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Ambient Dirac eigenvalue estimates and the Willmore functional

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Abstract. We prove a lower bound for the Willmore functional with the help spinors and the Dirac operator. As a corollary we verify the Willmore conjecture in a certain region of the spin-conformal moduli space for tori with sufficiently small L^p-norms of the Gauss curvature.

Keywords: Dirac eigenvalue estimates, Willmore inequality, conformal immersions

1. Introduction

Willmore conjectured [**Wil65**] that for any immersion F of the 2-torus T^2 into Euclidean space $\mathbb{R}^3$, the Willmore functional

$$W(F) := \int_{T^2} H^2$$

is bounded from below by $2\pi^2$. The conjecture has been verified for a large class of manifolds (see e.g. [**Top98**] or [**Amm00b**] for an overview over the known cases). However, the general case remains still open until today.

Any such immersion induces a metric and a spin structure on T^2, hence for any immersion F we obtain an element of the spin-conformal moduli space, that we will denote by $[F]$. Li and Yau [**LY82**] proved that there is a compact subset A of positive measure in the (spin-)conformal moduli space such that the conjecture holds, if $[F] \in A$ (see Figure 1). Their methods have been extended in [**MR86**] to a larger compact subset of the (spin-)conformal moduli space.

In this paper we study the functional near one of the ends of the moduli space. The main theorem we prove is

Theorem 5.5. *Let $F : T^2 \to S^3$ be an immersion of the 2-dimensional torus in S^3. We endow S^3 with the standard metric g_{S^3}. Let F be regularly homotopic to an embedding. We set $g := F^* g_{S^3}$, and choose $(x, y) \in \mathcal{M}^{\mathrm{spin}}$ such that $T^2_{(x,y)}$ is spin-conformally equivalent to (T^2, g). Then*

$$\mathcal{W}(F) \geq \frac{\pi^2}{y} - \frac{1}{8} (\operatorname{osc} u) \, \|K_g\|_{L^1(T^2,g)} .$$

In particular, if $\|K_g\|_{L^1(T^2,g)} < 4\pi$, then for any $p > 1$

$$\mathcal{W}(F) \geq \frac{\pi^2}{y} - \frac{1}{8} \mathcal{S}_p \, \|K_g\|_{L^1(T^2,g)}$$

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where $\mathcal{S}_p := \mathcal{S}_p\left(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)}\,\mathrm{area}(T^2,g)^{1-(1/p)}, \frac{\mathrm{area}(T^2,g)}{\mathrm{sys}_1(T^2,g)^2}\right)$ *is an expression explicitely defined in Definition 2.2.* As a corollary we obtain the Willmore conjecture in a special case.

Another corollary states if we have a sequence of immersions F_i such that $[F_i]$ converges to a certain end of the spin-conformal moduli space in such a way that the L^p-norm of the Gaussian curvature is bounded by a sufficiently small number, then $W(F_i)$ tends to ∞ (Corollary 5.7).

In view of Theorem 5.2 it is tempting to conjecture that Theorem 5.3 could be generalized to the statement:

Naive Guess. *Let $F : (T^2, g, \sigma) \to (S^3, g_{S^3})$ be a conformal embedding. We denote by σ the spin structure on T^2 induced by F. Then the first positive eigenvalue μ_1 of the square of the Dirac operator D^2 on (T^2, g, σ) satisfies*

$$\mathcal{W}(F) \geq \mu_1 \mathrm{area}(T^2, g).$$

An example at the end of the article shows, that this naive guess does not hold. The example also shows that some of our curvature bounds are indeed necessary.

This article is the report of a talk given at the American University of Beirut in summer 2001. Many thanks to the organizers for this beautiful conference.

What we present here differs slightly from the talk. In this article, we omit several results published in previous preprints and articles. On the other hand, we include a detailed proof of Theorem 5.5. These estimates have not been published in this form previously. However, a weaker version already appeared in [**Amm98**].

The proof of our main statements rely on the fact that on a surface immersed into $\mathbb{R}^3$ a spinor is induced, a fact which plays an important role in several talks at the conference.

2. Uniformization theorem for the torus

Let us recall the famous uniformzation theorem.

Theorem 2.1 (Uniformization Theorem). *Let g be any metric on the two-dimensional torus. Then there is a conformal metric g_0 that is flat.*

We write $g = e^{2u} g_0$. We will recall an upper estimate for the oscillation $\operatorname{osc} u = \max u - \min u$.

