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Dense critical and vertex-critical graphs

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Abstract

This paper gives new constructions of k-chromatic critical graphs with high minimum degree and high edge density, and of vertex-critical graphs with high edge density.

Keywords: graph coloring, chromatic number, critical graph, vertex-critical, edge density.

1 Introduction

The graphs in this paper are finite and simple. Let G be a graph with vertex set V and edge set E. For $k \geq 0$ a *k-coloring* of G is a function $\varphi : V \to \{1, 2, \ldots, k\}$ satisfying $\varphi(x) \neq \varphi(y)$ whenever $xy \in E$. The smallest cardinality $k \geq 0$ for which a k-coloring of G exists is its *chromatic number* $\chi(G)$. The maximum cardinality of an independent set of vertices is denoted by $\alpha(G)$, the minimum degree of a vertex in G by $\delta(G)$, and the maximum degree by $\Delta(G)$. An element $t \in V \cup E$, or a subset $A \subset V \cup E$, is said to be *critical* if $\chi(G - t) < \chi(G)$, respectively if $\chi(G - A) < \chi(G)$. G is *vertex-critical* if every vertex of G is critical and *edge-critical*, or simply *critical*, if every edge and vertex of G is critical. Clearly every critical graph is also vertex-critical, and every vertex-critical k-chromatic graph G satisfies $\delta(G) \geq k - 1$. The *cycle* C_n is the 2-regular connected graph of order n.

For $k = 1, 2$ the only k-chromatic vertex-critical graphs are K_1 and K_2, respectively. The 3-chromatic vertex-critical graphs are the odd cycles. Thus, for $k \leq 3$ the set of vertex-critical graphs coincides with the set of edge-critical graphs. For k-chromatic graphs with $k \geq 4$ the two classes are different.

As the first main result of this paper, it is proved in Theorem 3 for $4 \leq k \leq d$ that there exists a constant $c_{k,d}$ such that infinitely many k-chromatic critical graphs have minimum degree at least d and at least $c_{k,d}n^2$ edges, where n is the order of the graph. Without any condition on the minimum degree this was first proved by B. Toft [11]. The bounds $c_{4,4} \geq \frac{1}{169}$ and $c_{4,5} \geq \frac{9}{2704}$ are derived from a construction of T. Gallai of an infinite class of 4-regular 4-chromatic critical graphs, and from a new construction of an infinite class of 5-regular 4-chromatic critical graphs, respectively. Also a general bound $c_{4,d} \geq cd^{-4}$ for infinitely many values of d is derived. Similar bounds for $k > 4$ can be derived from the examples constructed for $k = 4$ by forming complete joins with complete graphs of the appropriate orders.

Theorem 4 extends previous results of B. Zeidl [16] (for $k = 4$) and B. Toft [14] (for $k = 5, 6$) by showing for $k \geq 3$ that there exists an infinite family of vertex-critical k-chromatic graphs containing at least $\frac{(k-3)n^2+(k+1)n}{2(k-1)}$ edges, where n is the order. The simple construction used to prove Theorem 4 is generalized to give a proof of the existence of vertex-critical graphs containing 'rich' classes of critical subgraphs. Theorem 5 states that for every $k \geq 5$ and every $m > 0$ there exists a vertex-critical k-chromatic graph such that any subgraph remaining after the deletion of a set of m edges incident with the same vertex is again k-chromatic. This extends a result of J.I. Brown [1], who proved this statement for $k = 5$ and $m = 1$, thereby answering a problem asked by G.A. Dirac, whether there exists a vertex-critical graph without critical edges. J.J. Lattanzio [8] independently extended Brown's construction to give examples for all odd chromatic numbers $k \geq 5$ and $m = 1$.

Certain vertex-critical k-chromatic examples constructed in the proof of Theorem 5 have the interesting and substantially stronger property of containing edge-disjoint k-chromatic subgraphs. It would seem interesting to find a systematic way of constructing vertex-critical graphs of this type.

Dirac's original question for 4-chromatic vertex-critical graphs remains unsolved, however.

2 Critical graphs with many edges

Let $k \geq 4$. G.A. Dirac [3] proved that there exists a critical k-chromatic graph of order n if and only if $n \geq k$ and $n \neq k+1$. For any such n let $f_k(n)$ denote the largest number for which there exists a critical k-chromatic graph of order n and having $f_k(n)$ edges. The problem of determining $f_k(n)$ was raised by P. Erdős in the early 1950s and subsequently addressed by G.A. Dirac and B. Toft.

The main result on this problem, which for fixed k determines the exact order of magnitude for $f_k(n)$, is the following.

Theorem 1 (B. Toft [11]). *For every $k \geq 4$ there exists a positive constant c_k such that for all n, $n \geq k$ and $n \neq k+1$,*

$$f_k(n) > c_k n^2. \qquad \square$$

Furthermore, Toft obtained the particular bounds $c_4 \geq \frac{1}{16}$ and $c_5 \geq \frac{4}{31}$.

Prior to Theorem 1, G.A. Dirac [2] observed that $f_6(n) \geq \frac{1}{4}n^2 + n$ holds for infinitely many values of n, a bound obtained by completely joining two disjoint odd cycles of equal length. Such a critical 6-chromatic graph of order n is in fact $(\frac{1}{2}n + 2)$-regular. It remains open whether the three constants $\frac{1}{16}$, $\frac{4}{31}$ and $\frac{1}{4}$ are the best possible for $k = 4$, 5 and 6, respectively (see also the open problems collection by T.R. Jensen and B. Toft [7]).

The constructions of critical graphs that were used to prove Theorem 1 for $k = 4, 5$ give critical k-chromatic graphs of lowest possible minimum degree $k - 1$. The main result of this section implies that Theorem 1 also holds for the restricted class of graphs with minimum degree at least d, for any integer d.

The following result is noteworthy in this connection and it will be applied to prove Theorem 3. It shows that there exist classes of critical 4-chromatic graphs whose minimum degrees grow fairly rapidly as a function of their order. However, for fixed minimum degree d it only implies that the size can grow as fast as $cn^{4/3}$. The theorem was obtained independently and almost simultaneously by M. Simonovits [10] and B. Toft [13].

Theorem 2. *There exists, for infinitely many values of n, a critical 4-chromatic graph of order n and of minimum degree at least $cn^{1/3}$, where $c > 0$ is a constant.* $\square$

We will need the existence of small critical 4-chromatic graphs with minimum degree 4 and 5. A classical construction given by

T. Gallai yields an infinite family of 4-regular critical 4-chromatic graphs. The smallest member of this family has order 12 and also happens to be the smallest critical 4-chromatic graph of minimum degree 4.

Proposition 1 (T. Gallai [5]) *For every $n \geq 12$ such that n is divisible by 3, there exists a 4-regular critical 4-chromatic graph of order n.* $\square$

We proceed with a construction of a family of 5-regular critical 4-chromatic graphs. The smallest member of this family has order 24 and appears to be the presently smallest known critical 4-chromatic graph of minimum degree at least 5.

Proposition 2. *For every $n \geq 24$ such that n is divisible by 8, there exists a 5-regular critical 4-chromatic graph of order n.*

Proof. We construct for each $n \equiv 0$ modulo 8, $n \geq 24$, a graph G_n. Let $n = 8q$, $q \geq 3$, and let the vertex set of G_n be $\{u'_i, u''_i, v'_i, v''_i : i = 1, 2, \ldots, q+1\} \cup \{w'_i, w''_i : i = 1, 2, \ldots, 2q+1\}$, where vertices with subscripts $q+1$ and $2q+1$ respectively are identified with vertices with subscript 1 as follows: $u'_{q+1} = u''_1$, $u''_{q+1} = u'_1$, $v'_{q+1} = v''_1$, $v''_{q+1} = v'_1$, $w'_{2q+1} = w''_1$, $w''_{2q+1} = w'_1$. The edges of G_n are indicated in Figure 1. Note that the vertices can be partitioned naturally into three sets $\{u'_i, u''_i : i = 1, 2, \ldots, q\}$, $\{v'_i, v''_i : i = 1, 2, \ldots, q\}$ and $\{w'_i, w''_i : i = 1, 2, \ldots, 2q\}$ each inducing a subgraph isomorphic to a Möbius ladder (a 3-regular graph consisting of an even cycle together with all its longest chords). It is clear that G_n has order n and is 5-regular. The example of order $n = 64$ is shown in Figure 2.

The symmetry of G_n is such that an automorphism of G_n is obtained by composing the mapping of $(u'_i, u''_i, w'_{2i-1}, w''_{2i-1}, w'_{2i}, w''_{2i}, v'_i, v''_i)$ to $(u'_{i+1}, u''_{i+1}, w'_{2i+1}, w''_{2i+1} w'_{2i+2}, w''_{2i+2} v'_{i+1}, v''_{i+1})$, for all $i = 1, 2, \ldots, q$ (indices taken modulo q) with the mapping that swaps the vertices within each pair (u'_1, u''_1), (w'_1, w''_1), (w'_2, w''_2), and (v'_1, v''_1). In particular, the vertices lettered u belongs to the same orbit of the automorphism group, similarly for the vertices lettered v, and for those vertices lettered w which have equal parity of their indices. We will use these facts to simplify the case checking when we study the chromatic properties of G_n.

