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On the two-dimensional McKay correspondence

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Introduction

The theory of KLEIN *singularities* establishes a 1-1–correspondence between the conjugacy classes of finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ and the COXETER-DYNKIN-WITT *diagrams* of type ADE via the following scheme.

$$\begin{array}{ccc} \{\text{finite subgroups } \Gamma \text{ of } \mathrm{SL}(2, \mathbb{C})\}/\text{ conjugacy} & & \{\text{CDW - diagrams of type ADE}\} \\ \updownarrow & & \updownarrow \\ \{\text{Klein singularities } X_\Gamma = \mathbb{C}^2/\Gamma\}/\sim & \longleftrightarrow & \{\text{minimal resolutions } \widetilde{X}_\Gamma\}/\sim \end{array}$$

Here, the symbol $\sim$ in the last row denotes *complex–analytic equivalence*. The arrow in the second column is given in the upward direction by associating to a minimal resolution $\widetilde{X}_\Gamma$ of X_Γ the *dual graph* of its *exceptional set* $E \subset \widetilde{X}_\Gamma$.

In 1979, MCKAY [10] was able to construct directly via representation theory the resulting bijection in the first row in this diagram. To be more precise, according to this so called MCKAY *correspondence*, each *(nontrivial) irreducible complex representation* of Γ corresponds uniquely to an *irreducible component* of the exceptional set E.

Of course, geometers wanted to understand this phenomenon geometrically, and the first who succeeded in this attempt were GONZALES-SPRINBERG and VERDIER [6] in 1983. They associated to each nontrivial irreducible representation a vector bundle on $\widetilde{X}_\Gamma$ whose first Chern class hits precisely one component of E transversally. Their proof was not completely satisfying since they had to check the details case by case. But in 1985, ARTIN and VERDIER [1] gave a conceptual proof using only standard facts on rational singularities, and in combination with the paper [4] of KNÖRRER and HÉLÈNE ESNAULT from the same year it became clear how to understand the full strength of the correspondence completely in geometrical terms.

Research went then into different directions. One trend (from 1992 on) proposed to treat the *higher dimensional* case $\Gamma \subset \mathrm{SL}(n, \mathbb{C})$ under the slogan (c. f. the notes [11] of REID): if X_Γ has a *crepant* resolution $\widetilde{X}_\Gamma$, i. e. if the canonical sheaf of $\widetilde{X}_\Gamma$ is trivial, then there should be a bijection

$$\{\text{irreducible representations of } \Gamma\} \longleftrightarrow \text{basis of } H^*(\widetilde{X}_\Gamma, \mathbb{Z}) .$$

Another path started earlier: to understand the more general 2–dimensional case of finite subgroups $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$. Here, one has to name ESNAULT with the paper [3] in 1985, a

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doctoral thesis of WUNRAM written under my supervision in 1986 (published 1988 in Mathematische Annalen [15]) and some unpublished notes of myself [12] from 1987.

In June 1997 I learned from a lecture given by YUKARI ITO at RIMS in Kyoto that NAKAMURA constructed in joint work with her $\widetilde{X}_\Gamma$ in the case of finite subgroups of the *special* linear group $\mathrm{SL}\,(2,\,\mathbb{C})$ by invariant theory of Γ acting on a certain Hilbert scheme. They were able, again by checking case by case, to produce the correct representations from the irreducible components of $E \subset \widetilde{X}_\Gamma$ (and even more). You may find this in the beautiful article [8] (see also [7] and section 2 of the present notes). A year later, Nakamura lectured on this topic in Hamburg, and I became aware of how one should generalize the statement to (small) subgroups of the *general* linear group $\mathrm{GL}\,(2,\,\mathbb{C})$ and developed some vague ideas how to prove this without too many calculations. Up to now, this is still a conjecture verified only in the case of *cyclic* quotients by a simple calculation which heavily depends on the work of RIE KIDOH.

