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Partitions of graphs with high minimum degree or connectivity

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Abstract

We prove that there exists a function $f(\ell)$ such that the vertex set of every $f(\ell)$-connected graph G can be partitioned into sets S and T such that each vertex in S has at least ℓ neighbours in T and both $G[S]$ and $G[T]$ are ℓ-connected. This implies that there exists a function $g(\ell, H)$ such that every $g(\ell, H)$-connected graph contains a subdivision TH of H so that $G - V(TH)$ is ℓ-connected. We also prove an analogue with connectivity replaced by minimum degree. Furthermore, we show that there exists a function $h(\ell)$ such that the vertex set of every graph G of minimum degree at least $h(\ell)$ can be partitioned into sets S and T such that both $G[S]$ and $G[T]$ have minimum degree at least ℓ and the bipartite subgraph between S and T has average degree at least ℓ.

1 Introduction

It is well-known that the vertex set of every graph G of minimum degree at least $2\ell - 1$ can be partitioned into sets S and T such that the bipartite subgraph $(S,T)_G$ of G between S and T has minimum degree at least ℓ. Moreover, Hajnal [4] and Thomassen [10] proved that for every ℓ there exists $k = k(\ell)$ such that the vertex set of every graph of minimum degree at least k can be partitioned into sets S and T such that the graphs $G[S]$ and $G[T]$ induced by these sets both have minimum degree at least ℓ. They also proved an analogue where minimum degree is replaced by connectivity.

On the other hand, it is easily seen that we cannot simultaneously require that $G[S]$, $G[T]$ and $(S,T)_G$ have large minimum degree or connectivity if G has large minimum degree or connectivity (see Proposition 6). In Section 2 we show that we can nevertheless strengthen the results of Hajnal and Thomassen by requiring that each vertex in S has many neighbours in T:

Theorem 1 *For every $\ell \in \mathbb{N}$ there exists $k = k(\ell) \in \mathbb{N}$ such that the vertex set of every graph G of minimum degree at least k can be partitioned into non-empty sets S and T such that both $G[S]$ and $G[T]$ have minimum degree at least ℓ and every vertex in S has at least ℓ neighbours in T.*

Theorem 2 *For every $\ell \in \mathbb{N}$ there exists $k = k(\ell) \in \mathbb{N}$ such that the vertex set of every k-connected graph G can be partitioned into non-empty sets S and T such that both $G[S]$ and $G[T]$ are ℓ-connected and every vertex in S has at least ℓ neighbours in T.*

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Theorem 2 can be applied to show the existence of non-separating structures in highly connected graphs. A well-known result in this area is the theorem of Thomassen [9] which states that every $(\ell+3)$-connected graph G contains an induced cycle C such that $G - V(C)$ is ℓ-connected. In our case, if H is a structure whose existence is guaranteed by high connectivity, then Theorem 2 implies that in every highly connected graph G there exists a copy of H such that $G - V(H)$ is still highly connected. Indeed, if S, T is a partition as in Theorem 2, then any copy of H in $G[S]$ will do. For example, an application of the result of Mader that every graph of sufficiently large average degree contains a subdivision of a given graph H (see e.g. [3, Thm. 3.6.1]) immediately yields the following.

Corollary 3 *For every $\ell \in \mathbb{N}$ and every graph H there exists $k = k(\ell, H) \in \mathbb{N}$ such that every k-connected graph G contains a subdivision TH of H such that $G - V(TH)$ is ℓ-connected.*

Similarly, an 'induced analogue' of Corollary 3 follows from the result of [6] that for every $s \in \mathbb{N}$ and every graph H there exists $d = d(s, H)$ such that every $K_{s,s}$-free graph G with average degree at least d contains an induced subdivision of H:

Corollary 4 *For all $\ell, s \in \mathbb{N}$ and every graph H there exists $k = k(\ell, s, H) \in \mathbb{N}$ such that every k-connected $K_{s,s}$-free graph G contains an induced subdivision TH of H such that $G - V(TH)$ is ℓ-connected.*

Analogous results for graphs of large minimum degree can be deduced from Theorem 1.

Strengthening the theorem of Mader, Larman and Mani [7] showed that every sufficiently highly connected graph G contains a subdivision of a given graph H with prescribed branch vertices. Thus the following conjecture of Thomassen, which greatly strengthens Theorem 2, would even imply the existence of a non-separating such subdivision.

Conjecture (Thomassen [11]) *For every $\ell \in \mathbb{N}$ there exists $k = k(\ell) \in \mathbb{N}$ such that if G is a k-connected graph and $X \subseteq V(G)$ consists of at most ℓ vertices then the vertex set of G can be partitioned into non-empty sets S and T such that $X \subseteq S$, each vertex in S has at least ℓ neighbours in T and both $G[S]$ and $G[T]$ are ℓ-connected.*

If true, this would also imply a conjecture of Lovász (see [11]) that there exists a function $f(\ell)$ such that, for any two vertices x and y in an $f(\ell)$-connected graph G, there exists an induced x–y path P such that $G - V(P)$ is ℓ-connected. Kriesell [5] observed that Tutte's theorem on 3-connected graphs can be used to show that $f(1) = 3$ and proved that $f(2) = 5$.

Finally, for graphs of high minimum degree we can strengthen the result of Hajnal and Thomassen in a different direction. It turns out that instead of asking for high minimum degree on one side of the bipartite graph $(S, T)_G$ as in Theorem 1, one can require $(S, T)_G$ to have high average degree:

Theorem 5 *The vertex set of every graph G of minimum degree at least $2^{32}\ell$ can be partitioned into non-empty sets S and T such that both $G[S]$ and $G[T]$ have minimum degree at least ℓ, $(S,T)_G$ has average degree at least ℓ and $|S|, |T| \geq |G|/2^{18}$.*

In Section 4 we present some results about partitions of graphs of high average degree or chromatic number.

