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An Index Theorem For Open Manifolds Whose Ends Are Warped Products

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An Index Theorem For Open Manifolds Whose Ends Are Warped Products

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Abstract. We give an index formula for the Dirac operator acting on L^2-sections of the spinor bundle of an open spin manifold with a co-compact cylindrical end over which the metric is a warped product.

Keywords: Index Theory, Dirac operator, warped product.

MS Classification: 58G10

§ 1 Introduction Let M be a closed manifold and let D be an elliptic differential operator on M, then D is a Fredholm operator with finite index which was famously shown to be a topological invariant of M by Atiyah and Singer [AS63] in the early 1960's. In this way, many $\mathbb{Z}$-valued topological invariants of a compact manifold M (for instance its Euler characteristic $\chi(M)$ or the signature $\sigma(M)$ of a compact $4k$-manifold M) can be rediscovered as the indices of certain (twisted) Dirac operators which arise naturally in Differential Geometry.

The situation on manifolds with boundary and open manifolds is rather different. For an elliptic differential operator to be Fredholm and to have finite index requires suitable boundary conditions. If the index is well-defined and finite it will not generally have a topological interpretation, in contrast with the compact case; instead it will depend on the choice of metric and boundary conditions.

For compact manifolds M with boundary, Atiyah, Patodi and Singer [APS75] and, in slightly more general form, Gilkey [Gil93] have given natural boundary conditions for the Dirac operator which ensure that its index is finite and can be described in terms of the $\hat{A}$-form and certain invariants of the Dirac operator D_K on the boundary. More recently, the Dirac operator on open manifolds has been studied by Bunke [Bun91b, Bun92] who introduces the notion of "bounded geometry" to define an index which has a description in terms of suitable heat kernels.

In this paper we focus on the particularly simple class of open spin manifolds $\tilde{M}$ with cylindrical, co-compact ends $M \subset \tilde{M}$ over which the metric is a warped product. The underlying motivation is to study a situation simple enough to be treated in an elementary and explicit manner, while still capturing some non-trivial and interesting examples, such as real hyperbolic space $\mathbb{R}H^n$ and certain hyperbolic manifolds of finite volume. The Dirac operator on such cylinder manifolds can be studied in a fairly straightforward manner, and we obtain a simple index formula (Theorem 17) by combining Gilkey's index theorem for compact bounded manifolds with an explicit study of the warped product metric over the cylindrical end. As one expects, the behaviour of the index is similar to the case of compact manifolds with boundary if the warp function satisfies suitable upper growth bounds, while there are no harmonic spinors in L^2 if the warp function grows fast.

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§ 2 The Dirac operator on cylinder manifolds We begin with the definitions of cylinder manifolds with warped product metrics, and we collect the necessary facts about their spinor bundles and Dirac operators.

Definition 1 *A cylinder-manifold is a complete connected open manifold $\tilde{M}$ of the form $K \cup M$ where M is diffeomorphic to a cylinder $\mathbb{R}_0^+ \times N$ and K is a compact manifold with smooth boundary $\partial K = \{0\} \times N = \partial M$.*

We shall be interested in the case where $\tilde{M}$ and hence M and N are spin and the metric over M is a warped product:

Definition 2 *A* <u>*warped product metric*</u> *on a cylinder $I \times N$, where $I \subset \mathbb{R}$ is some interval, is a metric of the form*

$$g(t,p) = dt^2 + \rho(t)^2 g_N(p), \quad (t,p) \in I \times N. \tag{1}$$

Here $\rho : I \to \mathbb{R}^+$ is a smooth function and g_N is a fixed metric on N.

An important example is real hyperbolic space $\mathbb{R}H^n$ where we take $K = \overline{B_\varepsilon(x)}$ a closed ball around some $x \in \mathbb{R}H^n$ so that $N = \partial K \cong S^{n-1}$ and $M \cong [\varepsilon, \infty) \times S^{n-1}$. The hyperbolic metric on M is then a warped product

$$g_{\mathbb{R}H^n}|M = dt^2 + \sinh(t)^2 g_{S^{n-1}}.$$

We now refer to Bär [Bär96] for the following facts about the structure of the spinor bundle ΣM over M. Since we are interested in the index of the Dirac operator we assume that $\dim(M) = 2m$ is even so that

$$\Sigma M = \Sigma^+ M \oplus \Sigma^- M$$

is $\mathbb{Z}_2$-graded and a non-trivial index can be defined. Let

$$\mathrm{Cl}(\mathbb{R}^n) = \mathrm{Cl}^0(\mathbb{R}^n) \oplus \mathrm{Cl}^1(\mathbb{R}^n)$$

be the Clifford algebra of $\mathbb{R}^n$ and

$$\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$$

the complex spinor module where $\Sigma_n^\pm \cong \Sigma_{n-1}$. The complex spinor representation of $\mathrm{Cl}(\mathbb{R}^n)$ on Σ_n is Clifford multiplication which we write as

