short introduction of Niven's algorithm, [18, 1941] for determining the zeros of unilateral quaternionic polynomials p over the field of quaternions $\mathbb{H}$ with $n \geq 2$, where n is the degree of p. For the form of the polynomial p see (1.2) with $\mathcal{A} = \mathbb{H}$. Let

$$r(z) = z^2 - 2uz + v,\ u, v \in \mathbb{R},\ z \in \mathbb{H} \tag{4.1}$$

be a quadratic polynomial with real coefficients u, v and with a quaternionic variable z. Instead of $r(z)$ we also write $r(z; u, v)$ if we want to stress the dependence of r on u, v. We note that r defined in (4.1) vanishes for $u = \Re(z),\ v = ||z||^2$. And r remains zero if $u = \Re(z_0), v = ||z_0||^2$ for all $z_0 \in [z]$. See (2.11) in Lemma 2.6. Thus, instead of inserting $u = \Re(z),\ v = ||z||^2$, it is sufficient to insert $u = \Re(z_0),\ v = ||z_0||^2$, where z_0 is similar or quasi similar to z. Then, Niven in 1941, [18, p. 655] writes (with slightly different terminology)

$$p(z) = q(z)r(z; u, v) + R_0(u, v) + R_1(u, v)z, \tag{4.2}$$

where $R_0(u, v) + R_1(u, v)z$ is called the *remainder after division*, for short *remainder* of p. In order to find the representation (4.2) let q be defined by

$$q(z) = \sum_{j=0}^{n-2} b_j z^j,\ z, b_j \in \mathbb{H},\ 0 \leq j \leq n-2, b_0 \neq 0, b_{n-2} \neq 0, \tag{4.3}$$

and let $u, v \in \mathbb{R}$. Comparing the two sides of (4.2) yields

$$b_{n-2} = a_n. \tag{4.4}$$

$$\begin{aligned} &\text{If } n > 2: \\ b_{n-3} &= a_{n-1} + 2ub_{n-2}, \\ b_{n-k-1} &= a_{n-k+1} + 2ub_{n-k} - vb_{n-k+1}, \\ k &= 3, 4, 5, \ldots, n-1. \end{aligned} \tag{4.5}$$

The last loop is empty for $n = 3$. By this recursion, all $b_k,\ k = n-2, n-3, \ldots, 1, 0$ are determined uniquely, in this order. For the remainder terms we obtain

$$R_0 = a_0 - vb_0, \quad R_1 = \begin{cases} a_1 + 2ub_0 & \text{for } n = 2, \\ a_1 + 2ub_0 - vb_1 & \text{for } n > 2. \end{cases} \tag{4.6}$$

Actually, we only need the remainder terms $R_0,\ R_1$, most of the coefficients of q are not needed. Let $R_1 \neq 0$. Then the vanishing of the remainder can be expressed by

$$z = -R_1^{-1}R_0 = -\frac{\overline{R_1}}{||R_1||^2} R_0. \tag{4.7}$$

If $R_1 = 0$ the vanishing of the remainder implies $R_0 = 0$, and in this case $p(z_0) = 0$ for all $z_0 \in [z]$.

LEMMA 4.1. *For the space of quaternions $\mathbb{H}$ there is the equivalence*

$$p(z) = q(z)r(z; u, v) + R_0(u, v) + R_1(u, v)z = 0 \ \Leftrightarrow\ R_0(u, v) + R_1(u, v)z = 0. \tag{4.8}$$

Proof. Niven, [18, p. 655]. $\square$

It follows from (4.7) that

$$\overline{z} = -\overline{R_0}\,\overline{R_1^{-1}} = -\overline{R_0}\,\frac{R_1}{||R_1||^2}$$

and

$$2u := 2\Re(z) = z + \overline{z} = -\frac{1}{||R_1||^2}(\overline{R_1}R_0 + \overline{R_0}R_1), \tag{4.9}$$

$$v := ||z||^2 = z\overline{z} = \frac{||R_0||^2}{||R_1||^2}. \tag{4.10}$$

Equations (4.9), (4.10) require, therefore, to find $u, v \in \mathbb{R}$ such that

$$U\begin{pmatrix} u \\ v \end{pmatrix} := 2u||R_1||^2 + \overline{R_1}R_0 + \overline{R_0}R_1 = 0, \tag{4.11}$$

$$V\begin{pmatrix} u \\ v \end{pmatrix} := ||R_1||^2 v - ||R_0||^2 = 0. \tag{4.12}$$

