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On semiextensions and circuit double covers

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Abstract

We introduce a concept of a semiextension of a cycle, and we conjecture a simple necessary and sufficient condition for its existence. It is shown that our conjecture implies a strong form of the circuit double cover conjecture. We prove that the conjecture is equivalent to its restriction to cubic graphs, and we show that it holds for every cycle which is a spanning subgraph of the given graph.

Keywords: circuit double cover, cycles in graphs, semiextension.

1 Introduction

Graphs in this paper are finite and undirected, and they may possess multiple edges and loops. The number of edges (counting loops twice) incident to a vertex v in the graph G is its *degree*, denoted by $d_G(v)$. We use the term *circuit* for a connected 2-regular graph and the term *cycle* for a connected graph all vertices of which have positive even degrees. A *circuit double cover* (CDC) of a graph G is a tuple $(C_i : i \in I)$, such that C_i is a circuit in G for every $i \in I$, and such that $|\{i \in I : e \in E(C_i)\}| = 2$ holds for every $e \in E(G)$. In other words, every edge of G is covered twice by circuits, where the same circuit may appear repeated.

The following Circuit Double Cover Conjecture (CDCC) is a classical open problem in graph theory.

Conjecture, CDCC. [Szekeres [8] and Seymour [7]] *If G is an edge 2-connected graph, then G has a circuit double cover.*

The following strengthening of CDCC, which we will refer to as the strong circuit double cover conjecture (SCDCC), has been suggested. If true, it is perhaps more readily accessible to inductive proof attempts. In this paper we introduce one particular such line of approach.

Conjecture, SCDCC. [P.D. Seymour, [2], p. 237; and Goddyn [4]] *If G is an edge 2-connected graph, and C is any circuit in G, then there exists a circuit double cover of G which includes C.*

It is an interesting added feature of SCDCC that it implies a longstanding open conjecture of Sabidussi, to which we will return in the next section.

Standard reductions can be applied to show that both CDCC and SCDCC are equivalent to their restrictions to 3-connected cubic graphs. More precisely, if either of the conjectures is false, then at least one counterexample of minimal order is cubic and 3-connected. Thus, equivalently to SCDCC, if G is a 2-connected cubic graph, or any subdivision of such a graph, and C is any circuit in G, there should exist a set of circuits in G which cover every edge of C exactly once and every other edge of G exactly twice. We call the pair (G, C) a *minimal counterexample* to SCDCC if G is edge 2-connected, C is a circuit in G, no CDC of G contains C, and SCDCC holds for every proper subgraph of G. When it is clear from the context which circuit C in G is meant, we will also speak of G as a minimal counterexample to SCDCC. As remarked, if SCDCC is false, then there exists a minimal counterexample (G, C) such that G is a 3-connected cubic graph.

An *extension* of a circuit C is a circuit different from C which contains all the vertices of C. In an approach to SCDCC, Seymour asked during a workshop on cycles in graphs held in Barbados in 1990 whether every circuit of a 3-connected cubic graph G has an extension in G. If the answer to this question were affirmative, then the truth of SCDCC could be derived as follows. For graphs C and D we denote by $C \oplus D$ the subgraph of $C \cup D$ induced by the edges that belong to C or D but not to both.

Proposition 1. *Let G be a cubic graph and C a circuit in G. If there exists an extension of C in G, then G is not a minimal* *counterexample to SCDCC.*

Proof. Assume that D is an extension of C in G. Let $F = G-(E(C)\setminus E(D))$. Then F is, because of $D \neq C$, a proper subgraph of G. The two ends of every edge of C are connected in D by two disjoint paths, hence these paths connect them also in F. So F is 2-connected. We may assume that there exists a set S of circuits in F which cover the edges of D once and all remaining edges of F twice. Adding to S the components of $C \oplus D$, we obtain a set of circuits in G covering the edges of C once and all other edges twice, as desired. ∎

Unfortunately, not every circuit C of any cubic graph has an extension, as demonstrated by Fleischner [1] (see also [6]). Fleischner's example is the circuit C in the graph G of Figure 1. However, in the following we will define structures called semiextensions, of which extensions are special instances. They retain the property that using their existence inductively, as in the proof of Proposition 1, we can construct a CDC including C.