$$(x, \psi) \mapsto x \cdot \psi.$$

If we choose an orthonormal basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$ and identify $\mathbb{R}^{n-1} \cong \mathrm{Span}\{e_1, \ldots, e_{n-1}\}$, the map $x \mapsto x \cdot e_n$ extends to an isomorphism $\mathrm{Cl}(\mathbb{R}^{n-1}) \cong \mathrm{Cl}^0(\mathbb{R}^n)$. This allows us to consider Σ_n as a module for $\mathrm{Cl}(\mathbb{R}^{n-1})$ via Clifford multiplication which is now defined as

$$(x, \psi) \mapsto x \cdot e_n \cdot \psi.$$

However, Σ_n is not irreducible in this case, but the summands $\Sigma_n^\pm$ are the two irreducible spinor representations for $\mathrm{Cl}(\mathbb{R}^{n-1})$. Carrying this over to vector bundles, we may view

$$\Sigma M|N \cong \Sigma N \oplus (-\Sigma N)$$

as the direct sum of the two possible spinor bundles over N. This isomorphism being canonical we may extend this identification to each fibre $N_t = \{t\} \times N$ of the cylinder $\mathbb{R}_0^+ \times N$ and hence view both $\nabla^{\Sigma M}$, D_M and $\nabla^{\Sigma N}$, D_N, the spinor connections and Dirac operators over M and N, respectively, as operators acting on sections of ΣM.

In Proposition 3 to Proposition 5 we collect some elementary formulas which we will need later on; their proofs are obvious and will be omitted.

Proposition 3 *Let $M = I \times N$ with $I \subset \mathbb{R}$ be a cylinder with warped product metric $g = dt^2 + \rho(t)^2 g_N$. Let $\mathbf{n}$ be the "outward" unit normal vector field of N. Then the form operator of N in M is*

$$S(x) = -\nabla_x \mathbf{n} = -\frac{\dot{\rho}(t)}{\rho(t)} x. \tag{2}$$

The mean curvature of N in M depends only on $t \in I$ and is given by

$$H(t) = -\frac{\dot{\rho}(t)}{\rho(t)}. \tag{3}$$

The following identities can be found in [Bär96].

Proposition 4 *For each $\psi \in C^\infty(\Sigma M)$ and $x \in C^\infty(TM)$*

$$\nabla_x^{\Sigma M} \psi = \nabla_x^{\Sigma N} \psi + \frac{1}{2} S(x) \cdot \mathbf{n} \cdot \psi.$$

Combining these results with (2), (3) one obtains the formula

$$-\mathbf{n} \cdot D_M^{\pm} \psi = \pm D_{N_t} \psi - \frac{n-1}{2} H(t) \psi + \nabla_{\mathbf{n}}^{\Sigma M} \psi \tag{4}$$

where $H(t)$ denotes the mean curvature of N_t in M which is constant along the fibres.

See also [BB01] for a study of Dirac operators on cylinder manifolds.

Proposition 5 *For a vector field $x \in C^\infty(TM)$ let $x^* \in C^\infty(T^*M)$ be the (metrically) dual 1-form $x^* = g(x, \cdot) = \iota_x g$. Define $\alpha \in C^\infty(\Lambda^2 TM \otimes \mathrm{End}(TM))$ by*

$$\alpha(x, y) = x^* \otimes y - y^* \otimes x.$$

Then the $\mathrm{End}(TM)$-valued curvature forms of M and N satisfy

$$\Omega^M(x, y) = \begin{cases} \Omega^N(x, y) + \frac{\dot{\rho}(t)^2}{\rho(t)^2}\, \alpha(x, y) & \text{for} \quad x, y \in C^\infty(TN) \\ -\frac{\ddot{\rho}(t)}{\rho(t)}\, \alpha(\mathbf{n}, y) & \text{for} \quad x \perp N, y \in C^\infty(TN). \end{cases} \tag{5}$$

§ 3 Harmonic spinors on warped products Let $\mu_1, \mu_2, \mu_3, \ldots$ be the eigenvalues of D_N (operating on $\Sigma^+ M|N$) with corresponding eigenspinors $\sigma_1, \sigma_2, \sigma_3, \ldots$ (in any order). We assume the $(\sigma_j)_j$ to be normalised to $\|\sigma_j\|_{L^2} = 1$. Extending them by parallel transport from $N = N_0$ along the axis of our cylinder to N_t for $t > 0$ we get sections $(\sigma_j(t,p))_{j \in \mathbb{N}}$ of $\Sigma^+ M$ which, for all fixed $t \geq 0$, constitute an L^2-eigenbasis of D_{N_t} in $\Sigma^+ M|N_t$ with eigenvalues $\mu_j(t) = \mu_j/\rho(t)$, indexed by $j \in \mathbb{N}$. Dually we have $\sigma_{-j}(t,p) = \mathbf{n} \cdot \sigma_j(t,p)$ in $\Sigma^- M$ which restrict to eigenspinors for $-D_{N_t}$ (acting on $\Sigma^- M|N_t$) over each fibre N_t with eigenvalues $\mu_{-j}(t) = -\mu_j/\rho(t)$. The family $(\sigma_j(t,\cdot))_{j \in \mathbb{Z}-\{0\}}$ is an L^2-orthonormal basis for the sections of $\Sigma M|N_t$ for each $t > 0$ and therefore we may write any spinor $\psi \in C^\infty(\Sigma M)$ in the form

$$\psi(t,p) = \sum_{j \in \mathbb{Z}-\{0\}} \alpha_j(t) \sigma_j(t,p) \tag{6}$$

for certain functions $\alpha_j : \mathbb{R}_0^+ \to \mathbb{C}$.

