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Kähler Differentials and some Applications in Arithmetical Geometry

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Distinguendo come aritmetici i corpi generabili con un numero finito di elementi, definiamo l'aritmetica quale teoria dei corpi aritmetici, e il semplice problema di trovare gli elementi di un corpo aritmetico osservabili in tutte le sue prospettive abbastanza ampie, conduce ai numeri interi, i quali pertanto rispondono a una questione fondamentale che si pone, qualora si voglia ascendere dalla moltitudine degli aspetti di un oggetto alla conoscenza dell'oggetto in sé. La stessa questione, resa piú efficace coi mezzi del calcolo differenziale, diventa allora l'origine dell'aritmetica infinitesimale.
Erich Kähler [GA] p. 1
(Distinguishing as arithmetic the fields generated by a finite number of elements, we define as arithmetical the theory of arithmetic fields. And the basic problem to find the elements of an arithmetic field observable in all sufficiently wide perspectives leads to the integer numbers and thus answers a fundamental question which comes up if one wants to ascend from the multitude of aspects of an object to the cognizance of the object itself. The same question, made more efficient by the means of the differential calculus, becomes the origin of the infinitesimal arithmetic.)

Erich Kähler had an intimate knowledge of Cartan's calculus of differential forms which he enriched and developed further to give precise definitions for the notions of genera and plurigenera of algebraic varieties in [K10] and [K11]. Moreover, he used this calculus to lay the foundation of a general theory of partial differential equations (in the book *Einführung in die Theorie der Systeme von Differentialgleichungen* [K13]).

As his research got more and more involved with arithmetics, he felt the necessity to give a purely algebraic definition of a differential module resp. of a differential algebra associated to any ring by giving a precise meaning to the notion of *infinitesimally small*, often used successfully but not very precisely by physicists. Kähler's concept of a differential module of a ring had a great impact on commutative algebra and algebraic geometry and, " like the differentials of analysis, differential modules "linearize" problems, i. e. reduce questions about algebras (non-linear problems) to questions of linear algebra" (Kunz [KD], p. V).

In Grothendieck's *Eléments de la Géométrie Algébrique* [EGA] O_{IV} § 20 - 22 the required formalism is elaborated without using the name *Kähler differentials* which nowadays seems to be standard, as to be seen for instance in Hartshorne's *Algebraic Geometry* [Ha]. E. Kunz who still more enriched the system of Kähler's definitions and gave many applications in commutative algebra ([KD], [KE]) reports on the history of the name *Kähler differentials* ([Ku6]): F. K. Schmidt was very familiar with Kähler's research (as to be seen by Schmidt's review [SAD] of Kähler's *Algebra und Differentialrechnung* [AD]). Thus, the notion "Kähler differentials" was used by his students Berger, Kiehl, Kunz and Nastold. In 1962, these attended a talk given by Zariski in Frankfurt and afterwards had a discussion with him. It could be that, as an effect of this discussion and of the study of Kunz's papers by Zariski and his student Lipman, the notion was carried over to the USA.

Kähler used differentials to clarify the notion of inseparable field extensions and to introduce a different associated to a ring finitely generated over a subring, the *Kähler different*, appearing under this name also in Schmidt's review [SAD]. As I see it, this different found its most important application in a criterium for the regularity of a local ring (*Lemma di Mehner*) which extends another criterium valid only over a ground field (*Lemma di Zariski*). And finally, the different is essential for Kähler's discussion of his arithmetically integral differentials associated to a field finitely generated over $\mathbf{Q}$ (or a Galois field) which he introduced as a common generalization of the integral elements of a number field and the abelian differentials of the first kind from function theory.

It is natural to ask for relations between these integral differentials associated to a field or field extension and integral differentials belonging to models of the field, i. e. varieties resp. schemes having this field as the field of rational functions. This theme has been pursued by Kähler and has led to some further developments (see [Be 4] and [KW]). Moreover, it is natural to compare these arithmetically distinguished differentials with otherwise distinguished differentials, as for instance the Néron differentials in the case of abelian function fields and the differentials with integer Fourier expansion in the case of modular functions fields.

In this text, Kähler's definitions and main results will be reported as they appear in his Geometria Aritmetica ([GA]) from 1958 and, in sometimes more preliminary version, in his papers [RAK] and [CPA] from 1951 resp. 1952 as well as in the book [AD] from 1953. Some later developments are sketched which concern the themes mentioned above. For more about Kähler's impact in commutative algebra than recounted here, we refer to the books [KD] and [KW] and to the text by J.-B. Bost. The newest developments in global algebraic and arithmetical geometry concerning an aritmethic Hodge theory will be commented upon in the text by U. Jannsen.

With the exception of the theme of Zeta functions which will be treated apart, this text is also meant to give a survey of Kähler's work in Algebraic and Arithmetic Geometry. So, while the topic of differentials is pursued in the main part of the article, the other topics, namely his terminology for local and commutative algebra, and his system to describe and study algebraic and arithmetic varieties, their spaces and subvarieties, as well as other nowadays standard tools, are assembled in two Appendices. For an excellent overview the reader may also consult Kähler's *Infinitesimal-Arithmetik* [IA].

It's a pleasure to thank P. Slodowy for numerous hints concerning the substance and the linguistics of this article.

Contents

1 Kähler Differentials

Kähler formulated his definitions in various ways. We will reproduce here two of them and later compare these to the now usual ones.

1. 1 The Differential Module

Aiming at the study of purely algebraical fields, in the paper [RAK] from 1951, he looks at *purely algebraic* rings R, i. e. those generated as a ring by a finite number of elements. For these he defines on p. 71

Definition 1 : The *differentiation* of R is a homomorphism of R as an R-module into an R-module RdR consisting of the elements

$$adb + a'db' + \dots \quad \text{with} \quad a, a', \dots, b, b', \dots, \in R\ ,$$

where da means the image (=: *differential*) of the element $a \in R$ and where the *differentiation rules*

$$\begin{aligned} d\,(a+b) - da - db \;&=\; 0 \\ d\,(a \cdot b) - a \cdot db - b \cdot da \;&=\; 0 \quad \text{for all} \quad a, b \in R \end{aligned}$$

are the defining relations of the R-module.
In the *differential module* RdR there are no other relations (*differential equations*) than those which can be obtained from the differentiation above by linear combination with coefficients from R. This differential module is uniquely determined by R up to R-isomorphy . Using several copies

$$Rd_1R\,,\ Rd_2R\,,\ Rd_3R, \dots$$

of this module, Kähler defines by a sort of direct multiplication R-modules of higher degree

$$Rd_1Rd_2R\,,\ Rd_1Rd_2Rd_3R$$

which again are uniquely determined by R.

1.2 Modules and algebras of infinitesimal elements

In principle, this definition above contains everything to be said. Nevertheless in 1958, in his *Calcolo diffenziale* in [GA], 1. - 13., the definition looks a bit more formal (and has - in my opinion - a nice visual touch), so that we reproduce it here also. Doing this, we try to stick to Kähler's original notations as far as possible, but we have to face the frequently arising problem that Kähler sometimes uses different symbols (and even names) for the same objects in different papers, so that we have to change some symbols to get a unified treatment here.

If $\check{R}$ is a commutative ring, an element $x \in R$ will be called *infinitesimal of degree* n, if we have $x^n = 0$ and no power of smaller degree is zero. If $\check{R}$ is not necessarily commutative, $x \in \check{R}$ is called *infinitesimal*, if the bilateral ideal generated by x consists of nilpotent elements.

From now on a *ring* will always be a commutative ring. A subring R of $\check{R}$ is called *infinitesimally near* to another subring $\widetilde{R}$ of $\check{R}$ if there is an isomorphism σ of R onto $\widetilde{R}$ such that in $\check{R}$ we have the relations

$$(x^\sigma - x)\,(y^\sigma - y) \;=\; 0 \quad \text{for all} \quad x, y \in R\,.$$

Here Kähler uses the notations $x^\sigma = \sigma\,(x)\,,\ (x^\sigma)^\tau = x^{\sigma\tau}$, etc.. If 2 is invertible in $\check{R}$, the condition above simplifies to

$$(x^\sigma - x)^2 \;=\; 0 \quad \text{for all} \quad x \in R\,,$$

saying that the *differentials* $x^\sigma - x$ are infinitesimal of degree ≤ 2.

Definition 2 : An *h-th infinitesimal correspondence of* R is a ring generated by R and h infinitesimal near copies $R^{\sigma_i}(i = 1, \ldots, h)$ of R

$$[\,R, R^{\sigma_1}, \ldots, R^{\sigma_h}\,]$$

Remark : Such a ring can be constructed as

$$[\,R, d_1 R, \ldots, d_n R\,]\,,$$

generated by R and the elements $d_i x$ (thought of as $d_i x = x^{\sigma_i} - x$), $i = 1, \ldots, h$, obeying only the relations

$$\begin{aligned} &d_i\,(x + y) - d_i x - d_i y \;=\; 0 \\ &d_i\,(x \cdot y) - x \cdot d_i y - y \cdot d_i x \;=\; 0 \\ &d_i x \cdot d_i y \;=\; 0 \quad \text{for all} \quad x, y \in R\,. \end{aligned}$$

More precisely, we take the free R-algebra generated by R and all $d_i x$, $i = 1, \ldots, h, x \in R$ and divide out the ideal generated by the left sides of the relations given. This ring is *generic* in the sense that any other h-th infinitesimal correspondence $[R, R^{\tau_1}, \ldots, R^{\tau_h}]$ is simply a homomorphic image of the generic one constructed above.
Each h-th generic infinitesimal correspondence of R can be interpreted as a subring of an *infinite generic infinitesimal correspondence* I_R of R. Each permutation of the indices of this correspondence induces an automorphism π of this ring.

Definition 3 : A *differential form of degree* n of R is an element

$$\omega = \sum_{m\in\mathbf{Z},\, a,\dots,b\in R} m d_1 a \cdot \ldots \cdot d_n b + \sum_{e,f,\dots,g\in R} e \cdot d_1 f \cdot \ldots \cdot d_n g \ ,$$

of I_R (the first part won't appear if R has a unit element.)
An important special case is given, when all the rings $R^{\sigma_i}, R^{\sigma_j}$ in the correspondence are mutually infinitesimally near. We then have to add the relations

$$d_i x \cdot d_j y + d_i y \cdot d_j x = 0 \quad \text{for all} \quad x, y \in R$$

and we call the rings arising this way *generic differential correspondences* Ω_R. The reader can verify (or extract from [GA] 8.-12.) that the elements of Ω_R are sums of *exterior* differential forms of degree n subject to the rules

$$\begin{aligned} & d_i x \cdot d_j y + d_i y \cdot d_j x = 0 \\ & d_i \left(\sum m d_1 a \ldots d_n b + \sum e \cdot d_1 f \ldots d_n g\right) = \sum d_i e \cdot d_1 f \cdot \ldots \cdot d_n g \\ & d_i \left(d_j X\right) = 0 \ , \quad x, y, a, \ldots, b, e, f, \ldots, g \in R \ , \ m \in \mathbf{Z} \ , \quad X \in \Omega_R \ . \end{aligned}$$

In [GA] 12., Kähler compares this with the usual calculus of exterior forms for rings of C^∞-functions and also introduces the symbol $\wedge$ for the multiplication in Ω_R so that, for the multiplication of exterior differential forms ξ and η of degree h resp. h', we obtain

$$\xi \wedge \eta = (-1)^{hh'} \eta \wedge \xi \ , \ d\,(\xi \wedge \eta) = d\xi \wedge \eta + (-1)^h \, \xi \wedge d\eta \ , \ dd\xi = 0 \ .$$

Finally, in [GA] 13. it is stated that Ω_R is defined as a ring extension of R by the following equations

$$\begin{aligned} &(1) \quad d_i \,(x + y) - d_i x - d_i y = 0 \\ &(2) \quad d_i \,(x \cdot y) - x d_i y - y d_i x = 0 \\ &(3) \quad d_i x \cdot d_i y = 0 \\ &(4) \quad d_i x \cdot d_j y + d_i y \cdot d_j x = 0 \quad \text{for all} \quad x, y \in R \ . \end{aligned}$$

A Pfaffian equation

$$\sum_{n\in\mathbf{Z},\, a\in R} n d_i a + \sum_{e,f\in R} e \cdot d_i f = 0$$

is called a *differential equation of R*, if it is a consequence of the *differentiation rules* (1) and (2) (by ideal operations). The expression *the differential equations of R* is used for any system of differential equations such that the differentiation rules are consequences of these (even though, as Kähler remarks, this expression is not well defined).

As examples, we have for each ring R the differential equations

$$d1 = 0 \quad \text{and} \quad dx - 1dx = 0 \quad \text{for all} \quad x \in R\,.$$

Moreover, for any polynomial

$$f\,(x_1, \ldots, x_n) = \sum_{i_1,\ldots,i_n} a_{i_1\cdot\ldots\cdot i_m}\, x_1^{i_1} \cdot \ldots \cdot x_m^{i_m} \quad \text{with} \quad a_{i_1\ldots i_m} \in R\,,$$

using the notation f_{x_ν} for the *partial derivative*

$$f_{x_\nu} = \sum_{i_1,\ldots,i_m} i_\nu\, a_{i_1\ldots i_m} \cdot x_1^{i_1} \ldots x_\nu^{i_\nu - 1} \cdot \ldots \cdot x_m^{i_m}$$

and δf for the *variation* of f

$$\delta f = \sum_{i_1,\ldots,i_m} (da_{i_1\ldots i_m})\, x_1^{i_1} \cdot \ldots \cdot x_m^{i_m}\,,$$

we obtain the differential equation

$$df - \sum_\nu f_{x_\nu} \cdot dx_\nu - \delta f = 0\,.$$

If f and g are polynomials as above, and if the quotient $\dfrac{f(x_1 \ldots x_m)}{g(x_1 \ldots x_m)}$ exists in R (that is, g is not zero or zero divisor in R), we have the differential equations

$$d\left(\frac{f}{g}\right) - \sum_\nu \left(\frac{f}{g}\right)_{x_\nu} dx_\nu - \delta\left(\frac{f}{g}\right) = 0$$

with

$$\left(\frac{f}{g}\right)_{x_\nu} = \frac{g f_{x_\nu} - f g_{x_\nu}}{g^2}\,, \quad \delta\,(\frac{f}{g}) = \frac{g\delta f - f\delta g}{g^2}\,.$$

If R_0 is a subring of R, Kähler defines the *relative differentiation* $\dfrac{d}{R_0}$ *of* R *with respect to* R_0 in [GA] 14. by simply adjoining to the (absolute) differential equations the equations

$$dx_0 = 0 \quad \text{for all} \quad x_0 \in R_0\,.$$

1.3 Differential modules and universal derivations

In his address to the reunion of the Deutsche Mathematiker Vereinigung (DMV) in 1953 which appeared as the book [AD] , Kähler gave constructions similar to those reproduced here from the Geometria Aritmetica [GA]. F. K. Schmidt prepared a carefully written review [SAD] of this text [AD] in the Zentralblatt MATH which became very important for the later development of the algebraic theory of differentials. F. K. Schmidt points out a paper by A. Weil from 1943 which already expresses (part of) Kähler's ideas but is not mentioned by him in [AD] (which is understandable, as he already had preformulated most of his concepts before the war). F. K. Schmidt presents Kähler's notion in the following way (which is already very close to the formulation of the later treatments, as in the exposés of Cartier and Godement in the Séminaire Cartan-Chevalley in 1955/6 [CC], in Grothendieck [EGA] 0_{IV}, § 20.5, in Hartshorne [Ha], p. 172, or in Kunz [KD] p. 13):

Let R be a unitary commutative ring with a unitary subring R_0 and D a derivation of R/R_0 into an R-module M, i. e. D is a R_0-linear map from R to M with

$$D\,(x.y) \;=\; x\,Dy + yDx \quad \text{for all} \quad x, y \in R,$$

and thus

$$Dx \;=\; 0 \quad \text{for all} \quad x \in R_0\;.$$

If we compose D with a linear map Λ from M to another R-module M', we get a derivation $D' = \Lambda \cdot D$ from R to M' called a *specialization* of D. A derivation d of R/R_0 in an R-module is called *universal*, if each derivation D of R/R_0 into an R-module is a specialization of d. Then d is uniquely determined up to linear isomorphism. The R-module $R\frac{d}{R_0}R$ belonging to this universal derivation is called the module of differential forms of degree 1 (nowadays usually written as Ω^1_{R/R_0}).

The existence of Ω^1_{R/R_0} is clear for a free polynomial ring $R = R_0[X_1 \dots X_m]$, as the universal derivation is simply given by the formal differentiation of the polynomials. For an arbitrary ring extension of R_0 one reduces to this case by a Lemma (Kähler [GA] 287.) claiming that for a residue ring $R' = R/\mathfrak{a}$ and $R'_0 = (R_0 + \mathfrak{a})/\mathfrak{a}$ one has

$$R'\,\frac{d'}{R'_0}\,R' \;=\; R\,\frac{d}{R_0}\,R/(\mathfrak{a}\,\frac{d}{R_0}\,R + R\,\frac{d}{R_0}\mathfrak{a})\;.$$

In contemporary language one would say that Ω^1_{R/R_0} is the representing object of the functor $\mathrm{Der}_{R_0}(R, -)$.
We have a canonical isomorphism

$$\mathrm{Der}_{R_0}\,(R) \;:=\; \mathrm{Der}_{R_0}\,(R, R) \simeq \mathrm{Hom}_R\,(\Omega^1_{R/R_0}, R)\;.$$

Here we mention also that the dual $\mathrm{Der}_{R_0}\,(R)^*$ of the derivation module is called ([KD] p. 137) the module of *Zariski differentials*. As mentioned in [KD], its general relation to the module Ω^1_{R/R_0} of Kähler differentials is still an interesting object of study.

Moreover, we mention the following characterization of the module of Kähler differentials ([EGA] 0_{IV} 20.4.3))

$$\Omega^1_{R/R_0} = I/I^2$$

where I is the kernel of the map

$$\begin{aligned} R \otimes_{R_0} R &\longrightarrow R \\ a \otimes b &\longmapsto ab \,. \end{aligned}$$

Obviously, this description is not far from Kähler's original formulation via infinitesimal elements.

To come back to F. K. Schmidt's review of Kähler's algebraic introduction of differentials, he also remarks that, starting from the module Ω^1, one obtains the modules of differential forms of degree h resp. exterior differential forms by taking h-th tensor resp. exterior powers

$$\overset{h}{\otimes}\ \Omega^1 \quad \text{resp.} \quad \wedge^h \Omega^1 =: \Omega^h$$

(as is clear now, we also can take other symmetrizations to get further modules). By summation over h, we get the algebras of differentials resp. exterior differentials $\Omega_{R/R_0} = \oplus_h \Omega^h_{R/R_0}$ (with $\Omega^0_{R/R_0} := R$). In [KD] p. 39, we also find an introduction of this algebra by a universal property similar to the one described above defining the module Ω^1_{R/R_0}.

Some of the many applications of differentials as assembled in particular in Kunz's book will be mentioned later. Here we will only point to Kähler's exposition in [AD] § 18-21 resp. [GA] p. 19-37 where he studies the differential modules of field extensions K/K_0 in characteristic p. He defines the *inseparability degree*

$$\operatorname{insep}_{K_0} K = \operatorname{rank} \Omega^1_{K/K_0} - \operatorname{trdeg}_{K_0} K$$

and elaborates the fact that differentials of the elements of a family of elements of K are a basis of Ω^1_{K/K_0} exactly if this family constitutes a p-basis of K/K_0.

