HAMBURGER BEITRÄGE
ZUR MATHEMATIK

Heft 114

On Immersions of Uncountable Graphs

Thomas Andreae, Hamburg

März 2001

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Abstract

In his paper on well–quasi–ordering infinite trees [Proc. Cambridge Philos. Soc. 61 (1965) 697–720], Nash–Williams proposed the conjecture that the class of all graphs is well–quasi–ordered by the immersion relation (which is denoted here by $\leq_1$). In addition, in a subsequent paper, Nash–Williams discussed a weaker version of his original conjecture to the effect that the class of graphs is well–quasi–ordered with respect to a relation $\leq_2$ which, roughly speaking, is obtained by redefining $H \leq_1 G$ so that distinct vertices of H can be mapped into the same vertex of G. It is the purpose of the present note to disprove these conjectures by showing that, for either version of the immersion relation, infinite antichains of uncountable graphs exist. The question whether or not the conjectures hold for countable graphs remains open.

Keywords: immersion relation, infinite graphs, well–quasi–ordering, antichains

1 Introduction

For graph theoretic terminology, we refer to the definitions and notational conventions collected at the end of the introduction or to the textbook of Diestel [1]. The graphs considered in this note do not contain loops or multiple edges. For graphs G, H, we write $H \leq G$ if G contains a subgraph which is isomorphic to a subdivision of H. A classical result in infinite graph theory states that the class of trees is well–quasi–ordered by the *subdivision relation* $\leq$:

Theorem A (Nash–Williams [2]). If $T_1, T_2, \ldots$ is an infinite sequence of trees, then there exist positive integers i, j such that $i < j$ and $T_i \leq T_j$.

The following well–known conjecture of E. Vázsonyi apparently dates back to the nineteen–thirties. For finite graphs, the truth of the conjecture is an immediate consequence of the graph minor theorem of Robertson and Seymour [4]; cf. also [1]. In the general case, however, the conjecture remains far from proved to this day, although partial results on the infinite case can be obtained as consequences of results of Thomas [6].

Conjecture (Vázsonyi). If $G_1, G_2, \ldots$ is an infinite sequence of graphs in which every vertex has degree ≤ 3, then there exist positive integers i, j such that $i < j$ and $G_i \leq G_j$.

Theorem A and Vázsonyi's conjecture deal with restricted classes of graphs, namely, trees and graphs with maximum degree at most 3, respectively. This prompted Nash–Williams [2] to suggest another conjecture for the class of all graphs which, if true, would have both Theorem A and Vázsonyi's conjecture as corollaries. For a graph G, let $\mathcal{P}(G)$ denote the set of nontrivial paths of G. For graphs G, H, an *immersion* $\varphi : H \longrightarrow G$ is a mapping $\varphi : V(H) \cup E(H) \longrightarrow V(G) \cup \mathcal{P}(G)$ such that

(i) if $v \in V(H)$, then $\varphi(v) \in V(G)$

(ii) if $v, v' \in V(H),\ v \neq v'$, then $\varphi(v) \neq \varphi(v')$

(iii) if $e = vv' \in E(H)$, then $\varphi(e) \in \mathcal{P}(G)$ and path $\varphi(e)$ connects $\varphi(v)$ with $\varphi(v')$

(iv) if $e, e' \in E(H)$ are distinct, then $\varphi(e)$ and $\varphi(e')$ are edge–disjoint

(v) if $e = vv' \in E(H)$ and $v'' \in V(H)$, then $\varphi(v'') \notin V(\varphi(e)) \backslash \{\varphi(v), \varphi(v')\}$.

Writing $H \leq_1 G$ to indicate that there exists an immersion $\varphi : H \longrightarrow G$, Nash–Williams' conjecture reads as follows.

Conjecture A (Nash–Williams [2]). If $G_1, G_2, \ldots$ is an infinite sequence of graphs, then there exist positive integers i, j such that $i < j$ and $G_i \leq_1 G_j$.