Theorem 6 *Let $\psi_0 \in C^\infty(\Sigma M|N_0)$. Then there is a unique $\psi \in C^\infty(\Sigma M)$ with $D\psi = 0$ and $\psi|N_0 = \psi_0$.*

Proof. We make an ansatz (6) for ψ. Using (4), the equation $D\psi = 0$ is equivalent to the following family of ordinary differential equations for the coefficient functions:

$$\dot{\alpha}_j(t) + \left(\frac{\mu_j}{\rho(t)} + \frac{n-1}{2} \frac{\dot{\rho}(t)}{\rho(t)} \right) \alpha_j(t) = 0, \quad j \in \mathbb{Z} - \{0\}.$$

These can be integrated, and we obtain the unique solution

$$\alpha_j(t) = \alpha_j(0) \exp\left(-\mu_j \int_0^t \frac{ds}{\rho(s)} \right) \rho(t)^{-(n-1)/2} \tag{7}$$

where the constant $\alpha_j(0) = (\sigma_j, \psi_0)_{L^2(\Sigma M|N)}$ is determined by ψ_0. □

The following proposition is an immediate consequence of Theorem 6.

Proposition 7 *The L^2-norm of the spinor ψ over M is given by*

$$\|\psi\|_{L^2}^2 = \sum_{j \in \mathbb{Z}-\{0\}} |\alpha_j(0)|^2 \int_0^\infty \exp\left(-2\mu_j \int_0^t \frac{ds}{\rho(s)} \right) dt. \tag{8}$$

Proof. This follows from (7) and the fact that the volume elements of (N, g_N) and (M, g) with the warped product metric (1) are related by

$$\mathrm{vol}_M(t,p) = \varrho(t)^{\frac{n-1}{2}}\,\mathrm{vol}_N \wedge \mathrm{d}t.$$

$\square$

It is now easy to see that a necessary condition for this term to be finite is that $\alpha_j(0) = 0$ whenever $\mu_j \leq 0$. In addition, we need a bound on the growth of $\rho(t)$ so that the argument of the exponential function approaches $-\infty$ fast enough for the integral to converge. We illustrate this in the following remark.

Remark 8 Let us look at the example of a warp function

$$\rho(t) = (1 + Ct)^{\varepsilon}, \quad \text{with } C, \varepsilon > 0.$$

1. For $\varepsilon < 1$, a harmonic spinor on K has an L^2-extension to $\tilde{M}$ if and only if its decomposition (6) over $\partial K = N$ involves only eigenspinors with strictly positive eigenvalues $\mu_j > 0$.

2. If $\varepsilon > 1$ there are no harmonic spinors in $L^2(\Sigma\tilde{M})$.

3. Assume $\varepsilon = 1$. Then (8) becomes

$$\|\psi\|^2_{L^2} = \sum_{j \in \mathbb{Z}\setminus\{0\}} |\alpha_j(0)|^2 \int_0^\infty (1 + Cs)^{\frac{-2\mu_j}{C}}\, ds \tag{9}$$

and this converges precisely if $\alpha_j(0) = 0$ for all $j \in \mathbb{Z} \setminus \{0\}$ with $\mu_j \leq C/2$, and in this case the convergence of (9) is absolute with limit

$$\|\psi\|^2_{L^2} = \sum_{j \in \mathbb{Z}\setminus\{0\}} \frac{|\alpha_j(0)|^2}{2\mu_j - C}.$$

The Dirac operator D on a warped product is unfortunately not always Fredholm, but it still has a well-defined "index" $\mathrm{ind}_{L^2}(D^+)$ computed from the (finite!) multiplicities of the eigenvalue 0 in the point spectra of $\mathrm{spec}_p(D^{\pm})$. There appears to be no simple criterion to decide for an arbitrary given warp function ρ whether the corresponding Dirac operator is Fredholm, but it is possible to find conditions which are either necessary or sufficient.