For any graph H, an *H-path* means a path which intersects H precisely in its endvertices.

Definition. A *semiextension* in G of a cycle C is a cycle D in G satisfying $D \cap C \neq \emptyset$ and $E(D) \setminus E(C) \neq \emptyset$, with the further property that whenever G contains a $(C \cup D)$-path having endvertices x and y, with $x \in V(C) \setminus V(D)$, there exists a path between x and y also in $C \oplus D$.

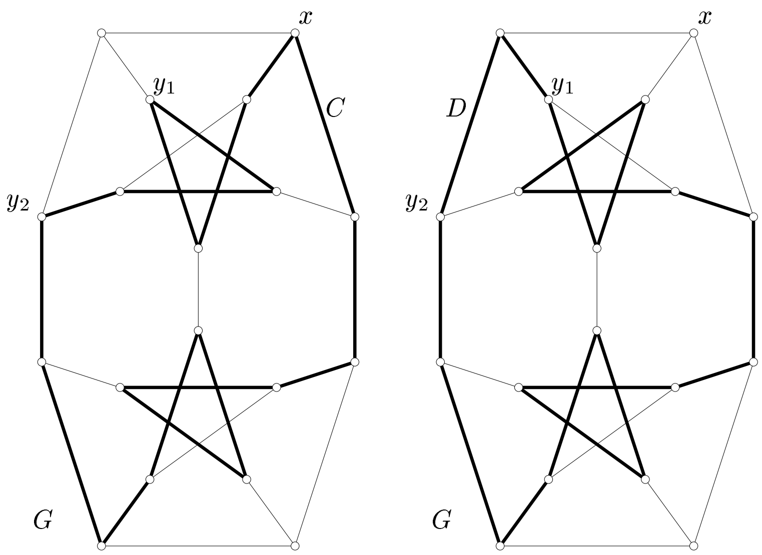

Figure 1. A semiextension D of a circuit C.

Figure 1 shows a graph G, a circuit C in G, and a semiextension D of C in G. In order to check that D is a semiextension of C in this figure, one needs only to check the pairs of vertices (x, y_1) and (x, y_2) in place of the (x, y) that appears in the definition, as x is the unique vertex of G that belongs to C but not to D.

It is clear that an extension of a circuit is a special case of a semiextension. We propose a conjecture which suggests that the trivial necessary conditions for the existence of a semiextension are sufficient, thus suggesting an easily checkable complete characterization of those cycles that allow semiextensions.

Conjecture 1. *There exists a semiextension in G of a cycle $C \subseteq G$, if and only if G contains a cycle C' satisfying both $C' \cap C \neq \emptyset$ and $E(C') \setminus E(C) \neq \emptyset$.*

In the following section, we will show that the truth of Conjecture 1 implies SCDCC.

Conjecture 1 easily implies the following conjecture. In our last section, we will prove that Conjectures 1 and 2 are actually equivalent, in other words, Conjecture 1 can be reduced to the special case of cubic graphs.

Conjecture 2. *For every circuit C in a 2-connected cubic graph G there exists a semiextension of C in G.*

2 From semiextensions to circuit double covers

We will now show that the truth of Conjecture 2 implies the truth of SCDCC.

Theorem 1. *Let G be a 2-connected cubic graph, and let C be a circuit in G. If there exists a semiextension of C in G, then G is not a minimal counterexample to SCDCC.*

Proof. Assume that a semiextension D of C exists in G, and assume that no proper subgraph of G is a counterexample to SCDCC. We will construct a CDC of G containing C, thereby proving our theorem.

Since G is a cubic graph, it is clear that D is a circuit. Let $A_1, \ldots, A_k$ be the components of $C \oplus D$; each such component is again a circuit. Because of the condition $E(D) \setminus E(C) \neq \emptyset$ we have $k > 0$.

