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Linear spaces with many projective planes

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Abstract

We assume that in a linear space $(P, \mathfrak{L})$ there is a non-empty set M of points with the property that every plane containing a point of M is a projective plane. In section 3 an example is given that in general $(P, \mathfrak{L})$ is not a projective space. But if M can be completed by two points to a basis of P, then $(P, \mathfrak{L})$ is a projective space.

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1 Introduction

As well known a projective space can be defined as a linear space with three points on every line satisfying the Veblen-Young-Axiom, or equivalently as a linear space in which every plane is a projective space (for definitions see below). In many papers the question is discussed which linear space $(M, \mathfrak{M})$ can be embedded into a projective space (cf. [1, 2, 3, 4, 5, 7, 8, 10]). The usual method for projective embeddings is the construction of the points P of the projective space $(P, \mathfrak{L})$ by line bundles of the given linear space $(M, \mathfrak{M})$. It is easy to show that $(P, \mathfrak{L})$ is a linear space and that $(M, \mathfrak{M})$ can be embedded into $(P, \mathfrak{L})$, the crucial problem is to prove that every plane of $(P, \mathfrak{L})$ is a projective plane. If for example $(M, \mathfrak{M})$ is locally projective and satisfies the Bundle Theorem, then $(P, \mathfrak{L})$ is a projective space (cf [4, 7]). For this proof one shows in a first step that every plane of P with a non-empty intersection with M is a projective space and in a hard second step that any plane of P is a projective plane.

This kind of problem related to projective embeddings suggest to consider the question, how many planes of a linear space $(P, \mathfrak{L})$ have to be known as projective planes to be sure that we have a projective space. In [9] it is shown that $(P, \mathfrak{L})$ is projective if every plane with a non-empty intersection with a locally complete subset $M \subset P$ is a projective plane. In this paper we will contribute a theorem to the problem. We assume that every plane containing a point of a subset M of P is a projective plane. We give an exact answer how many points have to be contained in M (cf. Theorem (2.8)). In particular we will give an example of a linear space with a lot of projective planes which is not a projective space. We deform for $n \geq 3$ the lines of an $(n+1)$-dimensional projective space $(P, \mathfrak{G})$ a little bit such one obtains an n- dimensional linear space $(P, \mathfrak{L})$. Nearly all planes of $(P, \mathfrak{L})$ are projective planes but one special type of 3-dimensional subspaces of $(P, \mathfrak{G})$ through a point z which become non-projective planes of $(P, \mathfrak{L})$ (cf. Theorem (3.10)).

Now we recall some definitions and notations (cf. [6]). Let $(P, \mathfrak{L})$ denote a *linear space* with the point set P and the line set $\mathfrak{L}$. We consider a line as a subset of P. A *subspace* is a subset $U \subset P$ such that for all distinct points $x, y \in U$ the unique line passing through x and y is contained in U. Let $\mathfrak{U}$ denote the set of all subspaces. For every subset $X \subset P$ we define the following *closure operation* :

$$^{-} : \mathfrak{P}(P) \to \mathfrak{U} : \ X \mapsto \overline{X} := \bigcap_{\substack{U \in \mathfrak{U} \\ X \subset U}} U \tag{1}$$

As well known, the *following finiteness* condition is satisfied in every linear space $(P, \mathfrak{L})$ (cf. [6, (8.4)]).

1.1 *Let $S \subset P$ and $x \in \overline{S}$. Then there exists a finite subset $R \subset S$ satisfying $x \in \overline{R}$.*

For $U \in \mathfrak{U}$ we call dim $U := inf\{|X|-1 : X \subset U \text{ and } \overline{X} = U\}$ the *dimension* of U. A subspace of dimension two is a *plane*. A linear space of dimension two is a projective plane, if every line contains three points and any two lines have a non-empty intersection.

2 Linear spaces with projective planes

In this section let $(P, \mathfrak{L})$ always be a linear space of $dim\ P \geq 2$ with a non-empty subset $M \subset P$ satisfying the property

(Z) Every plane E of $(P, \mathfrak{L})$ with $E \cap M \neq \varnothing$ is a projective plane.

In the following let U, T be always non-empty subsets of P. We denote the closure of $(P, \mathfrak{L})$ by $X \to \overline{X}$.

