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Existence of Solutions of the Classical Yang-Baxter Equation for a Real Lie Algebra

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Abstract

We characterize finite-dimensional Lie algebras over the real numbers for which the classical Yang-Baxter equation has a non-trivial skew-symmetric solution (resp. a non-trivial solution with invariant symmetric part). Equivalently, we obtain a characterization of those finite-dimensional real Lie algebras which admit a non-trivial (quasi-) triangular Lie bialgebra structure.

1 Introduction

Solutions of the classical Yang-Baxter equation (CYBE) play a role in many different areas of mathematics and physics. Very often their usefulness comes from the fact that solutions with invariant symmetric part give rise to quasi-triangular Lie bialgebra structures. In particular, solutions of the CYBE for Lie algebras over the real numbers are a main tool in solving and constructing completely integrable classical Hamiltonian systems.

In general, a classification of all solutions of the CYBE is very difficult. Nevertheless, Belavin and Drinfel'd succeeded in [1] to obtain for *any* finite-dimensional *simple* Lie algebra over the complex numbers a complete description of those solutions of the CYBE which have a non-zero symmetric part. The aim of this paper is much more modest in asking when there exist solutions of the CYBE which give rise to *non-trivial* Lie bialgebra structures on the underlying Lie algebra. Since in [5] we previously solved this problem over the complex numbers, here we restrict ourselves to the case of finite-dimensional Lie algebras over the real numbers. As in [5], it turns out that with the exception of a few cases occurring in dimension three, for every finite-dimensional non-abelian Lie algebra the CYBE has a non-trivial skew-symmetric solution.

Since in this paper we are only interested in solutions of the CYBE which give rise to Lie bialgebra structures, we will re-phrase our main result (see Theorem 1) in those terms and only need to prove the latter (see Theorem 2). All the necessary notation for this will be introduced in the next section. The proof of Theorem 2 which uses the methods of [3] in conjunction with those of [5] will be given in the last section. Elsewhere we will come back to discuss the existence of those solutions of the CYBE which do not give rise to Lie bialgebra structures. In the light of our main result, this shall have to be done only for the three exceptional cases.

2 Main Results

Let $\mathbb{F}$ be an arbitrary field and let $\mathfrak{a}$ be a Lie algebra over $\mathbb{F}$. For $r = \sum_{j=1}^{n} r_j \otimes r'_j \in \mathfrak{a} \otimes \mathfrak{a}$ set

$$r^{12} := \sum_{j=1}^{n} r_j \otimes r'_j \otimes 1, \quad r^{13} := \sum_{j=1}^{n} r_j \otimes 1 \otimes r'_j, \quad r^{23} := \sum_{j=1}^{n} 1 \otimes r_j \otimes r'_j,$$

where 1 denotes the identity element of the universal enveloping algebra $U(\mathfrak{a})$ of $\mathfrak{a}$. Note that the elements r^{12}, r^{13}, r^{23} are considered as elements of the associative algebra $U(\mathfrak{a}) \otimes U(\mathfrak{a}) \otimes U(\mathfrak{a})$ via the canonical embedding $\mathfrak{a} \hookrightarrow U(\mathfrak{a})$. Therefore one can form the commutators given by

$$\begin{aligned}
[r^{12}, r^{13}] &= \sum_{i,j=1}^{n} [r_i, r_j] \otimes r'_i \otimes r'_j, \\
[r^{12}, r^{23}] &= \sum_{i,j=1}^{n} r_i \otimes [r'_i, r_j] \otimes r'_j, \\
[r^{13}, r^{23}] &= \sum_{i,j=1}^{n} r_i \otimes r_j \otimes [r'_i, r'_j].
\end{aligned}$$

Then the mapping

$$\mathrm{CYB} : \mathfrak{a} \otimes \mathfrak{a} \longrightarrow \mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a}$$

defined via

$$r \longmapsto [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]$$

is called the *classical Yang-Baxter operator* for $\mathfrak{a}$. The equation $\mathrm{CYB}(r) = 0$ is the *classical Yang-Baxter equation* (CYBE) for $\mathfrak{a}$, and a solution of the CYBE is called a *classical r-matrix* for $\mathfrak{a}$ (cf. [2, Section 2.1B]).