In this lecture I would like to present this conjecture, to discuss it a little further and to prove it for *cyclic quotients*. I am convinced that it can be proven in full generality with these older results of my student and of myself which seem not to be known to many people.[1]

Let me add the remark that it has been and still is one of my dreams to describe the *deformation theory* of X_Γ by using only the representation theory of Γ. This is possible for the *infinitesimal* deformations according to a result of PINKHAM in the 70th which was used by BEHNKE, KAHN and myself for calculating the vector space T^1 of first order deformations yielding a nice *dimension formula*, but no clear image how to construct the *versal* deformation. At least in the cyclic case, I found a rather strange way to produce the deformation space over the ARTIN *component* combining a quiver construction and some invariant theoretic ideas [13].

Let me also mention at the end of these introductory remarks that CRAWLEY-BOEVEY released just before my departure to Japan a note [2] in which he proves the result of ITO-NAKAMURA without case by case checking; he uses the theory of *preprojective algebras* associated to MCKAY *quivers*. However, these methods are not available for general quotients.

1 McKay's observation

Let me first recall the construction of the MCKAY *quiver* associated to a finite small subgroup $\Gamma \subset \mathrm{GL}\,(2,\,\mathbb{C})$. Let $\mathrm{Irr}\,\Gamma := \{\,\rho_0, \rho_1, \ldots, \rho_r\,\}$ denote the set of irreducible complex representations of Γ, ρ_0 the trivial one, and c the natural representation on $\mathbb{C}^2$ given by the inclusion $\Gamma \subset \mathrm{GL}\,(2, \mathbb{C})$. Then,

$$\rho_i \otimes c^* \;=\; \sum_j a_{ij}\,\rho_j$$

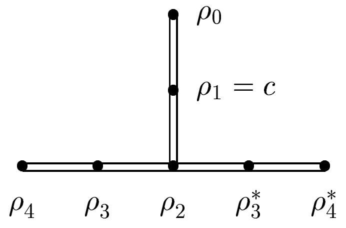
where c^* denotes the *dual* representation of c. The MCKAY quiver is formed in the following way: Associate to each representation a vertex and join the ith vertex with the jth vertex by a_{ij} arrows.

For the *binary tetrahedral* group $\mathbb{T}$, that is the preimage of the symmetry group $T \subset \mathrm{SO}\,(3,\,\mathbb{R})$ of a *regular tetrahedron* under the canonical group epimorphism

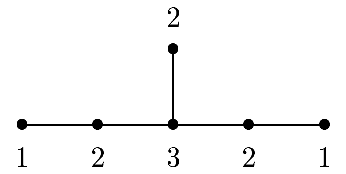
$$(0 \longrightarrow \{\pm 1\} \longrightarrow)\;\; \mathrm{SU}\,(2,\,\mathbb{C}) \;\longrightarrow\; \mathrm{SO}\,(3,\,\mathbb{R}) \;\longrightarrow\; 1\;,$$

[1] Note added August 2000: Akira Ishii informed me that he succeeded in proving the conjecture via the *multiplication formula* of WUNRAM in combination with an easy argument in the preprint *Mukai implies McKay: the McKay correspondence as an equivalence of derived categories* by T. Bridgeland, A. King and M. Reid, to appear in Jour. AMS

the resulting quiver looks as follows:



where a subgraph $\bullet\!\!=\!\!=\!\!\bullet$ stands for a *double arrow*, i. e. two arrows in opposite direction. Replacing such subgraphs by a simple line $\bullet\!\!-\!\!-\!\!\bullet$, forgetting ρ_0 and inserting the ranks of the corresponding representations, yields the new diagram



which, in fact, is not only the CDW diagram of type E_6 but also represents the *fundamental cycle* of the singularity $\mathbb{C}^2 / \mathbb{T}$.