The following facts can be found in [Bär00]. Let $\mu_0 \leq \mu_1 \leq \mu_2 \ldots$ be a listing of the positive eigenvalues of $D_N \oplus (-D_N)$ acting on $\Sigma M|_N \cong \Sigma N \oplus (-\Sigma N)$ (repeated according to their multiplicities), and consider the Hilbert space decomposition

$$L^2(\Sigma M|_N) \cong L^2(\Sigma N \oplus (-\Sigma N)) \cong \widehat{\bigoplus_{j\in\mathbb{N}_0} E(\mu_j) \oplus \mathbf{n} \cdot E(\mu_j)}.$$

Using $E(\mu_j) \oplus \mathbf{n} \cdot E(\mu_j) \cong \mathbb{C}^2$ this yields a unitary equivalence

$$L^2(\Sigma M) \cong \widehat{\bigoplus_{j\in\mathbb{N}_0} L^2([0,\infty)}, \mathbb{C}^2, dt)$$

(see [Bär00], Lemma 1) which transforms D_M^2 into an operator

$$D_M^2 \cong \bigoplus_{j\in\mathbb{N}_0} L_{\mu_j}. \tag{10}$$

Here L_μ acts on $L^2([0,\infty), \mathbb{C}^2, dt)$ via

$$L_\mu = -\frac{d^2}{dt^2} + \frac{\mu\dot\rho(t)}{\rho(t)^2} + \frac{\mu^2}{\rho(t)^2} = -\frac{d^2}{dt^2} + V_\mu(t). \tag{11}$$

Proposition 9 *Let ρ define a warped metric as in (1).*

1. *The associated Dirac operator is Fredholm if one of the following conditions hold:*

   (a) *There is a $c > 0$ so that*

   $$\liminf_{t\to\infty} V_\mu(t) \geq c \tag{12}$$

   for all $\mu \in \operatorname{spec}(D_N)$.

   (b) *There is a $c \geq 0$ so that*

   $$\lim_{t\to\infty} \rho(t) = c,\ \lim_{t\to\infty} \dot\rho(t) = 0,\ \ker(D_N) = \{0\}.$$

   (c) *$\rho(t) = (1 + Ct)^\varepsilon$ with $\varepsilon \leq 0$ and $\ker(D_N) = \{0\}$.*

2. *By contrast, D is not Fredholm if one of the following conditions hold:*

(a) $\ker(D_N) \neq \{0\}$.

(b) $\lim_{t\to\infty} V_\mu \leq 0$ *for some* $\mu \in \operatorname{spec}(D_N)$.

(c) $\rho(t) = (1 + Ct)^\varepsilon$ *with* $\varepsilon > 0$.

Proof. (1) We clearly have $(c) \Rightarrow (b) \Rightarrow (a)$, so it is enough to prove (a). We already know that $D_{\tilde{M}}$ is a self-adjoint operator with finite-dimensional kernel, so we only need to show that $0 \notin \operatorname{spec}_e(D_{\tilde{M}})$. Note that

$$0 \in \operatorname{spec}_e(D_{\tilde{M}}) \Rightarrow 0 \in \operatorname{spec}_e(D^2_{\tilde{M}}) \Rightarrow 0 \in \operatorname{spec}_e(D^2_M),$$

so it is enough to show that $0 \notin \operatorname{spec}_e(D^2_M)$. By [DS63], Theorem XIII 7.5 (p. 1438), $\operatorname{spec}_e(D^2_M)$ is contained in the smallest closed convex set $C \subset \mathbb{R}$ with the property that $\langle D^2_M \sigma, \sigma \rangle \in C$ for all $\sigma \in C_0^\infty(\Sigma M)$ of unit length $\|\sigma\| = 1$.

Proposition 10 *Suppose (12) holds. Then for all* $\varepsilon > 0$ *there is a* $t_0 \geq 0$ *so that*

$$V_\mu(t) \geq c - \varepsilon \quad \text{for all } t \geq t_0 \text{ and all } \mu \in \operatorname{spec}(D_N). \tag{13}$$

Proof. This follows from the following two elementary observations:

(i) The function $v_{a,b}(\mu) = (a\mu + \mu^2)/b^2$ has a minimum at $\mu = -a/2$.

(ii) There is an upper bound $\kappa > 0$ with $|\dot{\rho}(t)| \leq 2\kappa$, because otherwise, for any given $\mu \in \operatorname{spec}(D_N)$ and $t_0 > 0$, we could find a $t \geq t_0$ so that $|\dot{\rho}(t)| \geq |\mu|$ and hence either $V_\mu(t) \leq 0$ or $V_{-\mu}(t) \leq 0$, contradicting (12).

Now choose $j \in \mathbb{N}_0$ with $\mu_j \geq \kappa$. Then (i) and (ii) imply

$$V_{\pm\mu_k}(t) \geq V_{\pm\mu_j}(t) \;\text{ for all }\; k \geq j \;\text{ and all }\; t \geq 0. \tag{14}$$

Now from (12) we can choose $t_0 > 0$ so that the inequality in (13) holds for the finitely many eigenvalues $\mu_0, \ldots, \mu_j$. But then (14) implies that (13) holds for all $\mu \in \operatorname{spec}(D_N)$. $\square$

Let $0 < \varepsilon < c$. By modifying the decomposition $\tilde{M} = K \cup M$ if necessary, we may assume that $t_0 = 0$ in Proposition 10 and thus $V_\mu(t) \geq c - \varepsilon$ for all $\mu \in \operatorname{spec}(D_N)$ and $t \geq 0$. From our assumption (12) and from equation (11), we conclude for every $\sigma \in C_0^\infty(\Sigma M)$

$$\langle D^2_M \sigma, \sigma \rangle = \langle \sigma', \sigma' \rangle + \sum_{j \in \mathbb{N}_0} \langle V_{\mu_j} \sigma_j, \sigma_j \rangle \geq (c - \varepsilon)\|\sigma\|^2.$$

We conclude

$$\mathrm{spec}_e(D_M^2) \subset [c - \varepsilon, \infty)$$

and thus $0 \notin \mathrm{spec}_e(D_M^2)$. Then $D_{\tilde{M}}$ is Fredholm. Note that (12) implies in particular that $0 \notin \mathrm{spec}(D_N)$.