Assume for the moment that the characteristic of the ground field $\mathbb{F}$ is zero. For any vector space V over $\mathbb{F}$ and every natural number n the *symmetric group* S_n of degree n acts on the n-fold tensor power $V^{\otimes n}$ of V via

$$\sigma \cdot (v_1 \otimes \ldots \otimes v_n) := v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)} \quad \forall\, \sigma \in S_n; v_1, \ldots, v_n \in V.$$

The $\mathbb{F}$-linear mapping $\mathcal{S}_n : V^{\otimes n} \to V^{\otimes n}$ defined by $t \mapsto \sum_{\sigma \in S_n} \sigma \cdot t$ is called the *symmetrization mapping*. The elements of the image $\mathrm{Im}(\mathcal{S}_n)$ of $\mathcal{S}_n$ are just the *symmetric* n-tensors, i.e., elements $t \in V^{\otimes n}$ such that $\sigma \cdot t = t$ for every $\sigma \in S_n$. Moreover, $\mathrm{Im}(\mathcal{S}_n)$ is canonically isomorphic to the n-th *symmetric power* $S^n V$ of V. The $\mathbb{F}$-linear mapping $\mathcal{A}_n : V^{\otimes n} \to V^{\otimes n}$ defined by $t \mapsto \sum_{\sigma \in S_n} \mathrm{sign}(\sigma)(\sigma \cdot t)$ is called the *skew-symmetrization* (or *alternation*) *mapping*. The elements of the image $\mathrm{Im}(\mathcal{A}_n)$ of $\mathcal{A}_n$ are just the *skew-symmetric* n-tensors, i.e., elements $t \in V^{\otimes n}$ such that $\sigma \cdot t = \mathrm{sign}(\sigma) t$ for every $\sigma \in S_n$. Since $\mathrm{Im}(\mathcal{A}_n)$ is canonically isomorphic to the n-th *exterior power* $\bigwedge^n V$ of V, we identify skew-symmetric n-tensors with elements of $\bigwedge^n V$; e.g., we write

$$v_1 \wedge v_2 = (1 - \tau)(v_1 \otimes v_2)$$

in the case $n = 2$, where $\tau : V^{\otimes 2} \to V^{\otimes 2}$ is given by $v_1 \otimes v_2 \mapsto v_2 \otimes v_1$, and

$$v_1 \wedge v_2 \wedge v_3 = \sum_{\sigma \in S_3} \mathrm{sign}(\sigma) v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}$$

in the case $n = 3$. As a direct consequence of our identifications, we also have that $V^{\otimes 2} = S^2 V \oplus \bigwedge^2 V$.

If $\mathfrak{a}$ is a Lie algebra and M is an $\mathfrak{a}$-module, then the set of $\mathfrak{a}$-*invariant elements* of M is defined by

$$M^{\mathfrak{a}} := \{ m \in M \mid a \cdot m = 0 \quad \forall\, a \in \mathfrak{a} \}.$$

It follows from the remark after [5, Proposition 2] that *every* $\mathfrak{a}$-invariant skew-symmetric 2-tensor is a solution of the CYBE. Therefore a skew-symmetric classical r-matrix for $\mathfrak{a}$ is called *trivial* if it is $\mathfrak{a}$-invariant. But note that *not* every $\mathfrak{a}$-invariant symmetric 2-tensor satisfies the CYBE (see [5, Example 1]). Nevertheless, we say that an arbitrary classical r-matrix for $\mathfrak{a}$ is *trivial* if it is $\mathfrak{a}$-invariant. In particular, every classical r-matrix with $\mathfrak{a}$-invariant symmetric part is trivial if and only if its skew-symmetric part is $\mathfrak{a}$-invariant (and thus trivial). We will see below that *trivial* classical r-matrices for $\mathfrak{a}$ are exactly those which give rise to a *trivial* Lie bialgebra structure on $\mathfrak{a}$.

