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LAPLACE AND DIRAC OPERATORS ON NON-COMPACT MANIFOLDS

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Résumé

Les opérateurs de Dirac et de type Laplace sont fondamentaux tant pour les mathématiques que pour la physique. Si leur spectre fournit des informations importantes sur la géométrie de la variété, ils apparaissent aussi dans de nombreuses équations physiques (équation de la chaleur, de propagation, dynamique de l'électron...).

Notre (modeste) objectif est de décrire les opérateurs de Laplace et Dirac généralisés, et de prouver leur caractère essentiellement auto-adjoint dans le cas de variétés complètes mais non nécessairement compactes. Nous prouvons également, dans un cas particulier, le caractère essentiellement auto-adjoint de l'opérateur de Dirac spinoriel avec un potentiel Coulombien. Ces résultats sont établis en appliquant des théorèmes généraux d'analyse fonctionnelle. Le fait d'être essentiellement auto-adjoint est d'une importance cruciale, car cela implique de fortes propriétés pour le spectre de tels opérateurs. Nous prouvons par ailleurs le principe de décomposition, qui affirme l'invariance du spectre essentiel sous des perturbations à support compact.

L'article contient des introductions à l'analyse fonctionnelle ainsi qu'à la géométrie différentielle et spinorielle. Ces sections visent à présenter les concepts et résultats nécessaires aux preuves, si bien que l'article est, pour l'essentiel, auto-suffisant.

Abstract

Laplace and Dirac operators are fundamental operators in mathematics and physics. If their spectrum carries much information on geometrical data of the manifold, they also appear in many physical equations (heat equation, propagation equation, dynamics of the electron ...).

Our (modest) purpose is to briefly describe generalized Laplace and Dirac operators and to prove their essentially self-adjointness in the case of complete but non necessarily compact manifolds. Furthermore, we prove in a particular case the essentially self-adjointness of the spinorial Dirac operator with a Coulomb potential. These results are achieved by applying general theorems of functional analysis. Essentially self-adjointness is a property of crucial importance since it implies strong results for the spectrum of such operators. We also prove the decomposition principle, which states the invariance of the essential spectrum under compactly supported perturbations.

This article contains outlines on functional analysis as well as differential and spin geometry. These sections aim at introducing concepts and results needed for the proofs, so that this article is essentially self-sufficient.

Contents

Introduction

The article is structured as follows.

In section 1 we give a short review on linear operators in Hilbert spaces. Many of the propositions proved there are used only in this very section to prove further results.

In section 2 we recall fundamental concepts of differential geometry, and prove some useful formulas. Indisputably, the reader should have some background on differential geometry in order to understand this section and the rest of the article. A classical reference is [8], but there are a lot of other good books on the subject.

In section 3 we introduce some basic elements of spin geometry. Here there is no particular prerequisite, and the reader can refer to [9] for a (far!) more extended review on the subject.

In section 4 we describe Laplace and Dirac operators, using results from the first three sections (which are more or less independent). The reader having some background on these three subjects may read it independently.

Most of the time, we give explicit references to propositions or formulas that are used in the proof. In this way, $\ldots \underset{(5)}{=} \ldots$ (resp. $\ldots \underset{1.10}{\leq} \ldots$) means that we used formula (5) (resp. proposition 1.10) to prove the (in)equality. Furthermore, equalities and equivalences which are set by definition are denoted by ":=" and ":⇔". An index and a list of notations are to be found at the end of the article.

Of course, since we use functional analysis, differential/spin geometry and differential operators, we could not write down a complete (say rather extensive) review on these subjects (and we do not have the knowledge to do so, either). However, we tried to introduce the fundamental concepts, so that the reader who does not know much about some part may find the essential requirements. We also proved many of our claims, so that the paper may not turn out being just a list of results. Very often, we do not describe in detail all of our notations, but we tried to use coherent and "expressive" ones, so that the context should make clear what they mean. Let us hope that the article is not too far from what was intended, and that the reader may find some interest in reading it.

Last but not least, I want to express my grateful thanks to Christian Bär and all the members of the differential geometry team at Universität Hamburg, with a special mention to Sergiu for his sense of humor and his patience towards ceaseless questions.

1 Linear Operators

In this section, we introduce the concepts and theorems that are needed in the last section. Our only reference throughout the section is the book of Weidmann [12]. Almost all proofs given here are taken from it. We kept also many of the notations, so the reader who wants to read more should not be lost. Unfortunately, some proofs requiring to much work for the extent of this article, some results are just stated.

conventions: H, H_1 and H_2 are always vector spaces over the same field $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We will often speak of "scalar product", even in the complex case. Our convention for Hermitian products is left anti-linearity and right linearity (sorry for the reader who uses the other one).

1.1 Basic definitions

Definition 1.1. Let H_1 and H_2 be two vector spaces over $\mathbb{K}$. A *linear operator* (or simply *operator*) T from H_1 to H_2 is a linear map from a vector subspace $\mathrm{dom}(T)$ of H_1 (the *domain* of T) to H_2. The *range* of T is the vector subspace $\mathrm{ran}(T) := T(\mathrm{dom}(T))$ of H_2. The *kernel* of T is the vector subspace $\ker(T) := T^{-1}(\{0\})$ of H_1. T is called *densely defined* when $\mathrm{dom}(T)$ is dense in H_1 (with respect to the ambient topology).

When H_1 and H_2 are two normed spaces, the operator T is continuous (or bounded) if and only if

$$\|T\| := \sup\{\|Tf\| \mid f \in \mathrm{dom}(T),\ \|f\| \leq 1\} < \infty$$

We will note $L(H_1, H_2)$ the $\mathbb{K}$-vector space of all linear operators from H_1 to H_2 and $B(H_1, H_2)$ the $\mathbb{K}$-vector space of all continuous linear operators from H_1 to H_2 defined on *all* H_1 (i.e. $T \in B(H_1, H_2) \Rightarrow \mathrm{dom}(T) = H_1$). We set also $L(H) := L(H, H)$ and $B(H) := B(H, H)$. An operator T is called *bijective* when it is injective and surjective. Let us stress that this is abusive in the sense that T may not be a bijection.

Remarks.
- The domain of definition of T need *not* be equal to H_1!
- We will use the simplified notation $Tf := T(f)$ for $f \in \mathrm{dom}(T)$.

Definition 1.2. For two operators T and S in $L(H_1, H_2)$, we say that T is an *extension* of S (or that T extends S), and we will note $S \subset T$, when

$$\mathrm{dom}(S) \subset \mathrm{dom}(T) \quad \text{and} \quad \forall f \in \mathrm{dom}(S),\ Sf = Tf$$

Definition 1.3. Let H_1 and H_2 be two pre-Hilbert spaces, $T \in L(H_1, H_2)$, $S \in L(H_2, H_1)$. T and S are said to be *formal adjoints* when:

$$\forall f \in \mathrm{dom}(T),\ g \in \mathrm{dom}(S) \quad \langle Tf, g \rangle_{H_2} = \langle f, Sg \rangle_{H_1}$$

An operator in H which is formal adjoint to itself is called *symmetric*[1].

Definition 1.4. Let $T \in L(H_1, H_2)$. The *graph* of T is the vector subspace of $H_1 \times H_2$ defined by:

$$G(T) := \{(f, Tf) \mid f \in \mathrm{dom}(T)\}$$

It is easy to show that a subset G of $H_1 \times H_2$ is a graph of an operator if and only if it is a vector subspace with the property that $(0, g) \in G \Rightarrow g = 0$. In this situation, G is said to be a graph.

The following isomorphisms from $H_1 \times H_2$ to $H_2 \times H_1$ will be useful:

$$U(f_1, f_2) := (f_2, -f_1) \qquad V(f_1, f_2) := (f_2, f_1)$$

$H_1 \times H_2$ has a natural scalar product given by

$$\langle (f_1, f_2), (g_1, g_2) \rangle_{H_1 \times H_2} := \langle f_1, g_1 \rangle_{H_1} + \langle f_2, g_2 \rangle_{H_2}$$

with a similar formula for $H_2 \times H_1$. Note that for these scalar products, the isomorphisms U and V are isometries.

Definition 1.5. Let $T \in L(H_1, H_2)$. For $f, g \in \mathrm{dom}(T)$, $\langle f, g \rangle_T := \langle f, g \rangle_{H_1} + \langle Tf, Tg \rangle_{H_2}$ defines a scalar product. The associated norm (defined on $\mathrm{dom}(T)$), noted $\|.\|_T$, is called the *T-norm* or *graph norm*.

Definition 1.6. Let $T \in L(H_1, H_2)$, $S \in L(H_1, H_3)$ with $\mathrm{dom}(T) \subset \mathrm{dom}(S)$. S is called *T-bounded* when $\exists C \geq 0$ such that $\forall f \in \mathrm{dom}(T)$, $\|Sf\| \leq C\|f\|_T$. The *T-bound* of S is:

$$\inf \{b \geq 0,\ \mid \exists a \geq 0,\ \forall f \in \mathrm{dom}(T),\ \|Sf\| \leq a\|f\| + b\|Tf\|\}$$

1.2 Closed and closable operators

In this subsection we always assume $H, H_1, H_2, \ldots$ to be Hilbert spaces. When we consider a general operator on a Hilbert space, we cannot expect this operator to be bounded. However, we can still require (or hope ...) that it is not "too wild". Namely, an operator may not be continuous because:

- There is a sequence f_n in $\mathrm{dom}(T)$ converging to zero such that Tf_n does not converge.
- There is a sequence f_n in $\mathrm{dom}(T)$ converging to zero such that Tf_n converges, but not to zero.

Obviously we cannot do much to prevent the first "pathology" (without restricting to the case of bounded operators!), but it is also clear that the second case is particularly unpleasant. This is the reason why we introduce the notion of a closed (or closable) operator.

[1] Contrary to [12], we do not assume a symmetric operator to be densely defined

Definition 1.7. An operator T is said to be *closed* when its graph $G(T)$ is closed. It is said *closable* when $\overline{G(T)}$ is a graph. For a closable operator T, the *closure* $\overline{T}$ of T is the operator given by $G(\overline{T}) := \overline{G(T)}$.