In his paper [3], Nash–Williams subsequently presented a weaker version of Conjecture A which is still strong enough to imply Theorem A and Vázsonyi's conjecture. Let $\mathcal{C}(G)$ denote the set of cycles of a graph G. For graphs G, H, an *immersion* $\varphi : H \longrightarrow G$ *in the weak sense* is a mapping $\varphi : V(H) \cup E(H) \longrightarrow V(G) \cup \mathcal{P}(G) \cup \mathcal{C}(G)$ such that (i), (ii′), (iii′), (iv), (v) hold, where (i), (iv) and (v) are as above and the statements (ii′), (iii′) are as follows.

(ii′) if $e = vv' \in E(H)$ and $\varphi(v) \neq \varphi(v')$, then $\varphi(e) \in \mathcal{P}(G)$ and path $\varphi(e)$ connects $\varphi(v)$ with $\varphi(v')$

(iii′) if $e = vv' \in E(H)$ and $\varphi(v) = \varphi(v')$, then $\varphi(e) \in \mathcal{C}(G)$ and $\varphi(v)$ is a vertex on cycle $\varphi(e)$.

Writing $H \leq_2 G$ if there exists an immersion $\varphi : H \longrightarrow G$ in the weak sense, the modified version of Conjecture A reads as follows.

Conjecture B (Nash–Williams [3]). If $G_1, G_2, \ldots$ is an infinite sequence of graphs, then there exist positive integers i, j such that $i < j$ and $G_i \leq_2 G_j$.

In [4], Robertson and Seymour announced a proof of Conjecture A for the case of finite graphs $G_1, G_2, \ldots$ (which, of course, implies the truth of Conjecture B for finite graphs). It is the purpose of the present note to show that, in the general case of arbitrary graphs, counterexamples to Conjectures A and B exist.

The paper is organized as follows. In Section 2, we establish a result which, to some extent, clarifies the relationship between the immersion relations $\leq_1$ and $\leq_2$: we show that for each family of graphs $G_i (i \in I)$ there exists a family of graphs $H_i (i \in I)$ such that, for all $i, j \in I$, $G_i \leq_1 G_j$ if and only if $H_i \leq_2 H_j$ (Theorem 1). In particular this means that, in order to disprove Conjecture B, it is enough to find a counterexample to Conjecture A. In Section 3, we modify some of the ideas of Thomas [5] in order to make them go with the immersion relation $\leq_1$. This, when taken together with Theorem 1 of the present note, results into Theorem 2 stating that there exists a sequence $H_1, H_2, \ldots$ of uncountable graphs such that $H_i \not\leq_2 H_j$ for all i, j with $i < j$. Theorem 3 is a sharpened version of Theorem 2 showing that the sequence $H_1, H_2, \ldots$ can be modified to obtain an antichain $H'_1, H'_2, \ldots$, that is, $H'_i \not\leq_2 H'_j$ holds for all i, j with $i \neq j$.

Our terminology is standard. For graph theoretic terms not defined here, we refer to Diestel [1]. Vertex set and edge set of a graph G are denoted $V(G)$ and $E(G)$, respectively. An edge joining vertices v, w is denoted vw. For $v \in V(G)$, the degree of v in G is denoted by $d_G(v)$. An edge is said to be *pendant* if one of its ends has degree one. A family of graphs $G_i (i \in I)$ is *edge–disjoint* if $E(G_i) \cap E(G_j) = \emptyset$ for all $i, j \in I$ with $i \neq j$. A *path* is a graph P consisting of $n+1$ distinct vertices $a_0, \ldots, a_n (n \geq 0)$ and edges $a_i a_{i+1} (i = 0, \ldots, n-1)$. For paths, we use notations like $P = (a_0, \ldots, a_n)$. A path is *trivial* if it consists of just one vertex. For a path $P = (a_0, \ldots, a_n)$, the vertices a_0, a_n are called *endvertices* of P. A path P is an *x,y–path* (or likewise P is said to *connect x with y*) if x and y are its endvertices. In order to stress the difference of the above defined notions of an 'immersion' and an 'immersion in the weak sense', the former will also be referred to as an *immersion in the strong sense.*

A binary relation $\preceq$ on a set Q is a *quasi–order* if it is reflexive and transitive. We say that Q is *well–quasi–ordered* by $\preceq$ if the relation $\preceq$ is a quasi–order on Q and if for each infinite sequence $q_1, q_2, \ldots$ of elements of Q there exist positive integers i, j such that $i < j$ and $q_i \preceq q_j$. Given a quasi–order $\preceq$ on Q, a family $q_i (i \in I)$ of elements of Q is called an *antichain* (with respect to $\preceq$) if $q_i \not\preceq q_j$ for all $i, j \in I$ with $i \neq j$.