($2a$) This is entirely analogous to Theorem 1 in [Bär00] which proves the statement for the particular warp function e^{-t}, however, the proof translates literally to any warp function $\rho(t)$.

($2b$) [DS63], Theorem XIII 7.16b, (p.1448), implies $0 \in \mathrm{spec}_e(L_\mu)$ and hence $0 \in \mathrm{spec}_e(D_M^2)$. Therefore D_M is not Fredholm.

($2c$) This is an example for (2b). □

§ 4 The $\hat{A}$ and $T\hat{A}$-forms on warped cylinder-manifolds If ∇ and Ω denote the Levi-Civita connection and associated curvature of a Riemannian manifold $\tilde{M}$, we define the $\hat{A}$-form of $\tilde{M}$ by

$$\hat{A}(\tilde{M}) = \sqrt{\det\left(\frac{\Omega/2}{\sinh(\Omega/2)}\right)} = \exp\mathrm{Trace}\left[\frac{1}{2}\log\left(\frac{\Omega/2}{\sinh(\Omega/2)}\right)\right].$$

This is to be understood as a formal power series associated to the function

$$\hat{a}(x) = \frac{1}{2}\log\left(\frac{x/2}{\sinh(x/2)}\right) = -\frac{1}{24}x^2 + \frac{1}{5760}x^4 - \frac{1}{362880}x^6 + \ldots . \tag{15}$$

We substitute the End (TM)-valued curvature two-form Ω into (15), take the trace to make it a real-valued form and then insert it into the exponential series. Convergence is trivial since $\wedge^k\Omega = 0$ for $k > \dim(\tilde{M})/2$. Explicitly,

$$\hat{A}(\tilde{M}) = 1 - \frac{1}{48}\mathrm{Trace}(\Omega\wedge\Omega) + \frac{1}{2560}\mathrm{Trace}(\Omega\wedge\Omega\wedge\Omega\wedge\Omega) - \ldots \ .$$

Proposition 11 *Let a cylinder-manifold $\tilde{M}$ carrying a warped product metric with warp function $\rho(t)$ be given. If $\|\ddot{\rho}\|_{L^1} < \infty$ then the $\hat{A}$-form is integrable over $\tilde{M}$.*

Proof. We only need to look at those components of $\hat{A}(\tilde{M})$ which are multiples of the volume element. Thus we let $\hat{A}_{2m}(t,p)\mathrm{vol}_M$ be the homogeneous part of maximal degree of $\hat{A}(\tilde{M})$. Since $\|\ddot{\rho}\|_{L^1} < \infty$,

$$|\dot{\rho}(t)| \leq |\dot{\rho}(0)| + \|\ddot{\rho}\|_{L^1} \quad \forall t \in \mathbb{R}_0^+ \tag{16}$$

is bounded. Using (5) and (16) it then follows that

$$|\hat{A}_{2m}(t,p)| \le C_1 \frac{|\ddot{\rho}(t)|}{\rho(t)^{2m-1}} \tag{17}$$

for some constant $C_1 > 0$. We integrate this and obtain

$$\begin{aligned}
\left| \int_{\tilde{M}} \hat{A} \right| &\le \left| \int_K \hat{A} \right| + \left| \int_M \hat{A}_{2m}(t,p)\mathrm{vol}_M \right| \\
&\le C_2 + \int_M |\hat{A}_{2m}(t,p)|\, \mathrm{vol}_M \\
&\le C_2 + \int_{\mathbb{R}_0^+} \int_N |\hat{A}_{2m}(t,p)|\, \rho(t)^{2m-1}\, \mathrm{vol}_N(p)\, dt \\
&\le C_2 + C_1\, \mathrm{Vol}(N) \int_{\mathbb{R}_0^+} |\ddot{\rho}(t)|\, dt \\
&= C_2 + C_1\, \mathrm{Vol}(N) \|\ddot{\rho}\|_{L^1} < \infty.
\end{aligned}$$

The $\hat{A}$-form is certainly integrable over the compact part K of a cylinder-manifold, and we bound its absolute value by C_2. □