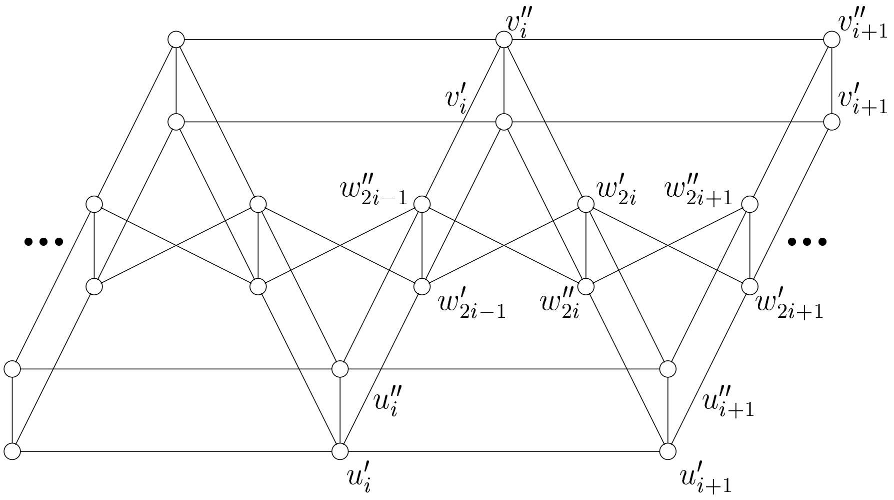


Figure 1. Construction of 5-regular critical 4-chromatic graphs.

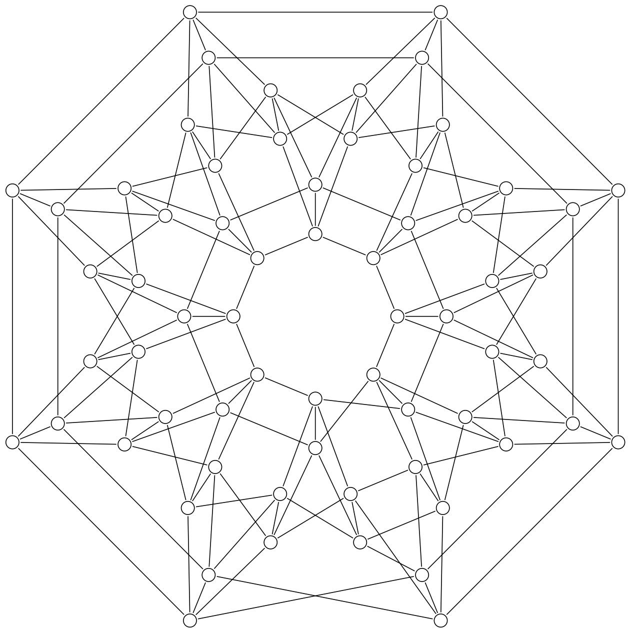


Figure 2. A 5-regular critical 4-chromatic graph of order 64.

We will now show that G_n has chromatic number at least 4. Suppose that a 3-coloring φ of G_n exists. For $i = 1, 2, \ldots, q+1$ define

$$c_i = (\varphi(u_i'), \varphi(u_i''), \varphi(w_{2i-1}'), \varphi(w_{2i-1}''), \varphi(v_i'), \varphi(v_i'')).$$

Claim 1. If the value of c_i can be obtained from $(1, 2, 2, 1, 3, 2)$ by a permutation of the colors $1, 2, 3$, then c_{i+1} is obtained from c_i by a cyclic permutation of colors.

Proof of Claim 1. Assume $c_i = (1, 2, 2, 1, 3, 2)$ and consider, in the following order, the colors of the vertices w_{2i}'', u_{i+1}', u_{i+1}'', w_{2i+1}'',

v''_{i+1}, v'_{i+1} and finally w'_{2i+1}. The only possibility is the one that leads to $c_{i+1} = (3, 1, 1, 3, 2, 1)$. This proves Claim 1.

We call φ *u-transposing at i* ($i = 1, 2, \ldots, q$) if $\varphi(u'_i) = \varphi(u''_{i+1})$ and $\varphi(u''_i) = \varphi(u'_{i+1})$, otherwise we call φ *u-cyclic at i*. Similarly, φ is called either *w-transposing* or *w-cyclic at i* ($i = 1, 2, \ldots, 2q$). Observe that φ is u-cyclic at i if and only if $(\varphi(u'_{i+1}), \varphi(u''_{i+1}))$ can be obtained from $(\varphi(u'_i), \varphi(u''_i))$ by a cyclic permutation of the colors $1, 2, 3$; similarly for the property of φ being w-cyclic at i. Since $\varphi(u'_1) = \varphi(u''_{q+1})$ and $\varphi(u''_1) = \varphi(u'_{q+1})$, we have

(i) φ is u-transposing at some $i = 1, 2, \ldots, q$.

Similarly,

(ii) φ is w-transposing at some $i = 1, 2, \ldots, 2q$.

Since φ is u-transposing at some i we may assume, by symmetry, that this happens at $i = 1$, and moreover that $\varphi(u'_1) = \varphi(u''_2) = 1$ and $\varphi(u''_1) = \varphi(u'_2) = 2$. At least one of w'_1, w''_1 must be colored with 3, since otherwise it would be impossible to extend φ to w'_2 and w''_2. We may therefore further assume, without loss of generality, that $\varphi(w'_1) = 3$. This leaves no other possibility than $\varphi(w''_1) = 1$, $\varphi(w'_2) = 2$, $\varphi(w''_2) = 3$, $\varphi(w''_3) = 2$, $\varphi(v''_1) = 3$ and $\varphi(v''_2) = 1$. In particular $c_1 = (1, 2, 3, 1, 1, 3)$ or $c_1 = (1, 2, 3, 1, 2, 3)$ depending on the color of v'_1.

Claim 2. $\varphi(v'_2) \neq 3$.

Proof of Claim 2. Suppose $\varphi(v'_2) = 3$. This leaves no other possibility than $\varphi(w'_3) = 1$. We then have $c_2 = (2, 1, 1, 2, 3, 1)$ and may observe that φ is w-cyclic at $i = 1, 2$. By repeated application of Claim 1 it follows that c_j is obtained from $(2, 1, 1, 2, 3, 1)$ by a cyclic permutation of the colors $1, 2, 3$ for $j = 3, 4, \dots, q + 1$. Thus φ is w-cyclic at every $i = 1, 2, \dots, 2q$. This contradiction to (ii) proves Claim 2.

Claim 2 implies $\varphi(v'_2) = 2$ and in turn $\varphi(v'_1) = 1$. Thus $c_1 = (1, 2, 3, 1, 1, 3)$.

Define for $i = 1, 2, \dots, q$

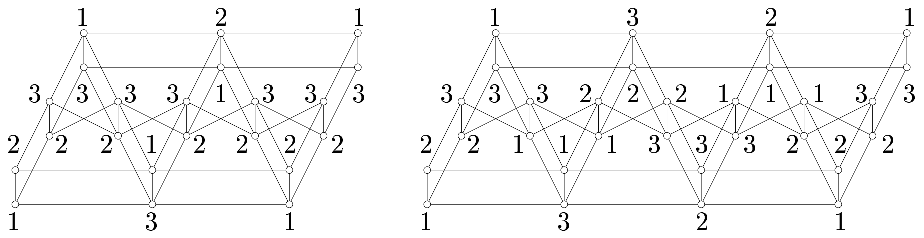
$$c'_i = (\varphi(u'_i), \varphi(u''_i), \varphi(v'_i), \varphi(v''_i)).$$

Claim 3. If c'_i is obtainable from $(1, 2, 1, 2)$ by permutation of the colors $1, 2, 3$, then c'_{i+1} arises from c'_i by a cyclic permutation of the colors.