McKay's observation may be formulated in the following way:

For every finite subgroup $\Gamma \subset \mathrm{SL}\,(2, \mathbb{C})$ *, one has* $a_{ji} = a_{ij} \in \{ 0, 1 \}$ *. Replacing each* double arrow *by a line, one finds exactly the (extended) CDW diagrams of correct type.*

2 The Ito - Nakamura construction

Let $\mathrm{Hilb}^n(\mathbb{C}^2)$ be the Hilbert scheme of all 0–dimensional subschemes on $\mathbb{C}^2$ of colength n. It is well–known that the canonical Hilbert-Chow–*morphism*

$$\mathrm{Hilb}^n\,(\mathbb{C}^2) \longrightarrow \mathrm{Sym}^n(\mathbb{C}^2) \;=\; (\mathbb{C}^2)^n/\mathfrak{S}_n$$

is a resolution of singularities (Fogarty), and $\mathrm{Hilb}^n(\mathbb{C}^2)$ carries a holomorphic symplectic structure (Beauville). Let $\Gamma \subset \mathrm{GL}\,(2, \mathbb{C})$ be a finite small subgroup of order $n = \mathrm{ord}\,\Gamma$, and take the invariant part of the natural action of Γ on $\mathrm{Hilb}^n\,(\mathbb{C}^2)$. The resulting space $\mathrm{Hilb}^n\,(\mathbb{C}^2)^\Gamma$ is smooth and maps under the canonical mapping

$$\mathrm{Hilb}^n\,(\mathbb{C}^2)^\Gamma \longmapsto \mathrm{Sym}^n(\mathbb{C}^2)^\Gamma \;\cong\; \mathbb{C}^2/\Gamma$$

to X_Γ. It may have several components, but there is exactly one which maps *onto* X_Γ and therefore constitutes a *resolution* of X_Γ which will be denoted by

$$\widetilde{X}_\Gamma \;=\; \mathrm{Hilb}^\Gamma(\mathbb{C}^2)\ .$$

This component is proven to be the *minimal* resolution in the cases

- $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ (Ito-Nakamura)

- cyclic subgroups of $\mathrm{GL}(2, \mathbb{C})$ (Kidoh)

and should also satisfy minimality in the case of general finite small subgroups $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$ as was claimed in Ginzburg-Kapranov [5].

Hence, a point on the exceptional set E of $\widetilde{X}_\Gamma$ may be regarded as a Γ–invariant ideal $I \subset \mathcal{O}_{\mathbb{C}^2}$ with support in 0. Now, let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{\mathbb{C}^2,0}$, $\mathfrak{m}_X$ that of $\mathcal{O}_{X,0} = \mathcal{O}^\Gamma_{\mathbb{C}^2,0}$ and $\mathfrak{n} = \mathfrak{m}_X \mathcal{O}_{\mathbb{C}^2,0}$. Put

$$V(I) := I/(\mathfrak{m}I + \mathfrak{n}) .$$

This is a (finite–dimensional) Γ–module. For a (nontrivial) irreducible representation $\rho \in \mathrm{Irr}^0\,\Gamma := \mathrm{Irr}\,\Gamma \setminus \{\rho_0\}$ with representation space $V(\rho)$ put

$$E_\rho = \{ I : V(I) \quad \text{contains} \quad V(\rho) \} .$$

In the case of Klein singularities, i. e. for finite subgroups $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$, one has the following beautiful result of Ito and Nakamura which opened up a new way to understand the McKay correspondence completely in terms of the binary polyhedral group Γ.

Theorem (Ito - Nakamura [7]) *For $\rho \in \mathrm{Irr}^0\,\Gamma$, $E_\rho \cong \mathbb{P}_1$. Moreover, $E_\rho \cap E_{\rho'}$ is empty or consists of exactly one point for $\rho \neq \rho'$, and*

$$E = \bigcup_{\rho \in \mathrm{Irr}^0 \Gamma} E_\rho .$$

More precisely, $V(I) = V(\rho)$ for the ideals $I \in E_\rho$ corresponding to smooth points of E, and

$$E_\rho \cap E_{\rho'} \ni I \iff V(I) = V(\rho) \oplus V(\rho') .$$

It is now natural to ask the following

Question *How can this result be generalized to finite small subgroups* $\Gamma \subset \mathrm{GL}(2, \mathbb{C})$?