Definition 2.2. For any $p > 1$, let $\mathcal{S}_p$ be the function given by the expression

$$\mathcal{S}_p(\mathcal{K}, \mathcal{K}', \mathcal{V}) := \frac{p}{p-1}\left[\frac{\mathcal{K}'}{4\pi} + \frac{1}{2}\left|\log\left(1 - \frac{\mathcal{K}}{4\pi}\right)\right| + \frac{\mathcal{K}}{8\pi - 2\mathcal{K}}\log\left(\frac{2\mathcal{K}'}{\mathcal{K}}\right)\right] + \frac{\mathcal{K}\mathcal{V}}{8}$$

for $\mathcal{K} \in (0, 4\pi)$, $\mathcal{K}' \in [\mathcal{K}, \infty)$ and $\mathcal{V} \in [0, \infty)$.

We extend continuously by setting

$$\mathcal{S}_p(0, \mathcal{K}', \mathcal{V}) := \frac{p}{p-1}\frac{\mathcal{K}'}{4\pi}.$$

Theorem 2.3 ([**Amm00a**]). *Let g be any metric on the two-dimensional torus such that the Gauss curvature K_g satisfies $\|K_g\|_{L^1(T^2,g)} < 4\pi$. Then for any $p > 1$ we obtain a bound for the oscillation of u*

$$\operatorname{osc} u \leq \mathcal{S}_p\Big(\|K_g\|_{L^1(T^2,g)}, \|K_g\|_{L^p(T^2,g)}\left(\mathrm{area}(T^2,g)\right)^{1-(1/p)}, \frac{\mathrm{area}(T^2,g)}{\mathrm{sys}_1(T^2,g)^2}\Big).$$

Equality is obtained if and only if g is flat.

3. Spin structures on 2-tori

We recall some facts about spin structures on 2-tori.

PROPOSITION 3.1. *For any spin structure on the two-dimensional torus the following properties are equivalent:*

(1) the associated Dirac operator has no kernel,
(2) the torus is spin corbordant zero,
(3) the spin structure is induced from an embedding into $\mathbb{R}^3$*,*

Obviously, the last property can be equivalently reformulated as "The spin structure is induced from an immersion into $\mathbb{R}^3$ that is regularly homotopic to an embedding."

We say that a spin structure is *nontrivial* if it satisfies any of these conditions.

4. The spin-conformal moduli space

Two manifolds with Riemannian metrics and spin structures are said to be *spin-conformally equivalent*, if there is a conformal diffeomorphism between them preserving the spin structure.

The *spin-conformal moduli space* $\mathcal{M}^{\mathrm{spin}}$ (see Figure 1) is defined to be the set of all $(x, y) \in \mathbb{R}^2$ satisfying

$$0 \leq x \leq \frac{1}{2}, \qquad \left(x - \frac{1}{2}\right)^2 + y^2 \geq \frac{1}{4}, \qquad y > 0. \tag{4.1}$$

For such a pair (x, y), let $T^2_{(x,y)} = \mathbb{R}^2/\langle(1,0),(x,y)\rangle$ be the 2-torus equipped with the (flat) metric induced from $\mathbb{R}^2$ and equipped with the spin structure that admits a parallel section along the generator $(1, 0)$ of the fundamental group, and that does not admit a parallel section along the generator (x, y).

LEMMA 4.1. *For any 2-torus T^2 with a Riemannian metric g and a non-trivial spin structure σ, there is a unique $(x, y) \in \mathcal{M}^{\mathrm{spin}}$ such that (T^2, g, σ) is spin-conformally equivalent to $T^2_{(x,y)}$.*

Hence, we $\mathcal{M}^{\mathrm{spin}}$ can be identified with the equivalence classes of Riemannian 2-tori with spin structures under spin-conformal equivalence.