Niven writes ([18, p. 654]) with respect to these two equation: "... we give a method for obtaining the roots of [the quaternionic polynomial in question], which is not very practical in the sense that it involves the simultaneous solving of two real equations...".

In Section 5 and in Section 6 we will show how to circumvent the described difficulties. For $\mathcal{A} = \mathbb{H}$ the real roots of c directly define real zeros of p and the (nonreal) complex roots $\zeta = a + b\mathbf{i}$ define a similarity class which contains a zero of p. Thus, we apply Niven's algorithm with $u = \Re(\zeta) = a,\ v = \mathrm{abs}_2(\zeta) = ||\zeta||^2 = a^2 + b^2$ and obtain a zero of p.

For the nondivision algebras $\mathbb{H}_{\mathrm{coq}},\ \mathbb{H}_{\mathrm{nec}},\ \mathbb{H}_{\mathrm{con}}$ the matter is a little more complicated. But also in this case the roots of the companion polynomial define real quantities $u,\ v$ which may lead to a zero of p.

5. Niven's algorithm in $\mathbb{H}$ applied to the roots of the companion polynomial. In this section only $\mathcal{A} = \mathbb{H}$ is admitted. Given is the polynomial p defined in (1.2) with $\mathcal{A} = \mathbb{H}$, and with degree $n \geq 2$. Let

$$\rho_1, \rho_2, \cdots, \rho_{2\kappa}, \quad \zeta_1, \zeta_2, \cdots, \zeta_{n-\kappa}, \tag{5.1}$$

be the roots of the companion polynomial c, where $\rho_j, 1 \leq j \leq 2\kappa$ are the real roots and $\zeta_j, 1 \leq j \leq n-\kappa$ are the complex roots with positive imaginary parts. The complex roots with negative imaginary parts are omitted because they do not create new similarity classes. See Lemma 2.11. For the real roots no algorithm is necessary. There is the following result:

THEOREM 5.1. *The real roots ρ of c appear always as double roots and ρ is a zero of p.*

Proof. See [15]. A simple argument is $c(z) = p(z)\overline{p(z)}$ for $z \in \mathbb{R}$. ◻

THEOREM 5.2. *Let ζ be a complex (nonreal) root of the companion polynomial c of p. Then the similarity class $[\zeta]$ contains a zero of p.*

Proof. See [15]. ◻

THEOREM 5.3. *Let $\zeta = a + b\mathbf{i}, b > 0$ be a complex root of the companion polynomial c. Apply Niven's algorithm to $u = a, v = a^2+b^2$. If the remainder term R_1 of Niven's algorithm is invertible, then*

$$z = -R_1^{-1}R_0$$

is an isolated zero of p. If $R_1 = 0$, ζ is a spherical zero of p.

Proof. One has to show that z and ζ are similar. Now, $\Re(\zeta) = u, ||\zeta||^2 = u^2 + v^2$. The quadratic polynomial $r(z) = z^2 - 2uz + u^2 + v^2$ also vanishes at z, which implies $\Re(z) = \Re(\zeta) = u$, $||z||^2 = ||\zeta||^2 = u^2 + v^2$. Thus, z and ζ are similar. If $R_1 = 0$, then $r(z) = 0$ for all $z \in [\zeta]$, which implies $p(z) = 0$. □