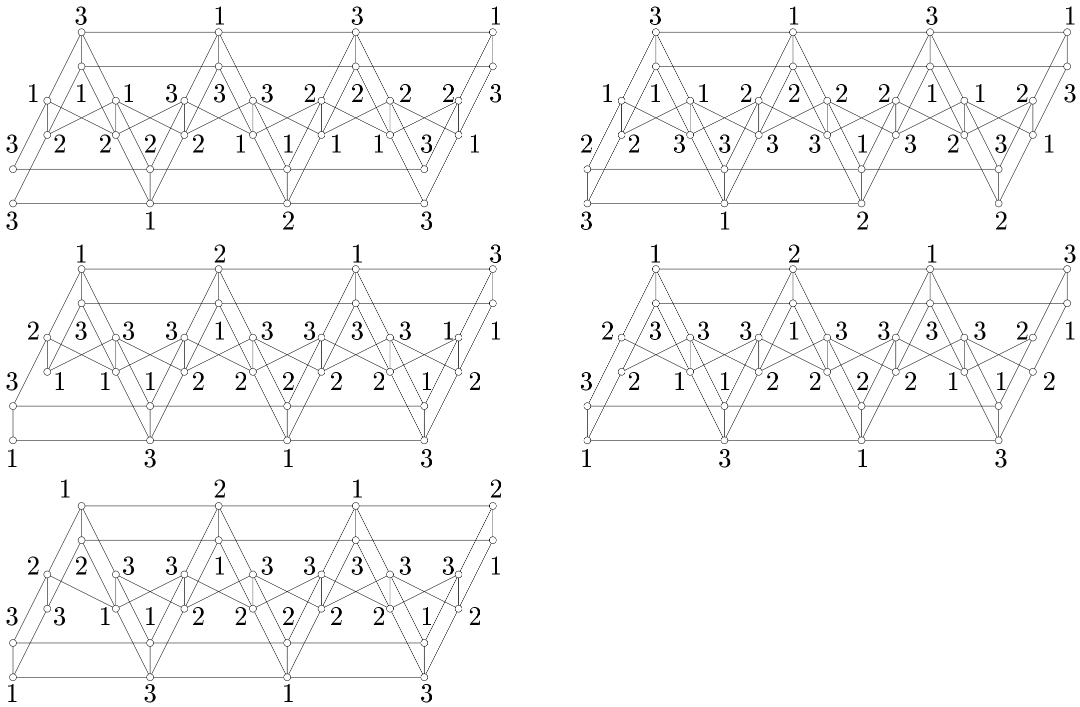
Proof of Claim 3. Assume $c'_i = (1, 2, 1, 2)$. First, one of u'_{i+1} and u''_{i+1} must be colored with 3, since otherwise it would be not be possible to extend φ to w'_{2i} and w''_{2i}. We may assume $\varphi(u'_{i+1}) = 3$ (the argument being similar in the case $\varphi(u''_{i+1}) = 3$). It easily follows that $\varphi(u''_{i+1}) = 1$, $\varphi(w''_{2i}) = 2$, $\varphi(w''_{2i+1}) = 3$ and $\varphi(v''_{i+1}) = 1$. Supposing $\varphi(v'_{i+1}) = 2$ holds, we deduce $\varphi(w'_{2i+1}) = 1$, and hence $c_{i+1} = (3, 1, 1, 3, 2, 1)$. But by Claim 1 this leads to contradiction with (ii) as in the proof of Claim 2. Therefore $\varphi(v'_{i+1}) = 3$, which yields $c'_{i+1} = (3, 1, 3, 1)$ and Claim 3 follows.

Finally, Claim 3 contradicts the previously established facts $c'_1 = (1, 2, 1, 3)$ and $c'_2 = (1, 2, 1, 2)$, and we have proved that G_n is not 3-colorable for $n \geq 24$.

We will next prove that G_n is 4-critical for all values of $n \geq 24$ divisible by 8, except $n = 32$. Thus it remains to prove that $G_n - e$ is 3-colorable for every edge e of G_n, $n \geq 24$, $n \neq 32$. Using the symmetry of G_n it suffices to consider the five cases $e = u'_1 u''_1$, $u'_q u'_1$, $u'_1 w'_1$, $w'_1 w''_1$ and $w''_1 w''_2$.



Partial colorings with $c_{i+2} = c_i$ and $c_{i+3} = c_i$.



Colorings of G_{24} minus one edge.

It follows that G_n is a critical 4-chromatic graph. This concludes the proof when $n \geq 24$, $n \neq 32$.

The remainder of the proof consists in showing that there exists a 5-regular critical 4-chromatic graph of order 32. To this end, we give Figure 3 as an example of such a graph.

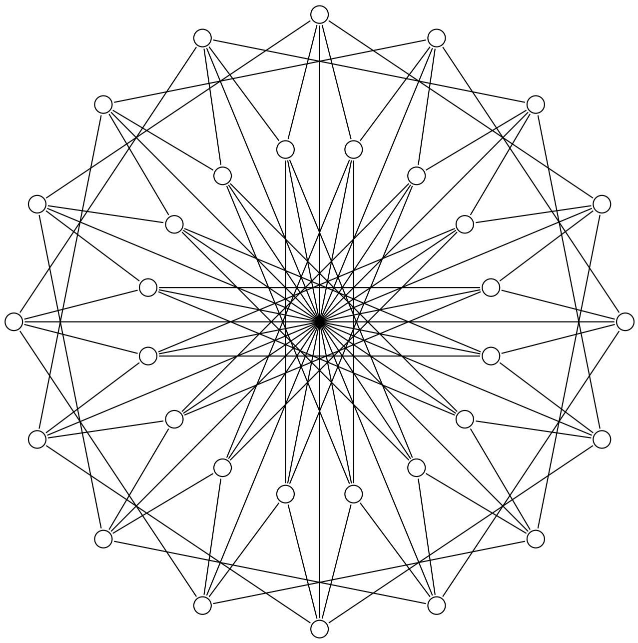


Figure 3. A 5-regular critical 4-chromatic graph of order 32.

Proving that this 5-regular graph is indeed critical 4-chromatic is left as (a difficult) exercise for the interested reader. Thus the proof is complete. □

The following lemma will be used in the proof of Theorem 3. The construction is very similar to the construction given by B. Toft in

[11] of critical 4-chromatic graphs with many edges, using odd wheels as the component graphs G_1 and G_2.

Lemma 1. *Let G_1 and G_2 be disjoint critical 4-chromatic graphs. Let x_i be a vertex of G_i of degree $d_i \geq 3$ with neighbors $y_1^i, y_2^i, \ldots, y_{d_i}^i$ in G_i, $i = 1, 2$. Obtain a new graph H by adding distinct new vertices $a_1^i, a_2^i, \ldots, a_{d_i}^i$ to $G_i - x_i$ and adding new edges joining a_j^i to y_j^i for $j = 1, 2, \ldots, d_i$, $i = 1, 2$, followed by adding $d_1 d_2$ new edges joining each pair $a_{j'}^1, a_{j"}^2$, $j' = 1, 2, \ldots, d_1$, $j" = 1, 2, \ldots, d_2$. Then H is critical 4-chromatic.*

Proof. Suppose H can be 3-colored. For any 3-coloring of H there is some $i = 1, 2$ for which the same color is assigned to $a_1^i, a_2^i, \ldots, a_{d_i}^i$. Such a 3-coloring can be extended to G_i by assigning this color to x_i, contradicting the fact that G_i is 4-chromatic. Thus H is at least 4-chromatic.

Let e be any edge of H. If e is an edge of G_1, then there exists a 3-coloring φ of $G_1 - e$ since G_1 is critical. Assume that $\varphi(x_1) = 1$. Color the vertices of H by assigning every vertex $x \in V(G_1) \setminus \{x_1\}$ the color $\varphi(x)$, and assign color 1 to each of $a_1^1, a_2^1, \ldots, a_{d_1}^1$. Continue by assigning the color 2 to the vertex a_1^2 and the color 3 to each vertex $a_2^2, a_3^2, \ldots, a_{d_2}^2$. Since the edge $x_2 y_1^2$ is critical in the 4-chromatic graph G_2, there exists a 3-coloring of $G_2 - x_2 y_1^2$ in which x_2 and y_1^2 both receive the color 3. The restriction of this coloring to the vertices of $G_2 - x_2$ completes the 3-coloring of $H - e$. A similar argument applies to the case when e is an edge of G_2.

Consider the case $e = y_i^1 a_i^1$, $1 \leq i \leq d_1$. Color $G_1 - x_1 y_i^1$ with three colors such that x_1 and y_i^1 are both colored with 1. In $H - e$, extend this coloring of the vertices of $G_1 - x_1$ by assigning the color 1 to each vertex $a_1^1, a_2^1, \ldots, a_{d_1}^1$. Complete the coloring of $H - e$ as in the previous case. We deal similarly with the case $e = y_i^2 a_i^2$, $1 \leq i \leq d_2$.

There remains the case $e = a_{j'}^1 a_{j"}^2$, $1 \leq j' \leq d_1$, $1 \leq j" \leq d_2$. In $H - e$, color both $a_{j'}^1$ and $a_{j"}^2$ with the color 1, color each vertex a_i^1 with the color 2, $i = 1, 2, \ldots, d_1$, $i \neq j'$, and each vertex a_i^2 with the color 3, $i = 1, 2, \ldots, d_2$, $i \neq j"$. Using that $x_1 y_{j'}^1$ and $x_2 y_{j"}^2$ are critical edges in G_1 and G_2, respectively, this coloring can be extended to a 3-coloring of $H - e$.

It follows that H is critical 4-chromatic. □

As a further tool we need the general construction of k-chromatic graphs due to G. Hajós [6] (incidentally, the construction first appeared in the book by G. Ringel [9]).

G is obtained from two disjoint graphs G_1 and G_2 by *Hajós's construction* as follows. Delete an edge $x_1 y_1$ from G_1 and delete an edge $x_2 y_2$ from G_2, identify x_1 and x_2 into a single vertex and add a new edge $y_1 y_2$. It is easy to see, for $k > 2$, that if G_1 and G_2 are both critical k-chromatic, then G is also critical k-chromatic. Moreover, the minimum degree of G is not smaller than the smallest minimum degree d of G_1 and G_2, providing $d > 1$.