In what follows let $\mathbb{R}$ denote the field of *real* numbers. We now describe (up to isomorphism) those Lie algebras which turn out to have *no* non-trivial skew-symmetric classical r-matrix. First, there is a *simple* Lie algebra, namely the *non-split three-dimensional simple Lie algebra* over $\mathbb{R}$ which is denoted by $\mathfrak{su}(2)$ and can be realized as the cross product algebra on three-dimensional Euclidean space, i.e.,

$$\mathfrak{su}(2) = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3; \quad [e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

Next, there is the (non-abelian) *nilpotent* three-dimensional *Heisenberg algebra*

$$\mathfrak{h}_1(\mathbb{R}) = \mathbb{R}p \oplus \mathbb{R}q \oplus \mathbb{R}\hbar$$

determined by the so-called *Heisenberg commutation relation*

$$[p, q] = \hbar.$$

Finally, there is the three-dimensional (non-nilpotent) *solvable* Lie algebra

$$\mathfrak{s}_\Lambda(\mathbb{R}) = \mathbb{R}h \oplus \mathbb{R}f_1 \oplus \mathbb{R}f_2;$$

$$[h, f_1] = \lambda_{11}f_1 + \lambda_{12}f_2, \quad [h, f_2] = \lambda_{21}f_1 + \lambda_{22}f_2, \quad [f_1, f_2] = 0,$$

where $\Lambda := (\lambda_{ij})_{1\leq i,j\leq 2}$ is an element of $\mathrm{GL}_2(\mathbb{R})$.

Remark. Let $\Lambda \in \mathrm{Mat}_2(\mathbb{R})$ be *singular*. If $\Lambda \neq 0$, then $\mathfrak{s}_\Lambda(\mathbb{R})$ is isomorphic either to the three-dimensional Heisenberg algebra or to the (trivial) one-dimensional central extension of the two-dimensional non-abelian Lie algebra. If $\Lambda = 0$, then, of course, $\mathfrak{s}_\Lambda(\mathbb{R})$ is abelian. It is elementary to show that therefore every *non-simple* three-dimensional real Lie algebra is isomorphic to $\mathfrak{s}_\Lambda(\mathbb{R})$ for a suitable choice of $\Lambda \in \mathrm{Mat}_2(\mathbb{R})$. Moreover, every *simple* three-dimensional real Lie algebra is isomorphic either to the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of traceless real 2-by-2 matrices or to $\mathfrak{su}(2)$.

We can now state the first version of the main result of this paper.

Theorem 1 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over $\mathbb{R}$. Then the following conditions are equivalent:*

(a) *The classical Yang-Baxter equation for $\mathfrak{a}$ has a non-trivial skew-symmetric solution.*

(b) *The classical Yang-Baxter equation for $\mathfrak{a}$ has a non-trivial solution with $\mathfrak{a}$-invariant symmetric part.*

(c) *$\mathfrak{a}$ is non-abelian and not isomorphic to $\mathfrak{su}(2)$, to $\mathfrak{h}_1(\mathbb{R})$ or to $\mathfrak{s}_\Lambda(\mathbb{R})$ where* $\mathrm{tr}(\Lambda) = 0$ *and* $\det(\Lambda) > 0$.

As an immediate consequence we obtain the following existence result:

Corollary 1 *If $\mathfrak{a}$ is a finite-dimensional non-abelian Lie algebra over $\mathbb{R}$ with* $\dim_{\mathbb{R}} \mathfrak{a} \neq 3$*, then the classical Yang-Baxter equation for $\mathfrak{a}$ has a non-trivial skew-symmetric solution.*

Before giving a proof of Theorem 1 we re-formulate its statement in terms of the existence of non-trivial (*quasi-*) *triangular Lie bialgebra structures*. For this purpose we first recall the relevant definitions.