We now introduce the transgression of the $\hat{A}$-form which describes how $\hat{A}(\tilde{M})$ is transformed under a change of connection. See [Gil93, Gil95] for details. Fix $t_0 \in \mathbb{R}_0^+$ and let $g_0 = dt^2 + \rho(t_0)^2 g_N$ be a product metric on the cylinder $\mathbb{R}_0^+ \times N$. Let $g_1 = g$ be our original warped product metric (1), and let ∇^i for $i = 0, 1$ be the respective Levi-Civita connections. We define a one-parameter family of interpolating connections

$$\nabla^s = s\nabla^1 + (1-s)\nabla^0, \quad s \in [0,1],$$

and we let Ω^s be the curvature form of ∇^s. Finally let $\vartheta = \nabla^1 - \nabla^0$ which is an $\mathrm{End}(TM)$-valued 1-form. Define the algebra $\mathcal{P}$ of invariant polynomials,

$$\mathcal{P} = \{f \in \mathbb{C}[X] \mid f(h^{-1}Ah) = f(A) \text{ for all } A \in M(n \times n, \mathbb{R}), h \in \mathrm{SO}(n)\}.$$

There is a natural graduation $\mathcal{P} = \oplus_j \mathcal{P}_j$ by the degree of polynomials.

Definition 12 *For every $f \in \mathcal{P}_j$ we define $\mathrm{Pol}_j(f)$ to be the coefficient $f_{(1,\dots,1)}$ with multi-index $\alpha = (1,\dots,1)$ in the expansion*

$$f(t_1A_1 + \cdots + t_jA_j) = \sum_{\alpha \in \mathbb{N}_0^j} f_\alpha(A_1,\dots,A_j) t_1^{\alpha_1} \cdots t_j^{\alpha_j}.$$

Definition 13 *In the notation above, the* $\underline{\text{transgression}}$ *of the* $\hat{A}$*-form with respect to* g_0*,* g_1 *is an element of* $C^\infty(\Lambda^* T N_{t_0})$ *defined by*

$$T\hat{A}(\nabla^1, \nabla^0) = \sum_{j=0}^{m} j \int_0^1 \mathrm{Pol}_j(\hat{a}_j)(\vartheta, \underbrace{\Omega^s, \dots, \Omega^s}_{j-1})\, ds, \tag{18}$$

where $\hat{a}(x) = \sum_{j=0}^m \hat{a}_i x^i$ *is the* $\hat{A}$*-polynomial defined in (15).*

One can then show (see [Gil93, Gil95]) that

$$\hat{A}^1 - \hat{A}^0 = d\{T\hat{A}(\nabla^1, \nabla^0)\}$$

This proves once more that for closed manifolds X (where $\partial X = \emptyset$) the integral $\hat{A}[X]$ is independent of the choice of metric, because with a different metric $\hat{A}[X]$ changes by the integral of an exact form, hence by zero.

Proposition 14 *Given a warped product as above with warp function* $\rho(t)$ *such that the limit* $\lim_{t\to\infty} \dot{\rho}(t)$ *exists, then so does*

$$\lim_{t\to\infty} \int_{N_t} T\hat{A}.$$

Proof. For $x, y \in C^\infty(TN)$, we find from (2)

$$\vartheta(x) \ : \ y \mapsto \left(\nabla^1_x y - \nabla^0_x y\right) = g_N(S(x), y)\, \mathbf{n} = -\frac{\dot{\rho}(t)}{\rho(t)}\, g_N(x,y)\, \mathbf{n}.$$

Moreover, we compute the restriction of Ω^s to the hyper-surface N_t as

$$\Omega^s | N_t = \Omega^N + s^2 \frac{\dot{\rho}(t)^2}{\rho(t)^2}\, \alpha.$$

From (18) and (5) in Proposition 5 it now follows that the transgression of the $\hat{A}$-form is a differential form whose top component of degree $2m-1$ has the form

$$(T\hat{A})_{2m-1}(t,p)\, \mathrm{vol}_N = \frac{\dot{\rho}(t)}{\rho(t)^{2m-1}} \sum_{i=0}^{m-1} c_i(p) (\dot{\rho}(t))^i\, \mathrm{vol}_N.$$

This is a polynomial in $\dot{\rho}(t)$ whose coefficients $c_j(p)$ are functions on N and independent of t. Thus,

$$\begin{aligned}
\int_{N_t} T\hat{A} &= \int_{N_t} (T\hat{A})_{2m-1}(t,p)\ \mathrm{vol}_{N_t} \\
&= \sum_{i=0}^{m-1} \int_N \frac{c_i(p)(\dot{\rho}(t))^{i+1}}{\rho(t)^{2m-1}} \rho(t)^{2m-1}\ \mathrm{vol}_N \\
&= \sum_{i=0}^{m-1} \left\{ \int_N c_i(p)\ \mathrm{vol}_N \right\} \dot{\rho}(t)^{i+1} \\
&= \sum_{i=0}^{m-1} C_i \dot{\rho}(t)^{i+1}. \qquad (19)
\end{aligned}$$

This is a polynomial in $\dot{\rho}(t)$ without constant term; it has a limit as $t \to \infty$ if $\dot{\rho}(t)$ has. In particular, this is true under the assumptions of Proposition 11. Note that the expression vanishes if $\rho \equiv$ const because the polynomial has no constant term. □

§ 5 The index of the Dirac operator on a warped cylinder-manifold

In [APS75], Atiyah, Patodi and Singer define a boundary condition for spinors on compact manifolds with boundary.