Let us now introduce some notation. Given a graph G, we write $e(G)$ for the number of its edges, $d(G) := 2e(G)/|G|$ for its average degree, $\delta(G)$ for its minimum degree and $\chi(G)$ for its chromatic number. If S and T are disjoint sets of vertices of G, then we denote by $G[S]$ the subgraph of G induced by S and write $(S,T)_G$ for the bipartite subgraph of G whose vertex classes are S and T and whose edges are all the S–T edges in G. We denote by $e_G(S,T)$ or, if this is unambiguous, by $e(S,T)$ the number of these edges and call $(S,T)_G$ the *bipartite subgraph of G between S and T*. We write $N(x)$ for the neighbourhood of a vertex $x \in G$ and $d_G(x)$ or $d(x)$ for the degree of x in G.

2 Proof of Theorems 1 and 2

Before proving Theorems 1 and 2, let us first observe that these results are best possible in the sense that one cannot additionally require that each vertex in T has many neighbours in S, i.e. that $(S,T)_G$ has large minimum degree.

Proposition 6 *For every $k \in \mathbb{N}$ there is a k-connected graph G whose vertex set cannot be partitioned into non-empty sets S and T such that each vertex of G has a neighbour in both S and T.*

Proof. Let G_k^n be the bipartite graph whose vertex classes are $X := \{1, \dots, n\}$ and the set $X^{(k)}$ consisting of all k-element subsets of X, and in which $x \in X$ is joined to $Y \in X^{(k)}$ if $x \in Y$. It is easy to see that G_k^n is k-connected if n is sufficiently large compared to k. Consider any partition of $V(G_k^n)$ into non-empty sets S and T. As we may assume that $n \geq 2k-1$, one of S, T contains at least k vertices from X. Suppose that this is true for S and let $Y \in X^{(k)}$ be a vertex of G_k^n corresponding to a set of k vertices in $X \cap S$. Then Y has no neighbour in T, which proves the proposition. □

In the proof of Theorems 1 and 2 we will apply the following quantitative versions (see [4, Remark 3 and Thm. 4.3]) of the results of Hajnal and Thomassen mentioned in Section 1.

Theorem 7 *The vertex set of every graph G of minimum degree at least 3ℓ can be partitioned into non-empty sets S and T such that both $G[S]$ and $G[T]$ have minimum degree at least ℓ.*

Theorem 8 *The vertex set of every 8ℓ-connected graph G can be partitioned into non-empty sets S and T such that both $G[S]$ and $G[T]$ are ℓ-connected.*

In the proof of Theorem 1 we will also use the following simple proposition (see e.g. [3, Prop. 1.2.2]) and a lemma which is an easy consequence of Theorem 7.

Proposition 9 *Every graph G contains a subgraph of minimum degree at least $d(G)/2$.*

Lemma 10 *The vertex set of every graph G of minimum degree at least 3ℓ can be partitioned into non-empty sets S and T such that both $G[S]$ and $G[T]$ have minimum degree at least ℓ and every subgraph of $G[S]$ has average degree less than 6ℓ.*

Proof. Theorem 7 implies that $V(G)$ can be partitioned into non-empty sets S and T such that both $\delta(G[S]) \geq \ell$ and $\delta(G[T]) \geq \ell$. Choose S and T satisfying these properties such that S is minimal. Then S, T are as desired. Indeed, if $G[S]$ had a subgraph of average degree at least 6ℓ, then by Proposition 9 it would also contain a subgraph H of minimum degree at least 3ℓ. Apply Theorem 7 to obtain a partition S', T' of $V(H)$. Let X be a maximal subset of $S \setminus T'$ such that $S' \subseteq X$ and $\delta(G[X]) \geq \ell$. Then each vertex in $S \setminus (T' \cup X)$ has at least $3\ell - \ell = 2\ell$ neighbours in G outside X. Hence X, $V(G) \setminus X$ is a partition contradicting the minimality of S. □

The following lemma shows that if S, T is a partition as provided by Lemma 10, then we can alter it by successively moving vertices from S to T to obtain a partition S', T' satisfying Theorem 1 except that $G[S']$ is only required to have large average degree. Theorem 1 will then immediately follow since $G[S']$, having large average degree, contains a subgraph of large minimum degree. The 'moreover' part of Lemma 11 will only be used in the proof of Theorem 2.

Lemma 11 *Let $c, k, \ell, r \in \mathbb{N}$ be such that $\ell \geq r$ and $k \geq 2^6 c\ell r$. Let G be a graph of minimum degree at least k and let S, T be a partition of $V(G)$ such that $d(G[S]) \geq \ell$, $\delta(G[T]) \geq \ell$ and every subgraph of $G[S]$ has average degree less than $c\ell$. Then there exists $S' \subseteq S$ such that, writing $T' := V(G) \setminus S'$, every vertex in S' has at least r neighbours in T', $d(G[S']) \geq \ell/8$ and $\delta(G[T']) \geq r$. Moreover, T' can be obtained from T by successively adding vertices having at least r neighbours in the superset of T already constructed.*

