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An accessibility theorem for infinite graph minors

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An Accessibility Theorem for Infinite Graph Minors

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We prove the following recent conjecture of Halin. Let Γ_0 be the class of all graphs, and for every ordinal $\mu > 0$ let Γ_μ be the class of all graphs containing infinitely many disjoint connected graphs from Γ_λ, for every $\lambda < \mu$. Then a graph lies in all these classes Γ_μ if and only if it contains a subdivision of the infinite binary tree.

1. Halin's conjecture

In his recent collection [2] of open problems about infinite graphs, Halin defines the following hierarchy of graphs. He defines Γ_0 to be the class of all graphs (finite or infinite), and for each ordinal $\mu > 0$ he defines Γ_μ as the class of all graphs containing, for each $\lambda < \mu$, infinitely many disjoint connected[1] graphs from Γ_λ. Thus, Γ_1 is the class of infinite graphs, Γ_2 is the class of all graphs containing infinitely many disjoint connected infinite graphs (or equivalently, either infinitely many disjoint rays or infinitely many disjoint infinite stars), and so on.

Halin then asks which graphs, if any, lie in every Γ_μ. He notes that $\Gamma_\lambda \supseteq \Gamma_\mu$ whenever $\lambda < \mu$, and that any graph with a minor in Γ_μ is itself in Γ_μ (induction on μ). We might add the observation that the sequence of Γ_μs does not become stationary, i.e. that for every ordinal μ there is a (connected) graph in $\Gamma_\mu \smallsetminus \Gamma_{\mu+1}$; this, too, follows easily by induction on μ.

Intuitively, it may be natural to consider not the graph properties Γ_μ themselves but their complements $\overline{\Gamma}_\mu$. As observed above, these classes are closed under taking minors, and they form an increasing sequence $\overline{\Gamma}_0 \subseteq \overline{\Gamma}_1 \subseteq \dots$. The question then arises whether every graph is captured by this sequence, and if not then which graphs are.

Following Halin, let us define as the *order* $o(G)$ of a graph G the least ordinal μ (if one exists) such that $G \notin \Gamma_\mu$. Note that if G has an order and H is a minor of G, then H too has an order and $o(H) \leqslant o(G)$.

Which graphs, then, have an order? At first glance, this may look like a difficult question. Indeed, it even seems unclear whether or not every graph has an order. As Halin observes, however, there is a simple sufficent condition for not having an order: if a connected graph G contains infinitely many disjoint copies of itself, then induction on μ shows at once that $G \in \Gamma_\mu$ for all ordinals μ.

[1] The connectedness requirement is essential: without it, every Γ_μ with $\mu > 0$ would just be the class of infinite graphs.

A simple example of such a graph is the infinite binary tree T_2. Halin conjectured that, up to taking minors (even topological minors), it is the unique canonical example:

Conjecture. (Halin [2])
A graph has an order if and only if it does not contain a subdivision of the infinite binary tree T_2.

The conjecture implies in particular that our simple sufficient condition for not having an order is also necessary (after closuring it 'upwards' under minors):

Corollary. *A graph has no order if and only if it has a connected minor that contains infinitely many disjoint copies of itself.*

The purpose of this note is to prove Halin's conjecture.

2. Terminology and prerequisites

Any notation not explained below is taken from [1]. Minors of infinite graphs are defined just as for finite graphs; note that the branch sets (the connected sets of vertices to be contracted) may now be infinite.

All trees we consider are rooted trees. If T is a tree with root r, we write $x \leqslant_T y$ for vertices $x, y \in T$ if $x \in rTy$, i.e. if x lies on the path in T from r to y. This is a partial ordering on $V(T)$ with r as its least element, in which every element other than r has a unique lower (= lesser) neighbour. The maximal elements in this partial order will be called the *leaves* of T. (Thus, r is a leaf if and only if $T = \{ r \}$; any other vertex of T is a leaf if and only if its degree is 1.) When κ is a cardinal, we denote by T_κ the tree in which every vertex has exactly κ upper (= greater) neighbours; the binary tree, for example, is denoted by T_2.

Note that if T' is a subtree of T then T' has a unique minimal vertex r', because incomparable (minimal) vertices are separated by their infimum. Unless otherwise stated, we take r' to be the root of T'; then its tree order $\leqslant_{T'}$ coincides with the order induced on T' by $\leqslant_T$. Given a vertex $t \in T$, we denote by T^t the subtree of T with vertex set $\{ t' \mid t \leqslant_T t' \}$ (and root t).