Definition 15 *Let K be a compact spin manifold with boundary $\partial K = N$. A spinor $\psi \in C^\infty(\Sigma K)$ satisfies the <u>APS-boundary condition</u> if the decomposition (6) over $\partial K = N$ has the property that*

$$\mu_j \leq 0 \quad \Rightarrow \quad \alpha_j = 0 \ \ \forall j > 0, \qquad (20)$$
$$\mu_j < 0 \quad \Rightarrow \quad \alpha_j = 0 \ \ \forall j < 0, \qquad (21)$$

that is, the spectral decomposition (6) in $\Sigma K|_N \cong \Sigma N \oplus (-\Sigma N)$ involves only D_N-eigenspinors of ΣN with positive eigenvalues and $(-D_N)$-eigenspinors of $(-\Sigma N)$ with nonnegative eigenvalues.

They then prove an index theorem for compact manifolds with boundary in the case where the metric of K is a product in a collared neighbourhood of N. Gilkey [Gil93] extends this to the general case.

Let $h = \dim(\ker(D_N))$ and recall the eta-invariant $\eta = \eta(D_N)$ which is a measure of the "spectral asymmetry" of the positive and negative eigenvalues of D_N (see [Gil93]) for definitions).

Theorem 16 (Atiyah-Patodi-Singer, Gilkey) *Let K be a compact spin manifold with boundary $\partial K = N$. The Dirac operator acting on spinors which satisfy the APS boundary condition has a finite index which can be expressed by the formula*

$$\mathrm{ind}_{\mathrm{APS}}(D_K^+) = \int_K \hat{A} + \int_{\partial K} T\hat{A} + \frac{\eta - h}{2}. \tag{22}$$

§ 6 An index theorem for warped product manifolds We would now like to compare the APS boundary condition for spinors on compact bounded manifolds in Definition 15 to the conditions for L^2-integrability on cylinder manifolds given in Proposition 7 and Remark 8. This will lead to an index formula for cylinder manifolds with warped product metrics. We define

$$\mathcal{H}_0^- = \left\{ \psi \in \ker(D_K^-) \,\middle|\, \psi|N \in \ker(-D_N) \right\}, \quad h_0^- = \dim(\mathcal{H}_0^-) \tag{23}$$

and use Gilkey's Theorem 16 with the notations introduced earlier.

Theorem 17 *Let $\tilde{M}$ be a $2m$-dimensional cylinder-manifold which is spin and carries a warped product metric over the cylindrical end $M \subset \tilde{M}$. Let $K = \tilde{M} \setminus M^\circ$ and $\partial K = N$.*

1. *If $\rho(t)/t^\varepsilon$ is bounded for some $\varepsilon < 1$ and all $t \gg 0$ then D^+ has a finite index in $L^2(\Sigma\tilde{M})$ which is given by*

$$\begin{aligned} \mathrm{ind}_{L^2}(D^+) &= \int_K \hat{A} + \int_{\partial K} T\hat{A} + \frac{\eta - h}{2} + h_0^- \\ &= \mathrm{ind}_{\mathrm{APS}}(D_K^+) + h_0^- . \end{aligned} \tag{24}$$

   Moreover, if $\|\ddot{\rho}\|_{L^1} < \infty$, we can write this as

$$\mathrm{ind}_{L^2}(D^+) = \int_{\tilde{M}} \hat{A} + \frac{\eta - h}{2} + h_0^- . \tag{25}$$

2. *If $\rho(t)/t^\varepsilon > C > 0$ for some $\varepsilon > 1$ and all $t \gg 0$, then $\tilde{M}$ has no harmonic spinors in $L^2(\Sigma\tilde{M})$ and we get*

$$\mathrm{ind}_{L^2}(D^+) = 0.$$

3. *If neither of the conditions on $\rho(t)$ in (1) or (2) holds then*

$$-\dim(\ker_{\mathrm{APS}}(D_K^-)) + h_0^- \leq \mathrm{ind}_{L^2}(D^+) \leq \dim(\ker_{\mathrm{APS}}(D_K^+)).$$

Proof. Observe that every harmonic spinor ψ on $\tilde{M}$ restricts to a harmonic spinor on M which satisfies the APS boundary condition, because we know from Proposition 7 that, with respect to the decomposition (6), L^2-integrability implies that $\alpha_j(0) = 0$ for all $j \in \mathbb{Z} \setminus \{0\}$ for which $\mu_j \leq 0$ which is a slightly stronger condition than the APS boundary condition (20) and (21). Conversely, whether a spinor ψ which satisfies the APS boundary condition is L^2-integrable and thus contributes to the index of D^+ acting on $L^2(\Sigma\tilde{M})$ depends on the warp function $\rho(t)$ and the space $\mathcal{H}_0^-$ in (23).