A *Lie coalgebra* over an arbitrary field $\mathbb{F}$ is a vector space $\mathfrak{c}$ over $\mathbb{F}$ together with a linear mapping

$$\delta : \mathfrak{c} \longrightarrow \mathfrak{c} \otimes \mathfrak{c},$$

such that

(1) $$\mathrm{Im}(\delta) \subseteq \mathrm{Im}(1 - \tau),$$

and

(2) $$(1 + \xi + \xi^2) \circ (1 \otimes \delta) \circ \delta = 0,$$

where 1 denotes the identity mapping on $\mathfrak{c}$, $\tau : \mathfrak{c} \otimes \mathfrak{c} \to \mathfrak{c} \otimes \mathfrak{c}$ denotes the *switch mapping* sending $x \otimes y$ to $y \otimes x$ for every $x, y \in \mathfrak{c}$, and $\xi : \mathfrak{c} \otimes \mathfrak{c} \otimes \mathfrak{c} \to \mathfrak{c} \otimes \mathfrak{c} \otimes \mathfrak{c}$ denotes the *cycle mapping* sending $x \otimes y \otimes z$ to $y \otimes z \otimes x$ for every $x, y, z \in \mathfrak{c}$. The mapping δ is called the *cobracket* of $\mathfrak{c}$, (1) is called *co-anticommutativity*, and (2) is called the *co-Jacobi identity*. Note that any cobracket on a one-dimensional Lie coalgebra is the zero mapping since $\mathrm{Im}(1 - \tau) = 0$. This is dual to the statement that every bracket on a one-dimensional Lie algebra is zero. For further information on Lie coalgebras we refer the reader to [6] and the references given there.

A *Lie bialgebra* over $\mathbb{F}$ is a vector space $\mathfrak{a}$ over $\mathbb{F}$ together with linear mappings $[\cdot,\cdot] : \mathfrak{a} \otimes \mathfrak{a} \to \mathfrak{a}$ and $\delta : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$ such that $(\mathfrak{a}, [\cdot,\cdot])$ is a Lie algebra, $(\mathfrak{a}, \delta)$ is a Lie coalgebra, and δ is a *derivation* from the Lie algebra $\mathfrak{a}$ into the $\mathfrak{a}$-module $\mathfrak{a} \otimes \mathfrak{a}$, i.e.,

$$\delta([x, y]) = x \cdot \delta(y) - y \cdot \delta(x) \qquad \forall\, x, y \in \mathfrak{a},$$

where the tensor product $\mathfrak{a} \otimes \mathfrak{a}$ is an $\mathfrak{a}$-module via the *adjoint diagonal action* defined by

$$x \cdot \left(\sum_{j=1}^{n} a_j \otimes b_j \right) := \sum_{j=1}^{n} ([x, a_j] \otimes b_j + a_j \otimes [x, b_j]) \qquad \forall\, x, a_j, b_j \in \mathfrak{a}$$

(cf. [2, Section 1.3A]). A Lie bialgebra structure $(\mathfrak{a}, \delta)$ on a Lie algebra $\mathfrak{a}$ is called *trivial* if $\delta = 0$.

A *coboundary Lie bialgebra* over $\mathbb{F}$ is a Lie bialgebra $\mathfrak{a}$ such that the cobracket δ is an *inner derivation*, i.e., there exists an element $r \in \mathfrak{a} \otimes \mathfrak{a}$ such that

$$\delta(x) = x \cdot r \qquad\qquad \forall\, x \in \mathfrak{a}$$

(cf. [2, Section 2.1A]). Let $r \in \mathfrak{a} \otimes \mathfrak{a}$ and define $\delta_r(x) := x \cdot r$ for every $x \in \mathfrak{a}$. Obviously, a coboundary Lie bialgebra structure $(\mathfrak{a}, \delta_r)$ on $\mathfrak{a}$ is *trivial* if and only if $r \in (\mathfrak{a} \otimes \mathfrak{a})^{\mathfrak{a}}$. Moreover, Drinfel'd observed that δ_r defines a Lie bialgebra structure on $\mathfrak{a}$ if and only if $r + \tau(r) \in (\mathfrak{a} \otimes \mathfrak{a})^{\mathfrak{a}}$ and $\mathrm{CYB}(r) \in (\mathfrak{a} \otimes \mathfrak{a} \otimes \mathfrak{a})^{\mathfrak{a}}$ (see [4, Section 4, p. 804] or [2, Proposition 2.1.2]). In particular, every solution r of the CYBE satisfying $r + \tau(r) \in (\mathfrak{a} \otimes \mathfrak{a})^{\mathfrak{a}}$ gives rise to a coboundary Lie bialgebra structure on $\mathfrak{a}$. Following Drinfel'd such Lie bialgebra structures are called *quasi-triangular*, and quasi-triangular Lie bialgebra structures arising from skew-symmetric classical r-matrices are called *triangular*.