Proof. Let $A \subseteq S$ be the set of all vertices in S having at least $4c\ell$ neighbours in S, and let $C := S \setminus A$. Thus $4c\ell|A| \leq 2e(G[S])$, and so, as every subgraph of $G[S]$ has average degree less than $c\ell$,

$$e(G[A]) \leq \frac{c\ell|A|}{2} \leq \frac{e(G[S])}{4}. \tag{1}$$

Define B to be the set of all those vertices in S which have less than r neighbours in T. (So if $B = \emptyset$, then $S' := S$ and $T' := T$ would be a partition as required in the lemma.) Since $\delta(G) \geq k \geq 4c\ell + r$, we have $B \subseteq A$. Also

$$\frac{k|B|}{2} \leq (k - r)|B| \leq 2e(G[S]). \tag{2}$$

Let B' be the set of all those vertices in B which have at least r neighbours in C. For every vertex $x \in B'$ choose a set N_x of r neighbours in C. Let $e_S(\bigcup_{x\in B'} N_x)$ denote the number of all edges in $G[S]$ which are incident to some vertex in $\bigcup_{x\in B'} N_x$. (So here we also count those edges with both endvertices in $\bigcup_{x\in B'} N_x$.) Then

$$e_S(\bigcup_{x\in B'} N_x) \leq |B'|4c\ell r \stackrel{(2)}{\leq} \frac{16c\ell r \cdot e(G[S])}{k} \leq \frac{e(G[S])}{4}. \tag{3}$$

Thus if we add $\bigcup_{x\in B'} N_x$ to T then we can ensure that every vertex in B' has at least r neighbours in the resulting superset of T while maintaining high average degree in the resulting subgraph of $G[S]$. We now have to deal with the vertices in $B \setminus B'$. As some of these vertices may have (nearly) all their neighbours in $B \setminus B'$, but we nevertheless want to ensure that every vertex in S' has many neighbours in T', we will move all of $B \setminus B'$ into T'. For each vertex of $B \setminus B'$ which has most of its neighbours outside $B \setminus B'$ (and thus in $A \setminus (B \setminus B')$), we will also move some of these neighbours into T' in order to ensure that $G[T']$ has large minimum degree. The main difficulty is that the removal of these neighbours should not decrease the average degree of the resulting subgraph of $G[S]$ too much.

So let $A_1 := B \setminus B'$, $A_2 := A \setminus A_1$ and let $A_1' \subseteq A_1$ be the set of all those vertices which have at least $k/2$ neighbours in A_2. Put $C' := C \setminus \bigcup_{x\in B'} N_x$ (Fig. 1). We will show that we can add $A_1 \cup \bigcup_{x\in B'} N_x$ to T together with a set $A_2' \subseteq A_2$ such that

(i) each vertex in A_1' has at least r neighbours in A_2',

(ii) $e(A_2', C') \leq e(A_2, C')/2$.

As we shall see, the partition thus obtained is as required in the lemma. However, let us first show that there exists a set A_2' satisfying (i) and (ii).

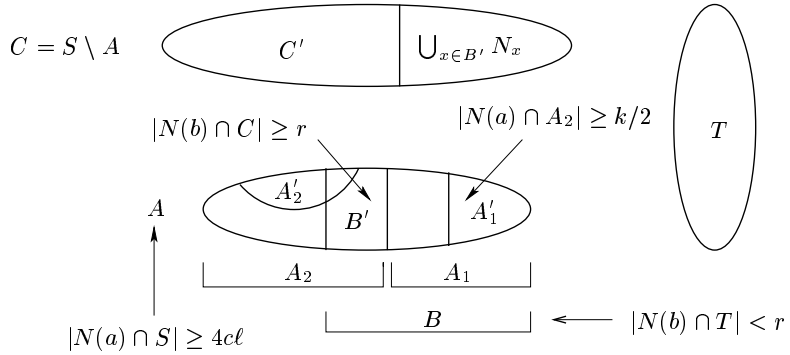

Figure 1: The set-up of the proof of Lemma 11.

Let $a_1, a_2, \ldots$ be any enumeration of the vertices in A_2 such that their degrees in $(A_2, C')_G$ form a non-increasing sequence. For every $x \in A_1'$ let R_x be the set consisting of the r rightmost neighbours of x in $a_1, a_2, \ldots$. Note that, writing $[a_i]$ for the set $\{a_1, \ldots, a_i\}$,

If $a \in R_x$ then x has at least $k/2 - r$ neighbours in $[a]$. $(*)$

Let $A_2' := \bigcup_{x \in A_1'} R_x$. Then clearly A_2' satisfies condition (i). To show that A_2' also satisfies (ii), we first prove the following

Claim. *For all $a \in A_2$ at most half of the vertices in $[a]$ are contained in A_2'.*

Suppose not and let a be a vertex in A_2 such that more than half of the vertices in $[a]$ are contained in A_2'. Consider the bipartite subgraph H of $G[S]$ between $[a]$ and the set A_1^* consisting of all those vertices $x \in A_1'$ for which $[a] \cap R_x \neq \emptyset$. Since $[a] \cap A_2'$ contains at most r vertices lying in a common set R_x, it follows that $|A_1^*| \geq |[a]|/(2r)$. Moreover, $(*)$ implies that each vertex in A_1^* has degree at least $k/2 - r \geq k/4$ in H. So if $|A_1^*| \geq |[a]|$, then

$$e(H) \geq \frac{k|A_1^*|}{4} \geq \frac{k|H|}{8} \geq \frac{c\ell|H|}{2};$$

while if $|A_1^*| \leq |[a]|$, we have

$$e(H) \geq \frac{k|A_1^*|}{4} \geq \frac{k|[a]|}{8r} \geq \frac{k|H|}{16r} \geq \frac{c\ell|H|}{2}.$$

Thus in each case H is a subgraph of $G[S]$ of average degree at least $c\ell$, contradicting our assumption. This proves the claim.