Proposition 1.8. *Let $T \in L(H_1, H_2)$. We have:*

T is closed if and only if for all sequence f_n of $\mathrm{dom}(T)$ converging in H_1, such that Tf_n converges in H_2, we have $\lim f_n \in \mathrm{dom}(T)$ and $T(\lim f_n) = \lim Tf_n$.

T is closable if and only if for all sequence f_n in $\mathrm{dom}(T)$ converging to zero, such that Tf_n converges in H_2, we have $Tf_n \to 0$. In this case we have

$$\mathrm{dom}(\overline{T}) = \{f \in H_1 \mid \exists f_n \textit{ sequence in } \mathrm{dom}(T) \textit{ converging to } f \textit{ such that } Tf_n \textit{ converges}\}$$

We have also another criterion:

Proposition 1.9. *T is closed if and only if $(\mathrm{dom}(T), \langle .,. \rangle_T)$ is complete.*

Proposition 1.10. *Every bounded operator T is closable. For such an operator we have* $\mathrm{dom}(\overline{T}) = \overline{\mathrm{dom}(T)}$.

Proof. If $f_n \to 0$, $\|Tf_n\| \le \|T\|\|f_n\| \Rightarrow Tf_n \to 0$. It is clear that $\mathrm{dom}(\overline{T}) \subset \overline{\mathrm{dom}(T)}$. Let f_n be a sequence in $\mathrm{dom}(T)$ converging to f. From $\|Tf_n - Tf_m\| \le \|T\|\|f_n - f_m\|$ we see that Tf_n is a Cauchy sequence, so converges (H_2 is a Hilbert space), and $f \in \mathrm{dom}(\overline{T})$. □

Proposition 1.11. *Let T be a closable and injective operator. Then T^{-1} is closable if and only if $\overline{T}$ is injective. In this case $\overline{T^{-1}} = \overline{T}^{-1}$.*

Proof. We have $VG(\overline{T}) = \overline{VG(T)} = \overline{G(T^{-1})}$, which yields the result. □

Proposition 1.12. *Let H_1, H_2 be two Hilbert spaces, $T, S \in L(H_1, H_2)$, S T-bounded with T-bound smaller than 1. Then $T + S$ is closed (resp. closable) if and only if T is closed (resp. closable). We have* $\mathrm{dom}(\overline{T+S}) = \mathrm{dom}(\overline{T})$.

Proof. From the hypothesis, $\exists b < 1, a \ge 0,\ \forall f \in \mathrm{dom}(T),\ \|Sf\| \le a\|f\| + b\|Tf\|$. So:

$$\begin{aligned}
\|(T+S)f\| &\le \|Tf\| + \|Sf\| \\
&\le a\|f\| + (1+b)\|Tf\| \\
\|(T+S)f\| &\ge \|Tf\| - \|Sf\| \\
&\ge -a\|f\| + \underbrace{(1-b)}_{>0}\|Tf\|
\end{aligned}$$

So we can find $C > 0$ such that

$$\|Tf\| \le C(\|f\| + \|(T+S)f\|) \tag{1}$$

$$\|(T+S)f\| \le C(\|f\| + \|Tf\|) \tag{2}$$

So $\exists K_1, K_2 > 0$ with

$$K_1\|f\|_T \le \|f\|_{T+S} \le K_2\|f\|_T$$

Since the norms are equivalent, $(\mathrm{dom}(T), \langle .,. \rangle_T)$ is complete if and only if $(\mathrm{dom}(T), \langle .,. \rangle_{T+S})$ is complete, so T is closed if and only if $T + S$ is closed.

The equations (1) and (2) shows in particular that for a converging sequence f_n of $\mathrm{dom}(T)$, Tf_n is a Cauchy sequence (i.e. converges) if and only if $(T+S)f_n$ is a Cauchy sequence (i.e. converges).

Let $T + S$ be closable, f_n a sequence in $\mathrm{dom}(T)$ converging to zero, such that Tf_n converges. Hence $(T+S)f_n$ is a Cauchy sequence. $T + S$ is closable, so $(T+S)f_n \to 0$, and using one more time the inequalities, we see that $Tf_n \to 0$. So T is closable. Similarly, T closable implies $T + S$ closable.

The equality of domains follows from the characterization of the domain of the closure. □

1.3 Adjoint operator

In order to define the adjoint operator, let us recall the following theorem:

Theorem 1.13 (Riesz). *Let H be a Hilbert space over $\mathbb{K}$, $B(H,\mathbb{K})$ the space of* continuous *linear functionals on H. Then*

$$\begin{array}{ccc} H & \to & B(H,\mathbb{K}) \\ f & \mapsto & (g \mapsto \langle f, g\rangle) \end{array}$$

is an anti-linear isomorphism.

Definition 1.14. Let $T \in L(H_1, H_2)$ a densely defined operator. The *adjoint operator* T^* of T is the operator of $L(H_2, H_1)$ defined by

$$\begin{aligned} \operatorname{dom}(T^*) &:= \{g \in H_2 \mid f \mapsto \langle g, Tf\rangle \text{ is continuous as operator defined on } \operatorname{dom}(T)\} \\ &:= \{g \in H_2 \mid \exists\, w(g) \in H_1 \text{ with } \langle w(g), f\rangle = \langle g, Tf\rangle \text{ for all } f \in \operatorname{dom}(T)\} \\ T^* g &:= w(g), \quad w(g) \text{ as above} \end{aligned}$$

Proof. We must show that this definition is consistent. Since $\operatorname{dom}(T)$ is dense, $w(g)$ is clearly unique (when it exists).

Suppose that $f \mapsto \langle g, Tf\rangle$ is continuous on $\operatorname{dom}(T)$. Then it admits a unique continuous extension L_g on $\overline{\operatorname{dom}(T)}$. L_g is continuous on a closed vector subspace of a Hilbert space, hence on a Hilbert space. By theorem 1.13, $w(g)$ exists. □

Proposition 1.15. *Let $T \in L(H_1, H_2)$ be a densely defined operator.*

(i) *If T^* is densely defined, $T \subset T^{**}$*
(ii) *T symmetric $\Rightarrow T \subset T^*$*
(iii) $\ker(T^*) = \operatorname{ran}(T)^\perp$.
(iv) $G(T^*) = U(G(T)^\perp) = (UG(T))^\perp$
(v) *T^* is closed.*
(vi) *T is closable if and only if T^* is densely defined. In this case we have $\overline{T} = T^{**}$*
(vii) *If T is closable, then $(\overline{T})^* = T^*$.*
(viii) *Every symmetric operator is closable. Its closure is also symmetric.*

Proof. (i) $\forall f \in \operatorname{dom}(T)$, $g \mapsto \langle f, T^* g\rangle = \langle Tf, g\rangle$ is continuous on $\operatorname{dom}(T^*)$, hence $f \in \operatorname{dom}(T^{**})$ and $Tf = T^{**} f$.

(ii) $\forall f \in \operatorname{dom}(T)$, $g \mapsto \langle f, Tg\rangle = \langle Tf, g\rangle$ is continuous on $\operatorname{dom}(T)$, hence $f \in \operatorname{dom}(T^*)$ and $Tf = T^* f$.

(iii) Since $\operatorname{dom}(T)$ is dense,

$$\begin{aligned} T^* g = 0 &\Leftrightarrow \forall f \in \operatorname{dom}(T),\ \langle f, T^* g\rangle = 0 \\ &\Leftrightarrow \forall f \in \operatorname{dom}(T),\ \langle Tf, g\rangle = 0 \\ &\Leftrightarrow g \in \operatorname{ran}(T)^\perp \end{aligned}$$

(iv) By definition, and using the fact that U is an isometry:

$$\begin{aligned} G(T^*) &= \{(g,h) \in H_2 \times H_1 \mid \forall f \in \operatorname{dom}(T),\ \langle g, Tf\rangle = \langle h, f\rangle\} \\ &= \big\{(g,h) \in H_2 \times H_1 \mid \forall f \in \operatorname{dom}(T),\ \langle (g,h), (Tf, -f)\rangle_{H_2 \times H_1} = 0\big\} \\ &= (UG(T))^\perp = U(G(T)^\perp) \end{aligned}$$

(v) $G(T^*) = (UG(T))^\perp$ is closed.

(vi) $\overline{G(T)} = G(T)^{\perp\perp} = (U^{-1} G(T^*))^\perp$ so

$$\overline{G(T)} = \{(f,g) \in H_1 \times H_2 \mid \forall h \in \operatorname{dom}(T^*),\ \langle f, T^* h\rangle - \langle g, h\rangle = 0\}$$

Hence $(0,g) \in \overline{G(T)} \Leftrightarrow g \in \operatorname{dom}(T^*)^\perp$. So $\overline{G(T)}$ is a graph if and only if $\operatorname{dom}(T^*)$ is dense. In this case, $G(T^{**}) = U^{-1}(G(T^*)^\perp) = U^{-1}U(G(T)^{\perp\perp}) = \overline{G(T)} = G(\overline{T})$.

(vii) $G(T^*) = U(G(T)^\perp) = U(\overline{G(T)}^\perp) = U(G(\overline{T}))^\perp = G((\overline{T})^*)$.

(viii) We have $G(T) \subset G(T^*) \underset{(v)}{=} \overline{G(T^*)}$, which implies that $\overline{G(T)} \subset G(T^*)$. So $\overline{G(T)}$ is a vector subspace of a graph, hence is a graph. We have $G(T) \subset G(\overline{T}) \subset G(T^*) \underset{(vii)}{=} G((\overline{T})^*)$, and so $\overline{T}$ is symmetric (a densely defined operator is symmetric if and only if $G(T) \subset G(T^*)$).

□

Theorem 1.16 (closed graph). *Let $T \in L(H_1, H_2)$. Then two of the following statements imply the third:*

(i) T *is closed*

(ii) T *is bounded*

(iii) $\mathrm{dom}(T)$ *is closed*

Proof. In each case, let f_n be a sequence in $\mathrm{dom}(T)$ converging to f_∞ in H_1.