For $i = 1, \ldots, k$, we call a $(C \cup D)$-path P in G an *i-ear* if at least one endvertex of P belongs to $V(A_i) \setminus V(D)$ (in particular it belongs to $V(C) \setminus V(D)$). We let H_i denote the (graph-)union of A_i with all i-ears. Since D is a semiextension of C, there exists for every i-ear a path in $C \oplus D$, hence a path in A_i, between its endvertices. Thus every i-ear intersects $C \cup D$ only in vertices of A_i, that is, $V(H_i) \cap V(C \cup D) \subseteq V(A_i)$ holds. Moreover, the following holds.

$$\text{If } u \in V(H_i) \setminus V(D) \text{ and } uv \in E(G), \text{ then } uv \in E(H_i). \tag{1}$$

To see this, we may consider any i-ear P containing u and any path Q in $G - uv$ from v to a vertex of $C \cup D$, and we observe that $P \cup Q + uv$ contains an i-ear with uv as an edge.

Since G is cubic and $C \cap D \neq \emptyset$, there exists an edge e common to C and D. This e is neither an edge of any A_i nor of any i-ear, hence e is not an edge of any H_i, thus H_i is a proper subgraph of G for every $i = 1, \ldots, k$. From the construction of H_i as the union of A_i with paths both ends of which belong to A_i it is immediate that H_i is 2-connected, for $i = 1, \ldots, k$. We will proceed to show that $H_1, \ldots, H_k$ are pairwise disjoint.

Assume $v \in V(H_i) \cap V(H_j)$, $1 \leq i, j \leq k$. If $v \in V(A_i) \cap V(A_j)$, then $i = j$ is clear from the definition of $A_1, \ldots, A_k$. Hence we may assume $v \notin V(A_i)$, implying $v \notin V(C \cup D)$ as noted above, and that there exists an i-ear P containing v. By construction of H_j there exists a (possibly trivial) path Q in H_j from v to a vertex of A_j, such that Q has no interior vertex in $C \cup D$. It follows that $P \cup Q$ contains a $(C \cup D)$-path with one endvertex x in $V(A_i) \setminus V(D)$ and another endvertex y in $V(A_j)$. Since D is a semiextension of C, there exists a path in $C \oplus D$ between x and y. This implies that x and y do not lie on two distinct circuits $A_1, \ldots, A_k$, so again $i = j$ holds. Hence we have $V(H_i) \cap V(H_j) = \emptyset$ for $1 \leq i, j \leq k$ whenever $i \neq j$.

We define

$$F = G - (\bigcup_{i=1}^{k} V(H_i) \setminus V(D)) - (\bigcup_{i=1}^{k} E(H_i) \setminus E(D)).$$

Each element of $E(C) \setminus E(D)$ belongs to some A_i and hence not to F, so F is a proper subgraph of G.

For any $e \in E(G) \setminus E(D)$ it is clear that e is an edge of at most one of the graphs $F, H_1, \ldots, H_k$. By (1), if e has an endvertex that belongs to $V(H_i) \setminus V(D)$ for some $i = 1, \ldots, k$, then e is an edge of H_i. Hence, by definition of F, e belongs to at least one of $F, H_1, \ldots, H_k$, and we conclude that

$$\begin{array}{l} E(G) \setminus E(D) \text{ is the disjoint union of the set} \\ E(F) \setminus E(D) \text{ with the set } \left(\bigcup_{i=1}^{k} E(H_i)\right) \setminus E(D). \end{array} \qquad (2)$$

Next we will show that F is 2-connected. Assume $x \in V(F) \setminus V(D)$ and let Q be any path in G from x to D with no interior vertex in D. To show that F is 2-connected, it suffices to show $Q \subseteq F$; this is sufficient since D is a 2-connected subgraph of F, and since we may then connect each vertex of F to D by two paths that are disjoint except for this common endvertex, since such paths exist in G. Let e be the edge of Q incident to x, and let y be the endvertex of e distinct from x. The fact that x belongs to F but not to D implies, by definition of F, that x does not belong to any H_i for $i = 1, \ldots, k$, hence neither does e, and we deduce from (2) that $e \in E(F)$ holds. It is then clear how to prove $Q \subseteq F$ by straightforward induction on the length of Q. Thus we have shown that F is a 2-connected proper subgraph of G.

We are now ready to construct a CDC of G containing C. By our assumption, F contains a set of circuits $\mathcal{C}_0$ such that every edge of D is contained in exactly one element of $\mathcal{C}_0$, and such that every remaining edge of F is contained in exactly two elements of $\mathcal{C}_0$. Similarly, for each $i = 1, \ldots, k$, the 2-connected proper subgraph H_i contains a set of circuits $\mathcal{C}_i$, such that every edge of A_i is contained in exactly one, all other edges of H_i in exactly two elements of $\mathcal{C}_i$.