Now we are able to formulate a second version of our main result.

Theorem 2 *Let $\mathfrak{a}$ be a finite-dimensional Lie algebra over $\mathbb{R}$. Then the following conditions are equivalent:*

(a) *$\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure.*

(b) *$\mathfrak{a}$ admits a non-trivial quasi-triangular Lie bialgebra structure.*

(c) *$\mathfrak{a}$ is non-abelian and not isomorphic to $\mathfrak{su}(2)$, to $\mathfrak{h}_1(\mathbb{R})$ or to $\mathfrak{s}_\Lambda(\mathbb{R})$ where $\mathrm{tr}(\Lambda) = 0$ and $\det(\Lambda) > 0$.*

As above we also obtain the following existence result (which, of course, is just a re-formulation of Corollary 1):

Corollary 2 *If $\mathfrak{a}$ is a finite-dimensional non-abelian Lie algebra over $\mathbb{R}$ with $\dim_{\mathbb{R}} \mathfrak{a} \neq 3$, then $\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure.*

Remark. There are only *two* (isomorphism classes of) three-dimensional Lie algebras which are not excluded in Theorem 2, namely the (trivial) one-dimensional central extension of the two-dimensional non-abelian Lie algebra and $\mathfrak{sl}_2(\mathbb{R})$ (cf. also the remark before Theorem 1). But it is an immediate consequence of [7, Theorem 3.2] that these Lie algebras admit non-trivial triangular Lie bialgebra structures (see also the proof of [5, Theorem 1]).

3 Proof of Theorem 2

Before we give a proof of Theorem 2 we recall two results obtained in [5] and [3], respectively. The first lemma follows from [5, Theorem 2 and the argument in the proof of Theorem 3].

Lemma 1 *Let $\mathfrak{a}$ be a finite-dimensional non-abelian Lie algebra over an arbitrary field $\mathbb{F}$ with non-zero center. If $\mathfrak{a}$ is not isomorphic to the three-dimensional Heisenberg algebra, then $\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure.*

The second lemma is an immediate consequence of the proofs of [3, Lemma 4.2 and Lemma 4.1].

Lemma 2 (de Smedt) *Let $\mathfrak{a}$ be a finite-dimensional solvable Lie algebra over $\mathbb{R}$ with trivial center. If $[\mathfrak{a}, \mathfrak{a}]$ is non-abelian or $\dim_{\mathbb{R}}[\mathfrak{a}, \mathfrak{a}] \geq 3$, then $\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure.*

We now prove Theorem 2. Since the implication (a)$\Longrightarrow$(b) is trivial, it is enough to show the implications (b)$\Longrightarrow$(c) and (c)$\Longrightarrow$(a).

(b)$\Longrightarrow$(c): Since it is clear that every coboundary Lie bialgebra structure on an abelian Lie algebra is trivial, it suffices to prove that $\mathfrak{su}(2)$, $\mathfrak{h}_1(\mathbb{R})$, and $\mathfrak{s}_\Lambda(\mathbb{R})$ (where $\mathrm{tr}(\Lambda) = 0$ and $\det(\Lambda) > 0$) do not admit any non-trivial quasi-triangular Lie bialgebra structure. For $\mathfrak{su}(2)$ and $\mathfrak{h}_1(\mathbb{R})$ this was already done in [5, Examples 1 and 2]. Let us now consider $\mathfrak{s} := \mathfrak{s}_\Lambda(\mathbb{R})$ where $\det(\Lambda) \neq 0$. Then a straightforward computation yields

$$(\mathfrak{s} \otimes \mathfrak{s})^{\mathfrak{s}} = \begin{cases} \mathbb{R}(f_1 \otimes f_2 - f_2 \otimes f_1) & \text{if } \mathrm{tr}(\Lambda) = 0 \\ 0 & \text{if } \mathrm{tr}(\Lambda) \neq 0 \end{cases}$$