Theorem 3. *For $4 \leq k \leq d$ there exists a constant $c_{k,d} > 0$ such that for infinitely many values of n there is a graph $G = (V, E)$ with $|V| = n$ such that G is critical k-chromatic of minimum degree at least d and satisfies*

$$|E| \geq c_{k,d} n^2.$$

Moreover,

i) $c_{4,d} \geq \left(\frac{d-2}{2(n_d+d-3)}\right)^2$, *where n_d denotes the smallest order of a critical 4-chromatic graph with minimum degree at least d,*

ii) $c_{4,4} \geq \frac{1}{169}$,

iii) $c_{4,5} \geq \frac{9}{2704}$, *and*

iv) *there exists a constant $c > 0$ such that $c_{4,d} \geq cd^{-4}$ for infinitely many values of d.*

Proof. The proof of the existence of $c_{k,d}$ for all $d \geq k$ is needed only for $k = 4$. For $k = 5$ the statement follows by completely joining a new vertex to one of the 4-chromatic critical graphs. For $k = 6$ the result follows from the previously mentioned construction of G.A. Dirac, and it can similarly be extended to $k \geq 7$.

Let $d \geq k = 4$. By Theorem 2 there exists a critical 4-chromatic graph G_1 of minimum degree at least d. Choose any vertex x_1 of G_1. Let G_1' and G_1'' be disjoint copies of G_1 with vertices x_1' and x_1'', respectively, corresponding to x_1. Let G_2 be obtained by Hajós's construction applied to G_1' and G_1'' in such a way that x_1' and x_1'' are identified into a new vertex x_2.

For $i = 3, 4, \ldots$ let G_i be obtained from disjoint copies of G_{i-1} and G_1 by Hajós's construction such that x_{i-1} and the copy of x_1 in G_1 become identified into a new vertex x_i. Each G_i thus constructed is critical 4-chromatic. It is easily checked that the minimum degree of G_i is at least d, that the order of G_i is $(n-1)i + 1$, where n denotes the order of G_1, and that the degree D_i of x_i in G_i is at least $(d-2)i + 2$.

Let A' and A'' be two disjoint sets each consisting of D_i independent vertices. Let G_i' and G_i'' be two copies of G_i, disjoint from each other and from A' and A'', where x_i' and x_i'' denote the respective copies of x_i. Delete vertex x_i' from G_i', and for all vertices y' adjacent to x_i' in G_i' join y' to a vertex in A' by a new edge in such a way that each vertex in A' is joined to a vertex of G'. Similarly for G_i'', x_i'' and A''. Finally obtain a graph H_i by adding all possible edges between A' and A''. Then H_i is critical 4-chromatic by Lemma 1.

For each $i = 1, 2, \ldots$, the minimum degree of H_i is at least d, and H_i has $2((n-1)i+1)+2D_i$ vertices and more than D_i^2 edges, where $D_i \geq (d-2)i+2$. Hence

$$\begin{aligned}
\frac{|E(H_i)|}{|V(H_i)|^2} &\geq \left(\frac{D_i}{2((n-1)i+1+D_i)}\right)^2 && (1)\\
&\geq \left(\frac{(d-2)i+2}{2((n-1)i+1+(d-2)i+2)}\right)^2 && (2)\\
&= \left(\frac{(d-2)i+2}{2((n+d-3)i+3)}\right)^2 && (3)\\
&= \left(\frac{(d-2)}{2(n+d-3)} + \frac{2n-d}{2((n+d-3)i+3)(n+d-3)}\right)^2 && (4)\\
&\geq \left(\frac{d-2}{2(n+d-3)}\right)^2 && (5)
\end{aligned}$$

from which $c_{4,d} \geq \left(\frac{d-2}{2(n+d-3)}\right)^2$ follows as desired.

The bounds ii) and iii) on $c_{4,d}$ for $d = 4, 5$ now follow from i) together with Propositions 1 and 2.

Finally, iv) follows from Theorem 2 combined with i). $\square$

For $k = 4, 5$ it is unknown whether there exists a constant γ_k and, for infinitely many values of n, a critical k-chromatic graph of order n having minimum degree at least $\gamma_k n$. Dirac's construction of dense critical 6-chromatic graphs shows that γ_k exists for every $k \geq 6$. The existence of a constant γ_4 would imply the first part of Theorem 3.

It seems difficult to obtain good upper bounds for the constants $c_{k,d}$. By Turán's theorem [15], if G has more than $((k-2)/2(k-1))|V(G)|^2$ edges, then G contains K_k as a subgraph. Hence we have

$$c_{k,d} \leq \frac{k-2}{2(k-1)},$$

which seems very weak, but may still be essentially the best known such bound.

A related open problem of P. Erdős asks if there exists, for every $r \geq 6$, a critical 4-chromatic r-regular graph. Such graphs would be useful for getting lower bounds for $c_{4,d}$ for $d \geq 6$; the explicit bounds iii) and iv) of Theorem 3 were both derived from the existence of small 4- and 5-regular critical 4-chromatic graphs. Erdős asked in addition if perhaps there even exists a constant c such that there exists a critical 4-chromatic r-regular graph of order at most cr for infinitely many r? The affirmative answer would imply, by i) of Theorem 3, the existence of a constant $c' > 0$ such that $c_{4,d} \geq c'$ for infinitely many d. It does not appear likely that such a class of graphs exists, but it may still be natural to ask whether ii) of Theorem 3 can be substantially improved.

3 Vertex-critical graphs with many edges

For integers $i < j$ let $[i, j]$ denote the interval $\{i, i+1, \ldots, j-1, j\}$. For $n > 0$ and $0 \leq i, j \leq n-1$ we define $d_n(i, j) = \min\{|i-j|, n-|i-j|\}$ – the *cyclic distance* between i and j in the cyclic order $0, 1, 2, \ldots, n-1, 0$. For any $D \subset \mathbb{Z}_+$ let $G(n; D)$ denote the *circulant graph* with vertex set $x_0, x_1, \ldots, x_{n-1}$ and edge set $\{x_i x_j \mid 0 \leq i, j \leq n-1, \text{ and } d_n(i, j) \in D\}$. (Imagine the vertices arranged cyclically $x_0, x_1, x_2, \ldots, x_{n-1}, x_0$ with pairs of vertices joined by an edge if and only if their cyclic distance belongs to the prescribed set D.) For short we write $G(n; d_1, d_2, \ldots, d_n)$ instead of $G(n; \{d_1, d_2, \ldots, d_n\})$. Observe that every graph $G(n; D)$ is vertex-transitive, but in general not edge-transitive. The *kth power* G^k of G is the graph on the same vertex set as G where two vertices are adjacent in G^k if and only if they have (graph theoretic) distance at most k in G. In particular the kth power C_n^k of the cycle of length n is isomorphic to the circulant graph $G(n; 1, \ldots, k)$.

For $n \geq k$ and $n \neq k+1$ let $F_k(n)$ denote the largest integer for which there exists a vertex-critical k-chromatic graph of order n with $F_k(n)$ edges. It is clear that $F_k(n) \geq f_k(n)$, where $f_k(n)$ is the similarly defined quantity for critical k-cromatic graphs, and hence by Theorem 1 there exist constants c'_k such that $F_k(n) \geq c'_k n^2$ for all $n \geq k$, $n \neq k+1$. B. Zeidl [16] obtained $F_k(n) \geq \frac{1}{6}n^2$ for $k = 4, 5$. B. Toft [14] applied the classical theorem of B. Toft [12], stating that G is vertex-critical if and only if every block of its complement $\bar{G}$ is the complement of a vertex-critical graph. Using odd cycles and, for $k = 6$, using the 4-chromatic examples of Zeidl, he then obtained the bounds $F_5(n) \geq \frac{1}{4}n^2$ and $F_6(n) \geq \frac{3}{10}n^2$ for infinitely many values of n. We extend the previously known results by showing the following.

Theorem 4. *For every $k \geq 3$,*

$$F_k(n) \geq \frac{(k-3)n^2 + (k+1)n}{2(k-1)}$$

for infinitely many values of n.

Proof. Let $k \geq 3$ and $m \geq 2$ be fixed integers. Let $n = m(k-1)+1$ and let $G = G(n; 1, \ldots, k-2)$ $(= C_n^{k-2})$. Consider any $(k-1)$-coloring φ of $H = G - x_0$. Since any $k-1$ consecutive vertices $x_i, x_{i+1}, \ldots, x_{i+k-2}$, $i = 1, 2, \ldots, n-k+1$, induce a complete subgraph in H, φ assigns $k-1$ different colors to these vertices. Consequently $\varphi(x_{i+k-1}) = \varphi(x_i)$ must hold for every $i = 1, 2, \ldots, n-k+1$, and the only possible partitioning $S_1 \cup S_2 \cup \ldots \cup S_{k-1}$ of the vertices of H into $k-1$ color classes is given by $S_t = \{x_t, x_{t+(k-1)}, x_{t+2(k-1)}, \ldots, x_{t+(m-1)(k-1)}\}$, $t = 1, 2, \ldots, k-1$. Since the cyclic distance between any pair of vertices x_i, x_j of H belonging to the same set S_t is either equal to $m(k-1)$ (if $|i-j| < n/2$) or equal to $m(k-1)+1$ (if $|i-j| > n/2$), for some integer $m > 0$, it follows that the classes S_t, $t = 1, 2, \ldots, k-1$, define a proper $(k-1)$-coloring φ of H.