2 Kähler differents and the comparison with other differents

We now follow again the presentation in Kähler's Geometria Aritmetica and come to his first application of the notion of differential modules.

2.1 Kähler differents

The Kähler differents $\mathfrak{d}_n\,(R/R_0)$ of a ring extension R/R_0 (with Ω^1_{R/R_0} finitely generated as an R-module) are Fitting determinantal ideals of Ω^1_{R/R_0} and thus are defined as follows.

Definition : For

$$\Omega^1_{R/R_0} \;=\; \sum_{i=1}^{m} R\,\omega_i$$

and $n \in \mathbf{N}_0$ with $n < m$ the n-th *different* $\mathfrak{d}_n\,(R/R_0)\,)$ is defined as the ideal in R generated by the determinants

$$\begin{vmatrix} a^{(1)}_{i_1} & \dots & a^{(1)}_{i_{m-n}} \\ \vdots & & \\ a^{(m-n)}_{i_1} & \dots & a^{(m-n)}_{i_{m-n}} \end{vmatrix}$$

which can be built out of the matrices $(a^{(k)}_i)$ belonging to any $m-n$ relations

$$\sum_{i=1}^{m} a^{(k)}_i\,\omega_i \;=\; 0 \quad k = 1,\dots,m-n$$

between the (ω_i). For $n \geq m$ one defines

$$\mathfrak{d}_n\,(R/R_0) \;=\; R\;.$$

It is clear that these ideals form an ascending chain

$$\mathfrak{d}_n\,(R/R_0) \subset \mathfrak{d}_{n+1}\,(R/R_0)$$

and by a theorem of Fitting the differents are well defined, i. e. independently of the choice of the basis (ω_i). In [GA] 15. we find a proof of the fact that the formation of differents as well as the formation of differential modules is consistent with localization.

Proposition 1 : Under the assumption that the rings $R_0 \subset R$ and $\overline{R_0} \subset \overline{R}$ are subrings of a ring $\check{R}$ and that each element of $\overline{R_0}$ resp. $\overline{R}$ can be represented in the form x/y with $x,y \in R_0$ resp. $x,y \in R$ where $y \neq 0$ and not a zero-divisor in R, we have

$$\mathfrak{d}_n\,(\overline{R}/\overline{R_0}) \;=\; \mathfrak{d}_n\,(R/R_0)\,\overline{R}\;.$$

Another very useful result is found in [AD] p. 126 and also in [GA] 288.

Proposition 2 : For an ideal $\mathfrak{a}$ in R we have

$$\mathfrak{d}_\nu\,(R/R_0) + \mathfrak{a}^n \;=\; \mathfrak{d}_\nu\,((R/\mathfrak{a}^{n+1})/(R_0 + \mathfrak{a}^{n+1}/\mathfrak{a}^{n+1}) + \mathfrak{a}^n\;.$$

Kähler applies this result to determine or at least to establish estimates for the differents in the following situation.

Let K be a field finitely generated as an algebra over a subfield K_0 (K/K_0 *algebraic*), s and S noetherian local rings with quotient field K, where S denotes a discrete valuation ring, and $\mathfrak{p}$ and $\mathfrak{P}$ the maximal ideals of s resp. S ($\mathfrak{P}$ a principal ideal now). Moreover, assume

$s_0 := s \cap K_0$ and $S_0 = S \cap K_0$ with maximal ideals $\mathfrak{p}_0$ resp. $\mathfrak{P}_0$

$k = S/\mathfrak{P}$ algebraic over $k_0 = S_0/\mathfrak{P}_0$

$\mathrm{trdeg}_{k_0} k = \nu$, $\mathrm{rank}\, \Omega^1_{k/k_0} = \nu + h$ $(h \geq 0)$

$\mathrm{char}\, k = p \geq 0$ (and similar assumptions for $s/\mathfrak{p}$ over $s_0/\mathfrak{p}_0$).

Then in [AD] p. 138/9, Kähler arrives at his final description tying the different to the *ramification index* e for $\mathfrak{P}_0 \neq 0$ defined by

$$\mathfrak{P}_0 S = \mathfrak{P}^e$$

(if $\mathfrak{P}_0 = 0$ or $e = 1$, K/K_0 is called *unramified*).

Theorem: Under the assumptions above, we have

$$\begin{array}{rcll}
\mathfrak{d}_\mu (S/S_0) & = & 0 \quad \text{for} \quad \mu < \nu, & \\
\mathfrak{d}_\nu (S/S_0) & = & 0 \quad \text{if} \quad \mathfrak{P}_0 = 0, & \\
 & \subset & \mathfrak{P}^{e-1+h} & \text{if} \quad \mathfrak{P}_0 \neq 0 , \\
 & = & \mathfrak{P}^{e-1} & \text{if} \quad \mathfrak{P}_0 \neq 0, e \notin \mathfrak{P} , \ h = 0, \\
 & \subset & e\mathfrak{P}^{e} & \text{if} \quad \mathfrak{P}_0 \neq 0, e \in \mathfrak{P} , \ h = 0, \\
\mathfrak{d}_{\nu+\lambda}(S/S_0) & \subset & \mathfrak{P}^{h-\lambda} \quad \text{for} \quad 0 < \lambda < h, & \\
\mathfrak{d}_{\nu+h+1} (S/S_0) & = & S \ . &
\end{array}$$

Nothing is said about $\mathfrak{d}_{\nu+h} (S/S_0)$. A slight extension of the previous result is the following

Corollary : For $n = \mathrm{trdeg}_{K_0} K$, $i = \mathrm{insep}_{K_0} K$ we have

$$\begin{array}{rl}
\mathfrak{d}_\mu (K/K_0) & = 0 \quad \text{for} \quad \mu < n+i \\
 & = k \quad \text{for} \quad \mu = n+i
\end{array}$$

and

$$\begin{array}{rcll}
\nu & < & n+i & \text{if} \quad \mathfrak{P}_0 = 0, \\
\nu & = & n+i & \qquad \mathfrak{P}_0 \neq 0 , \ h = 0, \\
\nu + h + 1 & \geq & n+i & \text{anyway} \ .
\end{array}$$

In the Geometria Aritmetica we find results for the more general situation of local rings $s \subset K$ with

$$\dim_{s/\mathfrak{p}} (\mathfrak{p}/\mathfrak{p}^2) = r \geq 1$$

and in [GA] p. 290 we find, for the smallest ν with

$$\mathfrak{d}_\nu (s/s_0) = s ,$$

the relation

$$\begin{aligned}&\dim_{s/\mathfrak{p}} (\mathfrak{p}_0 s + \mathfrak{p}^2/\mathfrak{p}) + \mathrm{trdeg}_{(s_0+\mathfrak{p})/\mathfrak{p}}\, s/\mathfrak{p} \leq \nu \leq \\ &\quad \dim_{s/\mathfrak{p}} (\mathfrak{p}_0 s + \mathfrak{p}^2/\mathfrak{p}) + \mathrm{trdeg}_{(s_0+\mathfrak{p})/\mathfrak{p}}\, s/\mathfrak{p} + \mathrm{insep}_{(s_0+\mathfrak{p})/\mathfrak{p}}\, s/\mathfrak{p}\ .\end{aligned}$$

As a refinement of the Proposition 2, [GA] p. 294 states that the different is compatible with completion .

Proposition 3 : If s^* is the $\mathfrak{p}$-adic completion of s, we have

$$\mathfrak{d}_n (s/s_0)\, s^* = \mathfrak{d}_n^* (s/s_0)\,.$$

Here ([GA] p. 292), $\mathfrak{d}_n^* (s/s_0)$ denotes the ideal in s^* generated by the determinants of order $m - n$ of the matrix (α_{ik}) of a system of relations

$$\sum_{k=1}^{m} \alpha_{ik} \frac{d}{s_0} \xi_k = 0$$

which define the differential module $\Omega^1_{s^*/s_0} = \sum_{k=1}^{m} s^* \dfrac{d}{s_0} \xi_k\,.$

The most important application of the different is the following regularity criterium (a result of the thesis by Mehner, reproduced in [GA] 303).

Lemma di Mehner. Let s be as above, a local ring essentially of finite type over $\mathbf{Z}$ with quotient field K , $\mathrm{trdeg}\, K = n$. Then s is regular, exactly if

$$\begin{aligned}\mathfrak{d}_{n+1} (s) &= s \quad \text{for} \quad \mathrm{char}\, K \neq \mathrm{char}\, s/\mathfrak{p} \in \mathfrak{p}^2, \\ \mathfrak{d}_n (s) &= s \quad \text{else}\ .\end{aligned}$$

As another consequence of the calculus of the different, Kähler states the following classical result in [GA] 304.

Lemma di Zariski: For a local ring s, essentially of finite type over K_0 with quotient field K and $s/\mathfrak{p}$ separable over $k_0 + \mathfrak{p}/\mathfrak{p}$, the first different $\mathfrak{d}_r (s/s_0)$ which is equal to s has index

$$r \geq \mathrm{trdeg}_{K_0} K\ .$$

Here we have equality exactly if s is regular.

One may be interested to compare this with the formulation of the Lemma in [EGA] O_{IV} 22.6. Numerous results on regularity and smoothness criteria are to be found in [KD] p. 138-164. A direct geometric application of the higher differents is elaborated in the thesis by Gerhard Quarg [Q] where the different $\mathfrak{d}_\nu$ is used to define a tangential subscheme of a projective variety $X \subset \mathbf{P}^n$ with respect to a linear subspace $H \subset \mathbf{P}^n$.

2.2 Comparison with other differents

In his historical remarks at the end of [AD], Kähler mentions that E. Noether had also defined differents by differential methods. F. K. Schmidt, in his review, points out that differents had been introduced moreover by Dedekind and A. Weil. In the sequel there has been a lot of work by Berger([Ber]), Kunz, Nastold, and others comparing and generalizing these notions. Here we take over part of the material collected in [KD] §10 and Appendix G.

The Noether different

Definition : For an arbitrary ring extension R/R_0 the Noether different $\mathfrak{d}_N$ is defined by

$$\mathfrak{d}_N\,(R/R_0) \;:=\; \varphi\,(\mathrm{Ann}_{R\otimes_{R_0}R}\, I)\ ,$$

where $I := \ker\,\varphi$ for $\varphi : R\otimes_{R_0} R \to R$ defined by $a\otimes b \mapsto ab$.

Remark : $\mathfrak{d}_N$ is an ideal in R compatible with localization.

The most elementary example is the case where R/R_0 is a *finite complete intersection*, i. e.

$$R \;=\; R_0\,[x_1,\ldots,x_n] \;=\; R_0\,[X_1,\ldots,X_n]/\mathfrak{a}\ , \qquad \mathfrak{a} = (f_1,\ldots,f_n)\ .$$

Then one has

$$\mathfrak{d}_N\,(R/R_0) \;=\; \frac{\partial(f_1,\ldots,f_n)}{\partial(x_1,\ldots,x_n)}\,R\ ,$$

where $\frac{\partial f}{\partial x}$ abbreviates the class of the functional determinant $\frac{\partial f}{\partial x} \bmod \mathfrak{a}$. Here we see immediately that, in this case, we have

$$\mathfrak{d}_N \;=\; \mathfrak{d}_K \;:=\; \mathfrak{d}_0\,(R/R_0)\ .$$

More generally we have [KD] Proposition 10.18 : Let R/R_0 be an algebra of finite type and let $m := \mu\,(\Omega^1_{R/R_0})$ be the minimal number of generators over R. Then we have

$$\mathfrak{d}_N\,(R/R_0)^m \subset (\mathrm{Ann}\,\Omega^1_{R/R_0})^m \subset \mathfrak{d}_K\,(R/R_0) \subset \mathfrak{d}_N\,(R/R_0) \subset \mathrm{Ann}\,\Omega^1_{R/R_0}\ .$$

The Dedekind Different

In the classical number theoretic situation we have a ring extension R/R_0 with quotient fields K and K_0 such that K/K_0 is finite with trace map $\sigma_{K/K_0} : K \to K_0$. We then define

Definition : The *complementary module* $\mathfrak{C}_{R/R_0}$ is given by

$$\mathfrak{C}_{R/R_0} := \{x \in K \, , \; \sigma_{K/K_0}(rx) \in R_0 \quad \text{for all} \quad r \in R\}$$

and the *Dedekind different* by its dual

$$(*) \quad \mathfrak{d}_D(R/R_0) := \{x \in K : x\,\mathfrak{C}_{R/R_0} \subset R\} \, .$$

In [KD] App. G, this notion is generalized for R/R_0 satisfying

a) R_0 is noetherian, and any non-zerodivisor of R_0 is also one of R.

b) There is a subalgebra R_1/R_0 of R/R_0 such that R_1/R_0 is finite, and the following conditions hold:

α) For each $\mathfrak{c} \in \operatorname{Spec} R$, *let*$\mathfrak{c}_1 := \mathfrak{c} \cap R_1$. Then the canonical homomorphism $R_{1\,\mathfrak{c}_1} \to R_{\mathfrak{c}}$ is bijective.

β) The canonical homomorphisms $Q(R_0) \otimes_{R_0} R_1 \to Q(R_1)$ and $Q(R_1) \to Q(R)$ are bijective ($Q(R)$ denotes the full quotient ring of R).

If moreover there is a map $\sigma : K := Q(R) \to K_0 := Q(R_0)$ with the usual properties of a trace map, one may write

$$R \cdot \operatorname{Hom}_{R_0}(R_1, R_0) \; = \; \mathfrak{C}^{\sigma}_{R/R_0} \cdot \sigma$$

with a fractional R-ideal $\mathfrak{C}^{\sigma}_{R/R_0}$ (which can be seen to be independent of R_1) and the Dedekind different $\mathfrak{d}^{\sigma}_D(R/R_0)$ with respect to σ can be defined as above by $(*)$.

In [KD] G. 10, G. 11 and G. 12 the following results are proved

i) If, under the assumptions above, R/R_0 is finite and flat the following assertions are equivalent.
   a) R/R_0 is a Gorenstein algebra.
   b) $\mathfrak{C}^{\sigma}_{R/R_0}$ is an invertible ideal.

ii) Let R/R_0 be finite, assume that R_0 is noetherian and that each non-zerodivisor of R_0 is a non-zerodivisor of R. If $K = Q(R)$ is étale over $K_0 = Q(R_0)$, then $\mathfrak{d}_D(R/R_0)$ is defined as above and we have

a) $\mathfrak{d}_N(R/R_0) \in \mathfrak{d}_D(R/R_0)$.

b) If R/R_0 is projective, then both differents are equal. (This is a Theorem going back to E. Noether [No]).

iii) If R/R_0 has a presentation

$$R = R_0[X_1, \ldots, X_n]/(F_1, \ldots, F_n) = R_0[x_1, \ldots, x_n]$$

as a complete intersection, then

$$\mathfrak{d}_D(R/R_0) = \mathfrak{d}_N(R/R_0) = \mathfrak{d}_K(R/R_0) = \left(\frac{\partial F}{\partial x}\right).$$

And a more general statement is [KD] Proposition 10. 22: Let R/R_0 be an algebra of finite type such that $K = Q(R)$ is étale over $K_0 = Q(R_0)$. Then

$$\mathfrak{d}_K(R/R_0) \subset \mathfrak{d}_D(R/R_0).$$

The differents $\mathfrak{d}_K$ and $\mathfrak{d}_D$ are equal if R/R_0 is a finite Gorenstein algebra where R_0 is noetherian, $K = Q(R)$ is étale over $K_0 = Q(R_0)$ and R/R_0 is locally a complete intersection ([KD] p. 170). In general, both differents are not equal (see the example by Waldi [Wa] for the case of an almost complete intersection).

2.3 The Different of a Differential

The notion of the Kähler different has been extended in [Be1] by associating an R-module to a differential $\omega \in \Omega_{R/R_0}$ in the following way.

Let R be of finite type over R_0 with a presentation

$$R = B/\mathfrak{b},$$

where B is a free polynomial ring over R_0

$$B = R_0[X_1, \ldots, X_n]$$

and $\mathfrak{b}$ an ideal in B

$$\mathfrak{b} = \sum_{j=1}^{l} f_j B.$$

Then, with

$$\overline{\Omega}^1 = \sum_{i=1}^{n} R dX_i, \quad \overline{\Omega} = \oplus \wedge^q \overline{\Omega}^1$$

and

$$N \;=\; \sum_{j=1}^{n} RW_j \;, \quad W_j \;=\; \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}\, dX_i \;, \quad \frac{\partial f_j}{\partial x_i} \;=\; \frac{\partial f_j}{\partial X_i} + \mathfrak{b}$$

we obtain the presentations

$$\Omega^1_{R/R_0} \;=\; \overline{\Omega}^1/N \quad \text{resp.} \quad \Omega_{R/R_0} \;=\; \overline{\Omega}/(N \wedge \overline{\Omega}) \;.$$

Let now r be the smallest index h such that the Kähler different $\mathfrak{d}_h\,(R/R_0)$ is non-zero, i. e. such that there is a non-zero exterior product of $n-r$ factors $W_j \in N$, and all products of greater length are zero.

Definition : With the notation above for a differential $\omega \in \Omega_{R/R_0}$ its *different* $\mathfrak{d}\,(\omega|R/R_0)$ is defined as the ideal in R generated by the coefficients of all products

$$\varpi \wedge W_{j_1} \wedge \ldots \wedge W_{j_{n-r}} \;, \quad W_j \in N$$

where ϖ is a representative of ω in $\overline{\Omega}$.

In [Be1] there is a proof that $\mathfrak{d}\,(\omega|R/R_0)$ is well defined (which for $\omega = 1$ includes Kähler's proof of the well-definedness of his differents and is only a slight extension of his proof). Moreover, the construction of this different is compatible with localization. For this we have to generalize the definition above slightly: If R' is a localization of R, one has an exact sequence

$$0 \to N' \;=\; N \otimes_R R' \to \overline{\Omega}'^1 \;:=\; \overline{\Omega}^1 \otimes_R R' \to \Omega^1_{R'/R_0} \to 0$$

where $\overline{\Omega}'^1 = \sum_{i=1}^{n} R'\, dX_i$, $N' = \sum_{j=1}^{l} R'\, W'_j$, with $W'_j = W_j \otimes_R 1$. Thus, one has as already remarked in 2. 1

$$\mathfrak{d}_n\,(R'/R_0) \;=\; \mathfrak{d}_n\,(R/R_0) \otimes_R R'$$

and moreover the construction of $\mathfrak{d}\,(\omega|R/R_0)$ works as well associating an R-module $\mathfrak{d}\,(\omega'|R/R_0)$ to a differential $\omega' \in \Omega_{R'/R_0}$ ($= \Omega_{R'/R'_0}$ by $[EGA]\,0_{IV}$ 20.5.9, if R'_0 is the localization of R_0 corresponding to R'). Finally, one gets

Proposition : If R' is a localization of R, we have for $\omega' \in \Omega_{R'/R_0}$

$$\mathfrak{d}\,(\omega'|R'/R_0) \;=\; \mathfrak{d}\,(\omega'|R/R_0) \otimes_R R' \;.$$

3 Locally Integral Differentials

The notions of Kähler differentials, Kähler different and different of a differential are compatible with localization and thus apt to define sheaves and birational invariants of finite generated field extensions. This will be discussed later. Here we will first prepare the ground by further studying the following *local* situation.