Lemma 2.1 *Let U be a subspace of $(P, \mathfrak{L})$ and let $x \in M \setminus U$. Then we have*

$$\overline{U \cup \{x\}} = \bigcup_{u \in U} \overline{x, u}$$

Proof. Let $T := \bigcup_{u \in U} \overline{x, u}$. Obviously $T \subset \overline{U \cup \{x\}}$. Since $U \cup \{x\} \subset T$ also $\overline{U \cup \{x\}} \subset \overline{T}$. We will prove that T is a subspace, hence $T = \overline{T} = \overline{U \cup \{x\}}$. Let $a, b \in T$ with $a \neq b$ and $c \in \overline{a, b}$. We may assume $c \neq x$, else $x = c \in T$. By definition there are $a', b' \in U$ with $a \in \overline{a', x}$ and $b \in \overline{b', x}$ (holds also for $a = x$ or $b = x$). If $a' = b'$, then $a, b, c \in \overline{a', x} \subset T$. Assume now $a' \neq b'$. Then by (Z) $\overline{a', b', x}$ is a projective plane and for $c, a, b \in \overline{a', b', x}$ the point $c' = \overline{x, c} \cap \overline{a', b'}$ exists. Hence $c \in \overline{x, c'} \subset T$ and T is a subspace.

We define the following mapping

$$f_{U,x} : \overline{U \cup \{x\}} \setminus \{x\} \to U;\ u \to f_{U,x}(u) = u' := \overline{x, u} \cap U.$$

Lemma 2.2 *Let U be a subspace of $(P, \mathfrak{L})$, $x \in M \setminus U$ and let $T \subset \overline{U \cup \{x\}}$ be a subspace with $x \notin T$. Then*

$$\pi : T \to f_{U,x}(T);\ t \to t' := f_{U,x}(t)$$

is an isomorphism between T and $T' := f_{U,x}(T)$.

Proof. By definition π is surjective. Assume $a' = b'$ for $a, b \in T$, then $a, b \in \overline{x, a'}$. Hence $a \neq b$ would imply $\overline{a, b} = \overline{x, a'}$, i.e. $x \in \overline{a, b}$, a contradiction to $x \notin T$. Hence $a = b$ and π is injective. For distinct $a, b \in T$, $\overline{a, b, x}$ is a projective plane by (Z) with $\overline{a, b, x} \cap U = \overline{a', b'}$, since $x \notin U$. It follows $\pi(\overline{a, b}) = \overline{a', b'}$. In the same way for distinct $c', d' \in f_{U,x}(T)$ there are unique $c, d \in T$ with $\pi^{-1}(\overline{c', d'}) = \overline{c, d}$ and π maps collinear points exactly onto collinear points.

By Lemma 2.2 we obtain in particular.

Lemma 2.3 *Let U be a subspace of $(P, \mathfrak{L})$, $x \in M \setminus U$ and let $E \subset \overline{U \cup \{x\}}$ be a plane with $x \notin E$.*

(a) *Then E is isomorphic to the plane $f_{U,x}(E)$ of U.*

(b) *If U is a projective space, then also $\overline{U \cup \{x\}}$ is a projective space.*

Proof. (a) is Lemma 2.2 applied on planes.
(b) Let E be any plane of $\overline{U \cup \{x\}}$. If $x \in E$, then by (Z) E is a projective plane. If $x \notin E$ by (a) E is isomorphic to a projective plane $f_{U,x}(E)$ of U, hence a projective plane itself.

Lemma 2.4 *Let $M = \{x_1, x_2, \dots, x_{n-1}\}$ for a natural number $n \in \mathbb{N}$. Assume there are $a, b \in P$ such that $\overline{M \cup \{a, b\}} = P$. Then $(P, \mathfrak{L})$ is a projective space.*

Proof. Let $E \subset P$ be a plane with $x_1, a, b \in E$. By the property (Z), E is a projective plane. If $P \neq E$ there exists some $x_i \in M$, say x_2, with $x_2 \in P \setminus E$. By Lemma 2.3 (b) $\overline{E \cup \{x_2\}}$ is a projective space and therefore also $\overline{\{a, b, x_1, x_2\}} \subseteq \overline{E \cup \{x_2\}}$. By induction $\overline{\{a, b, x_1, \dots, x_{n-1}\}} = P$ is a projective space.