A *ray* is a one-way infinite path. A ray $t_0 t_1 \ldots$ in a tree T is *free* in T if t_{i+1} is the unique upper neighbour of t_i in T, for all $i = 0, 1, \ldots$.

We assume that the reader is familiar with the basic facts about tree-decompositions, e.g. as explained in [1, Ch. 12]. In particular, we shall use freely the basic separation lemma for tree-decompositions, [1, Lemma 12.3.1].

3. Proof of Halin's conjecture

In this section we prove that every graph not containing a subdivision of T_2 has an order. Clearly, a graph G contains a subdivision of T_2 if and only if it has a T_2 minor (which we write as $T_2 \preccurlyeq G$). The graphs without a T_2 minor have been characterized by Seymour and Thomas, as follows:

Theorem 1. (Seymour & Thomas [3])
A graph has no T_2 minor if and only if it has a tree-decomposition $(T, (V_t)_{t \in T})$ into finite parts such that $T_2 \not\preccurlyeq T$ and for every ray $t_0 t_1 \ldots \subseteq T$ there is an integer k such that $|V_i \cap V_{i+1}| < k$ for infinitely many i.

(In [3], this result is stated for $T_{\aleph_0}$ rather than for T_2. However, since $T_2 \preccurlyeq T_{\aleph_0} \preccurlyeq T_2$, the two versions are equivalent.)

Theorem 1 explains the structure of the graphs $G \not\succcurlyeq T_2$ in terms of the structure of the trees $T \not\succcurlyeq T_2$. This is a non-trivial step, for the trees without a T_2 minor are indeed easier to handle. For example, it is well known and easy to prove that a tree has no T_2 minor if and only if it can be pruned down to the empty graph by recursively deleting leaves or free rays. (We shall do this more formally in the course of the proof of Lemma 2.)

We shall need the following constructive analogue of this pruning lemma. Let $\mathcal{T}_0$ denote the 2-element set containing the one-vertex tree and the ray. For every ordinal $\mu > 0$, let $\mathcal{T}_\mu$ denote the set of all trees T (with root r, say) such that, for either $R := \{ r \}$ or R some ray in T starting at r, each component of $T - R$ lies in some $\mathcal{T}_\lambda$ with $\lambda < \mu$. Note that $\mathcal{T}_\lambda \subseteq \mathcal{T}_\mu$ whenever $\lambda < \mu$.

Lemma 2. *Let T be a tree. There exists an ordinal μ with $T \in \mathcal{T}_\mu$ if and only if $T_2 \not\preccurlyeq T$.*

Proof. For the forward implication, we show by induction on μ that no tree in any $\mathcal{T}_\mu$ contains a subdivision of T_2. This is certainly true for $\mu = 0$. Now let $\mu > 0$ and $T \in \mathcal{T}_\mu$ be given, with root r say, and let $R \subseteq T$ be as in the definition of $\mathcal{T}_\mu$. (Thus, either $R = \{ r \}$ or R is a ray in T starting at r.) Since deleting a ray from a subdivided T_2 leaves infinitely many components each containing another subdivided T_2, and no tree in any $\mathcal{T}_\lambda$ with $\lambda < \mu$ contains a subdivided T_2 (by the induction hypothesis), we also have $T \not\succcurlyeq T_2$ as required.

For the backward implication, let $T \not\succcurlyeq T_2$ be given, and let r be its root. As mentioned earlier, T can be pruned down to $\emptyset$ by deleting leaves or free rays recursively. Indeed, let $T_0 := T$, and for every ordinal $\mu > 0$ let T_μ be obtained from the tree $T_\mu^- := \bigcap_{\lambda < \mu} T_\lambda$ by deleting a leaf or a free ray; if T_μ^- has neither a leaf nor a free ray, we put $T_\mu := T_\mu^-$. Moreover, we label every vertex $t \in T_\mu^-$ deleted in the definition of T_μ with the ordinal $\mu =: \mu_t$. Now let μ^* be minimal such that $T_{\mu^*} = T_{\mu^*}^-$, and put $T_{\mu^*} =: T^*$. Note that since T^* has neither leaves nor free rays, any vertex of T^* lies below a vertex with two distinct upper neighbours, in which case we may construct a subdivided $T_2 \subseteq T^*$ inductively. Thus $T^* = \emptyset$, and every vertex t of T now has an ordinal label $\mu_t < \mu^*$.

Our aim is to show that $T \in \mathcal{T}_{\mu_r}$. To this end, we prove the following assertion for every ordinal μ, by induction on μ:

$$\text{For every } t \in T \text{ with } \mu_t = \mu, \text{ all the vertices } t' \in T^t \text{ have labels } \mu_{t'} \leqslant \mu_t, \text{ and } T^t \in \mathcal{T}_{\mu_t}. \tag{$*$}$$

For $t = r$ this yields the desired result.