Let R be an algebra essentially of finite type over R_0 (i. e. localization of a finitely generated algebra) and $R' = R_T$ localization of R in the multiplicatively closed system $T \subset R$. Then we have a canonical map

$$\Omega_{R/R_0} \overset{\varphi}{\to} \Omega_{R'/R_0} = R' \otimes_R \Omega_{R/R_0} = T^{-1}\Omega_{R/R_0}$$

and, following again Kähler, we introduce the notation

$$[R, \frac{d}{R_0} R] := \varphi(\Omega_{R/R_0}) .$$

Here, at least for Kähler, the most interesting case is where R is an integral domain and $R' = Q(R)$ its quotient field. In this, but also in the more general case above, one may try to distinguish in Ω_{R'/R_0} a bigger module than $[R, \frac{d}{R_0} R]$.

3.1 Integral differentials via the different

The following notion, using the different of a differential from 2.3, was proposed in [Be3] 4. It is inspired by Néron [Ne] .

Definition : Given a domain R of finite type over R_0 with quotient field $Q(R) = K$ a differential $\omega' \in \Omega_{K/R_0}$ is called *integral in* R, if

$$\mathfrak{d}(\omega' | R/R_0) \subset R .$$

We will write Δ_{R/R_0} for the R-module of these integral differentials for R/R_0. It can be shown ([Be3] 5.) that Δ_{R/R_0} is a finitely generated R-module. This notion has been used in [Be2] to give another formulation of a criterium of Néron characterizing an arithmetic minimal model of a finitely generated field of transcendence degree one. Moreover, this notion will reappear in the discussion of Kähler's *globally integral differentials of a field* below.

3.2 Integral and regular differentials à la Kunz-Waldi

Pursuing previous work by Kunz on generalized complementary modules and differential forms on singular varieties ([Ku 1-5]) and expecting to get a *dualizing sheaf* for most general varieties X/Y , Kunz and Waldi [KW] give another (essentially equivalent) definition of integral differentials for R/R_0 and, moreover, a still more refined definition of *regular* differentials.

Not striving for the high generality attained in [KW], we will only sketch here some of this.

Let K be again the quotient field of $R, moreover R/R_0$ essentially of finite type, $M = \Omega^1_{R/R_0}$ and

$$M_K \;=\; K \otimes \Omega^1_{R/R_0} \;=\; \Omega^1_{K/R_0}$$

with presentations

$$0 \longrightarrow N \overset{\alpha}{\to} F \;=\; R^m \overset{\beta}{\to} M \longrightarrow 0$$

resp.

$$0 \longrightarrow N_K \overset{\alpha_K}{\to} F_K \;=\; K^r \overset{\beta_K}{\to} M_K \longrightarrow 0\ .$$

Then $\mathfrak{d}_r\,(M) = \mathfrak{d}_r\,(R/R_0)$ is the first non-zero different, i. e. $\wedge^{m-r+1}\,\alpha = 0$ and for each $q \in \mathbf{N}$ there is a canonical commutative diagram

$$\begin{array}{ccc}
\wedge^q M & \xrightarrow{\varphi^q} & \mathrm{Hom}_R(\wedge^{m-r}N, \wedge^{m-r+q}F) \\
\downarrow & & \downarrow{\scriptstyle \xi^q} \\
\wedge^q M_K & \xrightarrow{\varphi^q_K} & \mathrm{Hom}_R(\wedge^{m-r}N_K, \wedge^{m-r+q}F_K)
\end{array}$$

where φ^q is defined by

$$\begin{array}{rcl}
\varphi^q\,(\omega) : \wedge^{m-r}\,N & \longrightarrow & \wedge^{m-r+q}\,F \\
u & \longmapsto & (\wedge^{m-r}\alpha)\,(u) \wedge \widetilde{\omega}
\end{array}$$

for $\widetilde{\omega} \in \wedge^q F$ a preimage of $\omega \in \wedge^q M = \Omega^q_{R/R_0}$ with respect to $\wedge^q\beta$; φ^q_K is defined analogously and ξ^q is the map that sends each linear map

$$\lambda : \wedge^{m-r}\,N \longrightarrow \wedge^{m-r+q}\,F$$

to its extension $\mathrm{id}_K \otimes \lambda$. Using this diagram, we can define, following [KD] 1. 14:

Definition : The R-algebra $\Delta_{(R/R_0)}$ of integral differentials of R/R_0 is defined as

$$\Delta_{R/R_0} \;:=\; \bigoplus_{q \in \mathbf{N}_0} \Delta^q\,(M)\ , \quad M \;=\; \Omega^1_{R/R_0}\ ,$$

with

$$\Delta^q\,(M) \;:=\; (\varphi^q_K)^{-1}\,(\mathrm{Im}\ \xi^q)\ .$$

In [KD] p. 7-23, this notion is studied from different angles. For instance, we have (repeating partially facts already stated in 3.1)

a) there is a canonical R-linear map

$$\gamma_{R/R_0} : \Omega_{R/R_0} \longrightarrow \Delta_{R/R_0} \ ,$$

which is bijective if Ω^1_{R/R_0} is a projective R-module.

b) Δ_{R/R_0} is compatible with localization.

c) Δ_{R/R_0} is a graded module over the graded ring Ω_{R/R_0} and as an R-module it is finitely generated.

d) $\Delta^0_{R/R_0} = R : \mathfrak{d}_r\,(R/R_0) := \{x \in K; x\,\mathfrak{d}_r\,(R/R_0) \subset R\}$

e) If $N \subset \Omega^1_{R/R_0}$ is a projective R-module with $N_K = \Omega^1_K$, then $\mathfrak{d}_0\,(\Omega^1_R/N) \neq \{0\}$ and

$$\Delta^r_{R/R_0} \ = \ (R : \mathfrak{d}_0\,(\Omega^1_R/N)) \ \wedge^r \ N \ .$$

The fractional R-ideals $R : \mathfrak{d}_0\,(\Omega^1_R/N)$ with N a free R-module as above, belong to the same ideal class of R.

f) A rational q-form $\omega \in \Omega^q_{K/R_0}$ belongs to Δ^q_{R/R_0} if and only if $\eta \wedge \omega \in \Delta^r_{R/R_0}$ for any $\eta \in \Omega^{r-q}_{R/R_0}$. The canonical map

$$\begin{array}{rcl} \Delta^q_{R/R_0} & \longrightarrow & \mathrm{Hom_R}\,(\Omega^{\mathrm{r-q}}_{\mathrm{R/R_0}}\,,\ \Delta^{\mathrm{r}}_{\mathrm{R/R_0}}) \\ \omega & \longmapsto & (\eta \mapsto \eta \wedge \omega) \end{array}$$

is bijective, and the R-module Δ^r_{R/R_0} is reflexive.

g) (Theorem 1.19 in [KW]). Suppose R is a (reduced) normal ring and $\Omega^1_{R_\mathfrak{c}}$ is a free $R_\mathfrak{c}$ module for each $\mathfrak{c} \in \mathrm{Spec}R$ with $h(\mathfrak{c}) = 1$. Then, in a canonical manner,

$$\Delta_{R/R_0} \ = \ (\Omega_{R/R_0})^{**}$$

and $\gamma_{R/R_0} : \Omega_{R/R_0} \to \Delta_{R/R_0}$ is the canonical map of Ω_{R/R_0} into its bidual (the module of Zariski differentials)

h) If e) is specialized to $K_0 = R_0$ and

$$R \ = \ K_0\,[X_1, \ldots, X_n]/(F_1, \ldots, F_h) \ = \ K_0[x_1, \ldots, x_n] \ ,$$

i. e. $r = \mathrm{trdeg}_{K_0} K = n - h$, and $x_1, \ldots, x_r$ are chosen with

$$\Omega^1_{K/K_0} \ = \ Kdx_1 + \ldots + Kdx_r \ ,$$

then we have

$$\begin{aligned}\Delta^r_{R/K_0} &= (R : \mathfrak{d}_0\,(R/K_0[x_1,\dots,x_r])\,dx_1 \wedge \cdots \wedge dx_r) \\ &= \Delta^{-1}\,R\,dx_1 \wedge \cdots \wedge dx_r\end{aligned}$$

with

$$\Delta := \frac{\partial(F_1,\dots,F_h)}{\partial(x_{r+1},\dots,x_n)}\;.$$

In [KW] p. 22, there is an example showing that, in the general situation, Ω_{R/R_0} may not be a subcomplex of the complex Ω_{K/R_0} contrary to what can be proved for the *regular* differential forms. To describe this, we follow [KW] p. 44 ff .

Assumption :

a) R_0 noetherian, R/R_0 essentially of finite type with a presentation

$$R = P/I\;, \quad P = R_0\,[X_1,\dots,X_n]_T\,, T \text{ a multiplicative system,}$$

I with a prime basis $F = \{F_1,\dots,F_n\}$, i. e. F is a quasiregular sequence and $I_\mathfrak{c} = (F_1,\dots,F_h)_\mathfrak{c}$ for each minimal prime divisor $\mathfrak{c}$ of I,

b) $\operatorname{Ass} R = \operatorname{Min} R$

c) $K = Q\,(R)$

d) Ω is an exterior differential algebra of R_0 where Ω^1 is a finite projective module of rank r_0, and Ω^1_K is a free K-module of rank $r := r_0 + n - h$.

Then we introduce

$$\begin{aligned}P_F &:= P/(F) \\ I_F &:= I/(F)\end{aligned}$$

and

$$A_F := \text{image of } \operatorname{Ann}_{P_F}(I_F) \text{ in } R\;.$$

One has the canonical exact sequence

$$(*) \quad 0 \to K \otimes_R I/I^2 \to K \otimes_P \Omega^1_P \to \Omega^1_K \to 0$$

and the associated canonical isomorphism

$$\varphi^r \;:\; \Omega^r_K \xrightarrow{\sim} \operatorname{Hom}_K\left(\wedge^h K(\otimes_R I/I^2)\,,\; K \otimes_P \Omega^{r_0+n}_P\right)\;.$$

The map

$$P_F \longrightarrow R \hookrightarrow K = Q(R)$$

induces a map

$$\xi_F^r : \mathrm{Hom}_{P_F}(\wedge^h(F)/(F)^2, P_F \otimes_P \Omega_P^{r_0+n}) \to \mathrm{Hom}_K(\wedge^h(K \otimes_R I/I^2), K \otimes_P \Omega_P^{r_o+n}) .$$

Let $i : \Omega_R \to \Omega_K$ be the functorial map. Then one can show that the following definition is independent of the choice of the presentation of R and the prime basis F of I.

Definition : We write

$$(\omega_{R/R_0}^{\Omega})^r := A_F \cdot (\varphi^r)^{-1}(\mathrm{im}\, \xi_F^r) .$$

For arbitrary $p \in \mathbf{N}_0$ let

$$(\omega_{R/R_0}^{\Omega})^p := \{\omega \in \Omega_K^p ; \quad i(\eta) \wedge \omega \in (\omega_{R/R_0}^{\Omega})^d \quad \text{for all} \quad \eta \in \Omega_R^{d-p}\}$$

and

$$\omega_{R/R_0}^{\Omega} := \sum_{p \in \mathbf{N}_0} (\omega_{R/R_0}^{\Omega})_p .$$

ω_{R/R_0}^{Ω} is called the *module of regular differential forms of* R/R_0 with respect to Ω. The elements of $(\omega_{R/R_0}^{\Omega})^p$ are called *regular* p-forms.

In the case, where Ω^{r_0} is a free R_0-module (of rank 1), i. e. $\Omega^{r_0} = R_0\omega_0$ (a case which for $r_0 = 0$ induces the special situation of the trivial presentation which was here used to describe the integral differentials above), a more explicit description of $(\omega_{R/R_0}^{\Omega})^*$ can be given. Namely, we have

$$K \otimes_P \Omega_P^{r_0+n} = K\, \widetilde{\omega}_0\, d\widetilde{X}_1 \wedge \ldots \wedge d\widetilde{X}_n ,$$

where the tilde denotes images in $K \otimes_P \Omega_P$. The canonical exact sequence $(*)$ shows that there is an equation

$$\omega^* d\widetilde{F}_1 \wedge \ldots \wedge d\widetilde{F}_n = \widetilde{\omega}_0 d\widetilde{X}_1 \wedge \ldots \wedge d\widetilde{X}_n$$

with some $\omega^* \in K \otimes_P \Omega_P^r$. Let ω^F be the image of ω^* in Ω_K^r. Then we have

$$\omega_{R/R_0}^r = A_F\, \omega^F .$$

If here we take the trivial differential algebra $\Omega = R_0$ and the case, where R/R_0 is generically étale, we have $\Omega_{K/K_0}^1 = \{0\}$ and ω_{R/R_0} turns out to be the fractional ideal

$$\omega_{R/R_0} = A_F \left(\frac{\partial(F_1, \ldots, F_n)}{\partial(x_1, \ldots, x_n)} \right)^{-1} .$$

Again, one can show that the definition of the regular differentials is compatible with localization. Moreover, in [KW] numerous results are obtained relating the regular to the integral differentials and to the generalized complementary modules. For instance, if R_0 is regular and R locally a complete intersection or normal, one has ([KW] Corollary 3. 10, Theorem 3. 17)

$$\Delta_{R/R_0} = \omega_{R/R_0},$$

and if R/R_0 is smooth and $\Omega = R$, then

$$\omega_{R/R_0} = \Delta_{R/R_0} = \Omega_{R/R_0} .$$

In general, under the assumptions given above, the regular differentials are always integral

$$\omega_{R/R_0} \subset \Delta_{R/R_0} ,$$

but ([KW] p. 91) there is a simple example showing that the image of the canonical map $i : \Omega_R \to \Omega_k$ need not be contained in ω^{Ω}_{R/R_0}. If this happens, however, the map

$$c^{\Omega}_{R/R_0} : \Omega_R \longrightarrow \omega^{\Omega}_{R/R_0}$$

is called the *fundamental class*. In Lipman ([Li] p. 39) one finds a justification of the name and, moreover, an extensive study relating to residues and duality in this context.

We will quit here the description of these far reaching developments and go back to another topic from Kähler's research.

4. Globally integral differentials

In the chapters III - VIII of the Geometria Aritmetica, Kähler develops and collects material from what is now called commutative and local algebra and constructs a system of notions for algebraic geometry which is now covered by the theory of schemes à la Grothendieck-Dieudonné. We will use here more or less freely the modern language (and give a description of Kähler's system in an Appendix) to introduce and present the central notion of the Geometria from Chapter IX ([GA] p. 345 ff)

La totale integritá di un corpo

which perhaps can best be translated to

The ring of integral differentials of a field.

Serre, in his letter to Grothendieck from November 13, 1959 ([CS] p.101), telling about A.Weil's work on a review of the [GA], points just to this notion as the only thing new to him. And in his review [WGA] of the Geometria Aritmetica, Weil estimates this Chapter as the most interesting of the book and translates its opening statement: *The major problem of arithmetic geometry is to describe the structure of a field, i. e. take cognizance of its essence starting from the multiplicity of its aspects.* This chapter IX is devoted to the study of the author's simultaneous generalization of the *differentials of the first kind* in classical geometry and of the integers of a number field:

The ring $\mathfrak{o}\,(= I\,(k))$ in Kähler's notation) of integers of an algebraic number field k can be characterized as the intersection

$$I\,(k) \;=\; \cap S$$

of all discrete valuation rings S of k. And the $\mathbf{C}$-vector space $D_1\,(K/\mathbf{C})$ of all differentials of the first kind of a Riemann surface $X/\mathbf{C}$ (or an algebraic curve) with a field of rational functions can be characterized as consisting of those rational 1-forms $\omega \in KdK$ which are holomorphic at each point $x \in X$ and thus integral for each local ring $\mathcal{O}_{X,x}$, $x \in X$ in the sense described in the section 3 above. Both notions are special cases of the following:

4.1 Absolutely Integral Differentials of a field

Let K be a field finitely generated over a ring A, and denote by

$$W\,(K/A)$$

the family of all discrete valuation rings S of K (i. e. regular with monogeneous maximal ideal $\mathfrak{P}$), essentially of finite type over A.

Definition : The ring of *integral differentials of* K over A is defined as the intersection

$$D\,(K/A) \;:=\; \bigcap_{S \in W(K/A)} [S, dS]\;.$$

As explained in Section 3 $[S, dS]$ is the subring of $\Omega_{K/A}$ generated by all $gdh_1 \wedge \ldots \wedge dh_q$, $g, h_1, \ldots, h_q \in S$. Kähler describes a ring of integral infinitesimals analogously, and it is clear that one can also look at notions, where certain symmetry conditions are imposed on the differentials (which then lead to the pluri-generi). But we will restrict here to the case above.

$D(K/A)$ is graduated

$$D \;=\; \oplus_{q=0}^{r} D_q\;, \quad r = \mathrm{trdeg}_{(A)}\,K\;, \quad D_0\,(K/A) \;=\; \bigcap_{S \in W(K/A)} S\;.$$

The case $A = \mathbf{Z}$ will be called the *arithmetic case* and for $A = \mathbf{Q}$ or $\mathbf{C}$ we are in the realm of algebraic geometry and function theory.

In the arithmetic case, D itself and each homogeneous part of D is an $I(k)$-module, where k denotes the largest number field contained in K. One has the relation

$$D(K/k) = kD(K) .$$

In [AD] p. 150-160, [GA] p. 359-380, and (in a somewhat more modern language) [Be4] the following crucial fact is shown :

Theorem : $D(K)$ is an $I(k)$-module of finite type.

Apparently $D(K)$ and $D(K/k)$ as well as all the variants using different symmetry conditions are birational invariants of K resp. K/k. As for many applications (for instance related to periods), one needs arithmetically distinguished differentials, this notion is very important and we will describe here more of this topic which at first looks merely field theoretic but has modifications reaching into algebraic and arithmetical geometry.

The central point of the *proof* of the theorem above ([GA] p. 369, [Be] p. 4) is the following statement (based on the analysis of the Kähler different $\mathfrak{d}_0$).

Let be

$C_0 = [A, x_1, \ldots, x_r]$ a subring of K such that its quotient field

$K_0 = (A, x_1, \ldots, x_r)$ has $\operatorname{trdeg}_{K_0} K = r$,

$y \in K$ generating K over K_0 with defining equation

$$f(y) = y^m + a_1 y^{m-1} + \ldots + a_m = 0, \quad a_i \in C_0 \quad (i = 1, \ldots, m) ,$$

S_0 a discrete valuation ring of K_0 and a localization of C_0, and

$S_i\,(i = 1, \ldots, g)$ discrete valuation rings of K dominating S_0 (i. e. with $S_0 \cap \mathfrak{P}_i = \mathfrak{P}_0$ for the maximal ideals $\mathfrak{P}_i$ of S_i and $\mathfrak{P}_0$ of S_0).

Then there is a $b \in S_0$ such that (using the notation from Section 1.2)

$$b f_y\, \omega \in \sum [S_0, y]\, dx_{i_1} \wedge \ldots \wedge dx_{i_h}$$

holds for each

$$\omega \in \bigcap_{i=1}^{g} [S_i dS_i] .$$

Here one can take $b = 1$, if all $S_i/\mathfrak{P}_k\,(i = 1,\ldots,g)$ are separable over $S_0/\mathfrak{P}_0$.