EXAMPLE 5.4. Let the quaternionic polynomial

$$p(z) = z^6 + \mathbf{j}z^5 + \mathbf{i}z^4 - z^2 - \mathbf{j}z - \mathbf{i}. \tag{5.2}$$

be given. This is Example 3.8 of [15, p. 251]. The 12 roots of the companion polynomial c are

$$\rho_1 = 1, \rho_2 = 1, \rho_3 = -1, \rho_4 = -1, \zeta_1 = \mathbf{i}, \zeta_2 = \mathbf{i}, \zeta_3 = (1+\sqrt{3}\,\mathbf{i})/2, \zeta_4 = (-1+\sqrt{3}\,\mathbf{i})/2,$$

where the four complex roots with negative imaginary part are not needed and therefore are not listed. According to Theorem 5.1 the four real roots ρ_1 to ρ_4 of c define two real zeros $1, -1$ of p. In order to compute the remaining zeros we have to compute the corresponding remainders R_0, R_1 for ζ_1 to ζ_4, which are contained in Table 5.5, and apply Theorem 5.3.

TABLE 5.5. Zeros of p, where p is defined in (5.2).

root of c	zero of p	type of zero	R_0	R_1
$\rho_1 = \rho_2 = 1$	1	real	not needed	not needed
$\rho_3 = \rho_4 = -1$	-1	real	not needed	not needed
$\zeta_1 = \zeta_2 = \mathbf{i}$	$\mathbf{i}$	spherical	0	0
$\zeta_3 = (1+\sqrt{3}\,\mathbf{i})/2$	$0.5(1,-1,-1,-1)$	isolated	$(2,-1,1,0)$	$(-1,-1,-2,0)$
$\zeta_4 = (-1+\sqrt{3}\,\mathbf{i})/2$	$0.5(-1,1,-1,-1)$	isolated	$(2,-1-1,0)$	$(1,1,-2,0)$

There are altogether 5 zeros. All possible types of zeros appear in this example.

We end this section with a more detailed example of a quaternionic polynomial p of degree 4.

EXAMPLE 5.6. We use example 7.6 of [11] as a polynomial over $\mathbb{H}$. In [11] it is used as a polynomial over $\mathbb{H}_{\text{coq}}$. This polynomial is

$$p(z) = z^4 + \frac{1}{6}\left((6,4,11,-1)z^3 + (-8,12,-2,2)z^2 + (2,2,-12,8)z + (3,-7,4,-8)\right). \tag{5.3}$$

The companion polynomial c of p is

$$c(z) = z^8 + 2z^7 + \frac{13}{6}z^6 - \frac{2}{3}z^5 + \frac{1}{3}z^4 + 5z^3 - \frac{4}{3}z^2 - \frac{20}{3}z + \frac{23}{6}.$$

It has only complex roots. The four roots with positive imaginary part are given in Table 5.7.

TABLE 5.7. Roots with positive imaginary part of the companion polynomial c.

$$\begin{aligned}
\zeta_1 &= -1.003718721498504 + 1.540059403619789\,\mathbf{i},\\
\zeta_2 &= -1.266027996825196 + 0.444261903953548\,\mathbf{i},\\
\zeta_3 &= 0.611025310795246 + 0.992525881224344\,\mathbf{i},\\
\zeta_4 &= 0.658721407528452 + 0.173064784357881\,\mathbf{i}.
\end{aligned}$$

The next step is to find the coefficients u, v of the quadratic polynomial r which are defined in (6.1). In the current case we have

$$u_j = \Re(\zeta_j), \quad v_j = (\Re(\zeta_j))^2 + (\Im(\zeta_j))^2, \; j = 1,2,3,4.$$

Since these values are easy to calculate, we omit the numerical values of u_j, v_j. Now we can compute the remainder terms R_0, R_1 using (4.4) to (4.6). They are given in Tables 5.8, 5.9.