By the claim there exists an injection which sends every vertex $a \in A_2'$ to a vertex in $A_2 \setminus A_2'$ preceding a in the sequence $a_1, a_2, \ldots$. The choice of the enumeration $a_1, a_2, \ldots$ now implies (ii).

Put $T' := T \cup A_1 \cup A_2' \cup \bigcup_{x \in B'} N_x$ and $S' := V(G) \setminus T'$. Let us first show that $G[S']$ has large average degree. Clearly,

$$e(G[S']) \geq e(A_2 \setminus A_2', C') \overset{\text{(ii)}}{\geq} e(A_2, C')/2.$$

Thus if $e(A_2, C') \geq e(G[S])/4$ then $e(G[S']) \geq e(G[S])/8$, as required. To verify the other case, note that

$$\begin{aligned} e(G[S']) &\geq e(G[S]) - e(G[A]) - e_S\Big(\bigcup_{x \in B'} N_x\Big) - e(A_1, C') - e(A_2', C') \\ &\overset{(1),(3)}{\geq} e(G[S]) - \frac{e(G[S])}{4} - \frac{e(G[S])}{4} - |B \setminus B'|r - e(A_2', C') \\ &\overset{(2),\text{(ii)}}{\geq} \frac{e(G[S])}{4} - \frac{e(A_2, C')}{2}. \end{aligned}$$

Thus $e(G[S']) \geq e(G[S])/8$ also holds in the case when $e(A_2, C') \leq e(G[S])/4$. This proves that $d(G[S']) \geq \ell/8$. Furthermore, every vertex in $S' \subseteq S \setminus A_1 = S \setminus (B \setminus B')$ has at least r neighbours in $(T \cup \bigcup_{x \in B'} N_x) \subseteq T'$.

Let us now verify that T' can be obtained from T by successively adding vertices having at least r neighbours in the superset of T already constructed. Clearly, as $B \cap \bigcup_{x \in B'} N_x = \emptyset$, $T_1 := T \cup \bigcup_{x \in B'} N_x$ can be constructed in this way. As $A_2' \subseteq (A \setminus B) \cup B'$, every vertex in A_2' has at least r neighbours in

T_1, and thus also $T_1 \cup A'_2$ can be constructed. Since by (i) every vertex in A'_1 has at least r neighbours in A'_2, we can now construct $T_1 \cup A'_2 \cup A'_1$. Finally, to construct T, we now successively move vertices from $A_1 \setminus A'_1$ to the other side of the partition as long as they have at least r neighbours there. Suppose that we are not able to exhaust all of $A_1 \setminus A'_1$ in this way, but are left with some non-empty set X. As every vertex in $X \subseteq A_1 \setminus A'_1 \subseteq B \setminus B'$ has less than r neighbours in C, less than $k/2$ neighbours in A_2, less than r neighbours in $T' \setminus X$ and as $X \cup C \cup A_2 \cup (T' \setminus X) = V(G)$, it follows that the graph $G[X]$ has minimum degree at least $k - k/2 - 2r \geq c\ell$, contradicting our assumption on $G[S]$. Thus T' can be constructed in the required way. In particular as $\delta(G[T]) \geq \ell \geq r$ this implies that $\delta(G[T']) \geq r$, so S' and T' are as desired.

□

Proof of Theorem 1. We will prove the theorem for $k \geq 2^{10} \cdot 6\ell^2 = 2^{11} \cdot 3\ell^2$. Apply Lemma 10 to obtain a partition S'', T'' with $\delta(G[S'']) \geq 16\ell$, $\delta(G[T'']) \geq 16\ell$ and such that every subgraph of $G[S'']$ has average degree less than $6 \cdot 16\ell$. Then apply Lemma 11 to find a partition S', T' such that $d(G[S']) \geq 2\ell$, $\delta(G[T']) \geq \ell$ and each vertex in S' has at least ℓ neighbours in T'. By Proposition 9, the graph $G[S']$ contains a subgraph H of minimum degree at least ℓ. Then $S := V(H)$, $T := V(G) \setminus S$ is a partition as required.

□

To prove Theorem 2, we need the following theorem of Mader (see e.g. [3, Thm. 1.4.2]) and an analogue to Lemma 10.

Theorem 12 *Every graph G has a $\lfloor d(G)/4 \rfloor$-connected subgraph.*

Lemma 13 *The vertex set of every 9ℓ-connected graph G can be partitioned into non-empty sets S and T such that $G[T]$ is ℓ-connected, the average degree of $G[S]$ is at least ℓ and every subgraph of $G[S]$ has average degree at most 32ℓ.*

Proof. Theorem 8 implies that $V(G)$ can be partitioned into non-empty sets S and T such that $d(G[S]) \geq \ell$, $G[T]$ is ℓ-connected and $|T| \geq 2\ell$. Choose S and T satisfying these properties such that S is minimal. We will show that S and T are as desired. First note that $G[S]$ cannot be 8ℓ-connected. For otherwise, by Theorem 8, S can be partitioned into two non-empty sets S_1 and S_2 such that each $G[S_i]$ is ℓ-connected. As G was 2ℓ-connected and $|S|, |T| \geq 2\ell$, there are at least 2ℓ independent S–T edges in G. So some S_i, S_2 say, must contain endvertices of at least ℓ of these edges. But then $G[T \cup S_2]$ is ℓ-connected, and so S_1, $T \cup S_2$ is a partition contradicting the minimality of S.