The set $\{0, 1, \ldots\}$ of natural numbers is denoted ω. The symbol $\mathfrak{c}$ denotes the cardinality of the continuum. Cardinals are identified with initial ordinals.

2 A result linking the two versions of the immersion relation

In this section, we establish a result (Theorem 1) on the relations $\leq_1$ and $\leq_2$ which in particular shows that, in order to disprove Conjecture B, it is enough to find a counterexample to Conjecture A. Most likely, Theorem 1 can also be applied in other situations to reduce a problem for $\leq_2$ to the analogous problem for $\leq_1$.

Theorem 1. For each family of graphs $G_i (i \in I)$ there exists a family of graphs $H_i (i \in I)$ such that, for all $i, j \in I$, $G_i \leq_1 G_j$ if and only if $H_i \leq_2 H_j$. In particular, $G_i (i \in I)$ is an antichain with respect to $\leq_1$ if and only if $H_i (i \in I)$ is an antichain with respect to $\leq_2$.

Proof. Let $G_i (i \in I)$ be a family of graphs. We may assume that $G_i \cap G_j = \emptyset$ for all $i, j \in I,\ i \neq j$. Let $\mu, \alpha, \beta, \gamma$ be cardinals such that

(1) $\omega \leq \mu < \alpha < \beta < \gamma$ and $d_{G_i}(v) < \mu$ for all $i \in I$ and $v \in V(G_i)$.

For the purpose of defining the graphs H_i, we introduce a graph F as follows. Let $V(F)$ be the union of disjoint sets $X_0, X_1, \ldots, X_5$ where X_0 consists of three elements a, b, c and $X_1, \ldots, X_5$ are infinite sets with $|X_1| = \mu,\ |X_2| = |X_3| = \alpha,\ |X_4| = \beta,\ |X_5| = \gamma$. Let further $\xi : X_2 \longrightarrow X_3$ be a bijection and define the edge set of F by

$$\begin{aligned} E(F) = \quad & \{ax : x \in X_1 \cup X_2 \cup X_3\} \cup \{bx : x \in X_1 \cup X_4 \cup \{c\}\} \\ & \cup \{cx : x \in X_5\} \cup \{xx' : x \in X_2,\ x' = \xi(x)\}. \end{aligned}$$

Observe that in F there are

– μ edge–disjoint paths of length two connecting a with b
– α edge–disjoint triangles attached at a
– β pendant edges attached at b
– γ pendant edges attached at c.

Let $V = \bigcup V(G_i)$, where the union is taken over all $i \in I$. For each $v \in V$, let F_v be a copy of F such that $F_v \cap F_w = \emptyset$ whenever $v \neq w$ and such that $F_v \cap G_j = \emptyset$ for all $v \in V$ and $j \in I$. For each $v \in V$, let $f_v : F \longrightarrow F_v$ be an isomorphism and put $a_v := f_v(a),\ b_v := f_v(b),\ c_v := f_v(c)$. For each $i \in I$, let H_i be the graph that results from G_i and the graphs $F_v (v \in V(G_i))$ by identifying each vertex $v \in G_i$ with vertex $a_v \in F_v$.

Since the so–defined graphs H_i are disjoint, we may write $d(x)$ rather than $d_{H_i}(x)$ to denote the degree of a vertex x in H_i. For each $i \in I$, we put $A_i := V(G_i) (= \{a_v : v \in V(G_i)\}),\ B_i := \{b_v : v \in V(G_i)\}$, and $C_i := \{c_v : v \in V(G_i)\}$. Further, we write $x \sim y$ to indicate that x and y are vertices of the same graph F_v.