TABLE 5.8. Values of R_0 belonging to $u_j, v_j,\ j = 1, 2, 3, 4$.

$$\begin{array}{r}
(9.590815703078643,\ -3.402734251027647,\ 14.229680534632516,\ -3.590344969380023),\\
(-0.842446669610599,\ -1.728260471738801,\ 9.623424283007916,\ -2.693097822106674),\\
(0.467851436993603,\ -4.990323339998328,\ -1.924041778888646,\ -1.509468794444240),\\
(-0.082568325300301,\ -2.501807973223255,\ -0.299090937824172,\ -1.386102433628440).
\end{array}$$

TABLE 5.9. Values of R_1 belonging to $u_j, v_j,\ j = 1, 2, 3, 4$.

$$\begin{array}{r}
(9.138108497227318,\ -3.247827653339322,\ -0.138140962533935,\ 0.555759044170659),\\
(1.203122049821211,\ -1.656703969346634,\ 7.297724047257375,\ -0.279204002372001),\\
(-2.656312182428501,\ 2.867400020779066,\ -2.159945235468775,\ 1.718192179463977),\\
(0.912928087493344,\ 3.816079135863114,\ -0.107532130875048,\ 1.560515895248309).
\end{array}$$

We find, that all four remainder terms R_1 are invertible and by applying (6.2) we obtain the four zeros of p.

TABLE 5.10. The four zeros $z_j,\ j = 1, 2, 3, 4$ of p derived from (6.2) in Theorem 6.1.

$$\begin{array}{l}
z_1 = (-1.003718721498503,\ -0.079115849118361,\ -1.535357907798967,\ -0.090552438151575),\\
z_2 = (-1.266027996825196,\ -0.234526677155184,\ -0.377305663619825,\ 0.002512609675671),\\
z_3 = (0.611025310795246,\ -0.233586955372734,\ -0.452816554678455,\ -0.851764008977247),\\
z_4 = (0.658721407528453,\ 0.144858931809357,\ 0.093444124119135,\ -0.015346176315263).
\end{array}$$

Let $\varepsilon_j = \frac{||p(z_j)||}{||z_j||}$ be the relative error of the zero z_j. Then $\varepsilon_j < 10^{-14}$ for all $j = 1, 2, 3, 4$. The norm $||\cdot||$ is the euclidean norm in $\mathbb{R}^4$.

The previous example of a quaternionic polynomial of degree 4 has four zeros. This is not by chance. If p has degree n, the real companion polynomial c has degree $2n$. There are $2n_1$ real double roots and $2n_2$ complex roots which always appear as pairs of complex conjugate roots and $2(n_1 + n_2) = 2n$. Altogether these roots define at most n zeros of p. This is in coincidence with a result of Gordon and Motzkin, [8, 1965], that a quaternionic polynomial of degree n has at most n zeros.

6. Niven's algorithm in $\mathbb{H}_{\rm coq}$ applied to the roots of the companion polynomial. Let the polynomial p be defined as in (1.2) have coefficients from the algebra of coquaternions $\mathbb{H}_{\rm coq}$. It will be shown, that the use of the roots of the companion polynomial c of p together with an application of Niven's method will lead to all zeros of p.

However, there is a principal difference between the space of quaternions $\mathbb{H}$ and the spaces $\mathbb{H}_{\rm coq}, \mathbb{H}_{\rm nec}, \mathbb{H}_{\rm con}$. For $\mathbb{H}$ there is formula (4.8) in Lemma 4.1, which reduces the problem of zero finding of p to finding a vanishing remainder. There is the phenomenon that the remainder term R_1 may be not zero but noninvertible. In this case one has to solve $R_0 + R_1 z = 0$ which may have no solution or various types of solutions. For $R_1 = 0$ one has to check whether there is a hyperbolic solution.

We will denote the roots of the companion polynomial in the same way as in (5.1), which means that we delete the complex roots with negative imaginary part. This is a consequence of Lemma 2.11. We have to distinguish between (nonreal) complex and real roots of the companion polynomial c. Each complex root will - essentially - define one zero of p. However, an individual real root of c is not good enough to define a zero of p. We start with the simplest case.