Let us now show that $G[S]$ does not even contain an 8ℓ-connected subgraph (and hence, by Theorem 12, no subgraph of average degree at least 32ℓ). Suppose that H is an 8ℓ-connected subgraph of $G[S]$. As $G[S]$ is not 8ℓ-connected, it has a cut-set X with $|X| < 8\ell$. Let C be the unique component of $G[S] - X$ such that $V(H) \subseteq V(C) \cup X$. Let $S' := V(C) \cup X$ and $T' := V(G) \setminus S'$. Then $G[T']$ is ℓ-connected. Indeed, if it has a cut-set Y with $|Y| < \ell$, then, as $G[T]$ is ℓ-connected, there is a component D of $G[T'] - Y$ such that $T \subseteq V(D) \cup Y$. Since there is no path from $F := T' \setminus (V(D) \cup Y)$ to C within $G[S]$ which avoids

X and no path in G from F to T which avoids $S' \cup Y = V(C) \cup X \cup Y$, it follows that $X \cup Y$ separates C from F in G, contradicting the fact that G is 9ℓ-connected (Fig. 2). We may now successively move vertices from S' to T'

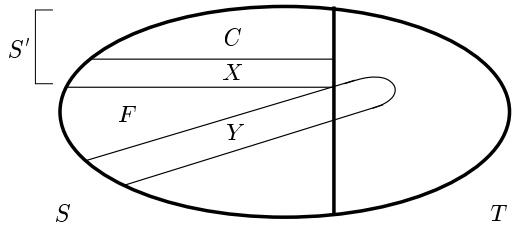

Figure 2: $X \cup Y$ separates C from F in G.

if they have less than 8ℓ neighbours in the subset of S' already constructed. (Thus each of these vertices has at least ℓ neighbours in the superset of T' and therefore adding it preserves ℓ-connectedness.) This process must terminate as H is a subgraph of $G[S']$ with minimum degree at least 8ℓ. The partition obtained in this way contradicts the minimality of S. □

Proof of Theorem 2. We will prove the theorem for $k \geq 2^{16}\ell^2$. First apply Lemma 13 to obtain a partition S'', T'' such that $d(G[S'']) \geq 2^5\ell$, $G[T'']$ is $2^5\ell$-connected and such that every subgraph of $G[S'']$ has average degree less than $32 \cdot 2^5\ell$. Now apply Lemma 11 to find a partition S' and T' such that each vertex in S' has at least ℓ neighbours in T', $d(G[S']) \geq 4\ell$, $\delta(G[T']) \geq \ell$ and T' can be obtained from T'' by successively adding vertices having at least ℓ neighbours in the superset of T'' already constructed. Thus $G[T']$ is ℓ-connected, since $G[T'']$ is. By Theorem 12, $G[S']$ contains an ℓ-connected subgraph H. As each vertex in $S' \setminus V(H)$ has ℓ neighbours in T', $G[T' \cup (S' \setminus V(H))] = G - V(H)$ is ℓ-connected. Thus $S := V(H)$, $T := V(G) \setminus S$ is a partition as required. □

3 Proof of Theorem 5

Similarly to Section 2, before proving the theorem, let us first show that it is best possible in the sense that we cannot always partition the vertex set of a graph of large minimum degree into non-empty sets S and T satisfying Theorems 1 and 5 simultaneously, i.e. we cannot always find S, T such that both $G[S]$ and $G[T]$ have large minimum degree, $(S,T)_G$ has large average degree and every vertex in S has many neighbours in T.

Proposition 14 *For every $k \in \mathbb{N}$ there is a k-connected graph G whose vertex set cannot be partitioned into non-empty sets S and T such that $\delta(G[S]) \geq 1$, $\delta(G[T]) \geq 1$, $d((S,T)_G)) \geq 1$ and each vertex in S has at least one neighbour in T.*

Note that Proposition 6 is a special case of Proposition 14, but has the advantage that its proof is less technical. Moreover, we will refer to the proof of

Proposition 6 again in Section 4 to deduce a result about partitions of graphs of large chromatic number and large minimum degree.

Proof of Proposition 14. Let G_k^n, X and $X^{(k)}$ be as defined in the proof of Proposition 6 and suppose that S, T is a partition of $V(G_k^n)$ contradicting Proposition 14. Then $0 < |X \cap S| < k$. (Indeed, if $|X \cap S| \geq k$ and Y is a set consisting of any k vertices in $X \cap S$, then all neighbours of the vertex $Y \in X^{(k)}$ are contained in S, contradicting our assumptions on S and T.) Moreover, as any vertex Y in $X^{(k)} \cap S$ must contain at least one of the at most $k-1$ points in $X \cap S$ (when Y is viewed as a k-element subset of X), we have

$$|X^{(k)} \cap S| \leq \sum_{i=1}^{k-1} \binom{k-1}{i} \binom{n}{k-i} \leq 2^k n^{k-1}.$$

If n is sufficiently large, this implies

$$\begin{aligned} e(S,T) &\leq |X \cap S| \cdot \binom{n}{k-1} + |X^{(k)} \cap S| \cdot k \\ &\leq k(1+2^k) n^{k-1} < \frac{1}{2} \cdot \binom{n}{k} \leq \frac{|G_k^n|}{2}. \end{aligned}$$

Thus $d((S,T)_{G_k^n}) < 1$, a contradiction. □

By Proposition 9, every graph G of large average degree contains a subgraph H of large minimum degree. The following lemma implies that if G has many vertices of large degree, then H can be chosen to contain a constant fraction of the vertices in G.