Theorem 2.5 *Let $(P, \mathfrak{L})$ be linear space with a subset M and points $a, b \in P$ such that $\overline{M \cup \{a, b\}} = P$. If M satisfies the property (Z) that every plane E with $E \cap M \neq \varnothing$ is a projective plane then $(P, \mathfrak{L})$ is a projective space.*

Proof. Let $p, q, r \in P$ be three points not on a line. By the finiteness condition (1.1) there exists a finite subset $\{a, b, x_1, \dots, x_{n-1}\} \subset M \cup \{a, b\}$ with $p, q, r \in \overline{\{a, b, x_1, \dots, x_{n-1}\}}$. By Lemma 2.4 $\overline{p, q, r}$ is a projective plane and therefore $(P, \mathfrak{L})$ is a projective space.

We notice some consequences of Theorem 2.5.

Corollary 2.6 *Let B be a basis of a linear space $(P, \mathfrak{L})$ and let $a, b \in B$ be any points. If $M := B \setminus \{a, b\}$ satisfies the property (Z), then $(P, \mathfrak{L})$ is a projective space.*

We show in the next section that this characterization of projective spaces is not longer true if M cannot be completed to a basis by two points.

3 Example

We construct in this section an example of a linear space $(P, \mathfrak{L})$ of dimension n which contains a projective space H of dimension $(n-2)$ as a subspace such that every point x of H except one point $z \in H$ satisfies condition (Z), i.e. we set $M := H \setminus \{z\}$. But $(P, \mathfrak{L})$ is not a projective space. The set M cannot be completed by only two points to a generating system of P. In particular, for $n = 3$ we obtain a 3-dimensional linear space which contains points satisfying (Z) but which is not a projective space.

Let $(P, \mathfrak{G})$ be a projective space of dimension $n+1$ for $n \in \mathbb{N}, n \geq 3$, with the closure $X \to \overline{X}$. Let $H \subset E \subset U \subset P$ a chain of hyperplanes, i.e., $dim\ H = n-2,\ dim\ E = n-1,\ dim\ U = n$.

We divide the line set $\mathfrak{G}$ in three disjunct parts

$\mathfrak{G}_E := \{G \in \mathfrak{G} : G \subset E\}$ lines of type I,
$\mathfrak{G}_U := \{G \in \mathfrak{G} : G \subset U$ and $G \not\subset E\}$ lines of type II,
$\mathfrak{G}_P := \{G \in \mathfrak{G} : G \not\subset U\}$ lines of type III.

Since for a line G of the projective space and the hyperplane U we have $G \cap U \neq \varnothing$, it follows:

Lemma 3.1 *For any line $G \in \mathfrak{G}$ it holds:*
(I) $G \in \mathfrak{G}_E \Leftrightarrow |G \cap E| \geq 2$;
(II) $G \in \mathfrak{G}_U \Leftrightarrow |G \cap U| \geq 2$ *and* $|G \cap E| = 1$;
(III) $G \in \mathfrak{G}_P \Leftrightarrow |G \cap U| = 1$.

Let $\varphi : E \to E$ be a central collineation with the hyperplane H of E as axis and the centre $z \in H$. Since $n \geq 3$ we have $dim\ E \geq 2$ and the existence of a central collineation $\varphi \neq id$ is obvious. We recall that $z \in G$ for every line

$G \subset E$ with $\varphi(G) = G$ and $G \not\subset H$. We introduce the following notation for a line $G \in \mathfrak{G}$

$e_G := G \cap E$ if G is a line of type II.

Now we define a new line set $\mathfrak{L}$ on P. For a line $G \in \mathfrak{G}_U$ of type II let

$G' := (G \setminus \{e_G\}) \cup \{\varphi(e_G)\}$,
$\mathfrak{G}'_U := \{G' : G \in \mathfrak{G}_U\}$ and
$\mathfrak{L} := \mathfrak{G}_E \cup \mathfrak{G}'_U \cup \mathfrak{G}_P$.

We notice the following easy properties.

Lemma 3.2 (a) *For every line $L \in \mathfrak{L}$ it holds $|L| \geq 3$.*
(b) *For every line $L \in \mathfrak{L}$ we have:*
$L \in \mathfrak{G}'_U \Leftrightarrow |L \cap U| \geq 2$ and $|L \cap E| = 1$.
(c) *$\mathfrak{L} = \mathfrak{G}_E \cup \mathfrak{G}'_U \cup \mathfrak{G}_P$ is a disjunct union.*
(d) *For every line $G \in \mathfrak{G}_U$ we have $G' \setminus E = G \setminus E$.*

We denote also the lines of $\mathfrak{G}'_U$ as lines of type II.