5. The Willmore functional

In this section S^3 always carries the metric g_{S^3} of constant sectional curvature 1. For any immersion $F : T^2 \to S^3$ we define the Willmore functional

$$\mathcal{W}(F) := \int_{(T^2, F^* g_{S^3})} \left(|H_{T^2 \to S^3}|^2 + 1\right)$$

where $H_{T^2 \to S^3}$ is the relative mean curvature of $F(T^2)$ in S^3 and integration is the usual integration of functions $T^2 \to \mathbb{R}$ over the Riemannian manifold $(T^2, F^* g_{S^3})$. Note that the mean curvature H of $F(T^2)$ in $\mathbb{R}^4$ satisfies

$$|H|^2 = |H_{T^2 \to S^3}|^2 + 1.$$

The following proposition says that the Willmore functional $\mathcal{W}$ is essentially the same as the functional W from the introduction.

PROPOSITION 5.1 ([**Tho23**],[**Wei78**]). *Let $S : S^3 \setminus \{N\} \to \mathbb{R}^3$ denote stereographic projection. Then for any immersion $F : T^2 \to S^3 \setminus \{N\}$*

$$W(S \circ F) = \mathcal{W}(F).$$

Willmore has conjectured that for any immersion $\mathcal{W}(F) \geq 2\pi^2$. This conjecture is still open in general, although it has been verified in many special cases. Li and Yau [**LY82**, Fact 3] proved that the conjecture holds, if F is not an embedding. They also proved:

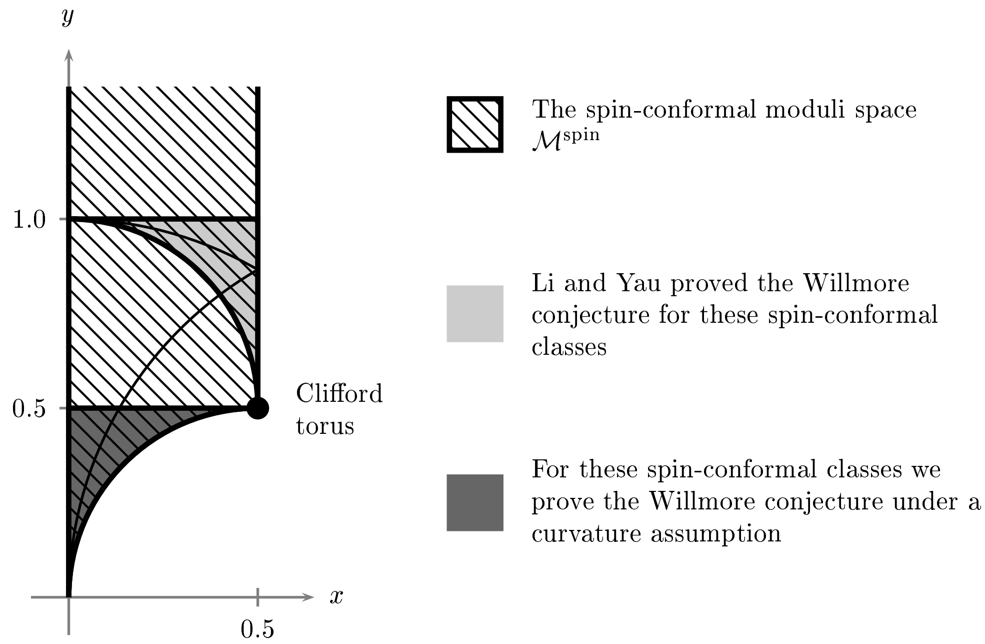


FIGURE 1. The spin-conformal moduli space

THEOREM 5.2 ([**LY82**, Theorem 1]). *Let $F : (T^2, g) \to (S^3, g_{S^3})$ be a conformal embedding. Let λ_1 be the first positive eigenvalue of the Laplacian Δ on (T^2, g) then*

$$\mathcal{W}(F) \geq \frac{1}{2}\,\lambda_1 \mathrm{area}(T^2, g).$$

From this theorem the conjectured inequality $\mathcal{W}(F) \geq 2\pi^2$ follows, if the spin-conformal equivalence class $[F]$ lies in a compact subset of $\mathcal{M}^{\mathrm{spin}}$ with positive measure (see Figure 1).

A similar lower bound for $\mathcal{W}(F)$ in terms of Dirac eigenvalues has been given by Bär.

THEOREM 5.3 ([**Bär98**]). *Let $F : (T^2, g, \varphi) \to (S^3, g_{S^3})$ be an* ***isometric*** *immersion preserving the spin structure. Then for the first eigenvalue μ_1 of the square of the Dirac operator on (T^2, g, φ) the inequality*

$$\mathcal{W}(F) \geq \mu_1 \mathrm{area}(T^2, g)$$

holds.