THEOREM 6.1. *Let p be the polynomial defined by (1.2) with coefficients from $\mathbb{H}_{\rm coq}$. Let c be the companion polynomial of p and assume that there exists a nonreal, complex, root $\zeta = a + b\mathbf{i}, b > 0$ of c. Define the quadratic polynomial r defined in (4.1) but over $\mathbb{H}_{\rm coq}$ by*

$$u = \Re(\zeta) = a, \quad v = \mathrm{abs}_2(\zeta) = a^2 + b^2, \tag{6.1}$$

and compute R_0, R_1 by Niven's algorithm (formulas (4.4) to (4.6)). If R_1 is invertible, then

$$z := -R_1^{-1}(u,v)R_0(u,v) \tag{6.2}$$

is an isolated zero of p. If $R_1 = 0$, then, ζ may be a hyperbolic zero of p.

Proof. The main argument is, that the root ζ as a root of the companion polynomial c has the property that the similarity class $[\zeta]$ in $\mathcal{A}$ may contain a zero of p. One has to show that z and ζ are similar. Now, by defintion (6.1) and by (4.9), (4.10) we have $\Re(\zeta) = a = u = \Re(z)$ and $\mathrm{abs}_2(\zeta) = a^2 + b^2 = v = \mathrm{abs}_2(z)$. This shows the similarity of ζ and z. In order to show that $r(z) = 0$ we use the connection between u, v and R_0, R_1 given in (4.9), (4.10) and use the inversion formula (2.4) which is valid in all four algebras of $\mathcal{A}$. We have (deleting the arguments of R_1 and of R_0)

$$\begin{aligned} r(z) &= (R_1^{-1}R_0)^2 + 2uR_1^{-1}R_0 + v \\ &= \frac{\overline{R_1}R_0\overline{R_1}R_0}{(\mathrm{abs}_2(R_1))^2} + 2u\frac{\overline{R_1}R_0}{\mathrm{abs}_2(R_1)} + v \\ &= \frac{\overline{R_1}R_0\overline{R_1}R_0}{(\mathrm{abs}_2(R_1))^2} - \frac{\overline{R_1}R_0 + \overline{R_0}R_1}{\mathrm{abs}_2(R_1)}\frac{\overline{R_1}R_0}{\mathrm{abs}_2(R_1)} + \frac{\mathrm{abs}_2(R_0)}{\mathrm{abs}_2(R_1)} \\ &= -\frac{\overline{R_0}R_1\overline{R_1}R_0}{(\mathrm{abs}_2(R_1))^2} + \frac{\mathrm{abs}_2(R_0)}{\mathrm{abs}_2(R_1)} = 0. \end{aligned} \tag{6.3}$$

For invertible R_1, formula (6.2) implies that the remainder vanishes and that $r(z) = 0$. Altogether, an invertible R_1 implies $p(z) = 0$. The term *may be a hyperboiic zero* means that either ζ is a hyperbolic zero, or it is not a zero at all. This has to be checked separately. ∎

It should be noted, that formula (6.3) is valid in all algebras. Thus, in all algebras an invertible R_1 generates an isolated zero of p. The above Theorem 6.1 does not cover the case of real roots of the companion polynomial c. And it also does not cover the case that R_1 is noninvertible but $R_1 \neq 0$.

THEOREM 6.2. *Let p be the polynomial defined by (1.2) over $\mathbb{H}_{\mathrm{coq}}$. Let c be the companion polynomial of p and assume that there exists a pair of real roots ρ_1, ρ_2 of c. Define*

$$a = \frac{1}{2}(\rho_1 + \rho_2), \quad b = \frac{1}{2}\big|\rho_1 - \rho_2\big|, \quad \zeta = a + b\mathbf{j}. \tag{6.4}$$

Put

$$u = \Re(\zeta) = a, \quad v = \mathrm{abs}_2(\zeta) = a^2 - b^2. \tag{6.5}$$

With these constants apply Niven's algorithm to compute R_0, R_1. If R_1 is invertible, z with the same formula as in (6.2) is an isolated zero of p. If $R_1 = 0$, then ζ may be a hyperbolic zero of p.