Theorem 3.3 *$(P, \mathfrak{L})$ is a linear space.*

Proof. Let $x, y \in P$ be distinct points and $G = \overline{x, y} \in \mathfrak{G}$. If G is a line of type I or III then $G \in \mathfrak{L}$ is also in $(P, \mathfrak{L})$ the unique line connecting x and y. Assume now that G is a line of type II. If $x, y \in U \setminus E$, then by Lemma 3.2 (d) the line $G' \in \mathfrak{L}$ is uniquely connecting x and y. Let $x \in E$. Then we have by definition of type II $y \in U \setminus E$. Let $x_0 := \varphi^{-1}(x)$ and $K := \overline{x_0, y} \in \mathfrak{G}_U$. Then $K' = (K \setminus \{x_0\}) \cup \{x\} \in \mathfrak{G}'_U$ contains x and y. If $L \in \mathfrak{L}$ is a line containing x and y, then $L \in \mathfrak{G}'_U$ is of type II and $L \cap E = \{x\}$. Therefore $L_0 := (L \setminus \{x\}) \cup \{\varphi^{-1}(x)\} = (L \setminus \{x\}) \cup \{x_0\}$ is a line of $\mathfrak{G}_U$ containing x_0, y, hence $L_0 = \overline{x_0, y} = K$ and $L = K'$. So any two points have a unique connecting line and $(P, \mathfrak{L})$ is a linear space.

By $X \to [X]$ we will denote the closure of $(P, \mathfrak{L})$ which is distinct to the closure $X \to \overline{X}$ of $(P, \mathfrak{G})$. To determine the structure of $(P, \mathfrak{L})$, we consider all planes of $(P, \mathfrak{L})$ and we will see that there are three types:

- planes which are also planes of $(P, \mathfrak{G})$, and therefore projective planes. (Cf. (3.4), (3.5))

- projective planes inside of E which we get from planes of $(P, \mathfrak{G})$ by a slight deformation under the collineation of φ. (Cf. (3.6))
- non-projective planes of $(P, \mathfrak{L})$ which coincide with the pointset of three-dimensional subspaces of $(P, \mathfrak{G})$. (Cf. (3.8); (3.9))

Now let $a, b, c \in P$ be three points not on a line, and $F = \overline{a, b, c}$ the plane of $(P, \mathfrak{G})$. We consider the plane $[a, b, c]$ of $(P, \mathfrak{L})$ for some cases.

Lemma 3.4 *If $F \cap U \subset E$, then $\overline{a, b, c} = F = [a, b, c]$.*

Proof. Because $F \cap U \subset E$, F contains no line of type II, hence all lines of F are exactly lines of $\mathfrak{L}$ and therefore F is a subspace of $(P, \mathfrak{L})$ with $F = [a, b, c]$.

Lemma 3.5 *If $F \cap E \subset H$, then $\overline{a, b, c} = F = [a, b, c]$.*

Proof. Assume for $x, y \in F$ that $\overline{x, y}$ is a line of type II, then $e = \overline{x, y} \cap E \subset F \cap E \subset H$ is a point of the axis H and $\varphi(e) = e$. It follows $\overline{x, y} = [x, y]$. Hence every line of F is exactly a line of $\mathfrak{L}$ and therefore $\overline{a, b, c} = [a, b, c]$.

Lemma 3.6 *Let $a, b \in E$ be distinct and $c \in U \setminus E$. Let $a_0 = \varphi^{-1}(a)$, $b_0 = \varphi^{-1}(b)$. Then $[a, b, c] = (\overline{a_0, b_0, c} \setminus \overline{a_0, b_0}) \cup \overline{a, b}$ is a projective plane of $(P, \mathfrak{L})$ and $[a, b, c] \subset U$.*

