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Abstract

We give an elementary proof for the fact that for all $\ell, s \in \mathbb{N}$ there exists $d = d(\ell, s)$ such that every graph of average degree at least d either contains $K_{s,s}$ as a subgraph or has an induced cycle of length at least ℓ.

1 Introduction

The following result, due to Kierstead and Penrice [3], is an analogue of the conjecture of Gyárfás and Sumner that given a tree T and $s \in \mathbb{N}$, every K_s-free graph of sufficiently large chromatic number contains an induced copy of T.

Theorem 1. *For every tree T and every $s \in \mathbb{N}$ there exists $d = d(T, s)$ such that every graph of average degree at least d contains either $K_{s,s}$ as a subgraph or an induced copy of T.*

Clearly, we cannot hope for an analogous result if we replace T by a cycle of fixed length ℓ; this follows (for example) from the fact that constant average degree is not sufficient to force the existence of a cycle of given length as a subgraph. However, we can require an induced cycle of length at least ℓ:

Theorem 2. *For all $\ell, s \in \mathbb{N}$ there exists $d = d(\ell, s)$ such that every graph G of average degree at least d contains either $K_{s,s}$ as a subgraph or an induced cycle of length at least ℓ.*

Theorem 2 is a special case of the following far more general result, which we prove in [4]: for every graph H and for every $s \in \mathbb{N}$ there exists $d = d(H, s)$ such that every graph of average degree at least d contains either $K_{s,s}$ as a subgraph or an induced subdivision of H. However, the case when H is a cycle admits a much simpler elementary proof. The main purpose of this note is to present this proof (Section 2).

The main tool used in our proof of Theorem 2 is the case of Theorem 1 where T is a path. In Section 3 we offer a proof of the entire Theorem 1 which is different from the one in [3].

We write $d(G) := 2|E(G)|/|V(G)|$ for the *average degree* of a graph G. We say that G is *H-free* if it does not contain H as a subgraph.

2 Induced cycles

For our proof of Theorem 2 we shall need the case of Theorem 1 when T is a path. This special case immediately follows from a theorem of Galvin, Rival and Sands [2, Thm. 4] that every sufficiently large $K_{s,s}$-free graph which has a Hamiltonian path contains a long induced path. The proof of the latter result—an elegant application of Ramsey's theorem—is much simpler than that of the general case of Theorem 1. The result itself is also used as a tool in [3] to prove Theorem 1.

Proof of Theorem 2. The Erdős-Stone theorem (see e.g. [1, Thm. 7.1.2]) implies that for all $\ell, s \in \mathbb{N}$ there is a $t_0(\ell, s)$ so that every graph with $t \geq t_0$ vertices and at least $t^2/(4\ell^2)$ edges contains a $K_{s,s}$. We will prove Theorem 2 for $d(\ell, s) := 4 \cdot \lceil \max\{d(P_{s\ell^2}, s), 4s^{s+1}\ell^{2s}, 4t_0(\ell, s)\}\rceil$, where $d(P_{s\ell^2}, s)$ is as defined in Theorem 1. By a theorem of Mader (see e.g. [1, Thm. 1.4.2]), every graph G has a $\lfloor d(G)/4\rfloor$-connected subgraph. So we may assume that we are given a k-connected graph G where $k \geq d(\ell, s)/4$. Thus in particular $d(G) \geq k$, and hence by Theorem 1 we may assume that G contains an induced path $P = x \dots y$ of length $s\ell^2$. Menger's theorem now implies that G contains at least k internally disjoint paths joining x to y. Let $\mathcal{Q}$ be the set consisting of all the paths not meeting $\mathring{P}$, where $\mathring{P}$ denotes the interior of P. So $|\mathcal{Q}| \geq k - s\ell^2 \geq k/2$. By short-cutting the paths in $\mathcal{Q}$ if necessary, we may assume that they are induced. Let $\mathcal{Q}'$ be the set consisting of all paths in $\mathcal{Q}$ of length at most ℓ.

Case 1. $|\mathcal{Q}'| \geq k/4$.

Suppose first that there exists a $Q \in \mathcal{Q}'$ such that G has less than $s\ell$ edges joining $\mathring{Q}$ to $\mathring{P}$. In this case, P has a segment S of length at least ℓ such that no edge in G joins $\mathring{S}$ to $\mathring{Q}$. Let S be a maximal such segment. It is easy to show that then G contains an induced cycle consisting of S together with two suitable edges e_1, e_2 joining the endpoints of S to $\mathring{Q}$ and the segment of Q between e_1 and e_2. Since this cycle has length at least ℓ, we may assume that for every $Q \in \mathcal{Q}'$ there are at least $s\ell$ edges joining $\mathring{Q}$ to $\mathring{P}$. Thus in the interior of every $Q \in \mathcal{Q}'$ there must be a vertex x_Q which has at least s neighbours on $\mathring{P}$. Since the number of s-element subsets of $V(\mathring{P})$ is at most

$$\binom{s\ell^2}{s} \leq (s\ell^2)^s \leq \frac{k}{4s} \leq \frac{|\mathcal{Q}'|}{s},$$

there are s paths in $\mathcal{Q}'$, $Q_1, \dots, Q_s$ say, and a set X of s vertices in $\mathring{P}$ such that each x_{Q_i} is joined to all of X. Thus G contains a $K_{s,s}$.