Let $\tilde{G}$ be the graph on the same set of n vertices as G, obtained from G by adding all possible new edges $x_i x_j$ between vertices at distance d, where $d \equiv 2, 3, \ldots, k-2$ modulo $k-1$, and $d > k-1$. By the preceding argument, φ remains a proper $(k-1)$-coloring of $\tilde{G} - x_0$, and clearly φ remains the only possible such coloring. One cannot extend φ to a $(k-1)$-coloring of $\tilde{G}$, since the neighbors of x_0 must receive all $k-1$ different colors, hence $\tilde{G}$ is k-chromatic. Combining the vertex-transitivity of $\tilde{G}$ with the existence of a $(k-1)$-coloring of $\tilde{G} - x_0$, it follows that $\tilde{G}$ is vertex-critical.

$\tilde{G}$ has precisely $((k-3)n^2 + (k+1)n)/(2(k-1))$ edges, as may easily be checked, giving the desired lower bound on $F_k(n)$. $\square$

Although the vertex-critical k-chromatic graphs of Theorem 4 are quite dense, they do not, in a certain sense, behave very robustly with respect to edge-deletion. In fact, one can obtain from any of these particular graphs a $(k-1)$-colorable graph by deleting just one edge (say the edge x_0x_1, which one can easily check). It is natural to ask if it is possible to obtain vertex-critical graphs with the property that the deletion of any prescribed size does not lower the chromatic number. Indeed, this question goes back to G.A. Dirac, who asked a little more modestly whether there exists a vertex-critical graph without any critical edges. Inspired by this question, P. Erdős asked in [4] if there exists a positive function $f : \mathbb{N} \to \mathbb{N}$ such that for every $k \geq 4$ there exists a k-chromatic vertex-critical graph for which no set of at most $f(n)$ edges is critical, where n is the order of the graph.

The fact that there exists a vertex-critical graph for which no single edge is critical was shown by J.I. Brown [1], who gave a 5-chromatic example of order 17. No other examples seem to have appeared in the literature. Some further examples with $k = 5, 6$ and 8 were announced by T.R. Jensen and B. Toft [7] in connection with the above problem raised by Erdős.

In the main part of this section we construct for every $k \geq 5$ an infinite family of vertex-critical graphs of chromatic number k having no critical edges. Indeed, these constructions show the stronger result that for every $m > 0$ there is a k-chromatic vertex-critical graph with the property that no set of m edges that are all incident with the same vertex is critical. However, it remains an interesting open problem if there exists any example of a 4-chromatic vertex-critical graph without a critical edge.

The constructions of the next theorem are similar to the constructions of Theorem 4, but the details are more complicated. The idea of the proof of Theorem 4 might be thought of as follows. For fixed $k \geq 3$ the order n of G is required to be congruent to 1 modulo $k-1$. The vertices of G are arranged in cyclic order, hence after deleting any vertex x, the remaining vertices are naturally arranged in linear order, and a proper $(k-1)$-coloring φ_x of $G-x$ is obtained by assigning colors in order:

$$\begin{gathered}1,2,3,\ldots,k-2,k-1,\\ 1,2,3,\ldots,k-2,k-1,\\ \vdots\\ 1,2,3,\ldots,k-2,k-1.\end{gathered}$$

The graph G can now be defined as the (unique) maximal graph of order n for which φ_x is a proper coloring of $G-x$ for every vertex x of G. It remains to prove that G with this definition is also k-chromatic.

The idea of the following constructions is analogous, except that the congruency restriction on n and the $(k-1)$-colorings φ_x have changed. For odd value of k the coloring looks as follows.

$$\begin{gathered}
1,2,\ldots,1,2,3,4,\ldots,3,4,(\ldots),k-2,k-1,\ldots,k-2,k-1,\\
1,2,\ldots,1,2,3,4,\ldots,3,4,(\ldots),k-2,k-1,\ldots,k-2,k-1,\\
\vdots\\
1,2,\ldots,1,2,3,4,\ldots,3,4,(\ldots),k-2,k-1,\ldots,k-2,k-1.
\end{gathered}$$

where each color occurs exactly m times in each repeated row, $m > 0$. Hence the same coloring is repeated with period $m(k-1)$, and this number therefore is required to divide $n-1$, the order of $G-x$.

For even k, the coloring becomes more complicated. Each period naturally becomes twice as long as in the case of odd k as shown:

$$\begin{gathered}
1,2,1,2,\ldots,1,2,\\
3,4,3,4,\ldots,3,4,\\
\vdots\\
k-3,k-2,k-3,k-2,\ldots,k-3,k-2,\\
k-1,1,k-1,1,\ldots,k-1,1,\\
2,3,2,3,\ldots,2,3,\\
\vdots\\
k-2,k-1,k-2,k-1,\ldots,k-2,k-1.
\end{gathered}$$

The graph G could now be defined as before as the maximal graph of order n for which the appropriate coloring of $G-x$, as just described, is a proper $(k-1)$-coloring for every $x \in V(G)$. This is not exactly what will happen, since it turns out that most of the edges of this maximal graph are redundant for the proof. Instead we will start by defining a graph with a smaller set of edges, but sufficiently many to ensure that it is a vertex-critical graph, where each vertex-deleted subgraph is uniquely $(k-1)$-colorable (with the coloring defined above). From this graph it is straightforward to add the necessary additional edges to complete the construction of the graph satisfying the requirements of the Theorem itself.

As it would expected from the discussion above, most of the proof of Theorem 5 falls into two cases, of odd and even k, where the latter tends to be the more tedious.

Theorem 5. *Let $k \geq 5$ and $m \geq 1$ be integers. Define a set of positive integers*

$$D_{k,m} = \begin{cases} \{1,3,5,\ldots,2m-3\} \cup [2m-1,(k-3)m+1] & \text{if } k \text{ is odd,} \\ \\ \{1,3,5,\ldots,2m-3\} \cup [2m-1,(k-4)m+2] & \\ \cup[(k+2)m-1, 2(k-2)m+1] & \text{if } k \text{ is even.} \end{cases}$$

Define

$$n_{k,m} = \begin{cases} (k-1)m & \text{for } k \text{ odd, and} \\ 2(k-1)m & \text{for } k \text{ even} \end{cases}$$

and let $N \equiv 1$ modulo $n_{k,m}$ and $N \geq 2(k-1)m\lceil \frac{m}{2} \rceil + 1$.

Then $G(N; D_{k,m})$ is vertex-critical and k-chromatic. Furthermore $G(N; D_{k,m})$ is a spanning subgraph of a vertex-critical k-chromatic graph $G_{N,k,m}$ with the additional property that no set of fewer than m edges incident to the same vertex is critical.

(Incidentally, the example of such a graph with $k = 5$ and $m = 2$ given by Brown [1] is precisely the graph $G(17; D_{5,2})$.)

Proof. Fix the numbers k, m and N as above, say with $N = qn_{k,m} + 1$, where q is a positive integer, and let $G = G(N; D_{k,m})$.

For $1 \leq i \leq N - n_{k,m}$ let H_i denote the subgraph of $G - x_0$ induced by the $n_{k,m}$ vertices $x_i, x_{i+1}, \ldots, x_{i+n_{k,m}-1}$. We show that the maximum size $\alpha(H_i)$ of a set A of independent vertices in H_i is at most $n_{k,m}/(k-1)$. By symmetry we may assume for notational convenience that $i = 1$ and hence that $V(H_i) = \{x_1, x_2, \ldots, x_{n_{k,m}}\}$. We may also assume $x_1 \in A$, for if s is the smallest index such that $x_s \in A$, then the set $\{x_{a-s+1} : x_a \in A\}$, which contains x_1, may serve in place of A.

Case 1: k is odd. Define for $j = 1, 3, 5, \ldots, 2m-1$ sets $B_j = \{j, j+1, 2m+j, 2m+j+1, \ldots, (k-3)m+j, (k-3)m+j+1\}$. By the definition of G, every vertex set $\{x_b : b \in B_j\}$ induces a complete subgraph in G, and it follows that $\{x_b \in A : b \in B_j\}| \leq 1$. Since the sets B_j partition the vertex set of H_1 we obtain the bound $|A| \leq m$ as desired.

Case 2: k is even. Note that if $x_a, x_b \in A$ where $a < b$, then $b-a > 1$ and $b-a \notin [2m-1, (k-4)m+2] \cup [(k+2)m-1, 2(k-2)m+1]$. In particular, if $x_b \in A$ for any $b > 1$, then $b \notin [2m, (k-4)m+3] \cup [(k+2)m, 2(k-2)m+2]$.