Let e be any edge of G. We consider the following five cases. It follows from (2) that this case distinction is exhaustive.

$e \in E(C) \setminus E(D)$,
$e \in E(C) \cap E(D)$,
$e \in E(D) \setminus E(C)$,
$e \in E(F) \setminus (E(C) \cup E(D))$, or
$e \in E(H_i) \setminus (E(C) \cup E(D))$ for some $i = 1, \ldots, k$.

We proceed to show in each case that e is covered by $\bigcup_{i=0,1,\ldots,k} \mathcal{C}_i$ exactly once if $e \in E(C)$, and exactly twice if $e \notin E(C)$.

If $e \in E(C) \setminus E(D)$, then $e \in E(A_i) \subseteq E(H_i)$ for a unique $i = 1, \ldots, k$, so e is covered once by $\mathcal{C}_i$ and by no $\mathcal{C}_j$ for $j \neq i$.

If $e \in E(C) \cap E(D)$, then $e \in E(F)$, but e is not an edge of any H_i. In this case e is covered once by $\mathcal{C}_0$.

If $e \in E(D) \setminus E(C)$, then $e \in E(A_i) \subseteq E(H_i)$ holds for a unique $i = 1, \ldots, k$, and we also have $e \in E(F)$. Now e is covered once by $\mathcal{C}_i$, once by $\mathcal{C}_0$, and by no $\mathcal{C}_j$ with $j \notin \{0, i\}$. Hence altogether e is covered twice.

If $e \in E(F) \setminus (E(C) \cup E(D))$, then e does not belong to any H_i, hence e is covered twice by $\mathcal{C}_0$ and is not covered by any $\mathcal{C}_i$ with $i \neq 0$.

If $e \in E(H_i) \setminus (E(C) \cup E(D))$ for some $i = 1, \ldots, k$, then $e \in$

$E(H_i) \setminus E(A_i)$ implies that e is covered twice by $\mathcal{C}_i$, and because of $e \notin E(F) \cup \bigcup_{j \neq i} E(H_j)$, there are no additional circuits that cover e.

Thus we have constructed a CDC of G containing C, as desired. ∎

A conjecture of Sabidussi is equivalent to the statement that any dominating circuit in a subdivision of a cubic graph belongs to some CDC. (For details on the statement of the original conjecture and the indicated equivalence see e.g. [3].) A circuit C is *dominating* in G if at least one endvertex of every edge of G lies on C. The following adaptation of the proof of Theorem 1 shows that the truth of this conjecture would follow from the existence of semiextensions for dominating circuits in cubic graphs. A *minimal counterexample* (G, C), respectively G, to Sabidussi's conjecture is defined analogously as in the case of SCDCC.

Theorem 2. *If C is a dominating circuit in a cubic graph G, and if C has a semiextension in G, then G is not a minimal counterexample to Sabidussi's conjecture.*

Proof. This is argued similarly as in the proof of Theorem 1. So we define subgraphs $D, A_1, \ldots, A_k, H_1, \ldots, H_k$, and F as before. It remains to show in addition that each A_i is a dominating circuit in H_i, for $i = 1, \ldots, k$, and that D is a dominating circuit in F.

To show that A_i is a dominating circuit in H_i, for $i = 1, \ldots, k$, assume $v \in V(H_i) \setminus V(A_i)$. Then, as in the proof of Theorem 1, $v \notin V(C \cup D)$ holds, hence in particular $v \notin V(C)$. Since C is dominating, every neighbour u of v in G satisfies $u \in V(C)$. We then have $v \in V(H_i) \setminus V(D)$ and $uv \in E(G)$, from which $uv \in E(H_i)$

follows from (1), so $u \in V(C) \cap V(H_i) \subseteq V(A_i)$. Thus we have shown that A_i dominates H_i.