The first part of 1. follows from Remark 8(1); with this sub-linear bound on the growth of $\rho(t)$ the space of harmonic spinors on K which satisfy the APS boundary condition but which have no extension to a harmonic L^2-spinor on $\tilde{M}$ is precisely $\mathcal{H}_0^- \subset \ker(D_K^-)$, hence (24). For the second part we let $K_t = K \cup [0,t] \times N$ and $M_t = [t, \infty) \times N$ and apply (24) to the decomposition $\tilde{M} = K_t \cup M_t$. We claim that (25) follows by letting $t \to \infty$. The left hand side of (24) is independent of t. Moreover, since $\|\ddot{\rho}\| < \infty$ Proposition 11 tells us that $\hat{A}(\tilde{M})$ is integrable, so the limit

$$\lim_{t\to\infty} \int_{K_t} \hat{A} = \int_{\tilde{M}} \hat{A}$$

exists and the right hand side of (25) is well-defined. It remains to show that

$$\lim_{t\to\infty} \int_{N_t} T\hat{A} = 0 \tag{26}$$

where $N_t = \{t\} \times N = \partial K_t$. Our assumption $\|\ddot{\rho}\|_{L^1} < \infty$ implies that

$$C = \frac{1}{2} \lim_{t\to\infty} \dot{\rho}(t)$$

exists. We claim $C = 0$. Suppose by contrary $C \neq 0$, thus $C > 0$ since otherwise ρ would eventually become negative. Then there is a $T > 0$ with

$$\dot{\rho}(t) > C > 0$$

for all $t > T$. Hence for $t > T$,

$$\begin{aligned} \rho(t) &= \rho(T) + \int_T^t \dot{\rho}(s)\,ds \\ &> \rho(T) + C(t-T). \end{aligned}$$

But then $\rho(t)t^{-\varepsilon} \to \infty$ as $t \to \infty$ for all $\varepsilon < 1$, contradiction. Hence $C = 0$. Then equation (19) in Proposition 14 implies (26). This proves (25).

2. is clear from Remark 8(2).

For 3. we use the remark made earlier that every harmonic spinor ψ on $\tilde{M}$ restricts to a harmonic spinor on M which satisfies the APS boundary condition, so $\mathrm{ind}_{L^2}(D^+)$ must satisfy the given bounds.

Thus for cylinder manifolds bounded volume growth is a partial substitute for compactness. □

Let us conclude by giving a few examples.

1. First, it is clear that any warped products whose warp function has polynomial growth of at least quadratic order (such as paraboloids) must have vanishing L^2-index.

2. In [GL83], Gromov and Lawson define the relative index of a pair $(\tilde{M}_1, \tilde{M}_2)$ of open manifolds which are isometric outside compact subsets $K_j \subset \tilde{M}_j$ with boundaries $\partial K_1 \cong \partial K_2 \cong N$. Suppose $\tilde{M}_1, \tilde{M}_2$ are cylinder manifolds carrying isometric warped product metrics with warp function $\rho(t) < Ct^{\varepsilon}$ for some $\varepsilon < 1$ so that $D^+_{\tilde{M}_j}$ has a finite index in $L^2(\Sigma_{M_j})$ for $j = 1, 2$. Then their relative index in the sense of [GL83] can be recovered as the difference

$$\mathrm{ind}_{\mathrm{rel}}(D^+_{\tilde{M}_1}, D^+_{\tilde{M}_2}) = \mathrm{ind}_{L^2}(D^+_{\tilde{M}_1}) - \mathrm{ind}_{L^2}(D^+_{\tilde{M}_2}).$$

3. In [Bär00] Bär studies hyperbolic manifolds of finite volume which have cusps $\mathbb{R}^+ \times N$ whose hyperbolic metric is of the form

$$g(t,p) = dt^2 + e^{-2t} g_N$$

where (N, g_N) is a flat Bieberbach manifold. They obviously satisfy the condition of bounded growth in Theorem 17 and the index is given by (25). Moreover, $\hat{A} = 0$ in this case and (25) reduces to the simple formula

$$\mathrm{ind}_{L^2}(D^+) = \frac{\eta - h}{2} + h_0^- .$$

In two dimensions, $\eta = 0$, in dimension four the η-invariants have been computed in [Pfä00]. The value of h depends on the spin structures on the connected components of the Bieberbach manifold N, see [Bär00].

4. It is well-known that real-hyperbolic space $\mathbb{R}H^n$ is a warped cylinder-manifold; it is isometric to $\mathbb{R}^n$ if we rescale the standard metric along the (Euclidean) distance spheres S_t^{n-1} by a factor $\rho(t)^2 = \sinh(t)^2$ and keep it fixed in radial direction. Since the warp-function $\rho(t) = \sinh(t)$ grows exponentially our computations in Paragraph 2 imply that for $n \equiv 0 \mod 2$ there are no harmonic spinors on $\mathbb{R}H^n$, in particular,

$$\mathrm{ind}_{L^2}(D^+) = 0.$$

The same is true in odd dimensions, however, since in this case the structure of the spinor bundle is different our techniques would require some modification in order to cover this case.

The spectrum of D on $\mathbb{R}H^n$ was computed by Bunke [Bun91a] and later by Camporesi-Higuchi [CH96]; note that there is a wrong statement in [Bun91a] about the eigenvalue $\lambda = 0$.

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