Proof. We have $\overline{a_0, b_0, c} \cap E = \overline{a_0, b_0}$ and every line of $\overline{a_0, b_0, c}$ but $\overline{a_0, b_0}$ is of type II. Since φ is a collineation it holds $\varphi(\overline{a_0, b_0}) = \overline{a, b} = [a, b]$. For every line G of $\overline{a_0, b_0, c}$ with $G \neq \overline{a_0, b_0}$ we have $G' = (G \setminus \{e_G\}) \cup \{\varphi(e_G)\} \subset (\overline{a_0, b_0, c} \setminus \overline{a_0, b_0}) \cup \varphi(\overline{a_0, b_0})$. Therefore for any two points x, y of $(\overline{a_0, b_0, c} \setminus \overline{a_0, b_0}) \cup \overline{a, b}$ the connecting line is a subset of $(\overline{a_0, b_0, c} \setminus \overline{a_0, b_0}) \cup \overline{a, b}$: If $x, y \in \overline{a_0, b_0}$, clearly $[x, y] = \overline{a, b}$. For $x, y \in \overline{a_0, b_0, c} \setminus \overline{a_0, b_0}$ and $e := \overline{x, y} \cap \overline{a_0, b_0}$ we have $[x, y] = (\overline{x, y} \setminus \{e\}) \cup \{\varphi(e)\}$ and for $x \in \overline{a_0, b_0, c} \setminus \overline{a_0, b_0}$, $y \in \overline{a_0, b_0}$ and $y_0 := \varphi^{-1}(y)$ it follows $[x, y] = (\overline{x, y_0} \setminus \{y_0\}) \cup \{y\}$.
Hence $(\overline{a_0, b_0, c} \setminus \overline{a_0, b_0}) \cup \overline{a, b}$ is a subspace of $(P, \mathfrak{L})$ containing a, b, c, obviously the plane $[a, b, c]$ which is generated by a, b, c. It is easy to see that $\overline{a_0, b_0, c}$ and $[a, b, c]$ are isomorphic, hence $[a, b, c]$ is a projective plane.

Lemma 3.7 *Let $D = [a, b, c]$ be a plane of $(P, \mathfrak{L})$ with $a, b, c \in U$. Then D is a projective plane with $D \subset U$.*

Proof. By Lemma 3.2 we have $[a, c] \cap E \neq \varnothing$ and $[b, c] \cap E \neq \varnothing$. Hence we may assume $a, b \in E$. For $c \in E$, by Lemma 3.4 $\overline{a, b, c} = [a, b, c] \subset E$ is a projective plane and for $c \in U \setminus E$ by Lemma 3.6 $[a, b, c] \subset U$ is a projective plane.

Lemma 3.8 *Let D be a plane of $(P, \mathfrak{L})$ which is not a projective plane, then there exist $a \in E \setminus H,\ b \in U \setminus E,\ c \in P \setminus U$ with $D = [a, b, c]$.*

Proof. Let $a'', b'', c \in D$ with $D = [a'', b'', c]$. By Lemma 3.7 we may assume $c \in P \setminus U$. Since U is a hyperplane, $b' = \overline{c, b''} \cap U = [c, b''] \cap U$ and $a' = \overline{c, a''} \cap U = [c, a''] \cap U$ exist and $[a'', b'', c] = [a', b', c]$. By Lemma 3.2 (b), $[a', b'] \cap E \neq \varnothing$ and there are distinct $a, b \in [a', b']$ with $a \in E$ and $[a, b, c] = [a', b', c]$. For $a, b \in E$ we have $\overline{a, b, c} \cap U \subset E$ and by Lemma 3.4, $[a, b, c]$ would be a projective plane. Hence $b \in U \setminus E$. If $a \in H$, then $\overline{a, b, c} \cap E = \{a\} \subset H$ and by Lemma 3.5 $[a, b, c]$ would be a projective plane. Hence $a \in E \setminus H$.

Lemma 3.9 *Let $a \in E \setminus H$, $b \in U \setminus E$, $c \in P \setminus U$ and let $a_0 = \varphi^{-1}(a)$. Then:*
(a) *$a_0 \in [a, b, c]$.*
(b) *$\varphi(\overline{a, a_0}) = \overline{a, a_0} = [a, a_0]$ and $z = [a_0, a] \cap H$ for the center z of φ.*
(c) *$\overline{a_0, a, b, c}$ is a subspace of $(P, \mathfrak{L})$.*
(d) *$[a, b, c] = \overline{a_0, a, b, c}$ is a plane of $(P, \mathfrak{L})$ which is not a projective plane.*
(e) *$[a, b, c] \cap H = z$.*

Proof. (a) Since $\overline{a_0, b}$ is a line of type II and since $\varphi(a_0) = a$ we have $[a, b] = (\overline{a_0, b} \setminus \{a_0\}) \cup \{a\}$. Because every line contains at least three points, $u \in [b, c] \setminus \{b, c\}$ and $v \in [a, b] \setminus \{a, b\}$ exist and $u, v \in [a, b, c]$.
Since $u, c \in P \setminus U, [v, c] = \overline{v, c}$ and $[a_0, u] = \overline{a_0, u}$ are lines of type III, and both are contained in the projective plane $\overline{a_0, b, c}$ of $(P, \mathfrak{G})$. Therefore $r := \overline{v, c} \cap \overline{a_0, u} = [v, c] \cap [a_0, u]$ exists and $r \in [v, c] \subset [a, b, c]$. But now it follows that $[a_0, u] = [r, u] \subset [a, b, c]$, in particular $a_0 \in [a, b, c]$.