The purpose of my lecture is to make the following conjecture plausible, in particular by proving it for *cyclic* quotients.

Conjecture *The Ito–Nakamura construction yields a minimal resolution of X_Γ, and one has the same result as above if the set of* all *nontrivial indecomposable representations is replaced by the subset of so–called* special *ones.*

Remark For the definition of *special* representations, see the last section.

3 Kidoh's calculation in the cyclic case

We are going to investigate in this section the case of *cyclic* groups Γ. This is merely a simple observation and follows easily by a calculation using Kidoh's determination of the invariant ideals in this case in terms of *continued fraction expansions*.

Recall that cyclic quotient surface singularities of $\mathbb{C}^2$ are determined by two natural numbers n, q with $1 \le q < n$ and $\gcd(n, q) = 1$. The cyclic group $C_{n,q}$ acting is generated by the linear map with matrix

$$\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^q \end{pmatrix} , \quad \zeta_n = \exp(2\pi i/n) \text{ an nth primitive root of unity}$$

that operates on the polynomial ring $\mathbb{C}[u, v]$ by $(u, v) \longmapsto (\zeta_n u, \zeta_n^q v)$. A monomial $u^\alpha v^\beta$ is invariant under this action if and only if

$$\alpha + q\beta \equiv 0 \mod n ,$$

e. g. for $(\alpha, \beta) = (n, 0)$, $(n-q, 1)$, $(0, n)$. The Hirzebruch–Jung continued fraction

$$\frac{n}{n-q} = a_1 - \frac{1}{a_2 - 1/\cdots} = a_1 - \underline{1 \rfloor} \overline{a_2} - \cdots - \underline{1 \rfloor} \overline{a_m} , \quad a_\mu \ge 2$$

gives a strictly decreasing sequence

$$\alpha_0 = n > \alpha_1 = n - q > \alpha_2 = a_1 \alpha_1 - \alpha_0 > \cdots$$

stopping with $\alpha_{m+1} = 0$, and a strictly increasing sequence

$$\beta_0 = 0 < \beta_1 = 1 < \beta_2 = a_1\beta_1 - \beta_0 < \cdots < \beta_{m+1} = n .$$

It is well–known that the monomials

$$u^{\alpha_\mu} v^{\beta_\mu} , \quad \mu = 0, \ldots, m+1$$

generate the invariant algebra

$$A_{n,q} := \mathbb{C}\langle u, v\rangle^{C_{n,q}} = \mathcal{O}_{\mathbb{C}^2/C_{n,q},0}$$

minimally. In particular, $\operatorname{embdim} A_{n,q} = m+2$, hence, $\operatorname{mult} A_{n,q} = m+1$. The numbers a_μ are exponents in canonical equations for $A_{n,q}$. On the other hand, the continued fraction expansion

$$\frac{n}{q} = b_1 - \underline{1 \rfloor} \overline{b_2} - \cdots - \underline{1 \rfloor} \overline{b_r} , \quad b_k \ge 2$$

gives invariants for the minimal resolution for $\mathbb{C}^2 / C_{n,q}$ whose exceptional divisor consists of a string of rational curves with *selfintersection numbers* $-b_k$.