Hence *every* 2-tensor with $\mathfrak{s}$-invariant symmetric part is already skew-symmetric, i.e., *every* quasi-triangular Lie bialgebra structure on $\mathfrak{s}$ is already triangular. It will thus follow directly from the argument below that $\mathfrak{s}$ does *not* admit *any* non-trivial quasi-triangular Lie bialgebra structure unless $\mathrm{tr}(\Lambda) \neq 0$ or $\mathrm{tr}(\Lambda) = 0$ and $\det(\Lambda) < 0$.

(c)$\Longrightarrow$(a): Suppose $\mathfrak{a}$ does not admit any non-trivial triangular Lie bialgebra structure. If $\mathfrak{a}$ is not solvable, then it was shown in [3, Lemmas 1 and 2] – by using the Levi decomposition theorem in conjunction with the classification of finite-dimensional simple Lie algebras over the complex numbers and the classification of three-dimensional simple Lie algebras over $\mathbb{R}$ – that $\mathfrak{a}$ is isomorphic to $\mathfrak{su}(2)$.

If $\mathfrak{a}$ is solvable, then by virtue of Lemmas 1 and 2, we can assume for the rest of the proof that the center $C(\mathfrak{a})$ of $\mathfrak{a}$ is zero and $[\mathfrak{a},\mathfrak{a}]$ is abelian of dimension at most 2.

Suppose that $\dim_{\mathbb{R}}[\mathfrak{a},\mathfrak{a}] = 1$. Since $C(\mathfrak{a}) = 0$, for any non-zero element $e \in [\mathfrak{a},\mathfrak{a}]$ there exists an element $a \in \mathfrak{a}$ such that $[a, e] \neq 0$. (Note that this means in particular that a and e are *linearly independent* over $\mathbb{R}$.) But because $\dim_{\mathbb{R}}[\mathfrak{a},\mathfrak{a}] = 1$, we have $[a, e] = \lambda e$ for some $0 \neq \lambda \in \mathbb{R}$. Set $r := a \otimes e - e \otimes a \in \mathrm{Im}(1 - \tau)$. Then it follows from [7, Theorem 3.2] that r is a solution of the CYBE and

$$\delta_r(a) = [a, a] \otimes e + a \otimes [a, e] - [a, e] \otimes a - e \otimes [a, a] = \lambda \cdot (a \otimes e - e \otimes a) \neq 0$$

implies that δ_r defines a non-trivial triangular Lie bialgebra structure on $\mathfrak{a}$.

Hence we can assume from now on that $C(\mathfrak{a}) = 0$ and that $[\mathfrak{a},\mathfrak{a}]$ is two-dimensional abelian. Since $\dim_{\mathbb{R}}[\mathfrak{a},\mathfrak{a}] = 2$, there exist $f_1, f_2 \in \mathfrak{a}$ such that

$$[\mathfrak{a},\mathfrak{a}] = \mathbb{R}f_1 \oplus \mathbb{R}f_2.$$

Next, we show that $\dim_{\mathbb{R}} \mathfrak{a}/[\mathfrak{a},\mathfrak{a}] = 1$. Suppose to the contrary that $\dim_{\mathbb{R}} \mathfrak{a}/[\mathfrak{a},\mathfrak{a}] \geq 2$. Because $C(\mathfrak{a}) = 0$, there is an element $a \in \mathfrak{a}$ such that $[a, f_1] \neq 0$. In particular, $a \notin \mathbb{R}f_1 \oplus \mathbb{R}f_2$, i.e., a, f_1, and f_2 are linearly independent over $\mathbb{R}$. It follows from $\dim_{\mathbb{R}} \mathfrak{a}/[\mathfrak{a},\mathfrak{a}] \geq 2$ that there also is an element $a' \in \mathfrak{a}$ such that a, a', f_1, and f_2 are linearly independent over $\mathbb{R}$. Moreover, for every $1 \leq i, j \leq 2$, there exist elements $\alpha_{ij}, \alpha'_{ij} \in \mathbb{R}$ such that