Next we consider the circuit $D \subseteq F$. Assume $v \in V(F) \setminus V(D)$. Then, by definition of F, we have $v \notin V(H_i)$ for any $i = 1, \ldots, k$, and therefore $v \notin V(C)$. Let u be any neighbour of v in G. Then we have $u \in V(C)$, since C is dominating. Now v does not belong to any H_i, so it follows from $uv \in E(G)$ combined with (1) that $u \notin V(C) \setminus V(D)$ holds, hence we have $u \in V(D)$, which shows that D dominates F.

Now the conclusion follows similarly to the proof of Theorem 1. ∎

3 Reductions and an existence result

The following classical result implies a result which is in a sense the best possible on extensions of long circuits in cubic graphs. The theorem generalizes to graphs all of whose vertices have odd degrees.

Theorem 3. [C.A.B. Smith, [9]] *For any edge e in a cubic graph G, the number of distinct Hamilton circuits in G containing e is even.*

Corollary 1. *For any Hamilton circuit C in a cubic graph G and for every edge e of C, there exists a Hamilton circuit in G which is different from C and contains e.*

Let C be a circuit in a cubic graph G. Corollary 1 implies that if C is a Hamilton circuit, then there exists an extension of C in G. If C contains all but a single vertex v of G, then the existence of an extension of C can be deduced as follows. We delete v from G and add a new edge between two of the neighbours of v in G. The remaining third neighbour of v in G is suppressed to obtain a new edge e. Applying Smith's theorem to the resulting cubic hamiltonian graph easily gives the desired extension of C in G.

Corollary 2. *If a circuit C in a cubic graph G contains all but at most one of the vertices of G, then C has an extension in G.*

Corollary 2 is best possible in the sense that "at most one" cannot be replaced by "at most two". This follows from the example shown in Figure 1.

We will show the truth of Conjecture 1 for spanning cycles. We will do this by reducing the statement of Conjecture 1 to its cubic case, in such a way that if C is a spanning cycle then the problem is reduced to another problem in which the new cycle is also spanning, following which we can apply Smith's theorem to find an extension.

Theorem 4. *If there exists a graph G having the property that G contains cycles C, C', such that $C' \cap C \neq \emptyset$ and $E(C') \setminus E(C) \neq \emptyset$ both hold, but there is no semiextension of C in G, then there exists a 3-connected cubic such graph.*

Proof. Assume that G is a graph with the property described in the statement of the theorem, and let C, C' be two cycles to witness it. If G is not 3-connected and cubic we will construct a graph $\hat{G}$ and cycles $\hat{C}, \hat{C}'$ in $\hat{G}$, such that $\hat{C}' \cap \hat{C} \neq \emptyset$ and $E(\hat{C}') \setminus E(\hat{C}) \neq \emptyset$ both hold, but there is no semiextension of $\hat{C}$ in $\hat{G}$. In addition $\hat{G}$ satisfies $\sum_{v \in V(\hat{G})} |d_{\hat{G}}(v) - 3| \leq \sum_{v \in V(G)} |d_G(v) - 3|$, and if

$\sum_{v \in V(\hat{G})} |d_{\hat{G}}(v) - 3| = \sum_{v \in V(G)} |d_G(v) - 3|$ holds, then $\hat{G}$ has strictly fewer edges than G. We shall in each of the various cases omit the details of checking that $\hat{G}$ satisfies this additional condition, since they are quite straightforward. The desired conclusion now follows by easy inductive reasoning. Moreover, if C is a spanning cycle in G, then our construction delivers a spanning cycle $\hat{C}$ of $\hat{G}$, a fact which is subsequently used to derive Corollary 3.

If G is not 2-connected, then there exists a block B of G which has an edge belonging to $E(C') \setminus E(C)$, and such that $B \cap (C' \cap C)$ is nonempty. Then $B \cap C$ consists of more than just a single vertex, otherwise $B \cap C'$ would be a semiextension of C in G. So we let $\hat{G} = B$, $\hat{C} = B \cap C$, and $\hat{C}' = B \cap C'$, so that $\hat{C}$ and $\hat{C}'$ are cycles in $\hat{G}$. It is now easy to check that any semiextension of $\hat{C}$ in $\hat{G}$ is also a semiextension of C in G. Hence we may assume that G is 2-connected, in particular G has no loop.