- $(i) + (ii) \Rightarrow (iii)$: T is bounded, so Tf_n is a Cauchy sequence in H_2, and converges in H_2. By proposition 1.8, $f_\infty \in \mathrm{dom}(T)$, hence $\mathrm{dom}(T)$ is closed.
- $(ii)+(iii) \Rightarrow (i)$: $\mathrm{dom}(T)$ is closed, so $f_\infty \in \mathrm{dom}(T)$. T being bounded, we have $Tf_n \to Tf_\infty$, hence $(f_n, Tf_n) \to (f_\infty, Tf_\infty) \in G(T)$, and $G(T)$ is closed.
- $(i) + (iii) \Rightarrow (ii)$: $\mathrm{dom}(T)$ is closed and so is a Hilbert space. For $g \in \mathrm{dom}(T^*)$ with $\|g\| \leq 1$ we set for all $f \in \mathrm{dom}(T)$,
$$L_g(f) := \langle T^* g, f\rangle$$
We have $|L_g(f)| = |\langle g, Tf\rangle| \leq \|Tf\|$. By the theorem of Banach-Steinhaus, we deduce that
$$\exists C,\ \forall g \in \mathrm{dom}(T^*),\ \|g\| \leq 1,\ \|L_g\| \leq C$$
but we have
$$\|T^*g\|^2 = \|L_g(T^*g)\| \leq \|L_g\|.\|T^*g\| \Rightarrow \|T^*g\| \leq C$$
hence T^* is bounded, and we already know it is closed. So $\mathrm{dom}(T^*)$ is closed (see above). Applying this result to T^* (which is closed with closed domain), we get that T^{**} is bounded. Since $T \subset T^{**}$, T is bounded.

□

Proposition 1.17. *Let $T \in L(H_1, H_2)$ be a bijective operator ($\ker T = \{0\}$, $\mathrm{ran}\, T = H_2$). Then:*
$$T \textit{ is closed} \Leftrightarrow T^{-1} \in B(H_2, H_1)$$

Proof. For an injective operator, we have $G(T^{-1}) = VG(T)$, hence T closed $\Leftrightarrow T^{-1}$ closed. Since T is surjective, $\mathrm{dom}(T^{-1}) = H_2$ is closed. The theorem of the closed graph yields the result. □

Definition 1.18. A densely defined operator $T \in L(H)$ such that $T = T^*$ is called *self-adjoint*. If T is symmetric and $\overline{T}$ is self-adjoint, T is called *essentially self-adjoint*.

1.4 Basis of spectral theory

Definition 1.19. Let $T \in L(H)$. $z \in \mathbb{K}$ is an *eigenvalue* of T when $z - T$ is not injective. In other words, $\exists f \in H,\ f \neq 0,\ Tf = zf$. Such a f is called an *eigenvector* of T associated to the eigenvalue z, and $\dim_{\mathbb{K}} \ker(z - T)$ (possibly ∞) is the *multiplicity* of z. The *point spectrum* of T is the set:
$$\sigma_p(T) := \{z \in \mathbb{K} \mid z - T \text{ is not injective}\}$$

When $z \notin \sigma_p(T)$ we set $R(z, T) := (z - T)^{-1}$

Definition 1.20. The *resolvent set*[2] of $T \in L(H)$ is

$$\varrho(T) := \left\{ z \in \mathbb{K} \mid z - T \text{ is injective, } \overline{\operatorname{ran}(z-T)} = H \text{ and } R(z,T) \text{ is bounded} \right\}$$

and the *spectrum* of T is

$$\sigma(T) := \mathbb{K} \setminus \varrho(T)$$

Definition 1.21. The *essential spectrum* $\sigma_e(T)$ of a self-adjoint operator T is the set of accumulation points of $\sigma(T)$ and eigenvalues with infinite multiplicity. The *discrete spectrum* $\sigma_d(T)$ of a self-adjoint operator T is the set $\sigma_d(T) := \sigma(T) \setminus \sigma_e(T)$. We say that T has a pure discrete spectrum when $\sigma_e(T) = \emptyset$.

Proposition 1.22. *For a closed operator $T \in L(H)$ we have*

$$\varrho(T) = \{ z \in \mathbb{K} \mid z - T \textit{ is bijective} \}$$

Proof. This is a simple consequence of the theorem of the closed graph (theorem 1.16), using the fact that for T closed, $z - T$ is closed. □

Proposition 1.23. *Let H_1, H_2 two Hilbert spaces, $T \in L(H_1, H_2)$ a closed bijective operator, $S \in L(H_1, H_2)$ with $\operatorname{dom}(T) \subset \operatorname{dom}(S)$ and $\|ST^{-1}\| < 1$. Then $T + S$ is also closed and bijective. We have:*

$$(T+S)^{-1} = \sum_{n=0}^{\infty} (-1)^n T^{-1} (ST^{-1})^n = \sum_{n=0}^{\infty} (-1)^n (T^{-1}S)^n T^{-1}$$

where these series converge in the sense of the norm of bounded operators in $B(H_2, H_1)$ (T^{-1} is bounded - see prop. 1.17).

Proof. The two series are obviously the same. Let us define

$$A_p := \sum_{n=0}^{p} (-1)^n T^{-1} (ST^{-1})^n$$

Since $\|ST^{-1}\| < 1$, this series is absolutely convergent, hence converge to $A \in B(H_2, H_1)$ ($B(H_2, H_1)$ is a Banach space because H_1 is complete). We have

$$\|Sf\| = \|ST^{-1}Tf\| \leq \|ST^{-1}\| \, \|Tf\|$$

so S is T-bounded with T-bound smaller than 1. Hence $T + S$ is closed. Furthermore,

$$\|(T+S)f\| \geq \|Tf\| - \|Sf\| \geq \underbrace{(1 - \|ST^{-1}\|)}_{>0} \|Tf\|$$

so $T + S$ is injective.

$$(T+S)A_p = Id + (-1)^{p+1} (ST^{-1})^{p+1} \to Id$$

$\forall g \in H_2$, $A_p g \to Ag$ and $(T+S)A_p g \to g$. Since $T + S$ is closed, $Ag \in \operatorname{dom}(T+S)$ and $(T+S)Ag = g$, i.e. $(T+S)A = Id_{H_2}$. In particular $\operatorname{ran}(T+S) = H_2$, which ends the proof. □

Theorem 1.24. *Let $T \in L(H)$ be a closed operator in the Hilbert space H. Then the resolvent $R(\cdot, T) \colon \varrho(T) \to B(H)$ is a continuous map. If $\sigma(T) \neq \emptyset$, we have*

$$\forall z \in \varrho(T), \ \|R(z,T)\|^{-1} \leq d(z, \sigma(T))$$

where $d(z, \sigma(T)) := \inf \{ |z - w| \mid w \in \sigma(T) \}$. If $z_0 \in \varrho(T)$, for $|z - z_0| < \|R(z_0, T)\|^{-1}$ we have $z \in \varrho(T)$ and:

$$R(z,T) = \sum_{n=0}^{\infty} (z_0 - z)^n R(z_0, T)^{n+1}$$

In particular, $\varrho(T)$ is open and $\sigma(T)$ is closed.

[2]Here we give a more general definition than that given in [12], so that $\varrho(T)$ is non necessarily empty for a non-closed operator.

Proof. Let $z_0 \in \varrho(T)$, $|z - z_0| < \|R(z_0,T)\|^{-1}$. Using proposition 1.23 with $T \equiv z_0 - T$ and $S \equiv (z - z_0)Id$, we obtain that $z - T$ is bijective, so $z \in \varrho(T)$, and the formula for $R(z,T)$ is a direct consequence. The resolvent $R(\cdot,T)$ is continuous since it is holomorphic. $\square$

Proposition 1.25. *Let T be a symmetric operator in the pre-Hilbert complex space H. Then $\sigma_p(T) \subset \mathbb{R}$, $\forall z \in \mathbb{C} \setminus \mathbb{R}$, $\|R(z,T)\| \leq |\operatorname{Im} z|^{-1}$.*

Proof. Let f be an eigenvector of T with eigenvalue z. $z^*\|f\|^2 = \langle Tf, f\rangle = \langle f, Tf\rangle = z\|f\|^2$. Since $f \neq 0$, this implies that $z = z^*$, i.e. $z \in \mathbb{R}$. With $z = x + iy$ we have

$$\begin{aligned}\forall f \in \operatorname{dom}(T),\ \|(z-T)f\|^2 &= \|(x-T)f + iyf\|^2 \\ &= \|(x-T)f\|^2 + |y|^2\|f\|^2 + iy\underbrace{\left(\langle (x-T)f, f\rangle - \langle f, (x-T)f\rangle\right)}_{=0} \\ &\geq |\operatorname{Im} z|^2\|f\|^2\end{aligned}$$

so $z - T$ is injective, and for $g \in \operatorname{dom}(R(z,T))$:

$$\|R(z,T)g\| \leq |\operatorname{Im} z|^{-1}\|(z-T)f\| = |\operatorname{Im} z|^{-1}\|g\|$$

so $\|R(z,T)\| \leq |\operatorname{Im} z|^{-1}$. $\square$

1.5 Criteria for (essentially) self-adjointness

Theorem 1.26. *Let T be a densely defined symmetric operator in a complex Hilbert space H. Then*

$$\begin{aligned}T = T^* &\Leftrightarrow \exists z_+,\ z_- \in \mathbb{C},\ \operatorname{Im} z_+ > 0,\ \operatorname{Im} z_- < 0,\ \operatorname{ran}(z_\pm - T) = H \\ &\Leftrightarrow \forall z \in \mathbb{C} \setminus \mathbb{R},\ \operatorname{ran}(z - T) = H\end{aligned}$$

Proof. "$\Rightarrow$" Suppose T is self-adjoint. Then all eigenvalues of T are real, i.e. $\forall z \in \mathbb{C} \setminus \mathbb{R}$, $\ker(z - T) = \{0\}$. So $\operatorname{ran}(z-T)^\perp = \ker((z-T)^*) = \ker(z^* - T) = \{0\}$. Hence $\operatorname{ran}(z - T)$ is dense. T is closed, so $R(z,T)$ is closed. From proposition 1.25, it is also bounded. Applying the theorem of the closed graph (theorem 1.16), $\operatorname{dom}(R(z,T)) = \operatorname{ran}(z - T)$ is closed, but also dense, thus $\operatorname{ran}(z - T) = H$.