Clearly, if $G_i \leq_1 G_j$ then $H_i \leq_1 H_j$ and thus $H_i \leq_2 H_j$. For the proof of the converse, let $i, j \in I$ and assume that $h : H_i \longrightarrow H_j$ is an immersion in the weak sense. Let $g : G_i \longrightarrow H_j$ denote the restriction of h to G_i. Clearly, Theorem 1 is proved if we show that

(2) g is an immersion $G_i \to G_j$ in the strong sense.

For the proof of (2), first observe that

(3) $h(c) \in C_j$ for all $c \in C_i$.

Indeed, as a consequence of the construction of the graphs H_i and H_j, together with the assumption (1), one obtains $d(c) = \gamma > d(v)$ for all $c \in C_i$ and all $v \in V(H_j)\backslash C_j$. Hence (3). For similar reasons, we have $h(b) \in B_j \cup C_j$ for all $b \in B_i$. However, for $b \in B_i, h(b) \in C_j$ is impossible since there is no cycle of H_j passing through a vertex of C_j, while each vertex of B_i is on a cycle of H_i. Hence

(4) $h(b) \in B_j$ for all $b \in B_i$.

Now, let $a \in A_i$. Note that there are α edge–disjoint cycles passing through a and, consequently, the same must be true for $h(a)$. From this, together with the fact that $\alpha > \mu$, one concludes that $h(a) \in A_j$, and thus we have found that

(5) $h(a) \in A_j$ for all $a \in A_i$.

We next show that the following holds.

(6) Let $a \in A_i,\ c \in C_i$ with $a \sim c$. Then $h(a) \sim h(c)$.

For the proof, consider $b \in B_i$ with $a \sim b \sim c$. We first show $h(a) \sim h(b)$. Suppose that $h(a) \sim h(b)$ does not hold. Note that there exists a family $(P_\nu)_{\nu<\mu}$ of μ edge–disjoint paths of H_j connecting $h(a)$ with $h(b)$ since there are μ such paths of H_i connecting a with b. For each $\nu < \mu$, let e_ν be the edge of P_ν which is incident with $h(a)$. From the supposition that $h(a) \sim h(b)$ does not hold, together with (5), it follows that all edges e_ν are in G_j and thus we have $d_{G_j}(h(a)) \geq \mu$, which contradicts (1). Hence $h(a) \sim h(b)$. Thus in order to obtain (6) it remains to show $h(b) \sim h(c)$. Suppose that this does not hold. Then one concludes from (4) and (5), together with $h(a) \sim h(b)$, that the path $h(bc)$ contains $h(a)$ as an inner vertex, which is impossible. Hence $h(b) \sim h(c)$.

We claim that

(7) $h(c) \neq h(c')$ for all $c, c' \in C_i,\ c \neq c'$.

Suppose $h(c) = h(c')$ for distinct $c, c' \in C_i$. By (3), we have $h(c) \in C_j$. Denote by D the component of $H_j - h(c)$ containing the vertices of B_j; let e be the unique edge connecting $h(c)$ with D. Let $b, b' \in B_i$ with $b \sim c,\ b' \sim c'$. By (4) we have $h(b),\ h(b') \in D$ and thus we conclude from $h(c) = h(c')$ that

e is an edge of both $h(cb)$ and $h(c'b')$, which is impossible. Hence (7).

As a consequence of (3), (5), (6), and (7) one obtains

(8) $h(a) \neq h(a')$ for all $a, a' \in A_i,\ a \neq a'$.

By (8), g is an immersion $G_i \to H_j$ in the strong sense. Further, by (5), g maps the vertices of G_i onto vertices of G_j. Moreover, note that it follows from the construction of H_j that a path of H_j is completely contained in G_j if its endvertices are in G_j. Hence (2). □

3 Counterexamples to Conjectures A and B

In this section, based on ideas of [5], we construct a counterexample to Conjecture A. The existence of a counterexample to Conjecture B then follows from Theorem 1.