(b) H is a hyperplane of E and $a \in E \setminus H$, hence $h = \overline{a, a_0} \cap H$ exists and it follows that $\varphi(\overline{a, a_0}) = \varphi(\overline{a_0, h}) = \overline{a, h} = \overline{a_0, a}$, in particular $h = z \in H$ is the center of φ. Since $\overline{a_0, a}$ is a line of type I, $\overline{a_0, a} = [a_0, a]$.

(c) By $b \in U \setminus E$, $c \in P \setminus U$ and $a, a' \in E$ we get $\overline{a_0, a, b, c} \cap E = \overline{a_0, a}$. Let $x, y \in \overline{a_0, a, b, c}$ be distinct points. If $\overline{x, y}$ is not of type II, $\overline{x, y} = [x, y] \subset \overline{a_0, a, b, c}$. If $\overline{x, y}$ is a line of type II let $p = \overline{x, y} \cap E \subset \overline{a_0, a}$. By (b) $p' := \varphi(p)$ and $p_0 := \varphi^{-1}(p) \in \overline{a_0, a}$ and $[x, y] = (\overline{x, y} \setminus \{p\}) \cup \{p'\} \subset \overline{a_0, a, b, c}$ if $x, y \notin E$, or $[x, y] = \{\overline{x, p_0} \setminus \{p_0\}\} \cup \{p\} \subset \overline{a_0, a, b, c}$ if $y = p \in E$ and $x \notin E$. Hence $\overline{a_0, a, b, c}$ is a subspace of $(P, \mathfrak{L})$.

(d) Since by (c) $\overline{a_0, a, b, c}$ is a subspace of $(P, \mathfrak{L})$ we have $[a, b, c] \subset \overline{a_0, a, b, c}$. On the other hand, by (a) $a_0 \in [a, b, c]$ and therefore $\overline{a_0, a, b, c} \subset [a, b, c]$. Since $\overline{a_0, a} \cap \overline{b, c} = [a_0, a] \cap [b, c] = \varnothing$, $[a, b, c]$ is not a projective plane.

(e) Since $a \in E \setminus H$, $b \in U \setminus E$ and $c \in P \setminus U$ and since $H \subset E \subset U \subset P$ is a chain of hyperplanes, $\overline{H \cup \{a, b, c\}} = P$ and $dim\,(H \cap \overline{a, a_0, b, c}) + dim\,P = dim\,H + dim\,\overline{a, a_0, b, c} = n - 2 + 3 = n + 1 = dim\ P$ it follows $dim\,(H \cap \overline{a, a_0, b, c}) = 0$ Hence by (b) $H \cap \overline{a, a_0, b, c} = z$.

We summarize:

Theorem 3.10 *(a) $(P, \mathfrak{L})$ is not a projective space.*

(b) $dim\,(P, \mathfrak{L}) = n$.

(c) A plane D of $(P, \mathfrak{L})$ is not a projective plane if and only if $z = D \cap H$ and if there are $a, b, c \in D$ with $a \in E \setminus H$, $b \in U \setminus E$ and $c \in P \setminus U$.

(d) For every $y \in H \setminus \{z\}$ and every plane F of $(P, \mathfrak{L})$ containing y, F is a projective plane.

Proof. (a), (c) and (d) by Lemma 3.8 and 3.9

(b) Complete in $(P, \mathfrak{G})$ the center z to a basis $\{z, d_1, \dots, d_{n-2}\}$ of H, then for $a \in E \setminus H$, $b \in U \setminus E$, $c \in P \setminus U$ by Lemma 3.9, $z \in [a, b, c]$ and $\{a, b, c, d_1, \dots, d_{n-2}\}$ is a basis of $(P, \mathfrak{L})$, hence $dim\,(P, \mathfrak{L}) = n$.

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