Applying this statement to variables $u_1,\ldots,u_r,v$ with

$$x_1 = u_1^{-1}\,,\; x_i = u_i\,u_1^{-1}\,(i=2,\ldots,r)\,,\; y = vu_1^{-1},$$

we get ([GA] p. 378-380, [Be 4] p. 7) quite explicit $\mathbf{Z}$-resp. $I(k)$-modules containing $D\,(K)$ (and k-modules containing $D\,(K/k)$).

Kähler illustrated his constructions by two concrete examples where

$$K = k\,(x,y) \quad \text{with} \quad f\,(x,y) = 0\,.$$

Example 1 (The *Fermat field*, [GA] 370., 394.): For $k = \mathbf{Q}$

$$f\,(x,y) = x^m + y^m - 1 = 0 \quad (m > 2)\,,\;\;,$$

one has

$$D_1\,(K) = \sum_{i+j\leq m-3} \mathbf{Z}\,\frac{x^i y^i}{y^{m-1}}\,dx\,.$$

Example 2 (an elliptic function field, [GA] 397.): For $k = \mathbf{Q}\,(\beta)$, $\beta^2 = -23$ and

$$f\,(x,y) = y^2 - x^3 + \beta\,,$$

one has with $I(k) = \mathbf{Z}[\alpha], \alpha = \frac{1+\beta}{2}$, i.e. with $\alpha^2 - \alpha + 6 = 0$

$$D_1\,(K) = 3\beta\,I\,(k)\,\frac{dx}{y}\,,$$

Other similar examples were studied by Kähler's students: Citing from a letter by Horst Schumann from November 2001, we reproduce the results obtained in Leipzig in 1958 and later: All concern elliptic function fields K over the same groundfield $k = \mathbf{Q}\,(\alpha)\,,\; \alpha^2 - \alpha + 6 = 0$ as in Kähler's example 2.

Example 3 (H. Schumann) : For

$$f\,(x,y) = y^2 - 2((3+2\alpha)x^4 + 9x^2 + (3+2\alpha))$$

one has

$$D_1\,(k) = \frac{2\mathfrak{c}_1\,dx}{y}$$

with the ideal

$$\mathfrak{c}_1 = (3+2\alpha)\,(31,\alpha+7)\,(127,\alpha-28) \subset I\,(k) = \mathbf{Z}[\alpha]\,.$$

Example 4 (W. Reutter) : For

$$f(x,y) = y^2 - 2(x^3 - 3x + \alpha - 12)$$

one has

$$D_1(K) = \frac{6\mathfrak{c}_2 dx}{y} \quad \text{with} \quad \mathfrak{c}_2 = (47, \alpha - 14) \ .$$

Example 5 (C.-P. Helmholz) : For

$$f(x,y) = y^2 - 6(x^3 + 3(2\alpha - 1) + 1)$$

one has

$$D_1(K) = \frac{3\mathfrak{c}_3 dx}{y} \quad \text{with} \quad \mathfrak{c}_3 = (3, \alpha)(64891, \alpha - 31388) \ .$$

Remark . The ideals $\mathfrak{c}_i \ (i = 1, 2)$ appearing in the examples 3 and 4 are not in the principal class and thus carry information about the field K.

Taking this into account, Kähler refined his notions further by defining, in [GA] 407. and 411., the birational invariants *rank* and *class* of the modules $\mathfrak{M} := D_n(K)$: Let

$$\mathfrak{M} = \sum_{j=1}^{m} I(k)\, \omega_j$$

and let r be the rank of $\mathfrak{M}$ as a k-module. Then each system of $m - r$ relations

$$\sum_{j=1} a_{ij}\, \omega_j = 0 \quad (i = 1, \ldots, m - r)$$

determines an ideal $\mathfrak{a}$ in $I(k)$ generated by the determinants of the $(m - r)$-submatrices of (a_{ij}) resp. $\mathfrak{a} = I(k)$ for $m = r$. By the usual Fitting process this ideal is well-defined and called the *class of* $D_n(K)$. In [GA] 406. and 408., it is shown that $\mathfrak{M}$ can be generated by r or $r + 1$ elements depending on whether $\mathfrak{a}$ is the principal class or not.

Another point of view to utilize the modules of integral differentials was proposed by Kähler around 1974: Let K be a field of modular functions. Then one can compare its integral differentials with those having an integral Fourier development. This will be discussed further below in Section 4. 4. Here we will state only two examples from Berndt-Schramm [BS].

Example 6 and 7 (the field of elliptic modular functions of level $N = 6$ and 7 [BS]): For $k = \mathbf{Q}(\sqrt{-3})$ and

$$f(x,y) = y^2 - x^3 - 1 = 0 \ ,$$

we have

$$D_1(K) = \sqrt{-3}\,\frac{dx}{y}\,I(k)$$

and for $k = \mathbf{Q}$ and

$$f(x,y) \;=\; x^3 + xy^3 + y \;=\; 0\;,$$

we have

$$D_1(K) \;=\; 7\sum_{i=1}^{3} w_i\,\mathbf{Z}\;, \quad w_1 \;=\; \frac{dx}{f_y}\;,\; w_2 \;=\; xw_1\;,\; w_3 \;=\; yw_1\;.$$

As a starting point for the determination of the absolutely integral differentials of a field K, one looks at the conditions arising from the localizations of a ring C with quotient field K, i. e. coming from an affine variety with function field K. In general (i. e. for $r > 1$) this will not be sufficient, so one looks at projective varieties and in all these examples, one has to manage a sometimes rather involved route of blow-up procedures.
All this proposes, as already hinted at in section 3, not only to associate integral differentials to the field K but also to varieties being models of the field. Kähler did this also and moreover he introduced integral differentials relative to divisors. We will follow these constructions here. Later, we will indicate a certain refinement of his notions.

4.2 Integral Differentials relative to a Divisor for a Field

Generalizing the usual notion of a divisor from number theory and Chevalley's concept of his generalized Riemann surface, Kähler proposes the following notion in [GA] 324. :

Definition : A divisor $\mathfrak{a}$ of a field K/A is a map associating to each discrete valuation ring S of K (essentially of finite type over A) an S-module $\mathfrak{a}(S) = \mathfrak{P}^{a(S)}$, where $\mathfrak{P}$ is the maximal ideal of S, i. e. $\mathfrak{a}$ is given by a map

$$\begin{array}{rcl} \mathfrak{a} =: W(K/A) & \longrightarrow & \mathbf{Z} \\ S & \longmapsto & \mathfrak{a}(S)\;. \end{array}$$

The examples of the divisors from number theory show that one usually imposes a finiteness condition. Kähler also does this (in [GA] 333.-345.), but we will leave aside these concepts of *finite divisors* and only mention two types of examples, namely the *principal divisors*

$$\mathfrak{a} \;=\; (x) \quad \text{with} \quad \mathfrak{a}(S) \;=\; xS \quad \text{for} \quad x \in K$$

and the *different divisor*

$$\mathfrak{a} \;=\; \mathfrak{d}_r \quad \text{with} \quad \mathfrak{a}(S) \;=\; \mathfrak{d}_r(S/A)\;, \quad r \;=\; \mathrm{trdeg}_{(A)}\,K\;.$$

Using this notion of a divisor $\mathfrak{a}$ of a field K, it is natural to introduce (as done in [GA] 412.) the following associated birational invariant of K.

Definition : The *A-module of integral differentials of K/A relative to the divisor* $\mathfrak{a}$ is given by

$$D\left(\frac{K/A}{\mathfrak{a}}\right) := \bigcap_{S \in W(K/A)} \mathfrak{a}(S)^{-1}[SdS] \, .$$

In particular, for the arithmetic case $A = \mathbf{Z}$ and $\mathfrak{a} = \mathfrak{d}_r$, we have here an object with a nice geometric interpretation which will become clearer after extending these purely field theoretic notions into the region of algebraic geometry.

4.3 Integral Differentials of a Variety

As explained in more detail in the Appendix B, the set V of stalks $\mathcal{O}_{X,x}$, $x \in X$, of a reduced irreducible scheme X of finite type over A with function field K is a *variety* in the sense of [GA] Ch. III. We will call, here and in the sequel, V a *model of* K/A. Using Kähler's notation $V(C)$ for the set of localizations $s = C_{\mathfrak{c}}$ of the ring C we also write

$$V = \bigcup_{i=1}^{\ell} V(C_i) \, , \; C_i \text{ of finite type over } A \, .$$

The model V is called *complete* (= *chiuse* in [GA]) if the scheme X/A is proper in the sense of [EGA]. Then, each discrete valuation ring $S \in W(K/A)$ of K containing A *dominates* an $s \in V$, i. e. we have $S \supset s$ and, for the maximal ideals $\mathfrak{P}$ resp. $\mathfrak{p}$, the equality $\mathfrak{P} \cap s = \mathfrak{p}$ holds.
The model V is called *regular*, if all local rings $s = \mathcal{O}_{X,x} \in V$ are regular.

If K is the quotient field of a ring R containing A, we have a canonical map $\varphi : \Omega_{R/A} \to \Omega_{K/A} = \Omega_{R/A} \otimes_R K$ and, in section 3, we have already introduced the notation

$$[R, dR] := \varphi(\Omega_{R/A}) \subset \Omega_{K/A}$$

(here we abbreviate $d := \dfrac{d}{A}$). Using this, we extend the objects from 4. 1 to the associated varieties.

Definition : For a model V/A of K we put

$$D^0(V/A) := \bigcap_{s \in V} [s, ds]$$

(s running over all local rings $s \in V$) and

$$D\,(V/A) \;:=\; \bigcap_{S \in V} \,[S, dS]\;.$$

(S running here over the discrete valuation rings $S \in V$).

Remark : We have the natural identification

$$D^0\,(V/A) \simeq \Gamma\,(X, \Omega_{X/A})\;.$$

4.4 Relations between Integral Differentials for a Variety and for its Function Field

Kähler relates the different notions of integrality on a variety and for a function field by proving the following statements. Here, for simplicity, we always assume $A = \mathbf{Z}$ or $A = k$, $d := \dfrac{d}{A}$, and V/A a model of the field K with $\mathrm{char} K = 0$ and $\mathrm{trdeg}_{(A)}\, K = r$.

Proposition 1 ([GA] 384.) : If V is complete over A, we have

$$D^0\,(V/A) \subset D\,(K/A)\;.$$

This derives simply from the fact that each valuation ring $S \in W\,(K/A)$ dominates a local ring $s \in V$, if V is complete over A. This fact is nowadays called the *valuative Criterium of Properness* for the scheme X belonging to V (as in [Ha] Theorem 4. 7)).

Proposition 2 ([GA] 389., 390.): Let s be a local ring with maximal ideal $\mathfrak{p}$ and quotient field K and essentially of finite type over A. In the case $A = \mathbf{Z}$ we have

$$D\,(K) \subset [s, ds] \quad \text{if} \quad \vartheta_r\,(s) \;=\; s,$$

and in the case $A = k$ we have

$$D\,(K/k) \subset [s, ds] \quad \text{if} \quad \mathfrak{d}_r\,(s/k) \;=\; s.$$

In both cases, s is regular (by the *Lemma di Mehner* resp. *di Zariski* reproduced here in Section 2. 1), and the statements result from the fact that, given the condition for the different, one may choose $c = 1$ in the following statement (a special case of [GA] pp. 388) : For $c, x_i \in s$ $(i = 1, \ldots, r)$ such that

$$cds \subset \sum_{i=1}^{r}\, sdx_i\,,$$

the following relation holds

$$c^n \bigcap_S [S, dS]_n \subset \wedge^n (\sum_{i=1}^{n} s\, dx_i)$$

where the intersection is running over all $S \in W(K/A)$ which are localizations of s.

This statement, in turn, follows from the fact that an integrally closed ring C is the intersection of all the valuation rings S localizing C.

Both Propositions together immediately lead to the following

Theorem 1 ([GA] 391.) : If V is a complete non-singular model of K over k with $\operatorname{char} K = 0$, we have

$$D(K/k) \;=\; D(V/k) \;=\; D^0(V/k)\ .$$

As Kähler remarks in [GA] 392., this Theorem reduces the determination of the integral differentials in the case $A = k$ to the resolution of singularities. As a consequence of this Theorem (and analogous ones for other types of symmetry of the differentials) we obtain the birational invariance of the ranks of the homogeneous parts of $D(V/k)$, i. e. of the genera resp. plurigenera as presented nowadays for instance in [Ha] Theorem 18.19. For the case $K = \mathbf{C}$ Kähler had already clarified these notions of birational invariant genera in 1932 (see [K10], [K11]).

In the arithmetic case $A = \mathbf{Z}$, Proposition 2 is not sufficient to fix $D(K)$ because there will be ramified local rings. But here we get a bit farther by the following sharpening of Proposition 2.

Proposition 3 ([GA] 393.): If for a regular local ring s of K the different $\mathfrak{d}_r(s)$ is a principal ideal, we have

$$D(K) \subset [sds] \;=\; \bigcap_{S \in V(s)} [SdS]\ ,$$

where the intersection is running over all localizations $S \in W(K)$ of s.

The final results of Kähler in this direction are the following two (obtained by using again the calculus of his differents and some more facts from ideal theory).

Proposition 4 ([GA], 399.): If V is a complete model of K over $\mathbf{Z}$ such that all regular rings $s \in V$ containing $\mathbf{Q}$ are regular, then there exists a rational number $c > 0$ with

$$D^0(V) \subset D(K) \subset (1/c)\, D^0(V)\ .$$

Theorem 2 ([GA], 400.) : We have

$$D(K/\mathbf{Q}) \;=\; \mathbf{Q} \cdot D(K)\ .$$

The problems in comparing $D(K)$ and $D(V)$ which arise from the ramification mentioned above can be avoided by the use of the notion of relatively integral differentials introduced in Section 3, where the different (and thus the arithmetic nature) is built in from the beginning.

On the scheme X corresponding to the model V, we introduce the subsheaf of the constant sheaf Ω_K with stalks $\widetilde{\Omega}_{X,x}$ consisting of the differentials $\omega \in \Omega_K$ integral in $s = \mathcal{O}_{X,x}$, i. e.

$$\widetilde{\Omega}_{X,x} \;=\; \Delta_{s/\mathbf{Z}}$$

in the notation of Section 3. Using moreover the language from 4.2, we can also write

$$\Gamma(X, \widetilde{\Omega}) \;=\; \{\omega \in [K, dK], \mathfrak{d}(\omega|s) \subset s \quad \text{for all} \quad s \in V\} \;=\; D^0\left(\frac{V}{\mathfrak{d}}\right)\ .$$

Here again, we arrive at a birational invariant notion:

Theorem 3 ([Be 3], 7.): If V is a complete regular model of K over $\mathbf{Z}$ we have

$$D^0\left(\frac{V}{\mathfrak{d}}\right) \;=\; D\left(\frac{V}{\mathfrak{d}}\right) \;=\; D\left(\frac{K}{\mathfrak{d}}\right)\ .$$

Unfortunately, the usefulness of this statement is strongly limited by the hypothesis of the existence of a regular model. At least, for the case of arithmetical dimension two we have such models and we can use our notions in the following independent formulation of Néron's minimality criterium ([Ne], Proposition 1 p. 89).

Proposition 5 ([Be 2], 10.): Let be

$A = \mathbf{Z}$ or $\mathbf{F}_p$, p prime $\neq 2, 3$

K a field finitely generated over (A) with $\mathrm{trdeg}_{(A)} K = r$, $r = 1$ for

$A = \mathbf{Z}$, $r = 2$ for $A = \mathbf{F}_p$ and

V a complete regular model of K over A.

Then V is minimal if there is an r-form ω with $\mathfrak{d}(\omega|S) = S$ for all valuation rings $S \in V$.

Néron's notions translate here as follows: For a local ring S of K with maximal ideal $\mathfrak{p}$, we have equivalences

$$\begin{aligned} \omega \in \Omega_{K/A} \qquad & \text{is } \mathfrak{p} - \text{morph} \iff \mathfrak{d}\,(\omega|s) \subset s \\ & \text{is zero in } \mathfrak{p} \iff \mathfrak{d}\,(\omega|s) \subset \mathfrak{p} \\ & \text{is non-zero in } \mathfrak{p} \iff \mathfrak{d}\,(\omega|s) \;=\; s\,. \end{aligned}$$

A concrete example is contained in the thesis of Radtke ([Ra]). Let $K = \mathbf{Q}\,(x,y)$ be an elliptic function field with minimal equation

$$f\,(x,y) \;=\; y^2 + a_1xy + a_3y - (x^3 + a_2x^2 + a_4x + a_6) \;=\; 0\;,$$

where $a_i \in \mathbf{Z}$, with minimal p-order of the discriminant of f for all primes p. Then we have

$$D\left(\frac{K}{\mathfrak{d}}\right) \;=\; \mathbf{Z}\,\omega$$

with the *Néron differential* $\omega = \dfrac{dx}{f_y}$.

4.5 Absolutely Integral Differentials and Arithmetic Modular Forms

Around 1970, Kähler proposed to compare the arithmetically distinguished differentials of the fields of modular functions with modular functions and forms with integer q-expansion. There are some results in this direction obtained by Berndt and Schramm ([Sch], [BS], [Be 5]) which will be sketched here.

An *elliptic modular form* of weight ℓ and level N (more precisely for $\Gamma := \Gamma\,(N) = \{M \in \mathrm{SL}\,(2,\mathbf{Z}), M \equiv E \bmod N\}$) is a meromorphic function

$$f : \mathfrak{H} \;=\; \{\tau = x + iy \in \mathbf{C}\,,\; y > 0\} \to \mathbf{C}$$

with

a) $f|_\ell\,[M]\,(\tau) \;:=\; f\left(\dfrac{a\tau + b}{c\tau + d}\right)\,(c\tau + d)^{-\ell} \;=\; f\,(\tau)$ for all $M \;=\; \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma\,(N)$.

b) There is a number $C > 0$ such that f has no singularity for $\mathrm{Im}\,\tau > C$.

c) f has a non-essential singularity in $i\infty$.

We write

$$f(\tau) = \sum c(n)\, q_N^n \ , \quad q_N := e(\tau/N) \ , \quad e(u) := \exp(2\pi u)$$

for the Fourier expansion *in the cusp* $i\infty$ of such a form. As usual, forms of weight $\ell = 0$ are called *modular functions*.

Definition 6 . For a subring A and a subfield k of $\mathbf{C}$ we denote by

$$K = \mathfrak{F}_N(k)$$

the field of elliptic modular functions f of level N with Fourier coefficients $c(n) \in k$ and by

$$\mathrm{Diff}_A K \quad \text{resp.} \quad \mathrm{Diff}_A^* K$$

the A-module of differential forms $\omega \in \Omega^1_{K/A}$ with Fourier expansion

$$\omega = \sum_n c(n)\, q_N^n \, dq_N, \; c(n) \in A \ ,$$

i. e. with A-coefficients in $i\infty$ resp. with Fourier expansion

$$\omega \circ M = \sum c^M(n)\, q_N^n \, dq_N \ , \quad c^M(n) \in A \quad \text{for all} \quad M \in \Gamma(1) \ ,$$

i. e. with A-coefficients in all cusps.