Next we consider any edge 2-cut $\{e_1, e_2\}$ in G. Since G is 2-connected, $G - e_1 - e_2$ has exactly two components G_1, G_2, where we choose the indices so that G_1 intersects C and contains at least one edge of C' that does not belong to C. Let the endvertices of e_1 be u_1 and u_2, and the endvertices of e_2 be v_1 and v_2, where $u_i, v_i \in G_i$, $i = 1, 2$. We obtain $\hat{G}$ from G_1 by adding a new edge $\hat{e}$ between u_1 and v_1. We then define $\hat{C}$ as the graph obtained from $G_1 \cap C$ by adding the edge $\hat{e}$ to $G_1 \cap C$ if C contains e_1, e_2, otherwise by letting $\hat{C} = C$. We define $\hat{C}'$ similarly. It follows from our choice of G_1 and G_2 that $\hat{C}$ and $\hat{C}'$ are cycles which satisfy $\hat{C}' \cap \hat{C} \neq \emptyset$ and $E(\hat{C}') \setminus E(\hat{C}) \neq \emptyset$. Suppose that there exists a semiextension

$\hat{D}$ of $\hat{C}$ in $\hat{G}$. If $\hat{e} \notin E(\hat{D})$, then let $D = \hat{D}$, otherwise let D be any cycle in G satisfying $D \cap G_1 = \hat{D} - \hat{e}$ and furthermore satisfying $D \cap G_2 = C \cap G_2$ if $C \cap G_2 \neq \emptyset$. Hence, if D is not a subgraph of G_1, then $e_1, e_2 \in D$ holds, and $D \cap G_2$ is a connected subgraph of G_2 in which u_2 and v_2 have odd degrees, and all other vertices have even degrees. We consider any $(C \cup D)$-path P in G with endvertices x, y where $x \in V(C) \setminus V(D)$.

First we assume $x \in G_2$. Then C intersects both G_1 and G_2, hence e_1, e_2 belong to C. This implies $P \subseteq G_2$, in particular $y \in G_2$. Moreover, $D \cap G_2 = \emptyset$ follows from $x \in V(C \cap G_2) \setminus V(D)$, since $C \cap G_2 = D \cap G_2$ holds by construction of D in the case when $C \cap G_2$ and $D \cap G_2$ are both nonempty. Since $C \cap G_2$ is connected, there exists a path in $C \cap G_2$ with endvertices x and y, and because of $D \cap G_2 = \emptyset$ this path is the desired path in $C \oplus D$ from x to y.

We may now assume $x \in G_1$. Since P and $C \cup D$ cannot both intersect G_2, it follows that also y belongs to G_1. If $C \cup D \subseteq G_1$ holds, then either P itself, or $P \cap G_1$ together with $\hat{e_1}$ in the case when P intersects G_2, forms a $(\hat{C} \cup \hat{D})$-path in $\hat{G}$ from $x \in V(\hat{C}) \setminus V(\hat{D})$ to y. So there exists a path Q in $\hat{C} \oplus \hat{D}$ from x to y. Then Q is also a path in $C \oplus D$ from x to y. This leaves the case $C \cup D \not\subseteq G_1$, from which $P \subseteq G_1$ follows, and P is thus a $(\hat{C} \cup \hat{D})$-path in $\hat{G}$ from $x \in V(\hat{C}) \setminus V(\hat{D})$ to y. Let $\hat{Q}$ be a path in $\hat{C} \oplus \hat{D}$ from x to y. If $\hat{e}$ is an edge of $\hat{Q}$, then $\hat{e}$ belongs to precisely one of $\hat{C}$ and $\hat{D}$, hence each of e_1 and e_2 belongs to either C or D, and not to both. Then e_1 and e_2 together with $\hat{Q} \cap G_1$ together with either $C \cap G_2$ or $D \cap G_2$ form a path in $C \oplus D$ from x to y. Altogether, the assumption that there exists a semiextension of $\hat{C}$ in $\hat{G}$ leads to the conclusion that there exists a semiextension of C in G, a contradiction. In the following we may therefore assume that G is edge 3-connected.

We assume that G is not cubic and we let v be a vertex of G satisfying $d_G(v) \neq 3$. Since G is edge 3-connected, the degree of v in G is equal to k for some $k \geq 4$. Let $e_1, e_2, \ldots, e_k$ be the distinct edges of G incident to v, and let v_i be the endvertex of e_i which is different from v, for $i = 1, 2, \ldots, k$.