Let μ and a vertex $t \in T$ with $\mu_t = \mu$ be given. Since no vertex of T is labelled zero, we have $\mu > 0$. Let R be the subgraph of T induced by all the vertices labelled μ. Then either $R = \{t\}$ and t is a leaf in T_μ^-, or R is a free ray in T_μ^- containing t; in the latter case, R has a tail in T^t. In either case $T^t \cap T_\mu^- \subseteq R$, so any vertex in $T^t - R$ has a label $\lambda < \mu$. This proves the first assertion in ($*$), and moreover the root t' of any component T' of $T^t - R$ has a label $\lambda < \mu$. Since $R \cap T^t$ is closed downwards in T^t, the components of $T^t - R$ are closed upwards. Thus $T' = T^{t'}$, and the induction hypothesis for ($*$) implies that $T' \in \mathcal{T}_\lambda$. Hence $T^t \in \mathcal{T}_\mu$, by the definition of $\mathcal{T}_\mu$ applied with $\{t\}$ or the ray tR. This completes the proof of ($*$), and hence of the lemma. $\square$

For our proof of Halin's conjecture, we want to show that for every graph $G \not\succcurlyeq T_2$ there is an ordinal τ such that $G \notin \Gamma_\tau$. By Theorem 1, G has a certain tree-decomposition $(T, (V_t)_{t \in T})$ with $T \not\succcurlyeq T_2$, and Lemma 2 tells us that $T \in \mathcal{T}_\mu$ for some ordinal μ. We now complete the proof of the conjecture by showing that τ exists, even as a function of μ (i.e. independently of G).

Given an ordinal μ, let us write $\mu =: \mu' + n$ with μ' a limit ordinal and $n < \omega$, and define $\tau(\mu) := \mu' + 2n + 2$.

Lemma 3. *Let $G \not\succcurlyeq T_2$ be a graph with a tree-decomposition $(T, (V_t)_{t \in T})$ as in Theorem 1, and let μ be an ordinal such that $T \in \mathcal{T}_\mu$. Then $G \notin \Gamma_{\tau(\mu)}$.*

Proof. We apply induction on μ. For $\mu = 0$, the tree T is either trivial or a ray. If $|T| = 1$ then G is finite and hence not even in Γ_1. So assume that T is a ray, $T = t_0 t_1 \ldots$ say. Since every V_t is finite, any infinite subgraph H of G meets V_{t_i} for infinitely many i. If H is connected, it therefore even meets every V_{t_i} with large enough i. Now if k is the integer associated with the ray $t_0 t_1 \ldots$ in Theorem 1, this can happen for no more than k disjoint connected graphs H. Hence $G \notin \Gamma_2$, as required.

Suppose now that $\mu > 0$. Let r be the root of T. Since $T \in \mathcal{T}_\mu$ there is an $R \subseteq T$, where either $R = \{r\}$ or R is a ray starting at r, such that every component T' of $T - R$ is a tree in $\mathcal{T}_\lambda$ for some $\lambda < \mu$.

Suppose, for a contradiction, that $G \in \Gamma_{\mu'+2n+2}$, and let $H_1, H_2, \ldots$ be infinitely many disjoint connected subgraphs of G in $\Gamma_{\mu'+2n+1}$. As in the case of $\mu = 0$, only finitely many of the H_i meet V_t for infinitely many $t \in R$. Let H

be one of the other H_i, and let U be the (finite) union of those V_t with $t \in R$ that meet H. Since $H \in \Gamma_{\mu'+2n+1}$, there are infinitely many disjoint connected graphs $H'_1, H'_2, \ldots \subseteq H$ in $\Gamma_{\mu'+2n}$. As U meets only finitely many of these, one of them, H' say, avoids U and hence lies in a subgraph G' of G such that

$$V(G') = \bigcup_{t \in T'} V_t$$

for some component T' of $T - R$. Then $T' \in \mathcal{T}_\lambda$ for some $\lambda < \mu$, and $(T', (V_t \cap V(H))_{t \in T'})$ is a tree-decomposition of H' as in Theorem 1. Our induction hypothesis therefore implies that $H' \notin \Gamma_{\tau(\lambda)}$. Since $\lambda < \mu = \mu' + n$, however, clearly

$$\tau(\lambda) \leqslant \mu' + 2n\,.$$

This contradicts the fact that $H' \in \Gamma_{\mu'+2n}$ by definition of H'. □

References

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