For an ordinal α, 2^α denotes the set of mappings $x : \alpha \longrightarrow \{0,1\}$. We put $2^{<\alpha} = \bigcup_{\beta<\alpha} 2^\beta$. If $x \in 2^\alpha$ and $\beta \leq \alpha$, then $x \restriction \beta \in 2^\beta$ denotes the restriction of x to β. For $x_0 \in 2^\omega$ and $n \in \omega$, we put

$$U_n(x_0) = \{x \in 2^\omega : x \restriction n = x_0 \restriction n\}\,.$$

The set 2^ω is considered as a topological space endowed with the product topology, i.e. the topology whose basic open sets are the $U_n(x)$.

Let $\kappa_0, \kappa_1, \ldots$ be cardinals with $\mathfrak{c} \leq \kappa_0 < \kappa_1 < \ldots$. For each $x \in 2^\omega$ and $n \in \omega$, let $A_{x,n}$ be a set of cardinality κ_n such that $A_{x,n} \cap 2^{<\omega+1} = \emptyset$ and $A_{x,n} \cap A_{y,m} = \emptyset$ whenever $(x,n) \neq (y,m)$. For each $X \subseteq 2^\omega$, we define a graph G_X by

$$V(G_X) := \{x \restriction \alpha : x \in X,\ \alpha \in \omega+1\} \cup \bigcup_{\substack{x\in X\\ n\in\omega}} A_{x,n}$$

$$E(G_X) := \bigcup_{\substack{x\in X\\ n\in\omega}} \{av : a \in A_{x,n},\ v \in \{x, x \restriction n\}\}\,.$$

For $X \subseteq 2^\omega$ and $v \in V(G_X)$, we denote by $d_X(v)$ the degree of v in G_X. As a consequence of the definitions, one obtains $d_X(v) = \sup\{\kappa_i : i \in \omega\}$ if $v \in X$, $d_X(v) = \kappa_n$ if $v = x \restriction n$ for $x \in X$ and $n \in \omega$, and $d_X(v) = 2$, otherwise. Hence, for $X, Y \subseteq 2^\omega$, we have

(9) $d_X(x) > d_Y(v)$ for all $x \in X$ and $v \in V(G_Y) \backslash Y$.

Lemma 1. For $X \subseteq 2^\omega$, let x, x_0 be distinct elements of X. Let further $n \in \omega$. Then there exists a system of κ_n edge–disjoint x, x_0–paths of G_X if and only if $x \in U_n(x_0)$.

Proof. Assume $x \in U_n(x_0)$. Then $x_0 \restriction n = x \restriction n$. Let $f : A_{x_0,n} \longrightarrow A_{x,n}$ be a bijection. For each $a \in A_{x_0,n}$, we define a path P_a of length four by putting $P_a = (x_0, a, x_0 \restriction n = x \restriction n,\ f(a), x)$. Then these paths form a system of κ_n edge–disjoint x, x_0–paths of G_X. For a proof of the 'only if' direction, assume that $x \notin U_n(x_0)$. Then $n \geq 1$ since $U_0(x_0) = 2^\omega$. Let P be an x, x_0–path of G_X. From the definition of G_X, together with the fact that $x \notin U_n(x_0)$, one readily obtains that P must contain a subpath $P' = (y, a, v, b, z)$ with $y \in X \backslash U_n(x_0),\ z \in U_n(x_0),\ v = y \restriction m = z \restriction m,\ a \in A_{y,m}, b \in A_{z,m}$ for some $m \in \omega$. Because $y \notin U_n(x_0)$ and $z \in U_n(x_0)$, we have $y \restriction n \neq z \restriction n$. Hence $m < n$ and thus we have proved that each x, x_0–path P of G_X must pass through a vertex v with $v = y \restriction m$ for some $y \in X$ and $m < n$. Note that there are only finitely many vertices of this kind and each such vertex has degree at most κ_{n-1}. Thus there cannot exist κ_n edge–disjoint x, x_0–paths of G_X. $\square$

Lemma 2. For $X, Y \subseteq 2^\omega$, let $g : G_X \longrightarrow G_Y$ be an immersion in the strong sense and denote by φ the restriction of g to X. Then φ is a continuous injective function $X \longrightarrow Y$.