Let

$$U = [1, 2m] \cup [(k-4)m+1, (k+2)m] \cup [2(k-2)m+1, 2(k-1)m],$$

and for each $j = 1, 3, 5, \ldots, 2m-1$ define subsets of U as follows.

$$\begin{aligned}
B_j^{(1)} &= \{j, j+1, (k-4)m+j, (k-4)m+j+1\}, \\
B_j^{(2)} &= \{(k-4)m+j, (k-4)m+j+1, (k-2)m+j, \\
&\quad\ (k-2)m+j+1\}, \\
B_j^{(3)} &= \{(k-2)m+j, (k-2)m+j+1, km+j, km+j+1\}, \\
B_j^{(4)} &= \{km+j, km+j+1, 2(k-2)m+j, 2(k-2)m+j+1\}, \\
B_j^{(5)} &= \{j, j+1, 2(k-2)m+j, 2(k-2)m+j+1\}.
\end{aligned}$$

Let

$$U_j = B_j^{(1)} \cup B_j^{(2)} \cup B_j^{(3)} \cup B_j^{(4)} \cup B_j^{(5)} \text{ for } j = 1, 3, 5, \ldots, 2m-1.$$

Observe that $\{x_b : b \in B_j^{(s)}\}$ induces a complete subgraph K_4 in G for all $s = 1, 2, 3, 4, 5$ and $j = 1, 3, 5, \ldots, 2m-1$, that $A \subset \{x_b : b \in U\}$, and that $U_1 \cup U_3 \cup \ldots \cup U_{2m-1}$ is a partitioning of U into m subsets.

For any $j \in \{1, 3, 5, \ldots, 2m-1\}$, an inspection of the sets $B_j^{(s)}$, $s = 1, 2, 3, 4, 5$, using the fact $|A \cap \{x_b : b \in B_j^{(s)}\}| \leq 1$, shows that $|A \cap \{x_b : b \in U_j\}| \leq 2$. Since the sets U_j partition the subscripts b with $x_b \in A$ into m subsets each of size at most 2, we conclude that $|A| \leq 2m$ in the case when k is even.

We have now established for all $k \geq 5$ that

$$\alpha(H_i) \leq n_{k,m}/(k-1) \text{ for all } i = 1, 2, \ldots, N - n_{k,m} \tag{6}$$

Consider any $(k-1)$-coloring φ of $G - x_0$ with corresponding partitioning $A_1, A_2, \ldots, A_{k-1}$ of the vertex set $\{x_1, x_2, \ldots, x_{qn_{k,m}}\}$ into color classes.

It follows from (6) that $|A_j \cap V(H_i)| = 2m$ for all $i = 1, 2, \ldots, N - n_{k,m}$ and $j = 1, 2, \ldots, k-1$, since H_i is a subgraph of $G - x_0$ of order $n_{k,m}$. In particular, since $|A_j \cap V(H_i)| = |A_j \cap V(H_{i+1})|$ for all $i = 1, 2, \ldots, N - n_{k,m} - 1$, we deduce

$$\varphi(x_i) = \varphi(x_{i+n_{k,m}}) \text{ for all } i = 1, 2, \ldots, N - n_{k,m} - 1 \tag{7}$$

In other words, φ is periodic with period $n_{k,m}$ on the vertex set of $G - x_0$, w.r.t. the linear ordering $x_1 < x_2 < \cdots < x_N$.

We are going to prove the uniqueness of φ as a $(k-1)$-coloring of $G-x_0$. In order to do this, we continue by studying a symmetric relation $\sim$ defined on the set of indices $\{1,2,\ldots,N-1\}$, by $a \sim b$ if and only if every $(k-1)$-coloring φ' of $G-x_0$ satisfies $\varphi'(x_a) \neq \varphi'(x_b)$.

First it is clear from the construction of G that

$$a \sim b \quad \text{if} \quad b-a \in D_{k,m} \tag{8}$$

It follows from (7) and (8) that $\varphi'(x_{a+n_{k,m}-d}) \neq \varphi'(x_{a+n_{k,m}}) = \varphi'(x_a)$ for any $a = 1,2,\ldots,N-n_{k,m}$ and $d \in D_{k,m}$, and therefore

$$a \sim b \text{ if } \; b-a = n_{k,m}-d, \;\; \text{where } d \in D_{k,m} \tag{9}$$

Denote for $i = 1,2,\ldots,n_{k,m}$ the vertex $x_{N-n_{k,m}+i}$ in $G-x_0$ by y_i. Then it follows from (7) that $\varphi(y_i) = \varphi(x_i)$ holds. For $1 \leq i,j \leq n_{k,m}$ the vertices x_i and y_j are adjacent in G, and hence in $G-x_0$, if and only if $n_{k,m}+i-j+1 \in D_{k,m}$. Thus for $d \in D_{k,m} \setminus \{1\}$ it follows that

$$\varphi'(x_{a+n_{k,m}-d+1}) = \varphi'(y_{a+n_{k,m}-d+1}) \neq \varphi'(x_a)$$

for every $a = 1,2,\ldots,d-1$, which implies

$$a \sim b \text{ if } \; b-a = n_{k,m}-d+1 \text{ and } a < d, \;\; \text{where } d \in D_{k,m} \setminus \{1\} \tag{10}$$

Using the definition of G together with (7)–(10) we will argue that the only possible pairs of distinct vertices x_a, x_b, where $1 \leq a,b \leq n_{k,m}$, that may satisfy $\varphi(x_a) = \varphi(x_b)$ are given by

$$|a-b| \in F_{k,m} \tag{11}$$

where

$$F_{k,m} = \begin{cases} \{2,4,\dots,2m-2\} & \text{for } k \text{ odd, and} \\ \\ \{2,4,\dots,2m-2\}\cup \\ [(k-4)m+3,(k+2)m-2] & \text{for } k \text{ even} \end{cases} \tag{12}$$

Assume that k is odd. It is clear from the definition of G that $1 \le a < b \le n_{k,m} = (k-1)m$ and $\varphi(x_a) = \varphi(x_b)$ can only hold if

$$b - a \in \{2,4,\dots,2m-2\} \cup [(k-3)m+1,(k-1)m].$$

Suppose that $b-a \in [(k-3)m+1,(k-1)m]$. By (7), $\varphi(x_b) = \varphi(x_{a+n_{k,m}})$ holds, and therefore x_b and $x_{a+n_{k,m}}$ are nonadjacent in G, hence $n_{k,m}-(b-a) \notin D_{k,m}$. This implies that $n_{k,m}-(b-a)$ is an even number, since $n_{k,m}-(b-a) \le (k-1)m-(k-3)m-1 \le 2m-1$. Let $d = n_{k,m}-(b-a)+1$; then d belongs to $D_{k,m} \setminus \{1\}$, and clearly $a < d$. By (10) this implies $a \sim b$, contradicting the assumption $\varphi(x_a) = \varphi(x_b)$. This shows (11) when k is odd.

The argument is similar when k is even.

We proceed to show that the restriction (11) on the color classes of any coloring of $G - x_0$ defines a unique proper coloring of $G - x_0$. This will again be dealt with by splitting into two cases. The following observations will also turn out useful when defining the graph $\tilde{G}$ at the last step of the proof.

Case 1: k is odd. By (6), each color class A_j, for $j = 1,2,\dots,k-1$, of φ contains precisely m vertices of $V(H_1) = \{x_1,x_2,\dots,x_{(k-1)m}\}$, and it follows from (11) and (12) that the only possibility, up to permutation of the color classes, is

$$A_1 \cap V(H_1) \;=\; \{x_1,x_3,x_5,\dots,x_{2m-1}\},$$

$$
\begin{aligned}
A_2 \cap V(H_1) &= \{x_2, x_4, x_6, \ldots, x_{2m}\},\\
A_3 \cap V(H_1) &= \{x_{2m+1}, x_{2m+3}, x_{2m+5}, \ldots, x_{4m-1}\},\\
A_4 \cap V(H_1) &= \{x_{2m+2}, x_{2m+4}, x_{2m+6}, \ldots, x_{4m}\},\\
&\vdots\\
A_{k-2} \cap V(H_1) &= \{x_{(k-3)m+1}, x_{(k-3)m+3}, \ldots, x_{(k-1)m-1}\} \text{ and}\\
A_{k-1} \cap V(H_1) &= \{x_{(k-3)m+2}, x_{(k-3)m+4}, \ldots, x_{(k-1)m}\}.
\end{aligned}
$$

To argue that there are no other choices of coloring, it is clear from (11) and (12) that $\varphi(x_i) = \varphi(x_1)$ implies $i \in \{1, 3, 5, \ldots, 2m-1\}$. Since this gives precisely $m = n_{k,m}/(k-1)$ possibilities for the value of i, (6) implies that the converse implication holds as well. Similarly for the remaining $k-2$ color classes.

By (7), the partitioning of the entire vertex set of $G - x_0$ into $k-1$ color classes is therefore, if it exists, uniquely determined.

In order to show that this uniquely determined partitioning corresponds to a proper coloring of $G - x_0$, it remains to check for $j = 1, 2, \ldots, k-1$ that the cyclic distance $d_N(a,b)$ between the indices of any pair $x_a, x_b \in A_j$ of vertices in the same color class satisfies $d_N(a,b) \notin D_{k,m}$.