Note that this estimate is non-trivial only if F is regularly homotopic to an embedding.

REMARK 5.4. At the end of this section we will show by example that in general "isometric" can not be replaced be "conformal" in this theorem.

In this article, our approach is to modify the techniques of the proof of Theorem 5.3. This yields together with Theorem 2.3 new results about the Willmore functional.

THEOREM 5.5. *Let $F : T^2 \to S^3$ be an immersion of the 2-dimensional torus in S^3. We endow S^3 with the standard metric g_{S^3}. Let F be regularly homotopic to an embedding. We set $g := F^* g_{S^3}$, and choose $(x, y) \in \mathcal{M}^{\mathrm{spin}}$ such that $T^2_{(x,y)}$ is* *spin-conformally equivalent to* (T^2, g). *Then*

$$\mathcal{W}(F) \geq \frac{\pi^2}{y} - \frac{1}{8}\,(\operatorname{osc} u)\ \|K_g\|_{L^1(T^2,g)}\,.$$

In particular, if $\|K_g\|_{L^1(T^2,g)} < 4\pi$, *then for any* $p > 1$

$$\mathcal{W}(F) \geq \frac{\pi^2}{y} - \frac{1}{8}\,\mathcal{S}_p\,\|K_g\|_{L^1(T^2,g)}$$

with $\mathcal{S}_p := \mathcal{S}_p\left(\|K_g\|_{L^1(T^2,g)}\,, \|K_g\|_{L^p(T^2,g)}\,\mathrm{area}(T^2,g)^{1-(1/p)}, \frac{\mathrm{area}(T^2,g)}{\mathrm{sys}_1(T^2,g)^2}\right)$.

Proof. We write the induced metric g on T^2 in the form $g = e^{2u}g_0$ with g_0 flat. Any Killing spinor on S^3 with the Killing constant $\alpha = (1/2)$ induces a spinor field ψ on (T^2, g) satisfying

$$D_g\psi = H\psi + \nu\psi,$$

where

$$\nu = \gamma(e_1)\gamma(e_2) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \in \mathrm{End}\left(\Sigma^+T^2 \oplus \Sigma^-T^2\right)$$

(see e. g. [**Bär98**]). There is an isomorphism of vector bundles [**Hit74**],[**Hij86**, 4.3.1] from the spinor bundle ΣT^2 associated to g to the spinor bundle $\Sigma_0 T^2$ associated to g_0

$$\begin{array}{ccc} \Sigma T^2 & \to & \Sigma_0 T^2 \\ \Psi & \mapsto & \widehat{\Psi} \end{array}$$

with

$$e^u\,\widehat{D_g\Psi} = D_{g_0}\widehat{\Psi} + \frac{1}{2}\gamma_{g_0}(\mathrm{grad}_{g_0} u)\widehat{\Psi}$$

and

$$|\widehat{\Psi}| = |\Psi|.$$

Here γ_{g_0} means Clifford multiplication corresponding to the metric g_0.

We apply this transformation to the induced spinor ψ from above. We obtain

$$D_{g_0}\widehat{\psi} = -\frac{1}{2}\gamma_{g_0}(\mathrm{grad}_{g_0} u)\widehat{\psi} + e^u H\widehat{\psi} + e^u\nu\widehat{\psi}.$$

As ν, $\gamma(V)$ and $\nu\gamma(V)$ are skew-hermitian for any vector V, this yields

$$\begin{aligned} \left|D_{g_0}\widehat{\psi}\right|^2 &= \frac{1}{4}\left|\gamma_{g_0}(\mathrm{grad}_{g_0} u)\widehat{\psi}\right|^2 + e^{2u}H^2\left|\widehat{\psi}\right|^2 + e^{2u}\left|\nu\widehat{\psi}\right|^2 \\ &= \frac{1}{4}|du|^2_{g_0} + e^{2u}H^2 + e^{2u}. \end{aligned}$$

Integration over (T^2, g_0) provides

$$\tilde{\lambda}_1 \mathrm{area}(T^2, g_0) \leq \frac{1}{4}\int_{T^2} |du|^2_{g_0}\ \mathrm{dvol}_{g_0} + \mathcal{W}(F), \tag{5.1}$$

where $\tilde{\lambda}_1$ denotes the smallest eigenvalue of the square of the Dirac operator on (T^2, g_0).