Proof. Since the number of real roots is even (see (3.4)), the existence of one real root implies the existence of another second real root. That ζ is similar to z has been shown in [11] where also the use of (6.5) is justified. For the term *may be a hyperbolic zero* see the proof of Theorem 6.1. ∎

We present an example which shows the effect of the two possibilities of R_1 being noninvertible. If $R_1 \neq 0$ but nevertheless noninvertible we may encounter a new type of zero of p which we have called *unexpected* zero of p. This is a type of zero which does not exist for quaternionic polynomials.

EXAMPLE 6.3. Let

$$p(z) = z^2 - 2az + a^2, \quad a \in \mathbb{H}_{\mathrm{coq}} \backslash \mathbb{R}. \tag{6.6}$$

Clearly, $p(a) = 0$. Let $a = (a_1, a_2, a_3, a_4)$ and $\mathrm{abs}_2(a) - a_1^2 = 0$. This means, that $a \overset{q}{\sim} a_1$. Let $\alpha \in \mathbb{R}$ be arbitrary and $A := (a_1, \alpha a_2, \alpha a_3, \alpha a_4)$. Thus, $A \overset{q}{\sim} a_1$. Now, we use the identity

$$r(z) = z^2 - 2\Re(z)z + \mathrm{abs}_2(z) = 0, \quad z \in \mathcal{A}, \tag{6.7}$$

valid in all algebras of $\mathcal{A}$ in order to show that

$$\begin{aligned}
p(A) &= A^2 - 2aA + a^2 \\
&= 2\Re(A)A - \mathrm{abs}_2(A) - 2aA + 2\Re(a)a - \mathrm{abs}_2(a) \\
&= 2a_1A - a_1^2 - 2aA + 2a_1a - a_1^2 \\
&= 2(a_1 - a)A + 2a_1a - 2a_1^2 \\
&= 2(a_1 - a)A + 2a_1(a - a_1) \\
&= 2(a - a_1)(a_1 - A) \\
&= -2\alpha(0, a_2, a_3, a_4)^2 = 0.
\end{aligned}$$

The last equation also follows from (6.7) since $\Re(0, a_2, a_3, a_4) = \mathrm{abs}_2(0, a_2, a_3, a_4) = 0$. Let $B = (a_1, a_2, a_4, a_3)$, (note the change of the enumeration of the indices in comparison to a) then $B \overset{q}{\sim} a_1$ in $\mathbb{H}_{\mathrm{coq}}$. However, B is in general not a zero of p. Thus, A is not a hyperbolic zero. The companion polynomial in this example is

$$c(z) = (z - a_1)^4.$$

If we compute R_0, R_1 from (4.6) using (6.4) and (6.5) we obtain $u = a_1, v = a_1^2$ and

$$R_0 = (0, 2a_1a_2, 2a_1a_3, 2a_1a_4), \quad R_1 = (0, -2a_2, -2a_3, -2a_4),$$

which means $R_0 = -a_1R_1$ or $R_0 + a_1R_1 = 0$ or in other words, R_0 and R_1 are linearly dependent as vectors in $\mathbb{R}^4$.

Why does this example not work in $\mathbb{H}$? If $\mathcal{A} = \mathbb{H}$, then the requirement $\mathrm{abs}_2(a) - a_1^2 = 0$ implies $a \in \mathbb{R}$ which was excluded in (6.6). The zeros of the type of A belongs to the type of *unexpected zeros*. How can we recognize unexpected zeros? According to the previous theorems a necessary condition is that the quantity R_1 of Niven's algorithm is not invertible and distinct from the zero element. In order to find all unexpected zeros one has to consider all solutions of $R_0 + R_1z = 0$. A solution technique is described in [14] where it is shown, that the equation $R_0 + R_1z = 0$ is equivalent to a real, linear 4×4 system.