The most interesting cases are

$$k = \mathbf{Q} \ , \quad A = \mathbf{Z} \qquad (\textit{case 1})$$

and

$$\begin{aligned} &k = \mathbf{Q}_N = \mathbf{Q}(\zeta_N) \ , \ \zeta_N = e(1/N) \ , \\ &A = \mathfrak{o}_N \ \text{the maximal order of } \mathbf{Q}_N \quad , \end{aligned} \qquad (\textit{case 2})$$

or

$$A = \mathfrak{o}'_N = \mathfrak{o}_N[1/N] \qquad (\textit{case 2'})$$

In the following, we restrict to these cases and abbreviate

$$K_N := \mathfrak{F}_N(\mathbf{Q}_N) \ , \quad K_N^0 := \mathfrak{F}_N(\mathbf{Q}) \ .$$

Remark . By the q-expansion principle (as in [DR] p. 313) one has

$$\mathrm{Diff}_{\mathfrak{o}'_N} K_N = \mathrm{Diff}^*_{\mathfrak{o}'_N} K_N$$

Using the results about the explicit description of $\mathfrak{F}_N(k)$ as in Shimura [Sh] 6.2, the most general statement to be made here is

Theorem 1 ([Sch], [BS] Satz 1): In all cases A with quotient field k one has

$$D_1(\mathfrak{F}_N(k)) \subset \mathrm{Diff}_A(\mathfrak{F}_N(k)) .$$

A certain sharpening of this inclusion is given if we allow for denumerators N:

Theorem 2 ([Sch] Satz 4.11): We have

$$D(K_N/\mathbf{Z}[1/N]) \;=\; \mathbf{Z}[1/N]\,\mathrm{Diff}_{\mathfrak{o}_N} K_N \;=\; \mathrm{Diff}^*_{\mathfrak{o}_{N'}} K_N .$$

More precise results come up if we look at the differentials on *Igusa's Kroneckerian models* V_N of K_N ([Ig]) . If J is the usual modular invariant generating the field K_1 of elliptic modular functions for $\Gamma(1) = \mathrm{SL}(2,\mathbf{Z})$, we put

$$V_N \;:=\; V\,(I(K_N/\mathbf{Z}[J])) \cup V\,(I(K_N/\mathbf{Z}[J^{-1}]))$$

where we use Kähler's notation $I(R/R_0)$ for the integral closure of a subring R_0 contained in a ring R and $V(R)$ for the set of localizations of R (in its prime ideals). This model is apparently complete and normal. Moreover, it is even regular (this is - as stated by Deligne - a consequence of [DR] and [Dr]). Then we can use Theorem 3 from Section 4. 4 which tells us that we have

$$D^0\left(\frac{V_N}{\mathfrak{o}}\right) \;=\; D\left(\frac{V_N}{\mathfrak{o}}\right) \;=\; D\left(\frac{K}{\mathfrak{o}}\right).$$

By an explicit analysis of the series appearing in the description of V_N one proves

Theorem 3 ([Sch] Satz 4.5) : We have

$$D_1(V_N) \;=\; \mathrm{Diff}^*_{\mathfrak{o}_N}$$

and

$$D\left(\frac{V_N}{\mathfrak{d}}\right) \;=\; \mathfrak{d}_0\,(\mathfrak{o}_N)^{-1}\,\mathrm{Diff}^*_{\mathfrak{o}_N} K_N .$$

Together with the last statement above, we get

Corollary . We have

$$D_1\left(\frac{K_N}{\mathfrak{d}}\right) \;=\; \mathfrak{d}_0(\mathfrak{o}_N)^{-1}\,\mathrm{Diff}^*_{\mathfrak{o}_N} K_N .$$

As already used in 4.1, explicit calculations for $N = 6$ and 7 are done in [BS]. Namely we can deduce from Klein-Fricke [KF] Bd I p. 682 ff and Bd II p. 391 resp. [KF] Bd II p. 277 ff, the descriptions

$$K = \mathfrak{F}_N(k) = k(x,y)$$

with

$$\begin{aligned} f(x,y) &= y^2 - x^3 - 1 = 0 \quad \text{for} \quad N = 6 \\ &= xy^3 + y + x^3 = 0 \quad \text{for} \quad N = 7 \end{aligned}$$

$$\begin{aligned} \mathrm{Diff}_A\, K &= w_1 A\,,\ w_1 = \frac{dx}{f_y} \quad \text{for} \quad N = 6 \\ &= \sum_{i=1}^{3} w_i A\,,\ w_1 = \frac{dx}{f_y}\,,\ w_2 = x w_1\,,\ w_3 = y w_1 \quad \text{for} \quad N = 7\,. \end{aligned}$$

Then we have for $N = 6$ ([BS] Satz 3)

$$\begin{aligned} D_1(K) = N\, D_1\left(\frac{K}{\mathfrak{d}}\right) &= 6 w_1 \mathbf{Z} \quad \text{in case 1} \\ &= 2\alpha\, \mathfrak{o}_N \quad \text{in case 2}\,, \end{aligned}$$

where

$$k = \mathbf{Q}_6 = \mathbf{Q}(\alpha)\,,\ \alpha = \sqrt{-3}\,,\ \mathfrak{o}_N = \mathbf{Z}[\sigma_6]\,,\ \sigma_6 = \eta_6 - 1\,,$$

and for $N = 7$ ([BS] Satz 4)

$$D_1(K) = N\, D_1\left(\frac{K}{\mathfrak{d}}\right) = \sum_{j=1}^{3} \overline{w}_j\, A\,, \quad \overline{w}_j = \sum_{j=1}^{3} a_{ij}\, w_i\,,$$

with

$$\begin{aligned} (a_{ij}) &= 7E_3 \qquad\qquad \text{in case 1} \\ &= \begin{pmatrix} \sigma^3 & 0 & 0 \\ -2\sigma^2 & 0 & \sigma^2 \\ 9\sigma & \sigma & -4\sigma \end{pmatrix} \quad \text{in case 2,} \end{aligned}$$

where

$$k = \mathbf{Q}_7 = \mathbf{Q}(\sigma_7)\,,\ \mathfrak{o}_N = \mathbf{Z}[\sigma_7]\,,\ \sigma = \sigma_7 = \eta_7 - 1\,.$$

It is tempting to pursue the theme proposed by Kähler and try to relate these arithmetically distinguished differentials to Shimura's construction of arithmetically distinguished automorphic forms and his canonical models - and moreover, via the Shimura varieties to the arithmetic automorphic representations discussed nowadays by Clozel, M. Harris and others. It may seem rather farfetched, but I have the feeling that even these developments fit into Kähler's basic intuition to create an arithmetic geometry subtle enough for future questions: For instance, some automorphic representations of the symplectic group $\mathrm{Sp}(n)$ have models spanned by differential forms on Siegel modular varieties. Then the arithmeticity of the representation may reflect in a model consisting of differentials with integral Fourier expansions...

4.6 Topologically Integral Differentials, Periods and Birational Invariants

Another nowadays rather familiar approach to define arithmetically distinguished differentials as those having integral periods was attacked in the first three paragraphs of the Chapter X, the last one of the Geometria Aritmetica [GA]. The idea for this and the construction of birationally invariant functions associated to an arithmetical function field K is already announced in last section of [RAK] from 1951. Moreover, these differentials are used to construct a Kähler metric on a complex r-dimensional manifold whose field of meromorphic functions is $K \otimes \mathbf{C}$. Apparently Kähler was very familiar with more or less simultaneous work by de Rham and Hodge (see [dR] and [Ho]), which he alludes to often but without precise citations and which forms a background for the following.

In [GA] 445., Kähler calls a connected r-dimensional holomorphic manifold R a $2r$*-dimensional Riemanniana* and in 446. he introduces $\Omega_c(R)$ as the subring of real $\mathcal{C}^\infty$-differentials with compact support in the ring $\Omega(R)$ of exterior $\mathcal{C}^\infty$-differentials on R. In 447., we find the notion central for this approach.

Definition. A form $\omega \in \Omega^p(R)$ is *topologically integral* if and only if ω is closed, i. e. $d\omega = 0$, and if there is an integer $(2r-p)$-cycle γ such that

$$\int_R \omega \wedge \theta = \int_\gamma \theta \, .$$

holds for each closed $\theta \in \Omega_c^{2r-p}(R)$.

In this case, Kähler calls ω homologous to γ and writes $\omega \sim \gamma$. Moreover, if $\omega' \in \Omega^p(R)$, $\omega \sim \omega'$ stands for $\omega - \omega' \in d\,\Omega(R)$, one can see that

$$\omega \sim \omega' , \quad \omega \sim \gamma \quad \text{implies} \quad \omega' \sim \gamma \, .$$

The topologically integral differentials of degree p form an additive group. Here we follow the notation from [IA] and don't write $I^p(R)$ as in [GA] but

$$T^p(R) \, .$$

The direct sum

$$T(R) := \bigoplus_{p=0}^{2r} T^p(R) \, , \quad T^0(R) = \mathbf{Z} \, ,$$

is a ring, the *ring of topologically integral differentials on* R. Apparently, via the usual isomorphisms, $T^p(R)$ induces a submodule of $H^p_{dR}(R)$ isomorphic to $H_{2r-p}(R, \mathbf{Z})$ mod torsion.
The complex structure on R allows for the introduction of the Hodge types $\Omega^{(r,s)}(R)$ resp. $H^{r,s}(R)$, $r + s = p$. In [GA] 450., Kähler mentions the problem to determine

$$T^{(r,s)}(R) = \Omega^{(r,s)}(R) \cap T(R) \, .$$

Following Kähler, we write

$$T^q(R) = \sum_{i=1}^{b_q} \mathbf{Z}\,\omega_i^q + d\,\Omega^{q-1}(R)$$

where for compact R the basis $\omega = (\omega_i^q)_{i=1,\ldots,b_q}$ can and will be chosen such that for $q \neq r$ we have

$$\int \omega_i^q \wedge \omega_j^{2r-q} = \delta_{ij} \, , \quad i, j = 1, \ldots, b_q$$

and for $q = r$ with $p := b_r/2$ odd, we have

$$\left(\int \omega_i^r \wedge \omega_j^r \right) = \begin{pmatrix} 0 & E_p \\ -E_p & 0 \end{pmatrix} =: J \, ,$$

while for $p = b_r/2$ even, we still have $\det\left(\int \omega_i^r \wedge \omega_j^r\right) \neq 0$.

In the special case $r = 1$, where R is a usual compact Riemann surface with genus p, we have a canonical system $(\gamma_i)_{i=1,\ldots,2p}$ of retrosections, and equivalent bases ω and ω' are related by

$$\omega' = \omega M \quad \text{with} \quad M \in \mathrm{Sp}(2r, \mathbf{Z}) \, .$$

In the Geometria Aritmetica, from 451. on, the manifolds R are "coming" from a model V of a field K finitely generated over $\mathbf{Q}$ with $\operatorname{trdeg}_{\mathbf{Q}} K = r$ in the following sense: $R := R(V)$ is, as a set, the set of complex points of V, i. e. the set of homomorphisms

$$P : s \longrightarrow \mathbf{C} \quad \text{with} \quad \ker P = \mathfrak{p}$$

for the local rings $s \in V$ with maximal ideal $\mathfrak{p}$. In the chapter V of the [GA] *Spazio di una Varietà*, this set R is equipped with the structure of a (Hausdorff-)topological space which is compact if V is a complete model of its function field K. Remembering that the model V is associated to a scheme X, this notion is analogous to the notion of the complex analytic space associated to X as in [Ha] p. 438 ff. Now, if V is algebraically regular, i. e. all $s \in V$ containing $\mathbf{Q}$ are regular local rings, as shown in [GA] 438., $R(V)$ carries the structure of a complex analytic manifold. And by [GA] 453., for $\deg K = g$, i. e. the degree of the maximal number field k contained in K is g, $R(V)$ is a disjoint union of g connected conjugate manifolds R_ν

$$R(V) = \bigsqcup_{\nu=1}^{g} R_\nu$$

which will be thought of as lying over the g complex embeddings

$$\sigma_i : k \hookrightarrow \mathbf{C} .$$

As to be expected, the arithmetically distinguished differentials come in here too: By [GA] 456. and 457., $\Theta \in D_q(K/\mathbf{Q})$ induces on each $R_\nu(V)$ and thus on R a holomorphic differential form, and this association is consistent with the differentiation d. From 458. on, Kähler combines the notions of arithmetically and topologically integral differentials in the following manner:
For a holomorphic r-form θ on a compact Riemanniana R, its periods relative to a basis (ω_i) of $T^r(R)$ are defined by

$$\pi_i := \int \omega_i \wedge \theta .$$

And by decomposing $\theta \sim \sum \wedge_k \omega_i$ in [GA] 458., Kähler deduces the *Theorem of Hodge* easily from the fact that the matrix

$$\left(i^n \int \omega_k \wedge \omega_\ell \right)$$

is hermitian with determinant $\neq 0$: If all periods of a holomorphic r-form θ on a compact $2r$-Riemanniana vanish, the form θ is zero.

In the sequel, the arithmetically distinguished differentials are used to define moduli and functions which are birational invariants of the field K.
Let

$$\theta = (\theta_j) , \quad j = 1, \ldots , \ell_q = g \, p_q ,$$

be a basis of the additive group $D_q (K)$ and

$$\omega = (\omega_i^{(\nu)}) \quad i = 1, \ldots , b_{2r-q} , \ \nu = 1, \ldots , g$$

a basis of $T^{2r-q} (R)$. Then there is the *period matrix*

$$\Pi := \left(\int \omega_i^{(\nu)} \wedge \theta_j \right) .$$

Kähler discusses the following special cases:

1. For $q = 0$ we have $D_0 (K) = \mathfrak{o}_k$, the ring of integers in the maximal number field k contained in K, generated as a $\mathbf{Z}$-module by $\theta_j = a_j \in k$ $j = 1, \ldots , g$. And the topologically integer $2r$-cohomology classes are represented by $2r$-differentials τ_ν with $\tau_\nu = 0$ on R_μ for $\mu \neq \nu$ and $\int_{R_\nu} \tau_\nu = 1$. Here, the period matrix is simply

$$\Pi = (\sigma_\nu a_j)$$

for the *points* $\sigma_\nu : k \hookrightarrow \mathbf{C}$, $\nu = 1, \ldots , g$. The square of the determinant of Π is the discriminant of k and thus a birational invariant of K.

2. For $r = 1$, i. e. the case of a field K of algebraic functions in one variable, all spaces R belonging to complete algebraically regular arithmetic models V of K are homeomorphic and decompose into $g = \deg K$ conjugate Riemannian surfaces R_ν of genus $p =$ rank of $D_1 (K)$ as a k-module. Here the basis ω of $T^1 (R)$ is chosen such that

$$\begin{gathered} \omega_{i+p(\nu-1)} = \omega_i^{(\nu)} , \quad \omega_{pg+i+p(\nu-1)} = \omega_{p+i}^{(\nu)} \\ \omega_i^{(\nu)} = \omega_{p+i}^{(\nu)} = 0 \quad \text{on} \quad R_\mu \quad \text{for} \quad \mu \neq \nu , \\ \int \omega_i^{(\nu)} \wedge \omega_{p+j}^{(\nu)} = \delta_{ij} , \quad i, j = 1, \ldots , p , \ \nu = 1, \ldots , g . \end{gathered}$$

Such a basis is called *canonical* . Two such bases ω and $\widetilde{\omega}$ are *equivalent* if there are matrices $A^{(\nu)} = (a_{ij}^{(\nu)}) \in M_p (\mathbf{Z})$ with $\det A^{(\nu)} = \pm 1$ and

$$\widetilde{\omega}_i (\nu) = \sum_{j=1}^{p} a_{ij}^{(\nu)} \, \omega_j^{(\nu)} , \quad i = 1, \ldots , p .$$

For a base $\theta = (\theta_1, \ldots, \theta_p)$ of $D_1(K)$, Kähler defines (in [GA] 463.-465.) for $m \in \mathbf{N}$

$$g_m(\theta) := \sum (\det \Omega)^{-2m}$$

and for $t \in \mathbf{C}$

$$\psi(\theta, t) := \sum |\det \Omega|^{-t} ,$$

where the sum is extended over all non-equivalent canonical bases of $T^1(R)$ and Ω is the square period matrix formed with the *first half* of the base ω, i. e. with ω_i, $i = 1, \ldots, pg$. These series converge, by a result of Hel Braun [Br], for $2m > p+1$ resp. $t \in \mathbf{C}$ with $\mathrm{Re}\, t > p+1$. Extending this result, obtained at first by Kähler for $g = 1$, G. Häuslein [Hä] showed in his thesis that g_m and ψ_K are birational invariants which decompose as products into factors of the type obtained in the case $g = 1$.

3. Another procedure to construct birational invariants suitable also in the cases $r > 1$ is proposed in [GA] 466.-471. Here again one takes a complete regular model V of K algebraic over $\mathbf{Q}$ with $\mathrm{trdeg}_{\mathbf{Q}} K = r$, $\deg K = g$, $\mathrm{char}\, K = 0$ and space $R = R(V) = \sqcup_{\nu=1}^{g} R_\nu$. Then Kähler shows that, for a differential $\Theta \in D_r(K/\mathbf{Q})$, the non-negative numbers

$$i^{n^2} \int_{R_\nu} \Theta \wedge \overline{\Theta} \quad \nu = 1, \ldots, g$$

don't depend on the choice of the model V for K. Abbreviating

$$\int_R \Theta \wedge \overline{\Theta} := \sum_{\nu=1}^{g} \int_{R_\nu} \Theta \wedge \overline{\Theta},$$

one thus gets, for $\mathrm{Re}\, t > gp_r$, a differential depending only on K and t

$$\Omega(t) := \sum_{\Theta \in D_r(K)} \frac{\Theta \wedge \overline{\Theta}}{(\int_R \Theta \wedge \overline{\Theta})^{1+t/2}}$$

and a function

$$\Psi_K(t) := \int_R \Omega(t) = \sum_{\Theta \in D_r(K)} \left(\int_R \Theta \wedge \overline{\Theta} \right)^{-t/2} .$$

These notions can be related to the periods in the following way: For a basis $\theta = (\theta_1, \ldots, \theta_\ell)$, $\ell = p_r g$, we get non-negative hermitian forms

$$F_\nu(x_1, \ldots, x_\ell) := i^{n^2} \int_{R_\nu} \left(\sum x_j \theta_j \right) \lambda \left(\sum \overline{x}_k \, \overline{\theta}_k \right) \quad \nu = 1, \ldots, g$$

which don't depend on the choice of V. To study the dependence of these forms on the moduli of the field K, one goes back to the topologically integral differentials and puts

$$\theta_j \sim \sum_{\mu=1}^{b_\nu} \omega_\mu^{(\nu)} \, b_{\mu j}^{(\nu)} \quad \text{on} \quad R_\nu \, .$$

Here the complex coefficients $b_{\mu j}^{(\nu)}$ can be expressed in terms of the periods, namely, using the matrix $\left(\int \omega_\lambda^{(\nu)} \wedge \omega_\mu^{(\nu)} \right) =: (e_{\lambda\nu}^{(\nu)})$, via

$$\pi_{\lambda j}^{(\nu)} \; = \; \int_{R_\nu} \omega_\lambda^{(\nu)} \wedge \theta_j \; = \; \sum_{\mu=1}^{b_\nu} e_{\lambda\mu}^{(\nu)} \, b_{\mu j}^{(\nu)}$$

we get

$$b_{\lambda j}^{(\nu)} \; = \; \sum_{\mu=1}^{b_\nu} m_{\lambda\mu}^{(\nu)} \, \pi_{\mu j}^{(\nu)} \quad m_{\lambda\mu}^{(\nu)} \in \mathbf{Z} \, , \quad \det\left(m_{\lambda\mu}^{\nu}\right) \; = \; \pm 1 \, ,$$

and thus

$$i^{n^2} \int \theta_j \wedge \overline{\theta}_k \; = \; i^{n^2} \sum_{\lambda,\mu} m_{\lambda\mu}^{(\nu)} \, \pi_{\lambda j}^{(\nu)} \, \overline{\pi}_{\mu k}^{(\nu)} \, .$$

In case n is odd, one can assume

$$\left(e_{\lambda\mu}^{(\nu)} \right) \; = \; \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \; = \; J \in M_{b_\nu}\left(\mathbf{Z}\right)$$

and therefore $(m_{\lambda\nu}^{(\nu)}) \; = \; -J$. The coefficients of $x_j \, \overline{x}_k$ in the form $F_\nu(x_1, \ldots, x_\ell)$ are then

$$c_{jk}^{(\nu)} \; = \; i \sum_{\lambda=1}^{b_\nu/2} \left(\pi_{\lambda j}^{(\nu)} \, \overline{\pi}_{b_\nu/2+\lambda,k}^{(\nu)} - \pi_{b_\nu/2+\lambda,j}^{(\nu)} \, \overline{\pi}_{\lambda j}^{(\nu)} \right) \, .$$

In [GA] 470., it is shown that

$$F\left(x_1, \ldots, x_\ell\right) \; = \; \sum F_\nu\left(x_1, \ldots, x_\ell\right)$$

is positive definite.