On the other hand

$$\begin{aligned} \int_{T^2} |du|^2_{g_0}\,\mathrm{dvol}_{g_0} &= \int_{T^2} u\Delta_{g_0} u\,\mathrm{dvol}_{g_0} \\ &= \int_{T^2} e^{2u} u K_g\,\mathrm{dvol}_{g_0} \\ &= \int_{T^2} u K_g\,\mathrm{dvol}_g \\ &\leq \frac{1}{2}\,(\operatorname{osc} u)\,\|K_g\|_{L^1(T^2,g)}. \end{aligned}$$

Together with Theorem 2.3 we obtain the statement. □

COROLLARY 5.6. *For any* $\kappa_1 \in \,]0, 4\pi[$, *any* $p > 1$ *and any* $\kappa_p > 0$ *there is a neighborhood* U *of the* $(y \to 0)$*-end of* $\mathcal{M}^{\mathrm{spin}}$ *with the following property:*
If $F : T^2 \to S^3$ *is an immersion such that* $[F] \in U$ *and if the curvature conditions*

$$\|K_g\|_{L^1(T^2,g)} < \kappa_1 \quad \textit{and} \quad \|K_g\|_{L^p(T^2,g)} \operatorname{area}(T^2,g)^{1-(1/p)} < \kappa_p$$

are satisfied, then the Willmore conjecture

$$W(F) \geq 2\pi^2$$

holds.

COROLLARY 5.7. *Let* $F_i : T^2 \to S^3$ *be a sequence of immersions. The induced metrics* $g_i := F_i^* g_{S^3}$ *together with the induced spin structures define a sequence* (x_i, y_i) *in the spin-moduli space* $\mathcal{M}^{\mathrm{spin}}$. *Assume that* $y_i \to 0$ *and that the curvature conditions*

$$\|K_{g_i}\|_{L^1(T^2,g_i)} < \kappa_1 < 4\pi \quad \textit{and} \quad \|K_g\|_{L^p(T^2,g)} \operatorname{area}(T^2,g)^{1-(1/p)} < \kappa_p$$

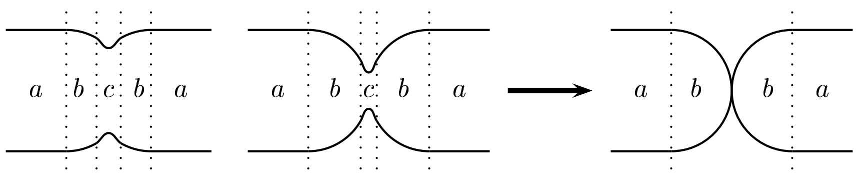
are satisfied for some $p > 1$ *and* $\kappa_p < \infty$. *Then*

$$\mathcal{W}(F_i) \to \infty.$$

The corollaries follow directly from the theorem.

6. An example showing the necessity of the curvature condition

The conclusion of the Corollary 5.7 is false if we drop the curvature conditions. To see this we construct a sequence of immersions with $y_i \to 0$ and $\mathcal{W}(F_i) < const.$ We start with an embedding $F : T^2 \to S^3$ which looks in a neighborhood of some point like a cylinder. Now we "strangle" the torus as in the picture below:



We get a sequence $F_i : T^2 \to S^3$ of C^1-embeddings with the following properties:

(i) $F_i(T^2)$ coincides with $F(T^2)$ in region a
(ii) $F_i(T^2)$ coincides with a part of a half-sphere in region b,
(iii) $F_i(T^2)$ coincides with a minimal surface in region c.

Note that the regions a, b and c depend on i. In the limit $i \to \infty$, region c disappears. After smoothing we get a family of smooth embeddings satisfying both $y_i \to 0$ and $\mathcal{W}(F_i) < const$ and $\operatorname{area}(T^2, F_i^* g_{S^3}) \to const.$

Hence, the first eigenvalue of D^2 is bounded from above. But the first eigenvalue of the spin-conformally equivalent flat torus with unit volume converges to ∞. This implies that there are spin-conformal classes in which the optimal constants in Lott's inequality [**Lot86**, Prop. 1] are not attained by flat metrics.

From this example we can also conclude that Theorem 5.3 does no longer hold, if we replace the condition "isometric spin immersion" by "conformal spin immersion".

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