Define correspondingly the decreasing sequence

$$i_0 = n > i_1 = q > i_2 = b_1 i_1 - i_0 > \cdots > i_r = 1 > i_{r+1} = 0$$

and

$$j_0 = 0 < j_1 = 1 < j_2 = b_1 j_1 - j_0 < \cdots < j_{r+1} = n .$$

Theorem (R. Kidoh [9]) *Let (n, q) be given. Then,* $\operatorname{Hilb}^{C_{n,q}}(\mathbb{C}^2)$ *consists of the following $C_{n,q}$–invariant ideals of colength $n = \operatorname{ord} C_{n,q}$:*

$$I_k(s_k, t_k) = (u^{i_{k-1}} - s_k v^{j_{k-1}}, v^{j_k} - t_k u^{i_k}, u^{i_{k-1}-i_k} v^{j_k - j_{k-1}} - s_k t_k),\ 1 \le k \le r+1 ,\ (s_k, t_k) \in \mathbb{C}^2 .$$

Remarks. 1. These are in fact $C_{n,q}$–invariant ideals, since $i_k \equiv q\, j_k \mathrm{mod} n$ and $u^{i_{k-1}-i_k}\, v^{j_k - j_{k-1}}$ are invariant functions.

2. The $(r+1)$ copies of $\mathbb{C}^2$ patch together to form the *minimal resolution* of $\mathbb{C}^2 / C_{n,q}$, i. e. $I_k\,(s_k,\, t_k) = I_{k+1}\,(s_{k+1},\, t_{k+1}) \Longleftrightarrow s_{k+1}\, t_k = 1$ and $t_{k+1} = t_k^{b_k}\, s_k$.

3. The exceptional divisor is equal to

$$I_1\,(0,\, t_1) \cup \bigcup_{k=2}^{r} \{I_k\,(s_k,\, t_k) : s_k\, t_k \;=\; 0\} \cup I_{r+1}\,(s_{r+1},\, 0)\ .$$

What about the representations of $C_{n,q}$ on the $V\,(I_k)$? For $I_1\,(0,\, t_1)$ the first generator $u^{i_0} = u^n$ is an invariant. The third is such in all cases anyway. So, $C_{n,q}$ acts on $V\,(I_1\,(0,\, t_1)) \cong \mathbb{C}$ as the *character*

$$\chi\,(\zeta_n^{i_1}) \;:\quad z \longmapsto \zeta_n^{i_1} z \quad (\text{recall that } qj_k \equiv i_k \bmod n)\ .$$

This remains automatically true for $I_2\,(s_2,\, t_2)$ with $t_2 = 0$, $s_2 \neq 0$. The first normal crossing point of the exceptional set is the ideal $I_2\,(0,\, 0)$ which is generated by u^{i_1}, v^{j_2} and an invariant. Therefore, the corresponding representation is the sum

$$\chi\,(\zeta_n^{i_1}) \oplus \chi\,(\zeta_n^{qj_2}) \;=\; \chi\,(\zeta_n^{i_1}) \oplus \chi\,(\zeta_n^{i_2})\ .$$

The ideal $I_2\,(0,\, t_2)$, $t_2 \neq 0$, is generated by u^{i_1}, $v^{j_2} - t_2\, u^{i_2}$ and the invariant $u^{u_1 - i_2}\, v^{j_2 - j_1}$. Now,

$$t_2\, u^{i_1} \;=\; v^{j_1}\,(u^{i_1-i_2}\, v^{j_2-j_1}) \;-\; u^{i_1-i_2}\,(v^{j_2} - t_2\, u^{i_2}) \in \mathfrak{m}\, I_2\,(0,\, t_2)\ .$$

Therefore, the representation is just the one-dimensional

$$\chi\,(\zeta_n^{i_2}) \;=\; \chi\,(\zeta_n^{qj_2})\ .$$

It should be clear how this game goes on: We get precisely the r representations $\chi\,(\zeta_n^{i_k})$, $k = 1, \ldots, r$, resp. the correct sum of two of them at the intersection points.

What is so *special* about these representations under all representations $\chi\,(\zeta_n^{i})$? The answer is known to me since a long time, although it is still rather mysterious.