In the classical case $r = g = 1$ of a curve over $\mathbf{Q}$, if the period matrix is normalized to

$$\pi \; = \; {}^t\left(Z, E\right) \, ,$$

where Z is an element of the Siegel upper half space $\mathfrak{H}_p$, we get

$$\Psi_K\left(t\right) \; = \; \sum_{x \in \mathbf{Z}^p} \left({}^t x (Z - \overline{Z}) x \right)^{-t} \, .$$

Remark. Moreover, the differential form $\Omega(t)$ defined above can be used to define a Kählerian metric ds^2 on the Riemanniana R_ν: If $x_1, \ldots, x_r$ are complex coordinates for R_ν in an open domain U, one writes

$$\Omega\,(t) \;=\; \Phi\,(t)\, dx_1 \wedge d\,\overline{x}_1 \wedge \ldots \wedge dx_r \wedge d\,\overline{x}_r$$

and then defines

$$ds^2 \;:=\; i \sum_{\nu,\ell=1}^{r} \frac{\partial^2 \log \Phi}{\partial x_j \partial \overline{x}_\ell}\, dx_j \cdot d\,\overline{x}_\ell \;,$$

resp. the Kähler form

$$\omega \;=\; i\partial\overline{\partial}\Phi \;.$$

As stated at the end of [IA], for geometric genus big enough, this comes out as a metric which is positive definite up to $(2r-2)$-dimensional subspaces, where the discriminant of the metric may vanish.

In [GA] 473., the last section of this paragraph, Kähler raises the question whether in general the Riemannianae for a given field K are homeomorphic and he proposes to look also at the period matrix π for $q = 1\,,\;\; r > 1\,$, and to put this together with its complex conjugate to get a square matrix with non vanishing determinant which up to a factor $\pm 1\,$, is determined by K alone.
Concerning Kähler's question, in his article on the topology of conjugate varieties [Re] p. 287, D. Reed states that there has been little published in this area since Serre [Se] and Abelson [Ab] have produced examples of conjugate varieties whose complex points constitute non homeomorphic topological spaces.

Appendix

A. Kähler's Perspectivism as his Version of Local Algebra

Inspired by Leibniz's *Monadologie*, Kähler developed a vocabulary to visualize the notions and relations from local algebra. This terminology is displayed in German in the article [RAK] and the book [AD], in French in the article [CPA], and in Italian in the chapters III and (partially) VII of the Geometria Aritmetica [GA]. Some of Kähler's notations are already translated into English by André Weil in his review [WGA]. We will here reproduce these and try to find appropriate translations for the other notions. Doing this, I will try to sketch the part of Kähler's system which seems to me relevant for the comprehension of the development of his mathematical work. Most of this resurfaces in Kähler's later work intended to reach into Natural Sciences, Philosophy and even Theology. The reader, interested to get an impression of this, will find material assembled in the appendix to this Collection of Kähler's Works.

A 1. Fundamental Notions: Homomorphisms as Perspectives

Starting from the fundamental fact that a field allows only two types of homomorphic images, namely the isomorphic image and the reduction of all elements to zero, Kähler is led to reproduce in mathematical terms the general situation that in one observes the objects reality in a certain perspective, i. e. by means of the aspects which these present from different perspectives. Following this idea, in [GA] 63., Kähler arrives at a collection of definitions translating the usual mathematical expressions.

- Let A be a ring.Then a homomorphism

  $$A \longrightarrow A/\mathfrak{a} \neq \{0\}$$

  determined by an ideal $\mathfrak{a}$ is a *perception* with *origin* $\mathfrak{a}$.

- The element $x + \mathfrak{a}$ of $A/\mathfrak{a} \neq \{0\}$ with representative $x \in A$ is the *perception* of x in $A/\mathfrak{a}$.

- If $x \in A \neq \{0\}$ and $xa \neq 0$ for each element $a \neq 0$ of A, x is *active* in A. And if $x \in A \neq \{0\}$ and $ax \neq 0$ for each $a \neq 0$ of A, x is *reactive* in A.

- If A is a commutative ring and has active elements, A is *objective* (= *ogettivo* in Italian).

- If R is the full quotient ring of the objective ring A, i. e. the set of quotients a/b, R is the *object* of A, in signs $R = (A)$. For $A = (A)$, A is called a (mathematical) *reality* ([AD] p. 62).

- If the origin $\mathfrak{a}$ of the perception $A \to A/\mathfrak{a}$ contains the origin of each perception, i. e. $\mathfrak{a}$ is a unique maximal ideal, the perception is *central*.

- If the ring S has a unit 1 and allows for a central perception $S \to S/\mathfrak{P}$, S is an *aspect* presented by the *perspective* of the *subject* $S/\mathfrak{P}$.

- For $R = (S)$, i. e. R is the object of the aspect S (and *of the perspective* $\mathfrak{P}$), S is an *aspect of* R and $\mathfrak{P}$ a *perspective of* R (and of no other ring).

Kähler convenes to denote the aspects by

$$S, s, S', s_1, \text{etc.}$$

while the corresponding symbols

$$\mathfrak{P}, \mathfrak{p}, \mathfrak{P}', \mathfrak{p}_1, \text{etc.}$$

indicate the respective perspectives presented by the aspects as well as the origins of the perceptions, i. e. the kernels of the homomorphisms

$$S \to S/\mathfrak{P}\,,\ s \to s/\mathfrak{p}\,,\ S' \to S'/\mathfrak{P}'\,,\ s_1 \to s_1/\mathfrak{p}_1$$

- The subject $S/\mathfrak{P}$ of a perspective $\mathfrak{P}$ is *individual* in the sense that it has a unit element and has no other perception than

  the identity. In the case of a commutative S the subject $S/\mathfrak{P}$ is a field.

- An aspect S is an *extension* of an aspect, and the perspective $\mathfrak{P}$ is an *extension* of $\mathfrak{P}$, if and only if s is contained as a subring in S and $\mathfrak{P} \cap S = \mathfrak{p}$ holds, i. e. in the language of local algebra, if S dominates s. Then as a consequence, one has an embedding

$$s/\mathfrak{p} \hookrightarrow S/\mathfrak{P}\ .$$

A 2. Local Algebra: Local Rings as Aspects of an Object

From now on, everything is restricted to commutative rings. In [GA] 68. Kähler proves the usual criterium for local rings: Let A be a commutative ring. Then A is an aspect exactly if A has an ideal $\mathfrak{a}$ such that from $x \in A$, $x \notin \mathfrak{a}$ one can conclude $x^{-1} \in \mathfrak{a}$. This $\mathfrak{a}$ is then the origin of a perspective presenting the object A.

And in [GA] 71. the generation of a local ring by localization of another ring is presented as follows.

- A commutative ring A is *base* of the aspect S or the perspective $\mathfrak{P}$ if and only if

1.) A is a subring of S, and

2.) each element of S is a quotient a/b of $a, b \in A$, $b \notin \mathfrak{P}$

The intersection $\mathfrak{P} \cap A$ is called the ideal *creating* (= *individuante* in Italian) S resp. $\mathfrak{P}$ in A. For an ideal $\mathfrak{c}$ in A to create a perspective with base A it is necessary and sufficient that $\mathfrak{c}$ is a prime ideal $\neq A$ containing all in A non-active elements of A. Then, we have

$$\mathfrak{P} \cap A = \mathfrak{c} \quad \text{and} \quad \mathfrak{P} = \mathfrak{c} S .$$

for the perspective created by $\mathfrak{c}$.

- A perspective $\mathfrak{P}$ (an aspect s) *is extended with elements* $x, y, \ldots$ of a ring A containing s signifies that there is an aspect S extending s and containing $x, y, \ldots$.

In the sections 77. - 90. of the [GA], this notion is discussed and related to the notion of integrality:

- The *integrity* (= integrità in Italian) of a ring extension R/A is defined in the following way:

  An element x of a primary ring R containing A is called *integral* in R if and only if each aspect intermediate between A and R can be extended with x. The set $I(R/A)$ of these elements is a ring which Kähler calls the *integrity*.

It is shown in [GA] 86., that $I(R/A)$ consists of the elements integral over A in the usual sense, namely that they annihilate a normed polynomial with coefficients in A.

To get a kind of geometric interpretation of this algebraic notion, Kähler refines the term aspect in two steps.

1.) An aspect s resp. a perspective $\mathfrak{p}$ is called *closed* if it has no inner extension. Here, an *inner* extension is an extension which has the same object, i. e. quotient ring, as the extended ring.

There are equivalent characterizations if the rings are noetherian. (In Italian, Kähler uses this usual term from our common algebraic language while, in the German text of [AD], he finds the - to my opinion - in this context nice expression *ergründlich* which might be translated into *explorable*). In [GA] 93. resp. 95., we have a proof that a noetherian aspect s is closed, if and only if

- the origin $\mathfrak{p}$ of s is a principal ideal, or, for the case $(s) = k$ a field,
- from $x \in k \setminus s$ follows $x^{-1} \in s$.

Then Kähler calls an aspect s *perfect*, if s is closed and noetherian with $(s) = K$ a field and $s \neq k$. Thus a perfect aspect of K is a discrete valuation ring of K (see for instance [Ha] Theorem 6.2 A).

The sections 98. - 108. of [GA] give a proof of the fact that the integrity of a finitely generated field K is a finite $\mathbf{Z}$-module and discuss methods for its determination. In this, several results by F. K. Schmidt are involved and reproduced. Here, we will only state the already announced *geometric* characterization of the integrity $I(K)$ of a finitely generated field with $\mathrm{char} K = 0$ ([GA] 346.)

$I(K)$ is the intersection

$$I(K) = \bigcap S$$

of all discrete valuation rings S which are arithmetical in the sense that they are essentially of finite type over $\mathbf{Z}$. This can be read as the set of elements of K which are contained in all perfect aspects S having a finitely generated base over $\mathbf{Z}$, or, more loosely, as the elements visible in all these special perspectives $\mathfrak{P}$ of K. In [CPA], Kähler calls these elements *les traits eternels of K*.

A 3. $\mathfrak{p}$-adic Completion as Environment of a Perception

Kähler uses the terms of his perspectivism in the following chapters of his Geometria Aritmetica for his formalism to describe Algebraic, Arithmetic and Analytical Geometry. We will report on these in a second Appendix and continue here immediately with the part of the perspectivism describing the process of $\mathfrak{p}$-adic completion, as to be found in [GA] 269. ff under the heading *Anelli individuali*.

- Every perception $A/\mathfrak{c}$ of a ring A *creates* (= *é il principio di* in Italian) an infinitely ascending sequence

  $$A/\mathfrak{c}\,,\ A/\mathfrak{c}^2\,,\ A/\mathfrak{c}^3, \ldots$$

  of perceptions of *precision* $n \in \mathbf{N}$. This sequence leads to (*sbocca* in Italian) a ring

  $$A/\mathfrak{c}^\infty$$

created by the perception (*individuale alla perceptione*) $A/\mathfrak{c}$ defined in the following manner:

- Every infinite sequence of the type

  $$a_0 + \mathfrak{c} \subset a_1 + \mathfrak{c}^2 \subset a_2 + \mathfrak{c}^3 \subset \cdots \quad (a_n \in A)$$

  determines an element α of $A/\mathfrak{c}^\infty$ and vice versa. The $(n+1)$-th term of the sequence determined by α and of precision n is denoted by

  $$(\alpha)_n$$

  and called the *n-th approximation* of α. $A/\mathfrak{c}^\infty$ is a ring with the usual operations.

- Every element $a \in A$ approximates one and only one element α of $A/\mathfrak{c}^\infty$ with infinite precision, namely the one given by the sequence

  $$a + \mathfrak{c} \subset a + \mathfrak{c}^2 \subset a + \mathfrak{c}^3 \subset \ldots$$

  which is called the *idea* of a in $A/\mathfrak{c}^\infty$.

Apparently, these ideas form a subring of $A/\mathfrak{c}^\infty$ isomorphic to A exactly if $\bigcap_{n\in\mathbf{N}} \mathfrak{c}^n = \{0\}$. This is the case if a noetherian aspect S with base A exists which satisfies $\mathfrak{P} \subset \mathfrak{c}$.

- In [AD] p. 127, Kähler calls the ring $A/\mathfrak{c}^\infty$ *die Umwelt der Anschauung* $A/\mathfrak{c}$ which might be translated into *the environment of the perception*. And in [GA] 270, he specializes to the situation $s/\mathfrak{p}^\infty$ for s an aspect, i. e. a local ring, and calls

  $$s^* := s/\mathfrak{p}^\infty$$

  the *individuality* (*individualità*) of $s/\mathfrak{p}$ resp. of $\mathfrak{p}$ or s.

In the sequel, he proves that s^* is again a local ring and he shows the usual facts about the completed rings. For instance, ([GA] 273.), s^* is regular (perfect), if s is regular (perfect). Moreover, in [GA] 276. - 282., facts are proved about norms, traces and lengths of ideal chains in relative situations which go into Kähler's proof of the finiteness of the module of integral differentials.

B. Kähler's Arithmetic and Algebraic Varieties

As Kähler often affirmed, his notions were already concieved of before 1939. But they got published only starting in 1951 ([RAK], [CPA], [GA], and the resumé in [IA]). Nowadays, they are only to be seen as a background to the larger system of the *Éléments de la Géométrie Algébrique* [EGA] by Grothendieck and Dieudonné, published starting in 1960, and the numerous texts in the sequel. But at the time of their appearance and several years after, they were competing with and related to, besides the works of the Italian School (see [Sev]), the work of A. Weil [WAG], Zariski, Samuel [ZS], Chevalley [Ch], Nagata [Na], Lang [La], van der Waerden [vdW] and Godement [Go] (here only books are cited and the choice of the enumeration is certainly influenced by a personal touch).

In the following sketch of Kähler's system, we will try to find an intermediate way between Kähler's terminology, as displayed partly in Appendix A and the usual notions from local and commutative algebra. Mostly, we follow the *Geometria Aritmetica* [GA], but sometimes we also use the *Infinitesimal-Arithmetik* which is the text of a lecture given in 1961 in Torino and which gives a kind of overview about the content of the Geometria Aritmetica.

B 1. Varieties

We take over from [GA] 110.

Definition.
A *variety* is a set of local rings fulfilling the following conditions :

I. Any two local rings belonging to the same variety have a common base, i. e. are localizations of a common subring.

II. If a local ring s belongs to the variety, every local ring which is a localization of s also belongs to the variety.

Varieties are usually denoted by $V, V', \dots$

- $s \in V$ means s is a local ring belonging to V
- $s \geq A$ is used as a sign that the local ring s is a localization of the ring A, or *the aspect s has A as a base.*
- For two local rings s and s', $s' \to s$ signifies that s' dominates (*extends*) s, i. e. $s \subset s'$ and $\mathfrak{p}' \cap s \subset \mathfrak{p}$.

Using these agreements, a collection V of local rings is a variety, if and only if

I. $s, s' \in V \implies \exists A$ such that $s \geq A$ and $s' \geq A$,

II. $s \in V, s' \geq s \Longrightarrow s' \in V$.

- From I., we conclude that all local rings s of V have the same full quotient ring (s) which Kähler calls the *object of the variety*. Thus, all local rings belonging to the variety *are aspects of the same object.*

- A variety is called *prime, primary resp. non-primary* if its object is a field, a primary ring or a non-primary ring.

As will become clearer later, a prime variety V with object K in this sense is simply the collection of local rings appearing as stalks in a scheme having the field K as field of rational functions. It is even more evident that the variety V is what is nowadays called a *model* of the field K.

From his definition Kähler can easily conclude the following facts :

- Two perspectives belonging to the same variety and having a common extension are equal, i. e. if a local ring s dominates s_1 and $s_2 \in V$, we have $s_1 = s_2$ ([GA] 111.).

- For $s_1, s_2 \in V$, from $\mathfrak{p}_2 \cap s_1 \subset \mathfrak{p}_1$, we can conclude $s_1 \subset s_2$ and vice versa ([GA] 112.).

- Each ring A consisting not only in zero-divisors is the base of a variety $V(A)$, namely the set of all local rings s with base A ([GA] 114.). Apparently, here we see $V(A) = \{s = \mathcal{O}_{X,x} \ , \ x \in X = \operatorname{Spec} A\}$ is the usual *affine variety*.

- A variety V is called *arithmetic*, if it is obtained as a collection of a finite number of affine varieties

$$V = \bigcup_{i=1}^{N} V(A_i) \ ,$$

  where the rings A_i are arithmetic, i. e. of finite type over $\mathbf{Z}$ and have the same quotient field. And V is called *algebraic over the field k*, if similarly the A_i are of finite type over k.

Synthesis of Varieties

The notion of a product of varieties appears in the following manner :

A subring A_0 in a ring A is called *active* if it has at least a non-zero-divisor and if every non-zero-divisor from A_0 stays non-zero-divisor in A. We take two subrings R_1, R_2, active in this sense in a ring R, and call the ring R_0 generated by these rings R_1 and R_2 in R the *synthesis of* R_1 *and* R_2 .

Definition. Let V_1 and V_2 be varieties with objects R_1 and R_2. Then the *synthesis* (V_1, V_2) of V_1 and V_2 is the set of local rings s coming up as localizations of the bases $[s_1, s_2]$, the subrings in R generated by all $s_1 \in V_1$ and $s_1 \in V_2$.

This definition produces again a variety. If V_1 and V_2 are generated as

$$V_1 = \bigcup_i V(A_i) , \quad V_2 = \bigcup_j V(B_j) ,$$

where the rings A_i and B_j are active in the same object R and contain the same unit element as R, we have ([GA] 117.)

$$(V_1, V_2) = \bigcup_{i,k} V([A_i, B_j]) .$$

This shows that Kähler's synthesis of varieties corresponds to the product of schemes (fibered over $\mathbf{Z}$).

Closed Varieties

The usual notion of a *complete* variety or model appears in [GA] 118. under the name of a closed variety :

- Rings $A, A', \ldots$ are called *compatible*, if they are subrings of the same primary ring.