Proof. By (9), we have $\varphi(X) \subseteq Y$. Clearly φ is injective and thus it remains to show that φ is continuous. For this purpose, let $x_0 \in X$. For some $n \in \omega$, consider the basic open neighborhood $U_n(\varphi(x_0))$. We have to show that there exists a natural number a_n such that

(10) $\varphi(U_{a_n}(x_0) \cap X) \subseteq U_n(\varphi(x_0))$.

We show that this is true for $a_n = n$. Let $x \in U_n(x_0) \cap X$. If $x = x_0$, then $\varphi(x) \in U_n(\varphi(x_0))$ clearly holds. Hence let $x \neq x_0$. By Lemma 1, x_0 and x are joined by κ_n edge–disjoint paths of G_X, and thus the same must hold for $\varphi(x_0)$ and $\varphi(x)$ in G_Y. Consequently, again by Lemma 1, $\varphi(x) \in U_n(\varphi(x_0))$. Hence (10). $\square$

By a result of Thomas [5, Theorem 2] there exists a sequence $X_1, X_2, \ldots$ of subsets of 2^ω (each of cardinality $\mathfrak{c}$) such that for all i, j with $i < j$ there is no continuous injective function $X_i \to X_j$. By combining this with Theorem 1 and Lemma 2 of the present paper, we obtain the following result.

Theorem 2. There exists a sequence $H_1, H_2, \dots$ of uncountable graphs such that $H_i \not\leq_2 H_j$ for all positive integers i, j with $i < j$.

Proof. Let $X_1, X_2, \dots$ be as in the paragraph before Theorem 2. Put $G_i := G_{X_i}$ $(i = 1, 2, \dots)$. Then, by Lemma 2, we have $G_i \not\leq_1 G_j$ for all i, j with $i < j$ and thus Theorem 2 follows by application of Theorem 1. (Note that the H_i constructed in the proof of Theorem 1 are uncountable graphs.) □

4 Antichains

The following is a sharpened version of Theorem 2.

Theorem 3. There exists a sequence $H'_1, H'_2, \dots$ of uncountable graphs such that $H'_i \not\leq_2 H'_j$ for all positive integers i, j with $i \neq j$.

Proof. Let $G_i (i = 1, 2, \dots)$ be as in the proof of Theorem 2. Then $G_i \not\leq_1 G_j$ for all i, j with $i < j$. Note that a graph G_X (as defined in Section 3) is infinite and connected if X is non–empty. Consequently, the G_i are infinite connected graphs. Let $\lambda_1 < \lambda_2 \dots$ be cardinals with $\lambda_1 > |V(G_i)|$ $(i = 1, 2, \dots)$. For $i = 1, 2, \dots$, let A_i be a set of vertices with $|A_i| = \lambda_i$ such that $V(G_i) \cap A_i = \emptyset$. Pick some $a_i \in A_i$ $(i = 1, 2, \dots)$. For $i = 1, 2, \dots$ define G'_i by

$$\begin{aligned} V(G'_i) &= V(G_i) \cup A_i \\ E(G'_i) &= E(G_i) \cup \{a_i v : \; v \in V(G'_i) \backslash \{a_i\}\}. \end{aligned}$$

We claim that

(11) $G'_i \not\leq_1 G'_j$ for all i, j with $i \neq j$.

For the proof, let $i \neq j$ and suppose that there exists an immersion $\varphi' : G'_i \longrightarrow G'_j$ (in the strong sense). Since the degree of a_i in G'_i is greater than the degrees of all $v \in V(G'_j) \backslash \{a_j\}$, we have $\varphi'(a_i) = a_j$. Hence $\lambda_i = d_{G'_i}(a_i) \leq d_{G'_j}(a_j) = \lambda_j$ and thus $i < j$. Let φ be the restriction of φ' to G_i. Since φ' is injective on $V(G'_i)$ and because G_i is an infinite connected graph, one obtains from $\varphi'(a_i) = a_j$ that φ is an immersion $G_i \longrightarrow G_j$ (in the strong sense). This contradicts $G_i \not\leq_1 G_j$ and thus (11) is proved. Theorem 3 then follows from (11) and Theorem 1, together with the fact that the graphs H_i of Theorem 1 are uncountable. □

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