4 Special representations

Let Γ be a finite small subgroup of $\mathrm{GL}\,(2,\, \mathbb{C})$ and ρ a representation of Γ on V. Γ operates on $\mathbb{C}^2 \times V$, and the quotient is a vector bundle on $(\mathbb{C}^2 \setminus \{0\})/\Gamma$ that extends to a *reflexive sheaf* on $\mathbb{C}^2/\Gamma = X_\Gamma$. In fact, one gets all reflexive modules M in this manner. M is indecomposable if and only if ρ is irreducible.

We can study more generally any *rational* surface singularity X and an arbitrary reflexive module M on it. Let $\pi : \widetilde{X} \longrightarrow X$ be a minimal resolution, and put

$$\widetilde{M} \;:=\; \pi^* M/\mathrm{torsion}\ .$$

Such sheaves on $\widetilde{X}$ were called *full sheaves* by Esnault.

Theorem (Esnault [3]) *A sheaf* $\mathcal{F}$ *on* $\widetilde{X}$ *is full if and only if the following conditions are satisfied*:

1. $\mathcal{F}$ *is* locally free, *i. e. a* vector bundle,
2. $\mathcal{F}$ *is* generated by global sections, *in particular* $H^1\,(\widetilde{X},\,\mathcal{F})\;=\;0\,,$
3. $H^1\,(\widetilde{X},\,\mathcal{F}^*\otimes\omega_{\widetilde{X}})\;=\;0\,.$

Under these assumptions, $M=\pi_*\mathcal{F}$ *is reflexive and* $\mathcal{F}=\widetilde{M}\,.$

Let $\widetilde{M}=\mathcal{F}$ be of rank r and full. Then one can construct an exact sequence

$$0\longrightarrow\mathcal{O}^r_{\widetilde{X}}\longrightarrow\mathcal{F}\longrightarrow N\longrightarrow 0$$

with $D:=\operatorname{supp}N$ a divisor in a neighborhood of the *exceptional* set E which cuts E transversally at regular points only. We call D the CHERN *divisor* $c_1\,(\mathcal{F})\,.$

Definition. A full sheaf / reflexive module / representation is called *special*, if and only if

$$H^1\,(\widetilde{X},\,\widetilde{M}^*)\;=\;0\,.$$

Theorem (Wunram [15], Riemenschneider [12])

a) $\widetilde{M}$ special $\Longleftrightarrow$ $\widetilde{M}\otimes\omega_{\widetilde{X}}\to[(M\otimes\omega_X)^{**}]^{\widetilde{\ }}$ *is an isomorphism.*

b) M special $\Longleftrightarrow$ $M\otimes\omega_X/$ torsion *is reflexive.*

c) ρ special $\Longleftrightarrow$ $(\Omega^2_{\mathbb{C}^2})^\Gamma\otimes(\mathcal{O}_{\mathbb{C}^2}\otimes V)^\Gamma\to(\Omega^2_{\mathbb{C}_2}\otimes V)^\Gamma$ *is surjective.*

Theorem (Wunram [15]) *There is a bijection*

$\{$ *special nontrivial indecomposable reflexive modules* $\}\leftrightarrow\{$ *irreducible components* E_j *of* $E\}$

via

$$M\;\longmapsto\;c_1(\widetilde{M})\,E_k=\delta_{jk}\,.$$

The rank of M_j *equals the multiplicity* r_j *of the curve* E_j *in the* fundamental cycle $Z=\sum r_j\,E_j\,.$

As a *Corollary*, one gets once more the McKAY *correspondence* since for the Klein singularities one has $\omega_X\cong\mathcal{O}_X$ (Gorenstein property) and $\omega_{\widetilde{X}}\cong\mathcal{O}_{\widetilde{X}}\,.$

WUNRAM has, in particular, computed the irreducible special representations in the case of cyclic quotients [14]. As it turns out, they coincide with the representations we have found in section 3. Thus, the conjecture in section 2 is true in this case.

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