THEOREM 6.4. *Let $\zeta = a + b\mathbf{i}$ be a nonreal, complex zero of the companion polynomial or let $\zeta = a + b\mathbf{j}$, where the construction is described in formula (6.4) of Theorem 6.2. Define in the first case $u = a,\ v = a^2 + b^2$ and in the second case $u = a,\ v = a^2 - b^2$. Assume in both cases that $b > 0$. With the quantities u, v compute R_0, R_1 by Niven's algorithm and assume that $R_1 \neq 0$ but noninvertible. Let there be a real constant γ such that*

$$R_0 + \gamma R_1 = 0.$$

Then, for all $\alpha \in \mathbb{R}$

$$z_0 = \alpha\overline{R_1} + \gamma$$

is a zero of p, provided z_0 is quasi similar to ζ.

Proof. We show, that the remainder $R_0 + R_1 z_0$ vanishes:

$$R_0 + R_1 z_0 = -\gamma R_1 + R_1(\alpha\overline{R_1} + \gamma) = \alpha\,\mathrm{abs}_2(R_1) = 0.$$

□

Theorem 6.2 has the following important consequence.

COROLLARY 6.5. *The maximal number of zeros of a coquaternionic polynomial of degree n is $\binom{2n}{2} = n(2n-1)$.*

Proof. The maximum number of real pairs out of $2n$ real numbers is $\binom{2n}{2}$. For more details see [11]. □

That a real double root can generate an unexpected zero can be seen in Example 6.6.

EXAMPLE 6.6. Let

$$p(z) = z^3 - (2a+1)z^2 + (a^2+2a)z - a^2. \tag{6.8}$$

It is easy to see that in all algebras $p(1) = p(a) = 0$. For $a = (-5, 10, 8, 6) \in \mathbb{H}_{\mathrm{coq}}$, which implies $\mathrm{abs}_2(a) = a_1^2 = 25$, the companion polynomial c is

$$c(z) = z^6 + 18z^5 + 111z^4 + 220z^3 - 225z^2 - 750z + 625.$$

The 6 roots of the companion polynomial c are $1, 1, -5, -5, -5, -5$. There are 15 pairs of real roots, but only 3 of them are distinct: $(1,1)$, $(1,-5)$, $(-5,-5)$. The zeros which belong to these pairs are listed in Table 6.7.

TABLE 6.7. Zeros of p, where p is defined in (6.8) with $a = (-5, 10, 8, 6) \in \mathbb{H}_{\mathrm{coq}}$.

real pairs of roots of c	zeros of p $\alpha \in \mathbb{R}$	type of zero	R_0	R_1
$(1,1)$	1	real	$(-36, 120, 96, 72)$	$(36, -120, -96, -72)$
$(1,-5)$	$(-2, 0, 3, 0)$	hyperbolic	0	0
$(-5,-5)$	$(-5, 10\alpha, 8\alpha, 6\alpha)$	unexpected	$(0, 600, 480, 360)$	$(0, 120, 96, 72)$

To find polynomials p with zeros different from isolated zeros requires a special construction of p. If we generate polynomials p with integer, but random coefficients, the resulting zeros are almost always isolated. This is the outcome of many numerical experiments.

7. Niven's algorithm in $\mathbb{H}_{\mathrm{nec}}$ and in $\mathbb{H}_{\mathrm{con}}$ applied to the roots of the companion polynomial. The construction proposed for coquaternions also works in the spaces $\mathbb{H}_{\mathrm{nec}}$ and in $\mathbb{H}_{\mathrm{con}}$. Apart from adapting the algebraic rules almost no changes are needed. Nevertheless, if we use the same polynomial for, say $\mathbb{H}_{\mathrm{coq}}$ and $\mathbb{H}_{\mathrm{nec}}$ the results will be different. If we choose Example 6.6 with the same $a = (-5, 10, 8, 6) \in \mathbb{H}_{\mathrm{nec}}$, then this a will not have the property $\mathrm{abs}_2(a) = a_1^2$ in $\mathbb{H}_{\mathrm{nec}}$, and we cannot expect an unexpected zero. The corresponding companion polynomial is $c(z) = (z^3 + 9z^2 - 57z + 47)^2$. It has three real double roots, $1, (6\sqrt{2} - 5), -(6\sqrt{2} + 5)$ and the 15 real pairs generate only two different zeros of p: 1 and a.