"$\Leftarrow$" Suppose there exist $z_\pm$ as specified. T being symmetric, the operators $z_\pm - T$ are injective, and by hypothesis, surjective. From proposition 1.25, $\|R(z_\pm,T)\| \leq |\operatorname{Im} z_\pm|^{-1}$. Theorem 1.24 tells us that $z - T$ is also bijective for all z with $|z - z_+| < |\operatorname{Im} z_+|$ or $|z - z_-| < |\operatorname{Im} z_-|$. Iterating the same argument, we get that $(z - T)$ is bijective for all $z \in \mathbb{C} \setminus \mathbb{R}$.

T is symmetric, so $T \subset T^*$. Let $f \in \operatorname{dom}(T^*)$ and $f_0 := (i - T)^{-1}(i - T^*)f$. $f_0 \in \operatorname{dom}(i - T) = \operatorname{dom}(T) \subset \operatorname{dom}(T^*)$. $Tf_0 = T^*f_0$ and

$$(i - T^*)(f - f_0) = (i - T^*)f - (i - T^*)f_0 = (i - T)f_0 - (i - T)f_0 = 0$$

Hence $f - f_0 \in \ker(i - T^*) = \operatorname{ran}(-i - T)^\perp = H^\perp = \{0\}$ and so $f = f_0 \in \operatorname{dom}(T)$. We have $\operatorname{dom}(T^*) \subset \operatorname{dom}(T)$, thus $T = T^*$.

$\square$

Corollary 1.27. *Let T be a densely defined symmetric operator in a complex Hilbert space H. Then*

$$\begin{aligned}\overline{T} = T^* &\Leftrightarrow \exists z_+,\ z_- \in \mathbb{C},\ \operatorname{Im} z_+ > 0,\ \operatorname{Im} z_- < 0,\ \overline{\operatorname{ran}(z_\pm - T)} = H \\ &\Leftrightarrow \forall z \in \mathbb{C} \setminus \mathbb{R},\ \overline{\operatorname{ran}(z - T)} = H\end{aligned}$$

Proof. It is enough to prove that $\forall z \in \mathbb{C} \setminus \mathbb{R}$, $\overline{\operatorname{ran}(z - T)} = \operatorname{ran}(z - \overline{T})$. We know that $z - T$ and $\overline{z - T} = z - \overline{T}$ are injective, and that $(z - T)^{-1}$ is bounded. Hence

$$\overline{\operatorname{ran}(z - T)} = \overline{\operatorname{dom}((z-T)^{-1})} \underset{1.10}{=} \operatorname{dom}\left(\overline{(z-T)^{-1}}\right) \underset{1.11}{=} \operatorname{dom}((\overline{z-T})^{-1}) = \operatorname{ran}(z - \overline{T})$$

$\square$

Corollary 1.28. *Let T be a densely defined symmetric operator in a complex Hilbert space H. Then T is essentially self-adjoint if and only if* $\ker(\pm i - T^*) = \{0\}$.

Corollary 1.29. *Let T be a densely defined symmetric operator in a complex Hilbert space H. Suppose that $\exists n \in \mathbb{N}$, $n \geq 2$, with* $\operatorname{ran}(i - T^n) = H$ or $\operatorname{ran}(-i - T^n) = H$ *(resp.* $\overline{\operatorname{ran}(i - T^n)} = H$ or $\overline{\operatorname{ran}(-i - T^n)} = H$*). Then T is (resp. essentially) self-adjoint.*

In particular, if T^n is (essentially) self-adjoint, so is T.

Proof. Suppose that $\overline{\operatorname{ran}(i - T^n)} = H$. We can find complex numbers $\gamma_\pm$ satisfying $\operatorname{Im}\gamma_+ > 0$, $\operatorname{Im}\gamma_- < 0$ and $\gamma_\pm^n = i$. Then

$$(T^n - i) = (T - \gamma_\pm)(T^{n-1} + T^{n-2}\gamma_\pm + \cdots + T\gamma_\pm^{n-2} + \gamma_\pm^{n-1})$$

Hence $\overline{\operatorname{ran}(T - \gamma_\pm)} \supset \overline{\operatorname{ran}(T^n - i)} = H$, so T is essentially self-adjoint from theorem 1.26.

Other cases are proved in the same way. □

Theorem 1.30 (Rellich-Kato). *Let T be a self-adjoint (resp. essentially self-adjoint) operator in the Hilbert space H, S a symmetric T-bounded operator with T-bound lower than 1. Then $T + S$ is self-adjoint (resp. essentially self-adjoint with $\overline{T + S} = \overline{T} + \overline{S}$ and* $\operatorname{dom}(\overline{T + S}) = \operatorname{dom}(\overline{T})$*)*

Theorem 1.31 (Wüst). *Let T be an essentially self-adjoint operator, S a symmetric operator with* $\operatorname{dom}(T) \subset \operatorname{dom}(S)$ *and assume that* $\exists C \geq 0$, $\forall f \in \operatorname{dom}(T)$, $\|Sf\| \leq C\|f\| + \|Tf\|$. *Then $T + S$ is essentially self-adjoint.*

1.6 Spectral theory for self-adjoint operators

1.6.1 Resolution of the identity

Let T_n be a sequence of bounded operators, $(T_n) \in B(H_1, H_2)^{\mathbb{N}}$, and $T \in B(H_1, H_2)$. We say that T_n converges *strongly* to T, and we will note $T_n \xrightarrow{s} T$ in the following case:

$$T_n \xrightarrow{s} T :\Leftrightarrow \forall f \in H_1,\ T_n f \xrightarrow[n\to\infty]{} Tf$$

Definition 1.32. A *resolution of the identity* in a Hilbert space H is a function $E\colon \mathbb{R} \to B(H)$ satisfying:

(i) $\forall t \in \mathbb{R}$, $E(t)$ is an orthogonal projection

(ii) $s \leq t \Rightarrow \operatorname{ran} E(s) \subset \operatorname{ran} E(t)$ (which we note by $E(s) \leq E(t)$)

(iii) $\forall t \in \mathbb{R}$, $E(t + \varepsilon) \xrightarrow[\varepsilon\to 0^+]{s} E(t)$

(iv) $E(t) \xrightarrow[t\to-\infty]{s} 0$, $\quad E(t) \xrightarrow[t\to+\infty]{s} Id$

Let E be a resolution of the identity in H. We set:

$$\forall f \in H,\ t \in \mathbb{R}, \quad \mu_f(t) := \langle f, E(t)f\rangle = \|E(t)f\|^2$$

$\mu_f\colon \mathbb{R} \to \mathbb{R}$ is bounded, not decreasing and right-continuous, with $\lim_{t\to-\infty}\mu_f(t) = 0$ and $\lim_{t\to\infty}\mu_f(t) = \|f\|^2$. Thanks to these properties, we can speak of μ_f-measurable functions and define $L^2(\mu_f)$. A function $u\colon \mathbb{R} \to \mathbb{K}$ is said to be E-measurable if it is μ_f-measurable for all $f \in H$. All Borel-measurable functions are E-measurable. The constant function 1 is in $L^2(\mu_f)$ for all $f \in H$, and so all bounded E-measurable functions are also in $L^2(\mu_f)$ for all $f \in H$.

We can show that a monotone sequence of orthogonal projections admits a limit (in the strong sense), which is still an orthogonal projection. Applying this result to a resolution of the identity E, we can define $E(t-)$ such that $E(t - \varepsilon) \xrightarrow[\varepsilon\to 0^+]{s} E(t-)$ (we have similarly $E(t+) = E(t)$ because E is right continuous). For $a, b \in \mathbb{R}$ we set (with $E(-\infty) = 0$ and $E(\infty) = Id$):

$$E((a,b]) := E(b) - E(a) \qquad E((a,b)) := E(b-) - E(a)$$
$$E([a,b]) := E(b) - E(a-) \qquad E([a,b)) := E(b-) - E(a-)$$

(note the similarity with the definition of a measure with a not decreasing right continuous function). So, "as usual" the next step is to define the integral of step functions: for $u(t) = \sum_{j=1}^{n} u_j \chi_{I_j}(t)$ we set

$$\int u(t)dE(t) := \int \sum_{j=1}^{n} u_j \chi_{I_j}(t) dE(t) = \sum_{j=1}^{n} u_j E(I_j)$$

For a step function, we can check that

$$\left\| \int u(t)dE(t)f \right\|^2 = \int |u(t)|^2 \, d\mu_f(t)$$

The property above shows that $u \to 0$ in $L^2(\mu_f)$ implies $\int u(t)dE(t)f \to 0$, where u is a step function. So a Cauchy sequence of step functions u_n in $L^2(\mu_f)$ gives rise to a Cauchy sequence $\int u_n(t)dE(t)f$ in H, thus convergent (let us recall that H is an Hilbert space). Every function u in $L^2(\mu_f)$ is a limit of step functions u_n in the sense of $L^2(\mu_f)$, and so we define

$$\int u(t)dE(t)f := \lim_{n\to\infty} \int u_n(t)dE(t)f$$

This definition is of course independent from the Cauchy sequence used to define it. The defined integral is linear and we have (this time for all function in $L^2(\mu_f)$):

$$\left\| \int u(t)dE(t)f \right\|^2 = \int |u(t)|^2 \, d\mu_f(t)$$

Theorem 1.33. *Let E be a resolution of the identity in H, $u \colon \mathbb{R} \to \mathbb{K}$ a E-measurable function. Then we can define an operator $\hat{E}(u)$ in H by:*

$$\begin{aligned} &\mathrm{dom}(\hat{E}(u)) := \left\{ f \in H, \ \mid u \in L^2(\mu_f) \right\} \\ &\hat{E}(u)f := \textstyle\int u(t)dE(t)f \quad \textit{for } f \in \mathrm{dom}(\hat{E}(u)) \end{aligned}$$

For all u, v E-measurable functions, $a, b \in \mathbb{K}$, $f \in \mathrm{dom}(\hat{E}(u))$, $g \in \mathrm{dom}(\hat{E}(v))$ and

$$\varphi_n(t) := \begin{cases} 1 & \textit{when } |u(t)| \le n, \\ 0 & \textit{otherwise}, \end{cases} \qquad \psi_n(t) := \begin{cases} 1 & \textit{when } |v(t)| \le n, \\ 0 & \textit{otherwise}, \end{cases}$$

it satisfies:

- *(i)* $\langle \hat{E}(v)g, \hat{E}(u)f \rangle = \lim_{n\to\infty} \int (\psi_n v^* \varphi_n u)(t) d\langle g, E(t)f \rangle$
- *(ii)* $\|\hat{E}(u)f\|^2 = \int |u(t)|^2 \, d\mu_f(t)$
- *(iii)* $\|u\|_\infty = C < \infty \Rightarrow (\mathrm{dom}(\hat{E}(u)) = H, \ \|\hat{E}(u)\| \le \|u\|_\infty)$
- *(iv)* $\hat{E}(1) = Id$
- *(v)* $\forall h \in H, \ \langle h, \hat{E}(u)f \rangle = \int u(t) d\langle h, E(t)f \rangle$
- *(vi)* $\forall t \in \mathbb{R}, \ u(t) \ge c \Rightarrow \langle f, \hat{E}(u)f \rangle \ge c \|f\|^2$
- *(vii)* $\mathrm{dom}(\hat{E}(|u| + |v|)) \subset \mathrm{dom}(\hat{E}(u) + \hat{E}(v)), \ a\hat{E}(u) + b\hat{E}(v) \subset \hat{E}(au + bv)$
- *(viii)* $\mathrm{dom}(\hat{E}(u)\hat{E}(v)) = \mathrm{dom}(\hat{E}(v)) \cap \mathrm{dom}(\hat{E}(uv)), \ \hat{E}(u)\hat{E}(v) \subset \hat{E}(uv)$
- *(ix)* $\mathrm{dom}(\hat{E}(u^*)) = \mathrm{dom}(\hat{E}(u)), \ \hat{E}(u^*) = \hat{E}(u)^*$

In the case of the preceding theorem, we will simply note:

$$\hat{E}(u) = \int u(t)dE(t)$$

1.6.2 The spectral theorem. Characterization of the spectra

Theorem 1.34. *Let T be a self-adjoint operator in the Hilbert space H. Then there exists a unique resolution of the identity E so that:*

$$T = \hat{E}(Id), \ ie \ T = \int t dE(t)$$

In the complex case, E is given by:

$$\langle g, (E(b) - E(a))f \rangle = \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{a+\delta}^{b+\delta} \langle g, (R(t - i\varepsilon, T) - R(t + i\varepsilon, T))f \rangle \, dt$$

When E is the resolution of the identity associated to the self-adjoint operator T, and for a E-measurable function u, we set $u(T) = \hat{E}(u)$. The following proposition justifies this notation:

Proposition 1.35. *For $u(t) = \sum_{j=0}^{n} u_j t^j$, with the usual convention that $T^0 = Id$, we have*

$$u(T) = \sum_{j=0}^{n} u_j T^j$$

Proposition 1.36. *Let T be a self-adjoint operator in the Hilbert space H, E its resolution of the identity. Then:*

(i) Let $\gamma \in \mathbb{R}$. $\forall f \in \mathrm{dom}(T), \ \langle f, Tf \rangle \leq \gamma \|f\|^2 \Leftrightarrow \forall t \geq \gamma, \ E(t) = Id$.

(ii) Let $\gamma \in \mathbb{R}$. $\forall f \in \mathrm{dom}(T), \ \langle f, Tf \rangle \geq \gamma \|f\|^2 \Leftrightarrow \forall t < \gamma, \ E(t) = 0$.

(iii) Let $t, s \in \mathbb{R}, \ s > 0$. $f \in \mathrm{ran}(E(t+s) - E(t-s)) \Rightarrow (f \in \mathrm{dom}(T)$ and $\|(T - t)f\| \leq s\|f\|)$.

In the following propositions, T is a self-adjoint operator in an Hilbert space H, E is its resolution of the identity and T_0 a restriction of T with $\overline{T_0} = T$.

Theorem 1.37. *The following statements are equivalent:*

(i) $\lambda \in \sigma(T)$

(ii) There is a sequence $(f_n) \in \mathrm{dom}(T)^{\mathbb{N}}$ with $\liminf \|f_n\| > 0$ and $(\lambda - T)f_n \to 0$.

(iii) There is a sequence $(g_n) \in \mathrm{dom}(T_0)^{\mathbb{N}}$ with $\liminf \|g_n\| > 0$ and $(\lambda - T)g_n \to 0$.

(iv) $\forall \varepsilon > 0, \ E(\lambda + \varepsilon) - E(\lambda - \varepsilon) \neq 0$.

Proof. $(i) \Rightarrow (ii)$ If $\lambda \in \sigma_p(T)$, it is obvious (take a constant sequence equal to an eigenvector).

If $\lambda \notin \sigma_p(T)$, since T is self-adjoint, the eigenvalues are real, hence $\lambda^* \notin \sigma_p(T)$. Thus,

$$\{0\} = \ker(\lambda^* - T) = \ker(\lambda^* - T^*) = \mathrm{ran}(\lambda - T)^{\perp}$$

But $\lambda \in \sigma(T)$, so $R(\lambda, T)$ is not bounded. (ii) follows easily.

$(ii) \Rightarrow (i)$ If $\lambda - T$ is not injective, then $\lambda \in \sigma_p(T) \subset \sigma(T)$. If $\lambda - T$ is injective, we set $h_n := (\lambda - T)f_n$. $\|h_n\| \to 0$ but $\liminf \|R(\lambda, T)h_n\| = \liminf \|f_n\| > 0$, so $R(\lambda, T)$ is not bounded.

$(ii) \Rightarrow (iii)$ Since $\overline{G(T_0)} = G(T)$,

$$\forall n \in \mathbb{N}, \ \exists g_n \in \mathrm{dom}(T_0) \text{ with } \|g_n - f_n\| \leq \frac{1}{n} \text{ and } \|T_0 g_n - T f_n\| \leq \frac{1}{n}$$

so we have $\liminf \|g_n\| > 0$ and $(\lambda - T)g_n \to 0$.

$(iii) \Rightarrow (ii)$ $\mathrm{dom}(T_0) \subset \mathrm{dom}(T)$ so we just take $f_n := g_n$.

$(ii) \Rightarrow (iv)$ Suppose $\exists \varepsilon > 0$ with $E(\lambda + \varepsilon) - E(\lambda - \varepsilon) = 0$. For the sequence f_n we have

$$\|(\lambda - T)f_n\|^2 = \int |\lambda - t|^2 d\|E(t)f_n\|^2 \geq \varepsilon^2 \int d\|E(t)f_n\|^2 = \varepsilon^2 \|f_n^2\|$$

This is a contradiction.

$(iv) \Rightarrow (ii)$ $\forall n \in \mathbb{N},\ E(\lambda + \frac{1}{n}) - E(\lambda - \frac{1}{n}) \neq 0$. So take $f_n \in \operatorname{ran}(E(\lambda + \frac{1}{n}) - E(\lambda - \frac{1}{n}))$ with $\|f_n\| = 1$. From proposition 1.36(iii) we have $\|(\lambda - T)f_n\| \leq \frac{1}{n}\|f_n\| \to 0$.

□

Corollary 1.38. *Let $a < b$. Then $E(b) - E(a) \neq 0 \Rightarrow (a, b] \cap \sigma(T) \neq \emptyset$. $(a, b) \cap \sigma(T) \neq \emptyset \Leftrightarrow E(b-) - E(a) \neq 0$*

Proof. If $(a, b] \subset \varrho(T)$, for all $\lambda \in (a, b]$, E is constant in a neighborhood of λ. Hence $E(b) = E(a+) = E(a)$. If $(a, b) \subset \varrho(T)$ we show in the same way that $E(b-) = E(a)$. If $\lambda \in (a, b) \cap \sigma(T)$, we choose $\varepsilon > 0$ small enough such that $a < \lambda - \varepsilon < \lambda + \varepsilon < b$. Then $E(\lambda + \varepsilon) - E(\lambda - \varepsilon) \neq 0 \Rightarrow E(b-) - E(a) \neq 0$. □

Theorem 1.39. *The following statements are equivalent:*

(i) $\lambda \in \sigma_p(T)$

(ii) There is a Cauchy sequence $(f_n) \in \operatorname{dom}(T)^{\mathbb{N}}$ *with* $\lim \|f_n\| > 0$ *and* $(\lambda - T)f_n \to 0$.

(iii) There is a Cauchy sequence $(g_n) \in \operatorname{dom}(T_0)^{\mathbb{N}}$ *with* $\lim \|g_n\| > 0$ *and* $(\lambda - T)g_n \to 0$.

(iv) $E(\lambda) - E(\lambda-) \neq 0$.

Furthermore we have $\ker(\lambda - T) = \operatorname{ran}(E(\lambda) - E(\lambda-))$.

Proof. $(i) \Rightarrow (ii)$ Obvious.

$(ii) \Rightarrow (iii)$ Construct g_n as in the part "$(ii) \Rightarrow (iii)$" of the proof of theorem 1.37. It is still a Cauchy sequence.

$(iii) \Rightarrow (i)$ Let $f := \lim g_n$. $\|f\| = \lim \|g_n\| > 0$ and T is closed, so $(\lambda - T)f = \lim(\lambda - T_0)g_n = 0$.

$(i) \Rightarrow (iv)$ Let $f \in \ker(\lambda - T)$. Then

$$0 = \|(\lambda - T)f\|^2 = \int |\lambda - t|^2 d\|E(t)f\|^2$$

so $|\lambda - t| = 0$ almost everywhere with respect to the measure μ_f. Hence $E(t)f$ is constant on $(-\infty, \lambda)$ and $(\lambda, +\infty)$, therefore $E(\lambda-)f = 0$ and $E(\lambda)f = f$. Thus $E(\lambda) - E(\lambda-) \neq 0$.

$(iv) \Rightarrow (i)$ By proposition 1.36 (iii), for $f \in \operatorname{ran}(E(\lambda) - E(\lambda-))$ we have $\forall \varepsilon > 0$, $\|(\lambda - T)f\| \leq \varepsilon\|f\|$ hence $f \in \ker(\lambda - T)$.