$$\begin{aligned} [a, f_1] &= \alpha_{11} f_1 + \alpha_{12} f_2, & [a, f_2] &= \alpha_{21} f_1 + \alpha_{22} f_2, \\ [a', f_1] &= \alpha'_{11} f_1 + \alpha'_{12} f_2, & [a', f_2] &= \alpha'_{21} f_1 + \alpha'_{22} f_2. \end{aligned}$$

If $\alpha_{12} = 0$, we have $[a, f_1] = \alpha_{11} f_1 \neq 0$, and we can argue as above (for the case $\dim_{\mathbb{R}}[\mathfrak{a},\mathfrak{a}] = 1$) that $\mathfrak{a}$ admits a non-trivial triangular Lie bialgebra structure. On the other hand, if $\alpha_{12} \neq 0$, we may set $h := \alpha'_{12} a - \alpha_{12} a'$ and $\lambda := \alpha'_{12}\alpha_{11} - \alpha_{12}\alpha'_{11}$. Then we obtain that $h \notin [\mathfrak{a},\mathfrak{a}]$ and that $[h, f_1] = \lambda f_1$. If we now put $r := h \otimes f_1 - f_1 \otimes h \in \mathrm{Im}(1 - \tau)$, we see as before that r is a solution of the CYBE. Since $h \notin [\mathfrak{a},\mathfrak{a}]$, and moreover, we may assume that f_1 and $[a, f_1]$ are linearly independent, we conclude that

$$\delta_r(a) = [a, h] \otimes f_1 + h \otimes [a, f_1] - [a, f_1] \otimes h - f_1 \otimes [a, h] \neq 0.$$

Hence δ_r defines a non-trivial triangular Lie bialgebra structure on $\mathfrak{a}$.

Finally, we can assume that $\mathfrak{a}$ is three-dimensional and $[\mathfrak{a}, \mathfrak{a}]$ is two-dimensional abelian. It then follows that $\mathfrak{a} \cong \mathfrak{s}_\Lambda(\mathbb{R})$ where $\det(\Lambda) \neq 0$. (The latter condition holds because $\dim_{\mathbb{R}}[\mathfrak{a}, \mathfrak{a}] = 2$.) If we identify skew-symmetric tensors with the corresponding elements in the appropriate exterior power, we obtain for an arbitrary skew-symmetric 2-tensor

$$r = \omega f_1 \wedge f_2 + \xi_1 h \wedge f_1 + \xi_2 h \wedge f_2$$

with $\omega, \xi_1, \xi_2 \in \mathbb{R}$ that

$$\mathrm{CYB}(r) = [\lambda_{12}\xi_1^2 - (\lambda_{11} - \lambda_{22})\xi_1\xi_2 - \lambda_{21}\xi_2^2] \cdot h \wedge f_1 \wedge f_2.$$

If $\mathrm{tr}(\Lambda) \neq 0$, then – as already established in the proof of the implication (b)$\Longrightarrow$(c) – there is *no* non-zero $\mathfrak{s}_\Lambda(\mathbb{R})$-invariant 2-tensor. Consequently, $r := f_1 \wedge f_2$ defines a non-trivial triangular Lie bialgebra structure on $\mathfrak{s}_\Lambda(\mathbb{R})$. On the other hand, if $\mathrm{tr}(\Lambda) = 0$, then the discriminant of the relevant homogeneous quadratic equation

$$\lambda_{12}X_1^2 - (\lambda_{11} - \lambda_{22})X_1X_2 - \lambda_{21}X_2^2 = 0$$

is the *negative* of $\det(\Lambda)$. Hence $\mathfrak{s}_\Lambda(\mathbb{R})$ admits a non-trivial triangular Lie bialgebra structure if and only if $\det(\Lambda) < 0$. This finishes the proof of Theorem 2.

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