Serôdio, Pereira, and Vitória, [25] have proposed an algorithm for finding the zeros of quaternionic polynomials, also based in Niven's algorithm. They define a quaternionic companion matrix and use the eigenvalues of this matrix as entries in Niven's algorithm. In [11], the authors Janovská and Opfer conjecture that these eigenvalues coincide with the roots of the companion polynomial. A variation of this algorithm, also applicable only for quaternions which involves also eigenvectors of the companion matrix has been proposed by De Leo, Ducati, and Leonordi, [6]. A survey on eigenvalue problems for quaternionic matrices has been presented by Zhang, [27]. A different approach for finding zeros for quaternionic polynomials has been chosen by Kalantari, [16].

If we have a look at the two Theorems 6.1, 6.2 we see, that a distinction between the various algebras has not to be made. By the two definitions (6.1), (6.5) of the quadratic polynomial r all cases can be accommodated. As such, this algorithm is a little simpler than the algorithms proposed previously by the authors of [11, 13, 15].

8. Niven's algorithm and the approach of Pogorui and Shapiro. Let $\mathcal{A}$ be one of the four algebras $\mathbb{H}, \mathbb{H}_{\mathrm{coq}}, \mathbb{H}_{\mathrm{nec}}, \mathbb{H}_{\mathrm{con}}$ and p be a unilateral polynomial over $\mathcal{A}$. See (1.2). By systematically using the identity

$$z^2 - 2\Re(z)z + \mathrm{abs}_2(z) = 0, \quad z \in \mathcal{A}, \tag{8.1}$$

Pogorui and Shapiro, [22, 2004], arrived (only for quaternions $\mathbb{H}$) at a representation of p in the form

$$p(z) = A(\Re(z), \mathrm{abs}_2(z)) + B(\Re(z), \mathrm{abs}_2(z))\, z, \quad A, B, z \in \mathcal{A}. \tag{8.2}$$

The two quantities A, B do depend on $\Re(z)$ and $\mathrm{abs}_2(z)$ but not fully on z. This allows to insert z_0 instead of z into A, B if $z_0 \in [z]_q$, without changing the values of A and B. This representation was used by D. Janovská and the present author [11, 13, 15] to extend this idea also to the remaining three algebras and to obtain an algorithm for finding all zeros of p in all algebras. The main tool was the use of the roots of the companion polynomial.

In order to compute A and B one needs $3n$ real multiplications plus $2(n+1)$ multiplications of an algebra element by a real number. See [15, p. 247].

Niven's algorithm (see(4.2)) is essentially based on the representation

$$p(z) = q(z)r(z; u, v) + R_0(u, v) + R_1(u, v)z, \tag{8.3}$$

$$r(z; u, v) = z^2 - 2uz + v, \quad R_0, R_1, z \in \mathcal{A},\ u, v \in \mathbb{R}, \tag{8.4}$$

where $R_0(u, v) + R_1(u, v)z$ is the *remainder after division* (short: *remainder*).

By using the recursion (4.4) to (4.6), we see that we need $2n - 2$ multiplications of an algebra element by a real number for computing R_0, R_1.

We found, that essentially the remainder $R_0 + R_1 z$ is the same as $A + Bz$. More precisely, we have the following statement.

THEOREM 8.1. *Let a unilateral polynomial p over $\mathcal{A}$ be given. For a given, arbitrary $z \in \mathcal{A}$ put $u = \Re(z), v = \mathrm{abs}_2(z)$ and compute both representations (8.2) and (8.3) of p. For the definition of r defined in (8.4), use the quantities u, v. Then,*

$$A = R_0, \quad B = R_1. \tag{8.5}$$

Proof. The construction of r implies $r(z; u, v) = 0$ which implies $p(z) = A + Bz = R_0 + R_1 z$. From here, (8.5) follows. □

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