Lemma 15 *Let $\ell \in \mathbb{N}$ and $0 < \alpha \leq 1$. Let G be a graph and let $X \subseteq V(G)$ be a set of at least $\alpha|G|$ vertices having degree at least ℓ in G. Then G has a subgraph of minimum degree at least $\alpha\ell/4$ which contains at least $|X|/4$ of the vertices in X.*

Proof. By Proposition 9, G contains a subgraph of minimum degree at least $\alpha\ell/2$. Let $H \subseteq G$ be maximal with minimum degree at least $\alpha\ell/4$ and suppose that it contains less than $|X|/4$ of the vertices in X. By the choice of H, every vertex of $G - V(H)$ has less than $\alpha\ell/4$ neighbours in H. Thus $G - V(H)$ still has at least $3|X|/4 \geq 3\alpha|G|/4$ vertices having degree at least $\ell - \alpha\ell/4 = (1-\alpha/4)\ell$ in $G - V(H)$. Thus

$$d(G - V(H)) \geq \frac{(3\alpha|G|/4) \cdot (1-\alpha/4)\ell}{|G|} \geq \frac{3\alpha}{4} \cdot \frac{3\ell}{4} \geq \frac{\alpha\ell}{2}.$$

By Proposition 9, $G - V(H)$ contains a subgraph H' of minimum degree at least $\alpha\ell/4$. But $H \cup H'$ contradicts the maximality of H. □

We will need the following special case of Chernoff's inequality (see [1, Thm. A.11 and A.13]).

Lemma 16 *Let $X_1, \dots, X_n$ be independent 0-1 random variables with $\mathbb{P}(X_i = 1) = p$ for all $i \le n$, and let $X := \sum_{i=1}^n X_i$. Then $\mathbb{P}(|X - \mathbb{E}X| \ge \mathbb{E}X/2) \le 2\mathrm{e}^{-\mathbb{E}X/16}$ and $\mathbb{P}(|X - \mathbb{E}X| \ge \mathbb{E}X/4) \le 2\mathrm{e}^{-\mathbb{E}X/64}$.*

Proof of Theorem 5. Let $k := 2^{32}\ell$ and let $(A, B)_G$ be a spanning bipartite subgraph of G of minimum degree at least $k/2$. Suppose that $|A| \ge |B|$. Delete edges if necessary to obtain a bipartite graph H in which each vertex in A has precisely $k/2$ neighbours in B. (Then B may contain some vertices of degree less than $k/2$.) Consider a random partition of B into sets B_1, B_2 where each vertex of B is included in B_1 with probability $1/2$ independently of all other vertices in B. Call a vertex $a \in A$ *good* if it has at least $k/8$ neighbours in each B_i, and *bad* otherwise. Given $a \in A$, write N_a for the number of those neighbours of a in H that lie in B_1. Lemma 16 together with the fact that $\mathbb{E}(N_a) = k/4$ implies that

$$\mathbb{P}(a \text{ is bad}) \le \mathbb{P}(|N_a - \mathbb{E}(N_a)| \ge \mathbb{E}(N_a)/2) \le 2\mathrm{e}^{-\mathbb{E}(N_a)/16} \le 1/4.$$

Hence the expected number of bad vertices in A is at most $|A|/4$. Thus there exists a partition B_1, B_2 of B such that the set A' of good vertices in A satisfies

$$|A'| \ge \frac{3|A|}{4}. \tag{4}$$

Let $H' := H - (A \setminus A')$. Now consider a random partition of A' into A'_1 and A'_2 where again each vertex of A' is included in A'_1 with probability $1/2$ independently of all other vertices in A'. Then, by Lemma 16,

$$\begin{aligned}\mathbb{P}\left(\big||A'_1| - |A'|/2\big| \ge |A'|/8\right) &= \mathbb{P}\left(\big||A'_1| - \mathbb{E}(|A'_1|)\big| \ge \mathbb{E}(|A'_1|)/4\right)\\ &\le 2\mathrm{e}^{-\mathbb{E}(|A'_1|)/64} \le 1/3.\end{aligned} \tag{5}$$

Let us say that a vertex $b \in B$ is *good* if $d_{H'}(b) \ge 3k/2^{17}$ and if each A'_i contains at least one quarter of the neighbours of b in H', otherwise call b *bad*. Given a vertex $b \in B$, let N'_b denote the number of those neighbours of b in H' that lie in A'_1. Using Lemma 16 again, if $d_{H'}(b) \ge 3k/2^{17}$ we have

$$\mathbb{P}(b \text{ is bad}) \le \mathbb{P}(|N'_b - \mathbb{E}(N'_b)| \ge \mathbb{E}(N'_b)/2) \le 2\mathrm{e}^{-\mathbb{E}(N'_b)/16} \le 1/2^{14}.$$

Call an edge of H' *good* if it is incident to a good vertex of B, and *bad* otherwise. Let Y denote the number of bad edges of H'. Let E' be the set of all those edges in H' whose endvertex in B has degree less than $3k/2^{17}$ in H'. As

$$|E'| \le \frac{3k|B|}{2^{17}} \le \frac{3k|A|}{2^{17}} \overset{(4)}{\le} \frac{k|A'|}{2^{15}} = \frac{e(H')}{2^{14}},$$

it follows that

$$\begin{aligned}\mathbb{E}(Y) &= \sum_{e \in E(H')} \mathbb{P}(\text{the endvertex of } e \text{ in } B \text{ is bad })\\ &\le |E'| + \sum_{e \in E(H') \setminus E'} \mathbb{P}(\text{the endvertex of } e \text{ in } B \text{ is bad})\\ &\le |E'| + e(H')/2^{14} \le e(H')/2^{13}.\end{aligned}$$

Now Markov's inequality implies that

$$\mathbb{P}(Y \geq e(H')/2^{12}) \leq \mathbb{P}(Y \geq 2\mathbb{E}Y) \leq 1/2.$$