First we consider the case $d_G(v) = d_C(v)$, that is, e_i is an edge of C for every $i = 1, 2, \ldots, k$. If v is a cutvertex of C, then we may choose distinct blocks B_1 and B_2 of C and fix the notation so that $e_1 \in B_1$ and $e_2 \in B_2$ hold. We may further assume that if v is a vertex of C', then e_1 and e_2 both belong to C'. This is clear if v is not a cutvertex of C. But if v is a cutvertex of C, and there is only one block of C which contains an edge of C' incident to v, say the block B_1, then we let H be the subgraph of G induced by the edges that belong to precisely one of C' and B_2, and we consider the component C'' of H that contains v. Now C'' is a cycle, and $C'' \cap C \neq \emptyset$ is clearly satisfied. Moreover, since C' is connected and not a subgraph of C, and therefore in particular not a subgraph of B_1, there exists a path in C' with v as its first vertex, such that precisely the last edge e of this path does not belong to B_1. Then e belongs to C'' but not to C, hence $E(C'') \setminus E(C) \neq \emptyset$ holds, and we may replace C' by C'' before continuing our argument.

Let $\hat{G}$ be the graph obtained from $G - e_1 - e_2$ by adding a new edge e between v_1 and v_2. Let $\hat{C}$ and $\hat{C}'$ be the subgraphs of $\hat{G}$ that correspond similarly to C and C' respectively. Then $\hat{C}$ is a cycle; if v is a cutvertex of C this follows from the choice of e_1 and e_2. Each component of $\hat{C}'$ is a cycle and contains at least one neighbour of v in G, hence the intersection of any such component with $\hat{C}$ contains a neighbour of v. It follows that $\hat{C}$ and $\hat{C}'$, possibly after replacing $\hat{C}'$ by a suitable component of $\hat{C}'$, satisfy $\hat{C}' \cap \hat{C} \neq \emptyset$ and $E(\hat{C}') \setminus E(\hat{C}) \neq \emptyset$. Suppose that there exists a semiextension $\hat{D}$ of $\hat{C}$ in $\hat{G}$. If e is an edge of $\hat{D}$, then let D be the subgraph of G obtained by replacing e in $\hat{D}$ by the edges e_1 and e_2, otherwise let $D = \hat{D}$. Then D is clearly a cycle which satisfies both $D \cap C \neq \emptyset$ and $E(D) \setminus E(C) \neq \emptyset$. Consider any $(C \cup D)$-path P in G with endvertices x and y, where $x \in V(C) \setminus V(D)$. We will exhibit a path Q in $C \oplus D$ with endvertices x and y. Since all edges incident to v belong to C, it is clear that P does not contain v, hence P is also a $(\hat{C} \cup \hat{D})$-path in $\hat{G}$ with endvertices x and y, where $x \in V(\hat{C}) \setminus V(\hat{D})$, so there exists a path Q in $\hat{C} \oplus \hat{D}$ with endvertices x and y. If Q does not contain e, then Q is the desired path in $C \oplus D$. Otherwise we obtain such a path as the union of the shortest subpath of Q which connects x to $\{v, v_1, v_2\}$ together with the shortest subpath which connects y to $\{v, v_1, v_2\}$ together with one or both of the edges e_1, e_2 (which both belong to $C \oplus D$ when e belongs to $\hat{C} \oplus \hat{D}$). This contradiction shows that there is no semiextension of $\hat{C}$ in $\hat{G}$.