Assume $x_a, x_b \in A_j$, where $1 \le a < b \le N$. It is clear from (7) and from the listing of the intersections of the color classes with $V(H_1)$, that if $b - a \le 2m$, then $d_N(a,b) \in \{2, 4, \ldots, 2m-2\}$. Moreover, if $2m < b - a \le \lfloor N/2 \rfloor = \frac{1}{2}q(k-1)m$, then there exists an integer r with $0 \le r \le \frac{1}{2}q - 1$ such that

$$d_N(a,b) - r(k-1)m \in \{(k-3)m+2, (k-3)m+4, \ldots, (k+1)m-2\}.$$

If q is even and $r = \frac{1}{2}q - 1$, the possible values of $d_N(a,b)$ are fewer, namely given by

$$d_N(a,b) - r(k-1)m \in \{(k-3)m+2, (k-3)m+4, \ldots, (k-1)m\},$$

since $d_N(a,b) \leq \lfloor N/2 \rfloor$ holds. Note that this is the only value of r which gives rise to fewer than $2m-1$ possible distances.

Finally, if $\lfloor N/2 \rfloor < b-a$, then there exists an integer r satisfying $0 \leq r \leq \frac{1}{2}q - 1$ such that

$$d_N(a,b) - r(k-1)m \in \{(k-3)m+3, (k-3)m+5, \ldots, (k+1)m-1\}.$$

Again when q is even and $r = \frac{1}{2}q - 1$ there is, as before, a single exceptional case, namely

$$d_N(a,b) - r(k-1)m \in \{(k-3)m+3, (k-3)m+5, \ldots, (k-1)m-1\}.$$

These observations can be summarized as follows:

$$x_a, x_b \in A_j \Rightarrow \begin{cases} d_N(a,b) \in \{2,4,\ldots,2m-2\}, \text{ or} \\ d_N(a,b) - r(k-1)m \in [(k-3)m+2, (k+1)m-1], \\ \qquad\qquad \text{where } 0 \leq r < \frac{1}{2}q - 1, \text{ or} \\ d_N(a,b) \in [\frac{1}{2}q(k-1)m - 2m + 2, \frac{1}{2}q(k-1)m], \text{ and } q \text{ is even.} \end{cases} \tag{13}$$

Comparing with the definition of $D_{k,m}$, it follows that the classes A_j indeed define a proper coloring of G.

<u>Case 2: k is even.</u> By (6), each color class A_j for φ satisfies $|A_j \cap V(H_1)| = 2m$, $j = 1, 2, \ldots, k-1$. Consider the intersection A of $V(H_1)$ with any fixed color class for φ; in particular $|A| = 2m$.

Let $a_1 = \min\{a : x_a \in A\}$, and define inductively for $\ell > 0$

$$h_\ell = |\{a : x_a \in A \text{ and } \ a_\ell \le a \le a_\ell + 2m - 2\}|,$$

and

$$a_{\ell+1} = \min\{a : x_a \in A \text{ and } \ a > a_\ell + 2m - 1\}$$

(if the minimum does not exist, define $a_{\ell+1} = a_{\ell+2} = \cdots = \infty$ and $h_{\ell+1} = h_{\ell+2} = \cdots = 0$).

It follows from (11, 12) that for all $\ell > 0$,

$$a_{\ell+1} \ge a_\ell + (k-4)m + 1 + 2h_\ell$$

with equality only if $\{x_{a_\ell}, x_{a_\ell+2}, \ldots, x_{a_\ell+2h_\ell-2}\} \subseteq A$. Since $a_1 \ge 1$, and since $h_\ell > 0$ if $a_\ell < \infty$, it follows inductively for $\ell \ge 0$ that

$$a_{\ell+1} \ge a_1 + \ell((k-4)m+1) + 2\sum_{i=1}^{\ell} h_i. \tag{14}$$

Fix $\ell \ge 0$ such that $a_{\ell+1} < \infty = a_{\ell+2}$ (which is possible, since $A \ne \emptyset$ implies $a_1 < \infty$), then by (11) and (12),

$$a_{\ell+1} + 2h_{\ell+1} - 2 \le a_1 + (k+2)m - 2. \tag{15}$$

Combining (14) with (15) gives

$$\ell((k-4)m+1) \le (k+2)m - 2\sum_{i=1}^{\ell+1} h_i = (k-2)m,$$

where the equality follows from $\sum_{i=1}^{\ell+1} h_i = |A| = 2m$. Since $k \ge 6$, we deduce that $\ell \le \frac{(k-2)m}{(k-4)m+1}$ and hence $\ell < 2$. Therefore $a_3 = a_4 = \cdots = \infty$ and $h_3 = h_4 = \cdots = 0$.

It now follows that there exist a_1, a_2 with $1 \le a_1 < a_2 \le 2(k-1)m$ such that $A \subseteq A^{(1)} \cup A^{(2)}$, where $A^{(\ell)} = \{x_a : a \in [a_\ell, a_\ell + 2m - 2]\}$,

$\ell = 1, 2$, and where $a_2 \geq a_1 + 2m - 1$. It is clear from (11) that $|A \cap A^{(\ell)}| \leq m$, and since $|A| = 2m$ we must have

$$A = \{x_a : a = a_1, a_1{+}2, a_1{+}4, \ldots, a_1{+}2m{-}2, a_2, a_2{+}2, \ldots, a_2{+}2m{-}2\} \tag{16}$$

It is easy to check from (11) that

$$a_2 - a_1 \in [(k-2)m + 1, km] \tag{17}$$

Consequently, if $a_2 \leq km$, then $a_1 \leq 2m - 1$, and if $a_1 \geq (k-2)m$, then $a_2 \geq 2(k-2)m + 1$.

The analysis holds with each set $A_j \cap V(H_1)$ in place of A. Furthermore the sets $A_j \cap V(H_1)$ partition $V(H_1)$, $j = 1, 2, \ldots, k-1$. By choice of indexing for the classes $A_1, A_2, \ldots, A_{k-1}$, we may assume $x_1 \in A_1$. Then, by (16), $\{x_1, x_3, x_5, \ldots, x_{2m-1}\} \subseteq A_1$, and by (17), $\min\{a : a \geq 2m \text{ and } x_a \in A_1\} \in [(k-2)m + 2, km + 1]$. Likewise $\{x_2, x_4, x_6, \ldots, x_{2m}\} \subseteq A_2$ follows, and similarly $\{x_{(j-1)m+1}, x_{(j-1)m+3}, \ldots, x_{(j+1)m-1}\} \subseteq A_j$ if j is odd, and $\{x_{(j-2)m+2}, x_{(j-2)m+4}, \ldots, x_{jm}\} \subseteq A_j$ if j is even, for $j = 3, 4, \ldots, k-2$. By (17), if $x_a \in A_j$, then $a \geq (j-2)m + (k-2)m + 3$ for each $j = 2, 4, \ldots, k-2$ if $a > jm$, and $a \geq (j-1)m + (k-2)m + 2$ for each $j = 3, 5, \ldots, k-3$ if $a > (j+1)m - 1$. In particular $x_{(k-2)m+1} \notin A_1 \cup A_2 \cup \ldots \cup A_{k-2}$, and therefore $\{x_{(k-2)m+1}, x_{(k-2)m+3}, \ldots, x_{km-1}\} \subseteq A_{k-1}$. Similarly $x_{(k-2)m+2} \notin A_2 \cup A_3 \cup \ldots \cup A_{k-1}$, and hence $\{x_{(k-2)m+2}, x_{(k-2)m+4}, \ldots, x_{km}\} \subseteq A_1$.

It is straightforward to complete the argument and to show that the color classes $A_1, A_2, \ldots, A_{k-1}$ necessarily have the following form, up to permutation of the indices.

$$A_j \cap V(H_1) = \begin{cases} \{x_{(j-1)m+1}, x_{(j-1)m+3}, \ldots, x_{(j+1)m-1}, \\ x_{(j+k-3)m+2}, x_{(j+k-3)m+4}, \ldots, x_{(j+k-1)m}\} & \text{for } j \text{ odd}, \\ \{x_{(j-2)m+2}, x_{(j-2)m+4}, \ldots, x_{jm}, \\ x_{(j+k-2)m+1}, x_{(j+k-2)m+3}, \ldots, x_{(j+k)m-1}\} & \text{for } j \text{ even}, \end{cases} \tag{18}$$

for $j = 1, 2, \ldots, k-1$.

Together with (7), we have by (18) that the coloring φ of $G - x_0$, if it exists, is uniquely determined.