Together with (5) this implies that there exists a partition A_1', A_2' of A' such that at most $e(H')/2^{12}$ of the edges of H' are bad and such that $||A_1'| - |A'|/2| \leq |A'|/8$. Thus

$$|A_i'| \geq \frac{3|A'|}{8} \tag{6}$$

for $i = 1, 2$. We will now show that there exist sets $A_i^* \subseteq A_i'$ and $B_i^* \subseteq B_i$ such that each $(A_i^*, B_i^*)_{H'}$ has large minimum degree and $(A_2^*, B_1^*)_{H'}$ contains a constant fraction of the edges of H'. Once we have obtained these sets, we will suitably extend $A_1^* \cup B_1^*$ to S and $A_2^* \cup B_2^*$ to T. To find A_1^* and B_1^*, consider the graph $H_1 := (A_1', B_1)_{H'}$. As every vertex in A_1' has at least $k/8$ neighbours in B_1 and

$$|A_1'| \overset{(6)}{\geq} \frac{3|A'|}{8} \overset{(4)}{\geq} \frac{|A|}{4} \geq \frac{|G|}{8} \geq \frac{|H_1|}{8}, \tag{7}$$

we may apply Lemma 15 with $X := A_1'$ and $\alpha := 1/8$ to obtain sets $A_1^* \subseteq A_1'$ and $B_1^* \subseteq B_1$ such that the minimum degree of $(A_1^*, B_1^*)_{H'}$ is at least $k/2^8$ and

$$|A_1^*| \geq \frac{|A_1'|}{4} \overset{(7)}{\geq} \frac{|G|}{2^5}. \tag{8}$$

Thus

$$e_{H'}(A_1^*, B_1^*) \geq \frac{k|A_1^*|}{2^8} \overset{(8)}{\geq} \frac{k|A_1'|}{2^{10}} \overset{(6)}{\geq} \frac{3k|A'|}{2^{13}}.$$

But as at most $e(H')/2^{12} = k|A'|/2^{13}$ edges in $H' \supseteq (A_1^*, B_1^*)_{H'}$ are bad, it follows that at least $k|A'|/2^{12}$ of the edges from $(A_1^*, B_1^*)_{H'}$ are good in H'. In other words,

$$\sum_{b \in B_1^*,\, b \text{ is good}} d_{H'}(b) \geq k|A'|/2^{12},$$

and therefore there are also at least $k|A'|/2^{14}$ edges in H' joining (the good vertices in) B_1^* to A_2'.

Now consider the graph $(A_2', B_1^*)_{H'}$ and let $A_2^\diamond$ be the set of all vertices in A_2' which have at least $k/2^{15}$ neighbours in B_1^*. Then

$$\frac{k|A_2^\diamond|}{2} + \frac{k|A'|}{2^{15}} \geq e_{H'}(A_2', B_1^*) \geq \frac{k|A'|}{2^{14}},$$

and therefore

$$|A_2^\diamond| \geq \frac{|A'|}{2^{14}} \overset{(4)}{\geq} \frac{|A|}{2^{15}} \geq \frac{|G|}{2^{16}}. \tag{9}$$

Now consider the graph $(A_2^\diamond, B_2)_{H'}$. As in this graph the degree of every vertex in $A_2^\diamond$ is at least $k/8$, we may apply Lemma 15 with $X := A_2^\diamond$ and $\alpha := 1/2^{16}$ to obtain sets $A_2^* \subseteq A_2^\diamond$ and $B_2^* \subseteq B_2$ such that $(A_2^*, B_2^*)_{H'}$ has minimum degree at least $k/2^{21}$ and

$$|A_2^*| \geq \frac{|A_2^\diamond|}{4} \overset{(9)}{\geq} \frac{|G|}{2^{18}}.$$

Since every vertex in $A_2^\diamond \supseteq A_2^*$ has at least $k/2^{15}$ neighbours in B_1^*, it follows that

$$e_{H'}(A_2^*, B_1^*) \geq \frac{k|A_2^*|}{2^{15}} \geq \frac{k|G|}{2^{33}}.$$

Let $S' := A_1^* \cup B_1^*$ and $T' := A_2^* \cup B_2^*$. Then $|S'| \overset{(8)}{\geq} |G|/2^5$, $|T'| \geq |G|/2^{18}$, $\delta(G[S']) \geq k/2^8 \geq \ell$, $\delta(G[T']) \geq k/2^{21} \geq \ell$ and $(S', T')_G$ has at least $k|G|/2^{33}$ edges. Choose $S \supseteq S'$ and $T \supseteq T'$ such that S and T are disjoint, both $G[S]$ and $G[T]$ have minimum degree at least $k/2^{21}$ and so that $S \cup T$ is maximal. Then $S \cup T = V(G)$. Indeed, otherwise every vertex in $G - (S \cup T)$ has at least $k - 2k/2^{21}$ neighbours outside $S \cup T$, i.e. $\delta(G - (S \cup T)) \geq k/2$. But then $S, V(G) \setminus S$ is a partition contradicting the choice of S, T. Note that $e_G(S,T) \geq e_G(S',T') \geq e_{H'}(A_2^*, B_1^*)$ and so $d((S,T)_G) \geq k/2^{32} = \ell$, as desired. □

It would be interesting to know whether a result analogous to Theorem 5 holds for graphs of high connectivity.

4 Partitions with constraints on the average degree or the chromatic number

We now derive a partition result for graphs of large average degree: the vertex set of every such graph can be partitioned into S and T such that each of $G[S]$, $G[T]$ and $(S,T)_G$ still has large average degree. This follows straightforwardly from the next theorem, which is a consequence of a result of Porter [8]. (The bounds in Porter's result were improved to best possible ones by Bollobás and Scott [2].)