From the two last points we have $\ker(\lambda - T) = \operatorname{ran}(E(\lambda) - E(\lambda-))$. □

Corollary 1.40. *An isolated point of $\sigma(T)$ is an eigenvalue of T. So $\sigma_d(T)$ is the set of all eigenvalues of T which are isolated points of the spectrum and have finite multiplicity, and $\sigma_e(T)$ is closed.*

Proof. Since λ is isolated, E is locally constant on the left and on the right of λ. But $\lambda \in \sigma(T)$ implies that $\forall \varepsilon > 0,\ E(\lambda + \varepsilon) - E(\lambda - \varepsilon) \neq 0$. Hence $E(\lambda) - E(\lambda-) \neq 0 \Rightarrow \lambda \in \sigma_p(T)$.

With this result the statement on $\sigma_d(T)$ is obvious. Furthermore, $\sigma_e(T)$ is obtained from the closed set $\sigma(T)$ by removing isolated points, hence remains closed. □

Theorem 1.41. *The following statements are equivalent:*

(i) $\lambda \in \sigma_e(T)$

(ii) There is a sequence $(f_n) \in \operatorname{dom}(T)^{\mathbb{N}}$ *with* $f_n \rightharpoonup 0$, $\liminf \|f_n\| > 0$ *and* $(\lambda - T)f_n \to 0$.

(iii) There is a sequence $(g_n) \in \operatorname{dom}(T_0)^{\mathbb{N}}$ *with* $g_n \rightharpoonup 0$, $\liminf \|g_n\| > 0$ *and* $(\lambda - T)g_n \to 0$.

(iv) $\forall \varepsilon > 0,\ \dim \operatorname{ran}(E(\lambda + \varepsilon) - E(\lambda - \varepsilon)) = \infty$.

where "$\rightharpoonup$" denotes weak convergence ($f_n \rightharpoonup f :\Leftrightarrow \forall h,\ \langle f_n, h\rangle \to \langle f, h\rangle$).

Proof. $(i) \Rightarrow (ii)$ If $\lambda \in \sigma_p(T)$ with infinite multiplicity, it is obvious, since we can take an orthonormal sequence in $\ker(\lambda - T)$.

If λ is an accumulation point of the spectrum, let us choose a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in the spectrum such that $n \neq m \Rightarrow \lambda_n \neq \lambda_m$, $\lambda_n \to \lambda$. We choose also $\varepsilon_n > 0$ such that the intervals $(\lambda_n - \varepsilon_n, \lambda_n + \varepsilon_n)$ are pairwise disjoint (and so $\varepsilon_n \to 0$). From $\lambda_n \in \sigma(T)$ we have (see theorem 1.37) $E(\lambda_n + \varepsilon_n) - E(\lambda_n - \varepsilon_n) \neq 0$. Let f_n be any element of norm 1 in $\operatorname{ran}(E(\lambda_n + \varepsilon_n) - E(\lambda_n - \varepsilon_n))$. We have

$$f_n \in \operatorname{dom}(T), \qquad \langle f_n, f_m \rangle = \delta_{nm} \Rightarrow f_n \rightharpoonup 0 \quad \text{and} \quad \|(\lambda - T)f_n\| \underset{1.36}{\leq} (|\lambda - \lambda_n| + \varepsilon_n)\|f_n\| \to 0$$

$(ii) \Rightarrow (iii)$ We choose g_n as always. The properties easily follow.

$(iii) \Rightarrow (iv)$ Suppose $\exists \varepsilon > 0$ with $\dim \operatorname{ran}(E(\lambda + \varepsilon) - E(\lambda - \varepsilon)) < \infty$. Let P be the projection

$$P := E(\lambda + \varepsilon) - E(\lambda - \varepsilon)$$

Let $h_n := \frac{Pg_n}{\|Pg_n\|}$. Since $\operatorname{ran}(P)$ is of finite dimension, we can take a subsequence h_{n_k} converging to $h \in \operatorname{ran}(P)$ with $\|h\| = 1$. In fact the subsequence g_{n_k} satisfies all the properties of g_n, so without loss of generality we can assume that h_n converges (and so we drop the index "n_k"). We have

$$0 = \lim \langle g_n, h \rangle = \lim \langle Pg_n, h \rangle = \lim \|Pg_n\| \langle h_n, h \rangle = \lim \|Pg_n\|$$

Then
$$\begin{aligned} \|(\lambda - T)g_n\|^2 &= \int |\lambda - t|^2 d\|E(t)g_n\|^2 \\ &\geq \varepsilon^2 \left(\int d\|E(t)g_n\|^2 - \int_{(\lambda-\varepsilon, \lambda+\varepsilon]} d\|E(t)g_n\|^2 \right) \\ &\geq \varepsilon^2 \left(\|g_n\|^2 - \|Pg_n\|^2 \right) \end{aligned}$$

so $\liminf \|(\lambda - T)g_n\|^2 \geq \varepsilon^2 \liminf \|g_n\|^2 > 0$. It is a contradiction.

$(iv) \Rightarrow (i)$ If $\dim \operatorname{ran}(E(\lambda) - E(\lambda -)) = \infty$ then λ is an eigenvalue with infinite multiplicity.

If $\dim \operatorname{ran}(E(\lambda) - E(\lambda -)) < \infty$, since $\forall \varepsilon > 0$, $\dim E(\lambda + \varepsilon) - E(\lambda - \varepsilon) = \infty$, we must have $E(\lambda + \varepsilon) - E(\lambda) \neq 0$ or $E(\lambda -) - E(\lambda - \varepsilon) \neq 0$. Hence, from corollary 1.38, for all $\varepsilon > 0$, $((\lambda - \varepsilon, \lambda) \cup (\lambda, \lambda - \varepsilon)) \cap \sigma(T) \neq \emptyset$, and λ is an accumulation point of the spectrum.

□

Definition 1.42. A sequence f_n satisfying the properties of (ii) (or (iii)) in theorem 1.41 is called a *Weyl sequence* associated to T, λ.

2 Differential geometry

In this section we want to introduce some basic (but fundamental) concepts of differential geometry, namely bundles and derivatives. We also prove some useful formulas and propositions. We do not aim at completeness, hence we refer to [8] or [4] for the elements discussed here.

All manifolds considered in this paper are finite dimensional, paracompact, connected, real manifolds without boundary. Unless stated otherwise (or obviously used in a different meaning), n always denotes the real dimension of the manifold M.

2.1 Bundles

A fundamental class of objects in differential geometry are the so-called fiber bundles. It is a generalization of the product of two spaces, where the total space may be "twisted".

Definition 2.1. A *fiber bundle* over a manifold M, with typical fiber E, is a manifold $\mathcal{E}$ together with a smooth map $\pi\colon \mathcal{E} \to M$ so that there is a covering of M by open sets U_α and diffeomorphisms $\phi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times E$, called *local trivializations*, such that $\pi|_{\pi^{-1}(U_\alpha)} = pr_1 \circ \phi_\alpha$, where pr_1 denotes the projection onto U_α in $U_\alpha \times E$.

So for a fiber bundle we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{E} \supset \pi^{-1}(U_\alpha) \arrow[r, "\phi_\alpha"] \arrow[d, "\pi"'] & U_\alpha \times E \arrow[d, "pr_1"] \\ M & U_\alpha \arrow[l, hook'] \end{tikzcd}$$

From this definition, we see that fibers $\mathcal{E}_x := \pi^{-1}(x)$ are diffeomorphic to E. However, this diffeomorphism is *not* canonical (in general). Note also that $\phi_{\alpha\beta} := \phi_\beta \circ \phi_\alpha^{-1}$ are diffeomorphisms of $(U_\alpha \cap U_\beta) \times E$ which act "fiberwise", ie we have *transition functions*:

$$\begin{array}{rrcl} \varphi_{\alpha\beta}: & U_\alpha \cap U_\beta & \to & \mathit{Diff}(E) \\ & x & \mapsto & (v \mapsto \phi_{\alpha\beta}(x,v)) \end{array}$$

which satisfy the *cocycle conditions*:

$$\begin{aligned} \phi_{\alpha\alpha} &= Id|_{U_\alpha \times E} \\ \phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} &= Id|_{(U_\alpha \cap U_\beta \cap U_\gamma)\times E} \end{aligned}$$

In fact such transition functions contain all the information on the bundle. Namely, given an open covering U_α of M with transition functions $\varphi_{\alpha\beta}$ which satisfy the cocycle conditions, we can construct a fiber bundle by "patching" local trivializations according to the transition functions.

A *section* s of a fiber bundle $\mathcal{E}$ over M is a map $s\colon M \to \mathcal{E}$ such that $\pi \circ s = Id|_M$. We will note $\Gamma(M,\mathcal{E})$ the space of all smooth sections in $\mathcal{E}$, and $\Gamma_c(M,\mathcal{E})$ the space of all smooth sections in $\mathcal{E}$ with compact support.

From this definition of fiber bundle we can construct different kinds of bundles, according to the nature of the fiber.

Definition 2.2. A *vector bundle* is a fiber bundle with a vector space E as typical fiber, and such that the diffeomorphisms ϕ_α may be chosen so that the transition functions $\varphi_{\alpha\beta}$ are vector space isomorphisms.

The canonical example of a vector bundle is TM, the *tangent bundle* of a manifold M. Another example is the *cotangent bundle* T^*M.

Definition 2.3. A *principal bundle* with structure group G is a fiber bundle P with a free and transitive right action of a Lie group G on the fibers:

$$\forall p \in P,\ \forall g,h \in G, \quad \begin{cases} (p{\cdot}g){\cdot}h = p{\cdot}(gh) \\ \pi(p{\cdot}g) = \pi(p) \\ p{\cdot}g = p \Leftrightarrow g = e \\ \forall q \in P,\ \exists g_{pq} \in G \mid p{\cdot}g_{pq} = q \end{cases}$$

Since the action is free and transitive, we can construct transition functions $\varphi_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ still satisfying the cocycle conditions.

Given a principal bundle P with structure group G and E a space with a left action of G by diffeomorphisms (i.e. there is an homomorphism $\rho\colon G \to \mathit{Diff}(E)$), we can construct an associated bundle with fiber E.