Case 2. $|\mathcal{Q} \setminus \mathcal{Q}'| \geq k/4$.

Let $\mathcal{Q}'' := \mathcal{Q} \setminus \mathcal{Q}'$. For every $Q \in \mathcal{Q}''$ let Q' denote the segment of $\mathring{Q}$ (directed from x to y) consisting of the first ℓ vertices. If there are distinct paths $Q_1, Q_2 \in \mathcal{Q}''$ such that no edge of G joins Q'_1 to Q'_2 then one can easily show that G has an induced cycle containing Q'_1 or Q'_2 (or both). Since the length of this cycle must be at least ℓ, we may assume that for every distinct $Q_1, Q_2 \in \mathcal{Q}''$ there is an edge joining Q'_1 to Q'_2. Consider $G' := G[\bigcup_{Q \in \mathcal{Q}''} V(Q')]$. Then

$$|G'| = \ell |\mathcal{Q}''| \geq k/4$$

and

$$e(G') \geq \binom{|\mathcal{Q}''|}{2} \geq \frac{|\mathcal{Q}''|^2}{4} = \frac{|G'|^2}{4\ell^2},$$

and thus G' contains a $K_{s,s}$ by the Erdős-Stone theorem and the choice of k.

$\square$

3 Induced trees

If T is a rooted tree and $i \in \mathbb{N}$ then the *ith level* of T is the set of all vertices of T whose distance to the root of T is exactly i. Given vertices $v, w \in T$ we say that w lies *above* v and v lies *below* w if v lies on the path in T which joins w to the root of T. A vertex w lying above v is said to be a *child* of v if it is adjacent to v in T. We denote by $\mathrm{br}(v)$ the subtree of T which is induced by all vertices above v and whose root is v. Two vertices of T are *incomparable* if none of them lies above the other. We denote by T^b_a the rooted tree in which every vertex which is not a leaf has exactly a children and in which every leaf has distance b from the root. Given a graph G and a rooted subtree T of G, we say that T is *pseudo-induced* in G if the endvertices of every edge in $E(G[V(T)]) \setminus E(T)$ are incomparable in T.

Proof of Theorem 1. Clearly, it suffices to prove the theorem for all trees of the form T^b_a. So let G be a $K_{s,s}$-free graph of average degree d where d is sufficiently large compared to a, b and s. It is easy to see that G contains every tree T with $|T| \leq d/2$ as a (not necessarily induced) subgraph. Thus the following assertion implies that G also contains large pseudo-induced trees:

> *For all $i, j \in \mathbb{N}$ there exist $h = h(i,j)$ and $r' = r'(i,j)$ so that for any $T^h_{r'} \subseteq G$, $G[V(T^h_{r'})]$ contains a pseudo-induced T^j_i whose leaves are leaves of the $T^h_{r'}$ and in which a vertex w lies above v if and only if w lies above v in the $T^h_{r'}$ (for all $v, w \in V(T^j_i)$).* $(*)$

The proof of $(*)$ proceeds by induction on j. The case $j = 1$ is trivial since every star is pseudo-induced. So suppose that $j > 1$ and that $(*)$ holds for $j - 1$. Let $h^* := h(i, j-1)$ and $r^* := r'(i, j-1)$; let r' and h be sufficiently large compared to r^*, h^* and s, and set r' to be much larger than h. Suppose that T is a copy of $T^h_{r'}$ in G.

It is easily seen that T contains a copy T' of $T^h_{\lceil r'/2 \rceil}$ such that in G either all or none of the leaves of T' are adjacent to the root x of T. Indeed, colour a leaf of T blue if it is adjacent to x and green otherwise. Now keep moving towards the root of T, colouring a vertex blue if at least half of its children are blue and green otherwise, until all the vertices of T are coloured. T must contain a monochromatic copy T' of $T^h_{\lceil r'/2 \rceil}$, which then has the desired property. By repeating this argument, it follows that T contains a copy U of T^h_r such that every vertex $u \in U$ is adjacent in G to either all or none of the leaves above u in U and where $r \geq r'/2^h$.