Theorem 17 *The vertex set of every graph G can be partitioned into non-empty sets S and T such that $(S,T)_G$ has at least $e(G)/2$ edges while both $G[S]$ and $G[T]$ have at most $e(G)/4 + |G|$ edges.*

Corollary 18 *The vertex set of every graph G can be partitioned into non-empty sets S and T such that $(S,T)_G$ has at least $e(G)/2$ edges while both $G[S]$ and $G[T]$ have at least $e(G)/4 - 3|G|$ edges.*

Proof. Put $n := |G|$ and $m := e(G)$. Let S, T be a partition satisfying the conditions in Theorem 17 such that $e(S,T)$ is minimal. Let us show that S, T is as required in the corollary. Suppose not. Then $e(G[S]) < m/4 - 3n$ (say). Thus

$$\sum_{t \in T} d_{(S,T)_G}(t) = e(S,T) > m/2 + 2n \geq 2e(G[T]) = \sum_{t \in T} d_{G[T]}(t),$$

and hence there exists a vertex $t \in T$ which has more neighbours in S than in T. But then $S' := S \cup \{t\}$ and $T' := T \setminus \{t\}$ contradicts the choice of S, T since $e(G[S']) \leq e(G[S]) + n < m/4 - 2n$, $e(G[T']) \leq e(G[T])$ and $e(S,T) > e(S',T') \geq e(S,T) - n > m/2 + n$. □

Note that Corollary 18 implies that for every $\varepsilon > 0$ the vertex set of every graph G with m edges and average degree at least $6/\varepsilon$ can be partitioned into S and T such that both $G[S]$ and $G[T]$ contain at least $m(1/4 - \varepsilon)$ edges and $(S,T)_G$ contains at least $m/2$ edges.

For a bipartite graph $G = (A, B)$ this (with $m/2$ replaced by $m(1/2 - \varepsilon)$) can also be proved using a straightforward probabilistic argument which we outline below. Consider a random partition of A into A_1 and A_2. Then with non-zero probability most of the edges of G will have the property that their endvertex b in B is *good* in the sense that about half of the neighbours of b lies in A_1 and the other half lies in A_2. Now consider any enumeration of the good vertices in B such that their degrees in G form a non-increasing sequence, and let B_1 be the set of all good vertices with odd indices. It is easily seen that $A_1 \cup B_1$, $A_2 \cup (B \setminus B_1)$ is a partition as desired. Since every graph G contains a spanning bipartite subgraph G' with $e(G') \geq e(G)/2$, this argument also gives Corollary 18 with much weaker bounds.

To conclude this section, let us prove a simple result about partitions of graphs of large chromatic number and large minimum degree: the vertex set of every such graph can be partitioned into sets S and T such that both $G[S]$ and $G[T]$ have large chromatic number and $(S,T)_G$ has large minimum degree. Note that we cannot additionally require $G[S]$ and $G[T]$ to have large minimum degree. In fact, there need not even exist a partition S,T such that each of $G[S]$, $G[T]$ and $(S,T)_G$ has minimum degree at least one. (Let H_k^n be the graph obtained from the graph G_k^n defined in the proof of Proposition 6 by making $G_k^n[X]$ complete. Then $\chi(H_k^n) \geq n$ and $\delta(H_k^n) = k$, but the proof of Proposition 6 shows that there is no partition S,T having the properties in question.)

Proposition 19 *The vertex set of every graph G can be partitioned into non-empty sets S and T such that $\chi(G[S]) = \lceil \chi(G)/2 \rceil$, $\chi(G[T]) = \lfloor \chi(G)/2 \rfloor$ and every vertex $x \in G$ has at least $d^*(x) := \min\{\lfloor \chi(G)/2 \rfloor, d(x)/2\}$ neighbours in the partition set not containing x.*

Proof. Let S, T be a partition with $\chi(G[S]) = \lceil \chi(G)/2 \rceil$ and $\chi(G[T]) = \lfloor \chi(G)/2 \rfloor$ and such that $e(S,T)$ is maximal under these conditions. (By partitioning $V(G)$ into the vertices in the first $\lceil \chi(G)/2 \rceil$ colour classes and the remaining $\lfloor \chi(G)/2 \rfloor$ colour classes, it is clear that such a partition exists.) Let x be a vertex in S and suppose that x has less than $d^*(x) \leq \chi(G[T])$ neighbours in T. Then the chromatic number of $G[T \cup \{x\}]$ is still $\chi(G[T])$. As $\chi(G) \leq \chi(G[S \setminus \{x\}]) + \chi(G[T \cup \{x\}])$, this implies that $\chi(G[S \setminus \{x\}]) = \chi(G[S])$. But as $d^*(x) \leq d(x)/2$, we have $e(S \setminus \{x\}, T \cup \{x\}) > e(S,T)$, contradicting the choice of S,T. Similarly it can be shown that each vertex $x \in T$ has at least $d^*(x)$ neighbours in S. Thus S,T is a partition as required. □

It is not hard to show that the vertex set of every graph G of large chromatic number and large average degree can be partitioned into S,T such that both $G[S]$ and $G[T]$ have large chromatic number and each of $G[S]$, $G[T]$ and $(S,T)_G$ has large average degree. The idea is to use an averaging argument to find a set $X \subseteq V(G)$ (consisting of some of the colour classes of G) such that $G[X]$

still has large chromatic number but only a small fraction of the edges of G is incident to vertices in X. Then one can apply Corollary 18 to $G - X$ to obtain a partition S', T'. Adding one half of the colour classes of $G[X]$ to each of S' and T' gives a partition of $V(G)$ as desired.

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