Definition 2.4. Let P be a principal bundle with structure group G, E a manifold, ρ an homomorphism from G into $\mathit{Diff}(E)$, the *associated bundle* is the product $P \times_\rho E$, where $P \times_\rho E$ is the quotient manifold of $P \times E$ under the equivalence relation

$$(p{\cdot}g, v) \sim (p, \rho(g)v)$$

We will also note $P \times_G E := P \times_\rho E$ when there is no ambiguity concerning the representation of G. We will systematically use the notation $\mathcal{E}$ (resp. $\mathcal{F}$) for the associated bundle to E (resp. F). The transition functions on the associated bundle are given by $\rho \circ \varphi_{\alpha\beta}$ where $\varphi_{\alpha\beta}$ are the transition functions of P. An element in $\mathcal{E}$ will be noted as $[\![p,v]\!]$ where $p \in P$, $v \in E$ and $[\![.]\!]$ denote the equivalence class of the given element. A particular case of associated bundles is when we have a representation of G on a vector space. From now on we may use the notations $P,\ M,\ \mathcal{E},\ E,\ \pi, \ldots$ without further specification.

Definition 2.5. Let $\mathcal{E}$ be a vector bundle over M with fiber $\mathbb{K}^N$. The *frame bundle* $GL(\mathcal{E})$ of $\mathcal{E}$ is the bundle whose fiber over $x \in M$ is the space of all invertible linear maps from $\mathbb{K}^N$ to the fiber $\mathcal{E}_x$. It is a principal bundle with structure group $GL(N,\mathbb{K})$ under the action

$$(p{\cdot}g)(v) := p(g(v))$$

where $p\colon \mathbb{K}^N \to \mathcal{E}_x$, $g \in GL(N,\mathbb{K})$ and $v \in \mathbb{K}^N$.

Associated bundles constructed from representations of $GL(M) := GL(TM)$ are called *tensor bundles*. Let us give a few examples (ρ denotes here the canonical representation of $GL_N(\mathbb{R})$ on $\mathbb{R}^N$):

- The trivial bundles $M \times E$ are obtained by taking the trivial action.
- The tangent bundle itself: $TM = GL(M) \times_\rho \mathbb{R}^N$. The isomorphism with TM is simply given by $[\![p, v]\!] \mapsto p(v)$. This is a valid definition since $(p{\cdot}g)(\rho(g^{-1})(v)) = (p \circ g)(g^{-1}(v)) = p(v)$.
- The cotangent bundle: $T^*M = GL(M) \times_{\rho^*} \mathbb{R}^N$, where $\rho^*(g) := \rho({}^t g^{-1})$.
- The density bundles: $|\Lambda|^s := GL(M) \times_{|\det \rho|^{-s}} \mathbb{R}^N$, where $|\det \rho|^{-s}(g) := |\det g|^{-s}$.
- The bundle of n-forms: $\Lambda := GL(M) \times_{\det \rho^{-1}} \mathbb{R}^N$, where $(\det \rho)^{-1}(g) := (\det g)^{-1}$.
- All classical tensor bundles: $(\otimes^r TM) \otimes (\otimes^s T^*M) := GL(M) \times_{(\otimes^r \rho) \otimes (\otimes^s \rho^*)} \otimes^{r+s}\mathbb{R}$, where $\otimes^r TM := \underbrace{TM \otimes \ldots \otimes TM}_{r \text{ times}}$ and $(\otimes^r \rho)(g)(v_1 \otimes \ldots \otimes v_r) := \rho(g)(v_1) \otimes \ldots \otimes \rho(g)(v_r)$ (with similar formulas for T^*M and ρ^*).

The following proposition gives a characterization of sections in an associated bundle in terms of equivariant maps.

Proposition 2.6. *Let $C^\infty(P, E)^G$ be the space of smooth* equivariant maps, *i.e. smooth maps $s\colon P \to E$ which satisfy $s(p{\cdot}g) = \rho(g^{-1})s(p)$. There is an isomorphism between $\Gamma(M, P \times_\rho E)$ and $C^\infty(P, E)^G$ given by:*

$$\begin{array}{lcll} C^\infty(P,E)^G & \to & \Gamma(M, P \times_\rho E) & \\ s & \mapsto & (x \mapsto [\![p, s(p)]\!]) & \quad \textit{where } p \textit{ is any element of } \pi^{-1}(x) \end{array}$$

Proposition 2.7. *Let P be a G-principal bundle over M, (E, ρ), (E_i, ρ_i), $i = 1 \ldots k$ representations of G, $\mathcal{E}$, $\mathcal{E}_i$, $i = 1 \ldots k$ the associated bundles, μ a G-equivariant map from $E_1 \times \cdots \times E_k$ to E, i.e. a map satisfying*

$$\mu(\rho_1(g)e_1, \ldots, \rho_k(g)e_k) = \rho(g)\mu(e_1, \ldots, e_k)$$

Then μ extends to a pointwise map $\tilde{\mu}\colon \Gamma(M, \mathcal{E}_1) \times \cdots \times \Gamma(M, \mathcal{E}_k) \to \Gamma(M, \mathcal{E})$. If μ is linear (resp. anti-linear) in the i^{th} variable, then $\tilde{\mu}$ is also linear (resp. anti-linear) in the i^{th} variable.

Proof. Let us consider sections s_i as equivariant maps (see proposition 2.6). We merely define $\tilde{\mu}$ to be:

$$(\tilde{\mu}(s_1, \ldots, s_k))(p) = \mu(s_1(p), \ldots, s_k(p))$$

The results follows immediately. □

Examples.
- The tensor and exterior products are GL_n-equivariant for the tensor representations of GL_n. Their extensions are the pointwise tensor/exterior products.
- The canonical scalar product on $\mathbb{R}^n$ is GL_n-equivariant for the canonical representation and the contragredient. Its extension on $T^*M \times TM$ is the contraction of one-forms by vector fields.
- The canonical scalar product on $\mathbb{R}^n$ is O_n-equivariant for the canonical representation. For a Riemannian manifold, its extension on $TM \times TM$ is the metric of the manifold.

2.2 Derivation on bundles

2.2.1 Lie derivative

A diffeomorphism ϕ of M gives rise to a diffeomorphism ϕ_* of TM by push-forward. It induces a diffeomorphism of $GL(M)$ by:

$$GL(\mathbb{R}^N, T_xM) \ni p \mapsto d\phi_x \circ p \in GL(\mathbb{R}^N, T_{\phi(x)}M) \quad \text{where } x := \pi(p)$$

This diffeomorphism commutes with the action of $GL_N(\mathbb{R})$ on $GL(M)$, and so induces a diffeomorphism on all tensor bundles by:

$$[\![p, v]\!] \mapsto [\![\phi{\cdot}p, v]\!]$$

For a differential form, this action is the inverse of the classical pull-back:

$$\phi\cdot\alpha = (\phi^{-1})^*\alpha$$

Definition 2.8. Let X be a smooth vector field, ie $X \in \mathfrak{X}(M) := \Gamma(M,TM)$, and ϕ_t the one-parameter group of diffeomorphisms generated by X. Then the *Lie derivative* of a section s is given by:

$$\mathcal{L}(X)s := \left.\frac{d}{dt}\right|_{t=0} \phi_t\cdot s$$

For example, in the case of TM, we obtain the Lie bracket: $\mathcal{L}(X)Y = [X,Y]$.

Proposition 2.9. *The Lie derivative satisfies* $\mathcal{L}(X)(s_1 \otimes s_2) = \mathcal{L}(X)s_1 \otimes s_2 + s_1 \otimes \mathcal{L}(X)s_2$

Proposition 2.10. *On differential forms, the Lie derivative satisfies:*

(i) $\mathcal{L}(X)\circ d = d \circ \mathcal{L}(X)$
(ii) $\mathcal{L}(X)(\alpha(Y)) = (\mathcal{L}(X)\alpha)(Y) + \alpha(\mathcal{L}(X)Y)$
(iii) $\mathcal{L}(X)(\alpha\wedge\beta) = (\mathcal{L}(X)\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}(X)\beta)$
(iv) $\mathcal{L}(X) = d\circ\iota(X) + \iota(X)\circ d$

for $\alpha \in \mathcal{A}^1(M)$, $\beta \in \mathcal{A}(M)$, $X, Y \in \mathfrak{X}(M)$, *where* $\iota(X)$ *denotes the interior product with* X.

Proof. (i) It follows from the fact that the exterior derivative d commutes with pull-backs.

(ii) Let us prove something stronger. Let μ be as in proposition 2.7 and assume that for each variable it is either linear or anti-linear (non necessarily (anti)linear in all variables). It is obvious that for $s_i \in C^\infty(M,E_i)^{GL_n}$, $i = 1\dots k$ and a diffeomorphism ϕ we have

$$\phi\cdot\tilde{\mu}(s_1,\dots,s_k) = \tilde{\mu}(\phi\cdot s_1,\dots,\phi\cdot s_k)$$

$\tilde{\mu}$ being multi(anti)linear, using the formula for ϕ_t and derivating at $t=0$ we obtain:

$$\mathcal{L}(X)\tilde{\mu}(s_1,\dots,s_k) = \sum_{i=1}^{k}\tilde{\mu}(s_1,\dots,\mathcal{L}(X)s_i,\dots,s_k)$$

The examples given after proposition 2.7 give (ii) and (iii).

(iii) See (ii).

(iv) The formula is obvious on functions, and a direct consequence of (ii) for one-forms. With a straightforward computation one can show that $d\circ\iota(X)+\iota(X)\circ d$ satisfies (iii). Hence, by recurrence on the degree, $\mathcal{L}(X) = d\circ\iota(X)+\iota(X)\circ d$.

□

2.2.2 Connections

If we want to generalize the usual derivation on a product $M\times E$ to a general bundle, we face the problem that there is no canonical way to identify two vectors of different fibers. In other words, the equality "$s(x) = s(y)$" for $x, y \in M$ does not make sense without a particular convention.

Let $\mathcal{E}$ be a fiber bundle over M and $V\mathcal{E}$ the *vertical tangent space* of $\mathcal{E}$, i.e. the subspace of tangent vectors which are tangent to the fibers:

$$X \in V\mathcal{E} \Leftrightarrow \forall f \in C^\infty(M),\ X(\pi^*f) = 0$$

So we have an injection $V\mathcal{E} \hookrightarrow T\mathcal{E}$. We have also a canonical surjection from $T\mathcal{E}$ onto the pull-back π^*TM by $X \mapsto \pi^*(\pi_*(X))$. Altogether, we have a short exact sequence:

$$0 \to V\mathcal{E} \to T\mathcal{E} \to \pi^*TM \to 0$$

In general, there is no canonical way to split such a sequence. This is achieved by giving a *connection* on $\mathcal{E}$, i.e. a distribution $y \mapsto H_y\mathcal{E}$ of n-dimensional vector spaces ($n := \dim M$) such that $\forall y \in \mathcal{E}$, $T_y\mathcal{E} = V_y\mathcal{E}\oplus H_y\mathcal{E}$.