Let us now assign to each leaf v of U a list of all those integers ℓ for which v is joined in G to the unique vertex below v that lies in the ℓth level of U. These lists never contain more than $s-1$ entries. Indeed, suppose that the list of v has (at least) s entries, $\ell_1 > \cdots > \ell_s$ say. Then for $k = 1, \ldots, s$ the unique vertex v_k lying below v in the ℓ_kth level is joined to v and so, by the choice of U, v_k must be joined to all leaves of U lying above it. In particular, v is a child of v_1 and each v_k is joined to all the $r \geq s$ children of v_1. Thus G contains a $K_{s,s}$, contradicting our assumption.

So the above assignment of lists to the leaves of U can be viewed as a colouring of these leaves with at most h^s colours. Since U has exactly r^h leaves, there must be a set R of at least r^h/h^s leaves which all have the same colour, L say. As L has less than s entries, there must be an entry $p \geq h/s$ of L such that $q \notin L$ for all q with $p - h/s < q < p$; and so if h is sufficiently large, this is true for all q with $p - h^* \leq q < p$. Colour a vertex u in the pth level of U red if u is adjacent in G to at least r^* leaves in R lying above u. Moving towards the root of U, colour a vertex red if at least r^* of its children are red and let R_ℓ be the set of red vertices on the ℓth level ($\ell \leq p$). Since every vertex in the pth level lies below r^{h-p} leaves of U, and since there are r^p vertices in the pth level of U, it follows that

$$|R_p|r^{h-p} + r^p r^* \geq |R| \geq \frac{r^h}{h^s}.$$

This in turn implies that $|R_p| \geq \frac{r^p}{2h^s}$, provided that r is sufficiently large. Similarly, for $q < p$ we have

$$|R_q|r + r^q r^* \geq |R_{q+1}|.$$

If r is sufficiently large, it follows inductively that $|R_q| \geq r^q/(2^{p-q+1}h^s)$ for all $q \leq p$. In particular, we have $|R_{p-h^*}| \geq 1$. Now pick any $u \in R_{p-h^*}$ and apply the induction hypothesis to a red copy of $T^{h^*}_{r^*}$ in $\mathrm{br}(u) \subseteq U$ to find a pseudo-induced T^{j-1}_i as in $(*)$. Since every leaf v of this T^{j-1}_i is a red vertex on the pth level of U, it follows that v is joined in G to $r^* \geq i$ red leaves above v. Let U' denote the extension of the T^{j-1}_i to the copy of T^j_i thus obtained. Recall that for each $\ell = p - h^*, \ldots, p-1$ no leaf $w \in R$ of U is joined in G to a vertex lying below w in the ℓth level of U. Thus U' must be as required in $(*)$, which completes the induction step.

We have shown that if $d = d(G)$ is sufficiently large, then G contains a pseudo-induced copy of $T^b_{a'}$. Thus the theorem will follow if we can show that for every $a, b \in \mathbb{N}$ there exists an a' such that any pseudo-induced copy of $T^b_{a'}$ in G contains a copy of T^b_a that is induced in G. But as already observed by Kierstead and Penrice [3], this can easily be verified by applying Ramsey's theorem and induction on b. Indeed, the case $b = 1$ follows immediately by applying Ramsey's theorem to the subgraph of G spanned by the leaves of a sufficiently large star. Now suppose that $b > 1$ and that the induction hypothesis holds for $b - 1$. Let $a' \in \mathbb{N}$ be sufficiently large and let $v_1, \dots, v_{a'}$ be an enumeration of the children of the root of $T^b_{a'} \subseteq G$. The induction hypothesis implies that $\mathrm{br}(v_i)$ contains a copy U_i of T^{b-1}_a that is induced in G ($i = 1, \dots, a'$). Label the vertices in each U_i, and for $i < j$ let B_{ij} denote the labelled ordered bipartite graph whose first vertex class is $V(U_i)$, whose second vertex class is $V(U_j)$ and whose edges are the U_i–U_j edges in G. Let us now consider the complete graph $K_{a'}$ in which the edge ij is coloured with the graph B_{ij}. So the number of colours only depends on a and b but not on a'. Thus if a' is sufficiently large, then by Ramsey's theorem there exists a monochromatic K_t where $t := \max\{a, 2s\}$. We may assume that the vertices of this K_t are $1, \dots, t$. Note that G cannot contain a U_i–U_j edge for any $1 \leq i < j \leq t$. For if there exists an edge joining the ℓth vertex of U_i to the kth vertex of U_j, then G contains a $K_{s,s}$ whose vertex classes are formed by the ℓth vertices of $U_1, \dots, U_s$ and the kth vertices of $U_{s+1}, \dots, U_{2s}$. Thus the copy of T^b_a formed by the root of the $T^b_{a'}$ and $U_1, \dots, U_a$ must be induced in G. $\square$

References

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