Definition. A variety V is called *closed* if each local ring compatible with the object of the primary variety V has a common extension together with a local ring belonging to V.

A variety V is called *closed over a ring* A, if each local ring containing A and compatible with the primary object of V has a common extension together with a local ring belonging to V.

Kähler's choice of the term *closed* is partly understandable by the relation to the special cases of integrally closed rings (it will be more understandable later by the fact that its space is compact).

- If $A_1 = I(R/A)$ denotes the ring of elements of a primary ring R which are integral over A, $V(A_1)$ is a closed variety ([GA] 119.).

- If the object R of a variety V which is closed over R contains the ring A, the elements of the intersection of all local rings $s \in V$ are integral over A

$$\bigcap_{s \in V} s \subset I\,(R/A)$$

([GA] 120.).

A very useful construction of *projective* models is given by the following

Proposition ([GA] 122.). If $x_1, \ldots, x_m$ are non-infinitesimal elements of a primary ring A (i. e. non nil-potent elements) and if $A_i = [x_1/x_i, \ldots, x_n/x_i]$, $i = 1, \ldots, m$, denotes the ring generated by the quotients x_j/x_i in the quotient ring of A, the variety

$$V = \bigcup_{i=1}^{m} V\,(A_i)$$

is a closed variety.

In ([GA] 124.) this is a bit more refined to the following statement :

If a ring A has a unit and is active in a primary ring as well as the elements $x_1, \ldots, x_m$, the variety

$$V = \bigcup_{i=1}^{m} V(A[\frac{x_1}{x_i}, \ldots, \frac{x_m}{x_i}])$$

is a closed algebraic variety over A, called the *projective variety generated by* $x_1, \ldots, x_m$ *over* A.

B 2. The Space of a Variety

In the world of schemes, we have the notion of a complex analytic space associated to a scheme (over **C**). Kähler's way to associate a space to his variety is essentially the same as in the scheme-theoretic context. It appears in the chapter V of the Geometria Aritmetica under the title *Lo spazio di una varietà* :

Definition. A *point of an object* R (i. e. a primary ring) is a homomorphism of an aspect s of this object (i. e. a local ring s with quotient ring $(s) = R$) into the field **C** of complex numbers.

This homomorphism factors via the perspective $s \to s/\mathfrak{p}$ and an isomorphism of the subject $s/\mathfrak{p}$ into $\mathbf{C}$. Each such homomorphism P of S into $\mathbf{C}$ is called a *point of the object* (s) *supported by the perspective* $\mathfrak{p}$ (or *by the aspect* s). The image $P(x)$ of an element $x \in s$ by the homomorphism P is called the *value of* x *in* and written as $x(P)$.

A is a it base of a point P is used as an abbreviation for *A is base of an aspect s supporting P*, and $A(P)$ signifies that P has A as a base in this sense.

Aspects s with residue characteristic $p \neq 0$ don't support points. An algebraic number field k with $\deg k = n$ has exactly n points, all supported by the aspect $s = k$.
The space of all the points supported by the local rings $s \in V$ is called $S(V)$, the *space of the variety* V.

This space is equipped with a topology in the following way :

Definition 2. Let P be a point with base A, where $1 \in A$ and $V(A) \subset V$. Then, a *neigbourhood* of P is given by the set of points Q having A as a base and fulfilling the finite system of inequalities

$$(*) \qquad |x_i(Q) - x_i(P)| < r_i \quad i = 1, \ldots, m$$

formed by arbitrary elements $x_1, \ldots, x_m \in A$ and any positive numbers $r_1, \ldots, r_m$.

In particular, a neighbourhood is called *arithmetic* if its base A is an arithmetic ring, i. e. of finite type over $\mathbf{Z}$. It is then natural to take for the x_i in the condition $(*)$ above the elements generating A. In [GA] 134., Kähler gives a proof that the space $S(V)$ of an arithmetic variety is thus a Hausdorff topological space.

Definition 3. A point P *lies over* a point p if the perspective $\mathfrak{P}$ supporting P is extension of the perspective $\mathfrak{p}$ supporting p (i. e. the local ring S supporting P dominates the local ring s sopporting p) and $x(P) = x(p)$ holds for all $x \in s$.

The main result in this context is the following

Proposition ([GA] 136.) : Let S be an extension of s with finitely many elements with

$$\operatorname{trdeg} \frac{S/\mathfrak{P}}{s + \mathfrak{P}/\mathfrak{P}} = 0\,, \quad \deg \frac{S/\mathfrak{P}}{s + \mathfrak{P}/\mathfrak{P}} = n\,.$$

Then, over each point p supported by s lie exactly n points P supported by S.

Each aspect supporting a point contains the prime field $\mathbf{Q}$. Thus, for the determination of the space of a variety V, it is sufficient to take the subset $(V, \mathbf{Q})$ of V consisting of the local rings s with $s \supset \mathbf{Q}$. And the explanation of the term *closed* variety is given by the following

Theorem ([GA] 139.) : Let V be algebraic and closed over the field k. Then, the space of all points which are lying over the same point p supported by k is a *compact* Hausdorff space, if the neighbourhoods are chosen by bases which are algebraic over k.

The proof of this statement takes four pages. Weil remarks in his review [WGA] that he doesn't know about any other written proof of this fact.

In [GA] 140.-147., Kähler studies the spaces for varieties V having as an object a field K finitely generated over a subfield k with

$$\operatorname{trdeg}_k K = n \quad \text{and} \quad \deg_k K := \deg_k I\,(K/k) \;=\; g$$

(here the field $I\,(K/k)$) of elements of K which are algebraic over k is the maximal algebraic extension field of k in K). Moreover, he uses the following terms ([GA] 146.) : If (C, K^σ) is the composition of the field C with K, the isomorphism σ is (or better *determines*) a *position of K with respect to C*. Two such positions σ and τ are equal if and only if there is an isomorphism ρ over C of the composition (C, K^σ) onto the composition (C, K^τ) with

$$x^{\sigma\rho} \;=\; x^\tau \quad \text{for all} \quad x \in K\ .$$

We reproduce here [GA] 147. restricted to the case of characteristic zero and $C = \mathbf{C}$.

Proposition : For K algebraic over k with

$$\operatorname{trdeg}_k K \;=\; n\ , \quad \deg_k K \;=\; g\ ,$$

there are exactly g positions of K with respect to $\mathbf{C}$ such that in every composition of $\mathbf{C}$ with K over k which has transcendence degree n over $\mathbf{C}$ the field K has one of these positions.

The sections [GA] 148.-167. are dedicated to attaching an analytic structure to the *arithemetic spaces*, i. e. the spaces of an arithmetic variety having as its object a field K (finitely generated over $\mathbf{Q}$). The following notions are fixed in [GA] 148. :

Definition 4. A variety V is called *analytic* if every $s \in V$ contains $\mathbf{C}$.
A point P is *analytic* if P is a point of an analytic variety with $c(P) = c$ for all $c \in \mathbf{C}$.
An *analytic space* is the set of analytic points of an analytic variety.
An *n-dimensional analytic field* is a field containing $\mathbf{C}$ with transcendence degree n over $\mathbf{C}$.
An *analytic realization* of a field K with $\text{trdeg} K = n$ is a composition $(\mathbf{C}, K^\sigma)$, n-dimensional over $\mathbf{C}$, Two such realizations $(\mathbf{C}, K^\sigma)$ and $(\mathbf{C}, K^\tau)$ are *equal*, if the positions σ and τ of K with respect to $\mathbf{C}$ are equal.

These definitions lead to the main result.

Theorem ([GA] 164) : The space $S(V)$ of an arithmetic variety V of a field K with $\text{trdeg} K = n$ and $\deg K = g$ decomposes

$$S(V) = \bigcup_{i=1}^{g} R((\mathbf{C}, V^{\tau_i}))$$

into g real $2n$-dimensional connected parts which identify with the analytic spaces $R_j := R((\mathbf{C}, K^{\tau_i}))$ corresponding to the different analytic realizations $(\mathbf{C}, K^{\tau_i})$ of K. This identification is given by mapping the point $P \in S(V)$ supported by $S \in V$ to the point $P_i \in R_i$ if and only if

$$x(P) = x^{\tau_i}(P_i)$$

holds for all $x \in S$.

In [GA] 166., Kähler explains that the g parts R_i into which the space $S(V)$ decomposes can have points in common. But they are disjoint if the arithmetic variety is *integral*, if its object is a field K and if each $s \in V$ contains the ring $I(K)$ of integral elements of K. In this case the space of the variety $V(I(K))$ consists of the g points $p_1, \ldots, p_g$ given by the g isomorphisms of $k = (I(K))$ into $\mathbf{C}$ and each R_i is the set of points of $S(V)$ lying over p_i in the sense of the Definition 3 above.

As to be expected, Kähler's proof of the Theorem uses power series methods. As I see it, the corner stone is the discussion of a polynomial Φ introduced in [GA] 151. as follows :

Let be

K the field generated by $x_1, \ldots, x_m$

K_0 the field generated by $x_1, \ldots, x_n$, $\deg_{K_0} K = h$,

K^* the Galois extension over K_0 determined by K,

$$\mathrm{Gal}(K^*/K_0) = \sum \mathrm{Gal}(K^*/K)\,\sigma_i$$

,

$\overline{K}$ the field generated by K^* and variables $y_{n+1}, \ldots, y_m$ and $u_{n+1}, \ldots, u_m$, i. e. with $\mathrm{trdeg}_{K^*}\overline{K} = 2\,(m-n)$.

Then Φ is defined as the product

$$\Phi\,(x_1, \ldots, x_n\,,\ y_{n+1}, \ldots, y_m\,,\ u_{n+1}, \ldots, u_m)\ :=$$

$$\textstyle\prod_{i=1}^{h}\left(u_{n+1}\,(y_{n+1} - x_{n+1}^{\sigma_i}) + \cdots + u_m\,(y_m - x_m^{\sigma_i})\right)\,.$$

The importance of this polynomial becomes clear from the following intermediate results :

Proposition ([GA] 152.) : There is a point P of the variety $V := V\,(\mathbf{Z}\,[x_1, \ldots, x_m])$ with $x_i\,(P) = c_i\ (i = 1, \ldots, m)$ exactly if the $c_i \in \mathbf{C}$ fulfill the equation

$$\Phi\,(c_1, \ldots, c_n\,,\ c_{n+1}, \ldots, c_m\,|\,u_{n+1}, \ldots, u_m) = 0\,.$$

Proposition ([GA] 159.) : The complete decomposition

$$\Phi\,(x, y, |u)\ =\ \prod_{i=1}^{g}\Phi_i\,(x, y|u)$$

of the polynomial Φ in the polynomial ring $\mathbf{C}\,[x, y, u]$ leads to a decomposition of the space $S\,(V)$ of the variety $V = V\,(\mathbf{Z}\,[x_1, \ldots, x_m])$ in $g = \deg K$ $2n$-dimensional connected parts which are defined by

$$P \in R_i \longleftrightarrow \Phi_i\,(x_1(P), \ldots, x_m(P)\,|\,u_{n+1}, \ldots, u_m) = 0\,.$$

If $(\mathbf{C}, K^{\tau_i})$ is the analytic realization of K satisfying

$$\Phi_i\,(x_1^{\tau_i}, \ldots, x_m^{\tau_i}\,|\,u_{n+1}, \ldots, u_m)\ =\ 0$$

the part R_i can be interpreted as the analytic space $A\,(V_i)$ of the variety $V_i = V\,(\mathbf{C}\,[x_1^{\tau_i}, \ldots, x_m^{\tau_i}])$ of $(\mathbf{C}, K^{\tau_i})$ by identifying the points $P \in A\,(V_i)$ and $P' \in R_i$ if and only if

$$x_j^{\tau_i}\,(P)\ =\ x_j\,(P')\,,\ j = 1, \ldots, m\,.$$

B 3 Kähler's notion of a figure

A subvariety V_0 of a given variety V is in the context of Kähler's work simply a subset (*varietà parziale*). For instance, in [GA] 170. he introduces the term $V(s)$ as the *star* (= *stella*) of $s \in V$ (or $\mathfrak{p}$), designating those $S \in V$ which are localizations of s. Subvarieties in the usual more restricted sense appear in [GA] 169. as *figure in una varietà* resp. in [IA] p. 18 as *arithmetische Gebilde* in the following manner :

Definition: A *figure* $\mathfrak{a}$ in a variety V is a prescription assigning an ideal $\mathfrak{a}(s)$ to each $s \in V$ such that

$$\mathfrak{a}(S) = \mathfrak{a}(s)\,S$$

holds for each $s \subset S \in V$.

Related notions are :

- A figure $\mathfrak{a}$ *passes through a perspective* $\mathfrak{p}$ and has $s/\mathfrak{a}(s)$ as *its aspect*, if $s \in V$ with $\mathfrak{a}(s) \neq s$. In this case $\mathfrak{p}$ is called *a perspective of* $\mathfrak{a}$.

The calculus of ideals translates into a calculus of figures, in which Kähler uses a terminology reflecting the geometric background.

Let $\mathfrak{a}, \mathfrak{b}, \ldots, \mathfrak{c}$ be figures in V.

- Their *intersection* is defined by the figure

$$\mathfrak{a} + \mathfrak{b} + \cdots + \mathfrak{c}$$

  with ideals

$$(\mathfrak{a} + \mathfrak{b} + \cdots + \mathfrak{c})(s) = \mathfrak{a}(s) + \mathfrak{b}(s) + \cdots + \mathfrak{c}(s) \quad \text{for} \quad s \in V.$$

- Their *union* is the figure

$$\mathfrak{a} \cap \mathfrak{b} \cap \cdots \cap \mathfrak{c}$$

  defined by

$$(\mathfrak{a} \cap \mathfrak{b} \cap \cdots \cap \mathfrak{c})(s) = \mathfrak{a}(s) \cap \mathfrak{b}(s) \cap \cdots \cap \mathfrak{c}(s).$$

- Their *product* is the figure

$$\mathfrak{a} \cdot \mathfrak{b} \cdot \ldots \cdot \mathfrak{c}$$

defined by

$$(\mathfrak{a} \cdot \mathfrak{b} \cdot \ldots \cdot \mathfrak{c})(s) = \mathfrak{a}(s) \cdot \mathfrak{b}(s) \cdot \ldots \cdot \mathfrak{c}(s) .$$

- The *division* of figures $\mathfrak{a}, \mathfrak{b}$ in V defines via

$$(\mathfrak{a} : \mathfrak{b})(s) := \mathfrak{a}(s) : \mathfrak{b}(s) = \{c \in s \,,\; c \cdot \mathfrak{b}(s) \subset \mathfrak{a}(s)\}$$

a figure if V is noetherian, i. e. all $s \in V$ are noetherian.

- A figure $\mathfrak{a}$ *contains* a figure $\mathfrak{b}$

$$\mathfrak{a} \geq \mathfrak{b}$$

if and only if

$$\mathfrak{a}(s) \subset \mathfrak{b}(s)$$

holds for all $s \in V$.

- A figure $\mathfrak{a}$ is called a *primary* or a $\mathfrak{P}$*-primary* figure if there is an $S \in V$ and a $\mathfrak{P}$-primary ideal $\mathfrak{Q} \subset S$ with

$$\begin{aligned} \mathfrak{a}(s) &= \mathfrak{Q} \cap s \quad \text{for} \quad s \subset S \\ &= s \qquad\quad \text{for} \quad s \not\subset S . \end{aligned}$$

Such a figure is called *prime* if and only if $\mathfrak{P} = \mathfrak{Q}$.
We write $\mathfrak{a} = \mathfrak{Q}$ resp. $\mathfrak{a} = \mathfrak{P}$.

- The figure $\mathfrak{t}$ defined by

$$\mathfrak{t}(s) = \{0\} \quad \text{for all} \quad s \in V$$

is called the *total figure*(*figura totale*).

Each ideal $\mathfrak{a}_0$ of $s_0 \in V$ generates a figure $(\mathfrak{a}_0)$ defined by

$$(\mathfrak{a}_0)(s) := [\mathfrak{a}_0 \cap s] \cdot s$$

([GA] 181.). The figure $(\mathfrak{q})$ defined by a $(\mathfrak{P} \cap s)$-primary ideal $\mathfrak{q}$ in $s \in V$ coincides with the $\mathfrak{P}$-primary figure $\mathfrak{Q}$ defined above (with $\mathfrak{Q}(S) = \mathfrak{q} \cdot S$). Let $\mathfrak{p}$ be a perspective of $\mathfrak{a}$, i. e. $\mathfrak{a}(s) \neq s$. Then in [GA] 186., the figure

$(\mathfrak{p})$ generated by $\mathfrak{p}$ is called a *prime figure of* $\mathfrak{a}$. Moreover,
$\mathfrak{p}$ is an *essential perspective* of $\mathfrak{a}$, if $\mathfrak{a}(s)$ is a genuine subset of

$$s \bigcap_{S>s} \mathfrak{a}(S)$$

and
$\mathfrak{p}$ is an *extremal perspective* of $\mathfrak{a}$, if

$$\mathfrak{a}(s) \neq s\,, \quad \mathfrak{a}(S) = S \quad \text{for} \quad S \in V(s) \quad \text{with} \quad S \neq s\,.$$

In [GA] 188., 189. and 193., we find the following criterion translating these notions into the usual ideal theory.
A necessary and sufficient condition for a primary noetherian local ring s to be essential for a figure $\mathfrak{a}$ is that its maximal ideal $\mathfrak{p}$ is a prime divisor of the ideal $\mathfrak{a}(s)$. And s is extremal exactly if $\mathfrak{a}(s)$ is $\mathfrak{p}$-primary, i. e. if there is an $\ell \in \mathbf{N}$ with

$$\mathfrak{p}^{\ell} \subset \mathfrak{a}(s) \subset \mathfrak{p}\,.$$

In [GA] 195. - 249., tools from local and commutative algebra are assembled under the titles *Cono tangente, Funzione caratteristica di una figura*, *Subordinazione di prospettive* and *Prospettive regolari*. We will reproduce here only some of this.

- The *width* (= *ampiezza*) of a perspective $\mathfrak{p}$ is the minimal number of elements needed to generate the maximal ideal $\mathfrak{p}$ of the local ring s. Let $p_1, \ldots, p_h$ be such elements generating $\mathfrak{p}$. They are then called a *minimal basis* of $\mathfrak{p}$ ([GA] 195.).

- A homogeneous form $\varphi(u_1, \ldots, u_h) \in s/\mathfrak{p}[u]$ of degree m is *tangential* to $x \in s$ if one has an $f \in s[u_1, \ldots, u_h]$ of degree m with

$$\varphi(u_1, \ldots, u_h) - f(u_1, \ldots, u_h) \in \mathfrak{p}[u_1, \ldots, u_h]$$

  such that

$$f(p_1, \ldots, p_h) \in x + \mathfrak{p}^{m+1}\,.$$

- Let $\mathfrak{a}(s)$ be an ideal in s. The ideal $\overline{\mathfrak{a}(s)}$ generated in $s/\mathfrak{p}[u]$ by the forms tangential to elements of $\mathfrak{a}(s)$ is homogeneous and consists entirely of elements tangential to elements of $\mathfrak{a}(s)$. It generates a figure $\overline{\mathfrak{a}(s)}$ in the affine variety $V = V(A)$, $A = s/\mathfrak{p}[u_1, \ldots, u_h]$ which is called the *tangential cone of the figure* $\mathfrak{a}$ *in* φ ([GA] 197.).