It remains to check that (18) and (7) together determine a proper coloring of $G - x_0$. Let $x_a, x_b \in V(G - x_0)$ be any pair of vertices belonging to the same class A_j, such that $d_N(a, b) \equiv s$ modulo $2(k-1)m$ $(0 \leq s < 2(k-1)m)$. Without loss of generality we can assume that $a < b$ and $x_a \in V(H_1)$. If $x_b \in V(H_1)$, we deduce from (18) that $d_N(a, b) \in \{0, 2, 4, \ldots, 2m-2\} \cup [(k-2)m+1, (k+2)m-3]$. If $2(k-1)m < b$ and $d_N(a, b) = b - a$, then either $s \in \{0, 2, 4, \ldots, 2m-2\} \cup [(k-2)m+1, (k+2)m-3]$ or $2(k-1)m - s \in \{0, 2, 4, \ldots, 2m-2\} \cup [(k-2)m+1, (k+2)m-3]$ by (18) and (7), and therefore $s \in \{0, 2, 4, \ldots, 2m-2\} \cup [(k-4)m+3, (k+2)m-3] \cup \{2(k-2)m+2, 2(k-2)m+4, \ldots, 2(k-1)m-2\}$. If $d_N(a, b) = N - (b-a)$, it similarly follows from (18) and (7) (when taking into account that $N \equiv 1$ modulo $2(k-1)m$) that either $s - 1 \in \{0, 2, 4, \ldots, 2m-2\} \cup [(k-2)m+1, (k+2)m-3]$ and $d_N(a, b) > 2(k-1)m$, or $2(k-1)m - s + 1 \in \{2, 4, \ldots, 2m-2\} \cup [(k-2)m+1, (k+2)m-3]$, implying $s \in \{1, 3, 5, \ldots, 2m-1\} \cup [km+1, (k+2)m-2]$ and $d_N(a, b) > 2(k-1)m$, or $s \in [(k-4)m+4, km] \cup \{2(k-2)m+3, 2(k-2)m+5, \ldots, 2(k-1)m-1\}$. We conclude that in any case

$s \in \{0, 2, 4, \ldots, 2m-2\} \cup [(k-4)m+3, (k+2)m-3] \cup [2(k-2)m+2, 2(k-1)m-1]$, or possibly $s \in \{1, 3, 5, \ldots, 2m-1\}$ if $d_N(a,b) > 2(k-1)m$, and hence the partitioning $A_1, A_2, \ldots, A_{k-1}$ of $V(G - x_0)$ given by (18) and (7) indeed determines a proper coloring of $G - x_0$ in Case 2.

Henceforth φ denotes the unique $(k-1)$-coloring of $G - x_0$. We will describe a supergraph $\tilde{G}$ of G with the following properties.

i) $V(\tilde{G}) = V(G)$,
ii) $\tilde{G}$ is vertex-transitive,
iii) φ is a $(k-1)$-coloring of $\tilde{G} - x_0$, and
iv) x_0 is joined in $\tilde{G}$ to at least m vertices from each color class A_j of φ, for $j = 1, 2, \ldots, k-1$.

Again we separate into the cases of odd and even k.

Case 1: k is odd. Recall that φ is given by

$$\begin{aligned}
A_1 \cap V(H_1) &= \{x_1, x_3, x_5, \ldots, x_{2m-1}\},\\
A_2 \cap V(H_1) &= \{x_2, x_4, x_6, \ldots, x_{2m}\},\\
A_3 \cap V(H_1) &= \{x_{2m+1}, x_{2m+3}, x_{2m+5}, \ldots, x_{4m-1}\},\\
A_4 \cap V(H_1) &= \{x_{2m+2}, x_{2m+4}, x_{2m+6}, \ldots, x_{4m}\},\\
&\vdots\\
A_{k-2} \cap V(H_1) &= \{x_{(k-3)m+1}, x_{(k-3)m+3}, \ldots, x_{(k-1)m-1}\},\\
A_{k-1} \cap V(H_1) &= \{x_{(k-3)m+2}, x_{(k-3)m+4}, \ldots, x_{(k-1)m}\},
\end{aligned}$$

together with (7). Since x_0 is adjacent in G to each of the $2m$ vertices

$$x_1, x_3, \ldots, x_{2m-3}, x_{2m-1}, x_{N-2m+1}, x_{N-2m+3}, \ldots, x_{N-1},$$

and since N satisfies $N \equiv 1$ modulo $(k-1)m$, we have that x_0 is adjacent in G to m elements of each of the classes A_1 and A_{k-1}. Furthermore, x_0 is adjacent in G to every vertex x_a with $a \in [2m, (k-3)m]$, and therefore to m elements of each set $A_3, A_4, \ldots, A_{k-3}$. However, the only elements of $A_2 \cup A_{k-2}$ adjacent to x_0 in G are $x_{2m-1}, x_{N-(k-3)m-1} \in A_2$ and $x_{(k-3)m+1}, x_{N-2m+1} \in A_{k-2}$.

If $m \leq 2$ we let $\tilde{G} = G$. Otherwise, let

$$D' = \bigcup_{1 \leq i \leq r} \{i(k-1)m + 2m - 1, i(k-1)m + (k-3)m + 1\},$$

where r is an integer such that $2r + 2 \geq m$. Since N satisfies $N > 2(k-1)m\lceil \frac{m}{2} \rceil$, it is possible to choose r such that, in addition, $\max D' = r(k-1)m + (k-3) + 1 < N/2$. Let $\tilde{G} = G(N; D_{k,m} \cup D')$.

It is straightforward to check that $\tilde{G}$ satisfies i), ii) and iv) above. From the argument showing that the classes $A_1, A_2, \ldots, A_{k-1}$ define a proper coloring of $G - x_0$ in the case when k is odd, it can be checked that if $x_a, x_b \in A_j$ $(1 \leq j \leq k-1)$ and $d_N(a,b) > (k-1)m$, then $d_N(a,b) \not\equiv 2m-1, (k-3)m+1$ modulo $(k-1)m$. Therefore iii) holds, and the proof of Case 1 is finished.

Case 2: k is even. Recall that φ is given by (18) together with (7). As in the previous case, x_0 is adjacent in G to m vertices $x_1, x_3, \ldots, x_{2m-1} \in A_1$ and m vertices $x_{N-2m+1}, x_{N-2m+3}, \ldots, x_{N-1} \in A_{k-1}$. Furthermore, x_0 is adjacent to every vertex x_a with $a \in [2m, (k-4)m] \cup [(k+2)m, 2(k-2)m]$ and therefore to at least m elements of each set $A_3, A_4, \ldots, A_{k-3}$. Finally, x_0 is adjacent to precisely 4 elements

$$x_{2m-1}, x_{(k+2)m-1}, x_{N-2(k-2)m-1}, x_{N-(k-4)m-2}$$

of A_2 and precisely 4 elements

$$x_{(k-4)m+2}, x_{2(k-2)m+1}, x_{N-(k+2)m+1}, x_{N-2m+1}$$

of A_{k-2}.

If $m \leq 4$, let $\tilde{G} = G$. Otherwise, let $D' = \{a > 2(k-1)m : a \equiv 2m-1, (k-4)m+2, (k+2)m-1,$ or $2(k-2)+1$ modulo $2(k-1)m$, and $a \leq 2r(k-1)m\}$, where $4r+4 \geq m$, and $2r(k-1)m + 2(k-2)+1 < N/2$ (such r exists, since $N > 2(k-1)m\lceil \frac{m}{2} \rceil$). The argument showing that i)–iv) hold is similar to the argument in Case 1 for odd k. This finishes the proof of Case 2 and of the Theorem. $\square$

4 Some remarks and open problems

Although we have not proved that the graph $G_{N,k,m}$ of Theorem 5 has no critical set of fewer than m edges, this property has been established by computer for some of the smaller examples of these graphs. This leads to the following

Conjecture 1. For every $k \geq 5$ and every $m \geq 2$ there exist infinitely many values of N for which no set of m edges of $G_{N,k,m}$ is critical.

Also, it seems likely that the class of graphs of Theorem 5 contains graphs with an even stronger property, namely the following.

Conjecture 2. For every $k \geq 5$ and every $m \geq 2$ there exist integers $N(m), M(m) > 0$ such that the vertex-critical k-chromatic graph $G_{N(m),k,M(m)}$ contains m edge-disjoint k-chromatic subgraphs.

The following theorem, which strengthens the result of J.I. Brown [1], gives some support to Conjecture 2.

Theorem 6. *There exists a vertex-critical graph G containing two edge-disjoint subgraphs each of chromatic number $\chi(G)$. In particular, $G_{65,5,4}$ is such a 5-chromatic graph.* $\square$

(The result has been obtained with a computer, and a proof of Theorem 6 will not be given here.)

The graph $G_{65,5,4}$ is not the smallest example satisfying Theorem 6. The graph $G(61; 1, 3, 5, 6, 7, 18, 19, 30)$ is a vertex-critical 5-chromatic 16-regular graph of order 61 which, rather remarkably, contains two edge-disjoint 5-chromatic 8-regular critical subgraphs $G(61; 1, 6, 7, 18)$ and $G(61; 3, 5, 19, 30)$ (they are nonisomorphic). However, the assertion that both 8-regular graphs are 5-chromatic has again been established with the aid of a computer (so has their criticality, but that is not the difficult part).

Finally, a question due to G.A. Dirac remains unanswered. Obtaining an answer seems beyond reach of the methods used in this paper.

Problem. *Is there a vertex-critical 4-chromatic graph without critical edges?*

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