Finally we assume $d_C(v) < d_G(v)$. We may assume that e_1 does not belong to C, and in addition that if v is a vertex of C, then e_2 belongs to C. We may further assume that if there exists an edge incident to v which belongs to C' but not to C then e_1 is such an edge, and that if there exists an edge incident to v which belongs to C but not to C' then e_2 is such an edge. Let $\hat{G}$ be the graph obtained from $G - v$ by adding two new vertices u, w together with a new edge uw, and let u replace v as endvertex of e_1, e_2 and w replace v as endvertex of $e_3, \dots, e_k$. We let $\hat{C}$ be the cycle in $\hat{G}$ that corresponds naturally to C. It then follows that uw is an edge of $\hat{C}$ if v is a vertex of C. We let $\hat{C}'$ be the similarly defined subgraph of $\hat{G}$ that corresponds to C'. That is, if v is a vertex of C', then uw is an edge of $\hat{C}'$ if and only if precisely one of e_1, e_2 is an edge of C', so that all vertex degrees in $\hat{C}'$ are even. Possibly $\hat{C}'$ is not connected. However, if $\hat{C}'$ is not connected, then e_1 and e_2 both belong to C', and there is no edge among $e_3, \dots, e_k$ which belongs to C but not to C', or we would have chosen such an edge to be the edge e_2. It follows that if $\hat{C}'$ is not connected, then at least one of $e_3, \dots, e_k$ belongs to $\hat{C} \cap \hat{C}'$. Hence, possibly after replacing $\hat{C}'$ by a component of $\hat{C}'$ which contains an edge in $E(C') \setminus E(C)$, we deduce that the conditions $\hat{C}' \cap \hat{C} \neq \emptyset$ and $E(\hat{C}') \setminus E(\hat{C}) \neq \emptyset$ are satisfied. Now suppose that $\hat{D}$ is a semiextension of $\hat{C}$ in $\hat{G}$. We let D be the corresponding subgraph of G, induced by $E(\hat{D}) \setminus \{uw\}$. Then D is a cycle with at least a vertex in common with C. If v is a vertex of C, then uw is an edge of $\hat{C}$, hence $E(\hat{D}) \setminus E(\hat{C}) \neq \{uw\}$ follows, so there exists an edge $\hat{e} \neq uw$ of $\hat{G}$ contained in $E(\hat{D}) \setminus E(\hat{C})$. Then $\hat{e}$ is also contained in $E(D) \setminus E(C)$, and we have $E(D) \setminus E(C) \neq \emptyset$. Consider any $(C \cup D)$-path P in G having endvertices x, y with $x \in V(C) \setminus V(D)$, and let $\hat{P}$ denote the corresponding path in $\hat{G}$. That is, if v is an endvertex of P, then uw does not belong to $\hat{P}$, and either u or w replaces v as the corresponding endvertex, and if not, then uw is an edge of $\hat{P}$ if and only if precisely one of e_1, e_2 belongs to P. We denote the endvertices of $\hat{P}$ corresponding to x and y by $\hat{x}$ and $\hat{y}$, respectively. If x is not equal to v, then $\hat{x} = x \in V(\hat{C}) \setminus V(\hat{D})$ is clear. If x is equal to v, then it follows from $v \in V(C)$ that $uw \in E(\hat{C})$, so $\hat{x}$ belongs to $\hat{C}$, and it follows from the construction of D from $\hat{D}$ that $\hat{x}$ does not belong to $V(\hat{D})$. Hence $\hat{x} \in V(\hat{C}) \setminus V(\hat{D})$ follows. Similarly, $\hat{y} \in V(\hat{C}) \cup V(\hat{D})$ holds. Let $\hat{Q}$ be a path in $\hat{C} \oplus \hat{D}$ with endvertices $\hat{x}, \hat{y}$, and denote the corresponding path in G by Q (after discarding the edges of any subpath of $\hat{Q}$ between u and w, if such exists). It is clear that the endvertices of Q are x and y, and, by the construction of $\hat{C}$ from C and of D from $\hat{D}$, that every edge of Q belongs to precisely one of C and D. It follows that D is a semiextension of C in G. This contradiction shows that $\hat{C}$ has no semiextension in $\hat{G}$. ▮

Corollary 3. *For every cycle C in G with $V(C) = V(G)$ and $C \neq G$, there exists a semiextension of C in G.*

Proof. G satisfies the assumption of Theorem 4 by letting C' be a circuit obtained by choosing an edge $e \in E(G) \setminus E(C)$, and adding e to any path in C that connects the endvertices of e. It follows inductively from the proof of Theorem 4, observing that if C is spanning in G, then also $\hat{C}$ is spanning in $\hat{G}$, that if there is no semiextension of C in G, then there exists a cubic graph with a Hamilton circuit which has no semiextension. This contradiction to Corollary 2 shows that C has a semiextension in G. ▮

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