- The characteristic function of a figure $\mathfrak{a}$ in a variety V is defined for $m \in \mathbf{N}_0$ by ([GA] 207.)

$$\chi(\mathfrak{a}(s), m) := \chi(\overline{\mathfrak{a}(s)}, m) ,$$

where, on the right, we have the classical characteristic function of the homogeneous ideal $\mathfrak{a}_h := \overline{\mathfrak{a}(s)}$ in A, i. e.

$$\chi(\mathfrak{a}_h, m) = \binom{m+h}{h} - \sum_{\ell=0}^{m} \varphi(\mathfrak{a}_h, \ell) ,$$

$\varphi(\mathfrak{a}_h, \ell) := \dim_{s/\mathfrak{p}} M_\ell(\mathfrak{a}_h), M_\ell(\mathfrak{a}_h)$ the space of ℓ-forms in $\mathfrak{a}_h$.

By Hilbert's famous theorem, reproduced in [GA] 206., the characteristic function of the homogeneous ideal $\mathfrak{a}_h$ is asymptotically a polynomial in m of degree $\dim \mathfrak{a}_h$. Here and in the sequel the following notions and symbols are used.

- The *dimension* $\dim \mathfrak{a}$ *of a figure* in a variety V (algebraic over a field k) is given by ([GA] 202.)

$$\dim \mathfrak{a} := \max \operatorname{trdeg}_{k+\mathfrak{P}/\mathfrak{P}} S/\mathfrak{P} \quad \text{for all} \quad S \in V \, , \ \mathfrak{a}(S) \neq S \, .$$

- The *dimension* of the tangential cone $\overline{\mathfrak{a}(s)}$ in $\mathfrak{p}$ is denoted by ([GA] 227.)

$$\dim \frac{\mathfrak{a}}{\mathfrak{p}}$$

and the *order* of a figure $\mathfrak{a}$ in $\mathfrak{p}$

$$\operatorname{ord} \frac{\mathfrak{a}}{\mathfrak{p}}$$

is defined by

$$\max \operatorname{ord} \frac{\mathfrak{P}}{\mathfrak{p}} \quad \text{for all} \quad S \in V(s) ,$$

where the *order of subordination* $\operatorname{ord} \dfrac{\mathfrak{P}}{\mathfrak{p}} = \ell$ denotes the maximal number of perspectives in an ascending chain ([GA] 213.)

$$\mathfrak{p} = \mathfrak{p}_0 < \mathfrak{p}_1 < \ldots < \mathfrak{p}_\ell = \mathfrak{P} .$$

- For two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in s with $\mathfrak{a} \subset \mathfrak{b}$,

$$\left(\frac{\mathfrak{a}}{\mathfrak{b}}\right) = \ell$$

denotes the length of a maximal non-refinable ascending chain of ideals ([GA] 208.)

$$\mathfrak{a} = \mathfrak{a}_0 \subsetneqq \mathfrak{a}_1 \subsetneqq \cdots \subsetneqq a_\ell = \mathfrak{b} ,$$

if such a chain exists, otherwise it is ∞.

Here we have the fundamental criterium ([GA] 209.)

$$\left(\frac{\mathfrak{a}}{\mathfrak{a} + xs}\right) = 1 \quad \text{for} \quad x \in s \setminus \mathfrak{a} , \quad x\mathfrak{p} \subset \mathfrak{a} .$$

And for local rings $s \subset S$, Kähler uses ([GA] 259.) the abbreviations

$$\begin{aligned} \deg \frac{S}{s} &:= \left(\frac{\mathfrak{P}}{S}\right)_s = \deg_{s+\mathfrak{P}/\mathfrak{P}} S/\mathfrak{P} , \\ \operatorname{mult} S &:= \left(\frac{0}{S}\right)_S \end{aligned}$$

and

$$\operatorname{rank} \frac{S}{s} := \left(\frac{0}{S}\right)_s .$$

where here the index s or S indicates that one takes chains of s-resp. S-modules.
The main results in this context relating these symbols, are the following :

Proposition ([GA] 210.). The characteristic function of a figure $\mathfrak{a}$ is

$$\chi\left(\mathfrak{a}\left(s\right), m\right) = \left(\frac{\mathfrak{a}(s) + \mathfrak{p}^{m+1}}{s}\right) .$$

Proposition ([GA] 229. and 230.). If s is noetherian and $\overline{\chi}$ the polynomial representing $\chi\left(\mathfrak{a}(s), m\right)$ for large m, we obtain for its degree

$$\deg \overline{\chi} = \dim \frac{\mathfrak{a}}{\mathfrak{p}} = \operatorname{ord} \frac{\mathfrak{a}}{\mathfrak{p}} .$$

A local ring s of width h, regular in the usual sense, is characterized by ([GA] 234.)

$$\chi\left(0\left(s\right), m\right) = \left(\frac{\mathfrak{p}^{m+1}}{s}\right) = \binom{m+h}{h} .$$

Here $0(s)$ denotes the ideal $\{0\}$ in s.

And as an example of the results obtained in the section about regular local rings, we cite :

Proposition ([GA] 242.). If $A = k[x_1, \ldots, x_m]$ generates a field K with $\mathrm{trdeg}_K K = m$, each local ring s with base A (i. e. localization of A) is regular with

$$\mathrm{ord}\, \frac{0}{\mathfrak{p}} = m - \mathrm{trdeg}_{k+\mathfrak{p}/\mathfrak{p}}\, s/\mathfrak{p}\ .$$

In [GA] 246., Kähler introduces the notion of *arithmetic dimension* of a finitely generated field K.

Definition.

$$\begin{aligned} \dim_a K &= \mathrm{trdeg}_{\mathbf{F}_p} K, && \text{if} \quad \mathrm{char}\, K = p \neq 0 \\ &= \mathrm{trdeg}_{\mathbf{Q}} K + 1, && \text{if} \quad \mathrm{char}\, K = 0\ . \end{aligned}$$

Then the proposition above has as its arithmetic counterpart

Proposition ([GA] 247.). If $A = \mathbf{Z}[x_1, \ldots, x_m]$ generates a field K of arithmetic dimension m, each local ring s with base A is regular with

$$\mathrm{ord}\, \frac{0}{\mathfrak{p}} = \dim_a K - \dim_a s/\mathfrak{p}\ .$$

And more generally, we have

Proposition ([GA] 248.). In an arithmetic variety

$$\mathrm{ord}\, \frac{\mathfrak{P}}{\mathfrak{p}} = \dim_a S/\mathfrak{P} - \dim_a s/\mathfrak{p}\ .$$

B 4. Kähler's extension of a variety

Chapter VII of the Geometria Aritmetica has the title

Varietà extensioni

In the French text [CPA] Kähler chose the term *amplification* and in the German text [IA] he uses *Überlagerung*. We will stick here to this and use the word *covering*. But, as Weil remarks in his review [WGA], the following concept corresponds to that of a proper morphism in modern algebraic geometry :

In [GA] 250., V_1 is a primary variety.

Definition. V_1 is a *covering* of a variety V_0 if it fulfills the following two conditions :

I. Every $s_1 \in V_1$ extends (i. e. dominates) an $s_0 \in V_0$.

II. Every local ring s which is an extension of an $s_0 \in V_0$ and compatible with the object of V_1 has a common extension s' together with an $s_1 \in V_1$.

Or, as Weil puts it : Let $V_i\,(i = 0,1)$ be two varieties such that their objects $R_i = R\,(V_i)$ (i. e. the full quotient rings of the local rings of the varieties V_0 resp. V_1) are primary rings with $R_1 \supset R_0$. Assume that to every $s_1 \in V_1$ there is $s_0 \in V_0$ (which is then necessarily unique) such that s_1 extends (= dominates) s_0 and write $s_0 = f\,(s_1)$. Assume also that, whenever a local ring $s \subset R_1$ extends a ring $s_0 \in V_0$, there is also a local ring $s_1 \in V_1$ and a local ring $s' \subset R_1$ that extends both s and s_1 . Then V_1 is said to extend V_0 or - as we will say here - V_1 *covers* V_0 .

Moreover, we will also use the following terminology

s_0 is the projection of s_1

$\mathfrak{p}_0$ is the perspective under $\mathfrak{p}_1$

as well as

V_1 lies over V_0 ,

and denote this by

$$s_1 \to s_0\,,\ \mathfrak{p}_1 \to \mathfrak{p}_0, \quad \text{resp.} \quad V_1 \to V_0\ .$$

In [GA] 256., Kähler extends these terms also to a more general situation : he calls V_1 a *complete extension* of V_0 , if V_1 and V_0 are not necessarily varieties but simply collections of local rings fulfilling the conditions I and II from the last definition and, additionally, the condition that each $s_1 \in V_1$ extends only one $s_0 \in V_0$. The *complete extension* $\cup s_i$ *of* $s_0 \in V_0$ consisting of those $s_i \in V_1$ which extend (= dominate) s_0 appears here as an example.

The most important fact about coverings is

Proposition 1 ([GA] 252.). The covering of a closed variety V_0 (over the ring A) is closed (over A).

Every figure $\mathfrak{a}$ in a variety V_0 extends uniquely to a figure in a covering variety V_1, again named $\mathfrak{a}$, defined by

$$a\,(s_1) \;=\; \mathfrak{a}\,(s_0)\cdot s_1 \quad \text{for} \quad s_1 \to s_0\ .$$

In the sections [GA] 259. - 268., Kähler discusses intensely facts about the coverings to be obtained for the symbols rank, deg and mult, already presented here in our section B 3. In a sense these are generalizations of the famous Hurwitz formula for the covering of a curve by another.

The starting point is the relation ([GA] 259.)

$$\operatorname{rank}\frac{S}{s} \;=\; \deg\frac{S}{s}\,\operatorname{mult} S \quad \text{for} \quad S \to s\ .$$

The following results are obtained by some substantial work with ideal lengths.

Proposition 2 ([GA] 261.). Let s_0 be a local ring contained in a primary ring A with finite chain length $\left(\dfrac{\mathfrak{p}_0 A}{A}\right)_{s_0}$. Then one has

$$\left(\frac{\mathfrak{p}_0 A}{A}\right)_{s_0} \;=\; \sum_s \left(\frac{\mathfrak{p}_0 s}{s}\right)_s\ ,$$

where the sum is extended over all extensions $s \to s_0$ with base A.

Proposition 3 ([GA] 266., 267.). Let $\cup_{i=1}^h s_i$ be the complete extension of a perfect local ring s_0 (i. e. regular of dimension one), contained in a variety V, algebraic over s_0 with object S which is primary of finite rank over the field (s_0). Let $\tau : S \to S/\mathfrak{P}$ be the residue homomorphism and s_0 arithmetic (i. e. essentially of finite type over $\mathbf{Z}$) or S^τ separable over (s_0^τ). Then there exists a ring A in S with finite basis over s_0 such that the variety $V\,(A)$ contains a complete extension of $\cup_i s_i$ and thus of s_0. And one has the relation

$$\sum_{i=1}^h \left(\frac{\mathfrak{p}_0 s_i}{s_i}\right)_{s_0} \;=\; \operatorname{rank}\frac{S}{(s_0)}$$

or equivalently

$$\sum_{i=1}^h \left(\frac{\mathfrak{p}_0 s_i}{s_i}\right) \deg_{s_0+\mathfrak{p}/\mathfrak{p}} s_i/\mathfrak{p}_i \;=\; \left(\frac{0}{s}\right) \deg\frac{s}{(s_0)}\ .$$

As a corollary to the proof of this Proposition in [GA] 268., Kähler mentions the following Nakayama-type statement.

Lemma. Let s_0 be a perfect (i. e. regular of dimension one) local ring, A an s_0-module with finite s_0-basis such that

$$A = \sum_{i=1}^{n} s_0\, a_i + \mathfrak{p}_0\, A \quad \text{with} \quad a_i \in A \quad (i = 1, \ldots, n) \; .$$

Then one concludes

$$A = \sum_{i=1}^{n} s_0\, a_i$$

and in case that A is a primary ring, moreover,

$$A = A \cdot (s_0) \; .$$

Results of the section *Anelli individuali* ([GA] 269.-282.) are already reported on in our section A 3. The section *Differenti relativi* ([GA] 283. - 304.) culminates in the criteria for the regularity of a local ring which are reproduced in the section 2.1 of the main part of this text.

The section *Molteplicità di una figura* ([GA] 305. - 323.) centers around the definition of a (relative) multiplicity and finishes with a theorem sharpening the Proposition 3 above :

Let $\mathfrak{a}$ be a figure in a variety V, containing the noetherian ring s, i. e. $\mathfrak{a}\,(s) \neq s$, and let s extend a regular local ring s_0

Definition ([GA] 314.). The *multiplicity* of $\mathfrak{a}$ in $\mathfrak{p}$ over $\mathfrak{p}_0$ is defined by

$$\frac{\mathfrak{a}}{\mathfrak{p}_0}\,(s) \;:=\; \lim_{n\to\infty} \left(\frac{\mathfrak{a}(s) + \mathfrak{p}_0^n\, s}{s} \right) \Big/ \left(\frac{\mathfrak{p}_0^n}{s_0} \right) \; .$$

This multiplicity is a natural number or zero. It is additive in the following sense

Proposition 4 ([GA] 315.). For a primary decomposition

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i \; , \quad \mathfrak{q}_i \quad \mathfrak{p}_i - \text{primary with} \quad \mathfrak{p}_i \neq \mathfrak{p}_j \quad \text{for} \quad i \neq j \; ,$$

we have

$$\frac{\mathfrak{a}}{\mathfrak{p}_0} = \sum_{i=1}^{h} \frac{\mathfrak{q}_i}{\mathfrak{p}_0} \; .$$

The multiplicity is simply a length of an ideal chain in the following special case :

Proposition 5 ([GA] 321.). If $\mathfrak{a}\,(s)$ is $\mathfrak{P}(s)$-primary and $\mathfrak{p}_0$ perfect (i. e. s_0 regular and one-dimensional), then

$$\begin{aligned}\frac{\mathfrak{a}}{\mathfrak{p}_0}\,(s) \;&=\; \left(\frac{\mathfrak{a}(s)+\mathfrak{p}_0 s}{s}\right) \quad \text{for} \quad \mathfrak{p} \neq \mathfrak{P}\,(s), \\ &= 0 \qquad\qquad\qquad\quad \text{for} \quad \mathfrak{p} \;=\; \mathfrak{P}\,(s)\;.\end{aligned}$$

As the final result in this topic in [GA] 322., Kähler proves the following :

Theorem. Let V be a primary variety covering the arithmetic variety V_0 and $s_0 \in V_0$ a perfect local ring contained in a local ring $S_0 \in V_0$. Then one has for a figure $\mathfrak{a}$ in V the relation

$$\sum_{\mathfrak{p}\to\mathfrak{p}_0} \frac{\mathfrak{a}}{\mathfrak{p}_0}\,(s)\deg_{s_0+\mathfrak{p}/\mathfrak{p}} s/\mathfrak{p} \;=\; \sum_{\mathfrak{P}\to\mathfrak{P}_0} \left(\frac{\mathfrak{a}(S)}{S}\right) \deg_{S_0+\mathfrak{P}/\mathfrak{P}} S/\mathfrak{P}\;,$$

provided that the terms of the sum on the left side are finite. The sums are running here over the complete extensions in V of s_0 resp. S_0.

As Kähler remarks at the end of his (rather long) proof, on the right side of the formula, the term $\left(\frac{\mathfrak{a}(S)}{S}\right)$ can be replaced by the multiplicity $\frac{\mathfrak{a}}{\mathfrak{P}_0}\,(S)$. In [GA] 323., there is also an analogous version for the case of varieties algebraic over a ground field. But unfortunately, as Weil justly complains in his review, Kähler gives no indication of the application he had in mind in deriving these results. At least, some light is shed on this by comparing with the nowadays usual introduction of the multiplicity of a local ring as initiated in 1960 in Chapter VIII of the second volume of the Commutative Algebra by Zariski and Samuel [ZS] :

Taking over the notation used here, the following terms are attached to a local ring s with maximal ideal $\mathfrak{p}$

- the characteristic function $P_{\mathfrak{p}}$ defined by

$$P_{\mathfrak{p}}\,(m) \;=\; \left(\frac{\mathfrak{p}^m}{s}\right)\;,$$

- the *Hilbert-Samuel polynomial* $\overline{P_{\mathfrak{p}}}$ defined

$$\overline{P_{\mathfrak{p}}}\,(m) \;=\; P_{\mathfrak{p}}\,(m) \quad \text{for large} \quad m$$

  and of degree $\deg \overline{P_{\mathfrak{p}}} = \dim s = n$,

- the multiplicity $e(\mathfrak{p})$ of s defined as

$$e(\mathfrak{p}) \;=\; n\,! \cdot \text{leading coefficient of } \overline{P_{\mathfrak{p}}}\;.$$

In [ZS] Vol II, p. 297, the treatment of the multiplicity culminates in a result comparable to the Proposition 3 above and the last Theorem. As is stated in [ZS], this theorem is the algebraic counterpart of the projection formula for the intersection cycles in Algebraic Geometry.

Theorem ([ZS] Theorem 24). Let s be a local ring, $\mathfrak{p}$ its maximal ideal, $\mathfrak{q}$ an ideal in s which is primary for $\mathfrak{p}$, and B an overring of s which is a finite s-module. Then B is a semi-local ring, and $B\mathfrak{q}$ is an open ideal in B, i. e. containing a power of the intersection of the maximal ideals $\mathfrak{p}_i, i = 1, \ldots, h$ of B. Let $\mathfrak{q}_i$ be the primary component of $B_{\mathfrak{q}}$ relative to $\mathfrak{p}_i$. If no element $\neq 0$ in s is a zero divisor in B, then the polynomials

$$[B : A]\,\overline{P_{\mathfrak{q}}}\,(m) \quad \text{and} \quad \sum_i\,[B/\mathfrak{p}_i : s/\mathfrak{p}]\,\overline{P_{\mathfrak{q}_i}}\,(m)$$

have the same degree and the same leading term.

Here $[B : s]$ denotes the maximum number of elements of B which are linearly independent over s. The ressemblance to Kähler's results becomes even more apparent by the Corollary 1 which Zariski and Samuel add here: Under the same hypothesis and notations as in the Theorem above, suppose furthermore that all the local rings $B_{\mathfrak{p}_i}$ have the same dimension as s. Then

$$[B : s]\,e(\mathfrak{q}) \;=\; \sum_i\,[B/\mathfrak{p}_i : s/\mathfrak{p}]\,e(\mathfrak{q}_i)\;.$$

In particular, this applies to the case where s is a local domain of dimension 1 and B is also a domain. Then, in the special case $\mathfrak{q} = \mathfrak{p}$ with

$$e(\mathfrak{q}_i) \;=\; e_i\;, \quad [B/\mathfrak{p}_i : s/\mathfrak{p}] \;=\; n_i\,,\; [B : s] \;=\; n\,,$$

the formula above reduces to the famous relation

$$n\,e(\mathfrak{p}) \;=\; \sum_i\, e_i\,n_i\;.$$

The remaining Chapters of the Geometria Aritmetica

Ch VIII	*Divisori*
Ch IX	*La totale integrità di un corpo*
Ch X	*Funzioni modulari ed il calcolo zeta*

are here to some extend covered by the main text about differentials resp. (the *Zeta Calculus*) by a separate exposition.

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