If $\mathcal{E}$ is a vector bundle over M, let $\mathcal{A}^k(M,\mathcal{E}) := \Gamma(M, \wedge^k T^*M \otimes \mathcal{E})$ be the space of k-differential forms with values in $\mathcal{E}$.

If P is a G-principal bundle, there is the following canonical identification between V_pP and $\mathfrak{g}$, the Lie algebra of G:

$$\begin{array}{ccl} \mathfrak{g} & \to & V_pP \\ X & \mapsto & X_P(p) := \frac{d}{dt}\big|_{t=0}(p\cdot\exp(tX)) \end{array}$$

For $g \in G$ we denote by R_g the diffeomorphism of P given by $R_g(p) := p\cdot g$.

Definition 2.11. Let P be a G-principal bundle over M. A *connection one-form* on P is a vertical G-invariant $\mathfrak{g}$-valued one-form $\omega \in \mathcal{A}^1(P,\mathfrak{g})^G$. It satisfies:

(i) $\forall X \in \mathfrak{g},\ \omega(X) = X$

(ii) $\forall X \in TP,\ g \in G,\ \omega((R_g)_*X) = Ad(g^{-1})\omega(X)$

Such a form determines a connection on P by $H_pP := \ker\omega$.

2.2.3 Covariant derivative

The covariant derivative is the generalization of the derivation which we were looking for.

Definition 2.12. A *covariant derivative* on a vector bundle $\mathcal{E}$ is a linear map

$$\nabla : \Gamma(M,\mathcal{E}) \to \Gamma(M, T^*M \otimes \mathcal{E})$$

which satisfies Leibniz's rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for $f \in C^\infty(M),\ s \in \Gamma(M,\mathcal{E})$.

The link between a covariant derivative and a connection lies in the following propositions:

Proposition 2.13. *There is a one-to-one correspondence between connections on the frame bundle $GL(\mathcal{E})$ and covariant derivatives on $\mathcal{E}$.*

Proposition 2.14. *Given a connection one-form ω and $p \in P$, let $\pi^*X(p)$ be the horizontal lift at p of $X \in TM$. Then*

$$(\nabla_X s)(p) := ds(\pi^*X(p))$$

defines a covariant derivative, where s and $\nabla_X s$ are thought as elements in $C^\infty(M,E)^G$ (and so ds makes sense).

Constructing a covariant derivative from a connection on a principal bundle is very useful because it provides a compatible family of covariant derivatives on all associated bundles.

Proposition 2.15. *Let $P, E, E_i, \mathcal{E}, \mathcal{E}_i$ be as in proposition 2.7, ω a connection form on P, $\nabla^{\mathcal{E}}$, $\nabla^{\mathcal{E}_i}$ the associated covariant derivatives. Then for all vector $X \in TM$:*

$$\nabla^{\mathcal{E}}_X(\tilde{\mu}(s_1 \dots s_k) = \sum_{i=1}^k \tilde{\mu}(s_1, \dots, \nabla^{\mathcal{E}_i}_X s_i, \dots, s_k)$$

Proof. Let $p \in P$ $\pi^*X(p)$ the horizontal lift of X at p and $\gamma(t)$ a curve in P such that $\gamma(0) = p$, $\gamma'(0) = \pi^*X(p)$.

$$\begin{aligned} \nabla^{\mathcal{E}}_X(\tilde{\mu}(s_1 \dots s_k))(p) &= \frac{d}{dt}\bigg|_{t=0} \tilde{\mu}(s_1 \dots s_k)(\gamma(t)) \\ &= \frac{d}{dt}\bigg|_{t=0} \mu(s_1 \circ \gamma(t) \dots s_k \circ \gamma(t)) \\ &= \sum_{i=1}^k \mu(s_1(p), \dots, \frac{d}{dt}\bigg|_{t=0} s_i(\gamma(t)), \dots, s_k(p)) \end{aligned}$$

$$
\begin{aligned}
\nabla_X^{\mathcal{E}}(\tilde{\mu}(s_1 \dots s_k))(p) &= \sum_{i=1}^{k} \mu(s_1(p), \dots, (ds_i)(\pi^* X(p)), \dots, s_k(p)) \\
&= \sum_{i=1}^{k} \mu(s_1(p), \dots, (\nabla_X^{\mathcal{E}_i} s_i)(p), \dots, s_k(p)) \\
&= \sum_{i=1}^{k} \tilde{\mu}(s_1, \dots, (\nabla_X^{\mathcal{E}_i} s_i), \dots, s_k)(p)
\end{aligned}
$$

□

Examples. Using the same examples given for the Lie derivative, the induced covariant derivatives satisfy:

- $\nabla_X(s \otimes t) = (\nabla_X s) \otimes t + s \otimes (\nabla_X t)$
- $\nabla_X(\alpha \wedge \beta) = (\nabla_X \alpha) \wedge \beta + \alpha \wedge (\nabla_X \beta)$
- $\nabla(\alpha(X_1, \dots, X_k)) = (\nabla\alpha)(X_1, \dots, X_k) + \sum_{i=1}^{k} \alpha(X_1, \dots, \nabla X_i, \dots, X_k)$

These examples show also that given $\mathcal{E}$ and $\mathcal{F}$ two vector bundles with covariant derivatives $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$, there is a canonical way to choose covariant derivatives on $\mathcal{E} \otimes \mathcal{F}$ and $\mathcal{E}^*$ by:

$$
\begin{aligned}
(\nabla_X^{\mathcal{E}^*} \alpha)(s) &:= X(\alpha(s)) - \alpha(\nabla_X^{\mathcal{E}} s) \\
\nabla_X^{\mathcal{E}\otimes\mathcal{F}}(s \otimes t) &:= (\nabla_X^{\mathcal{E}} s) \otimes t + s \otimes (\nabla_X^{\mathcal{F}} t)
\end{aligned}
$$

for $\alpha \in \Gamma(M, \mathcal{E}^*),\ s \in \Gamma(M, \mathcal{E}),\ t \in \Gamma(M, \mathcal{F}),\ X \in TM$.

Given a Riemannian metric on M, we can reduce the frame bundle to the bundle of isometric maps, where the scalar product on $\mathbb{R}$ is the canonical one.

Definition 2.16. The *orthonormal frame bundle* $O(M)$ is defined as follows:

$$
O(M) := \left\{ (x, \Omega_x) \mid \Omega_x \text{ is an isometry between } \mathbb{R}^N \text{ and } T_x M \right\}
$$

We see that the one-density bundle of a Riemannian manifold M is associated to a trivial representation of O_n. Hence, it is trivial, and M has a canonical one-density $|dx|$: this is the constant function 1 in $C^\infty(M, \mathbb{R})^{O_n}$. It follows that $\nabla|dx| = 0$.

The canonical scalar product on $\mathbb{R}^n \times \mathbb{R}^n$ is O_n-equivariant for the canonical representation of O_n. Its extension on $\mathfrak{X}(M) \times \mathfrak{X}(M)$ is the metric g used to build $O(M)$. Furthermore, if the covariant derivative on M is defined from a connection one-form on $O(M)$, then it is a *metric* covariant derivative:

$$
\nabla g = 0,\ ie\ \forall X, Y, Z \in \mathfrak{X}(M),\ \partial_X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)
$$

More generally, if we have a G-principal bundle P, a unitary representation ρ of G on E and a connection one-form ω on P, then the scalar product on E extends to a scalar product on $\Gamma(M, P \times_\rho E)$, and the covariant derivative induced by ω is metric with respect to this scalar product.

We will make a constant use of the fact that a metric on a (finite dimensional) vector bundle $\mathcal{E}$ gives rise to an isomorphism $\mathcal{E} \simeq \mathcal{E}^*$. The two isomorphisms are denoted by "$\sharp$" and "$\flat$". Schematically, in the particular case of TM, we have:

$$
\begin{tikzcd}
TM \arrow[r, shift left, "\flat"] & T^*M \arrow[l, shift left, "\sharp"]
\end{tikzcd}
$$

Note that in the case of complex vector bundles, these are in fact *anti*-linear isomorphisms.

If $\mathcal{E}$ and $\mathcal{F}$ are two vector bundles over M with metrics $g_{\mathcal{E}}$ and $g_{\mathcal{F}}$, we can define a canonical metric $g_{\mathcal{E}\otimes\mathcal{F}}$ on $\mathcal{E} \otimes \mathcal{F}$ and $g_{\mathcal{E}^*}$ on $\mathcal{E}^*$ by:

$$
\begin{aligned}
g_{\mathcal{E}\otimes\mathcal{F}}(s_1 \otimes t_1, s_2 \otimes t_2) &= g_{\mathcal{E}}(s_1, s_2) g_{\mathcal{F}}(t_1, t_2) \\
g_{\mathcal{E}^*}(\alpha, \beta) &= g_{\mathcal{E}}(\beta^\sharp, \alpha^\sharp)
\end{aligned}
$$

In the case of $\mathcal{E}^*$, for an orthonormal frame (e_i) of $\mathcal{E}$, we have $g_{\mathcal{E}^*}(\alpha, \beta) = \sum_i \overline{\alpha(e_i)}\beta(e_i)$. The exchange of the places of α and β in the other expression is just a trick in order to handle properly the case of Hermitian metrics.

Proposition 2.17. *Let $\mathcal{E}, \mathcal{F}, \nabla^{\mathcal{E}}, \nabla^{\mathcal{F}}, \nabla^{\mathcal{E}^*}, \nabla^{\mathcal{E}\otimes\mathcal{F}}, g_{\mathcal{E}}, g_{\mathcal{F}}, g_{\mathcal{E}^*}, g_{\mathcal{E}\otimes\mathcal{F}}$ be chosen as previously. Suppose that $\nabla^{\mathcal{E}}$ and $\nabla^{\mathcal{F}}$ are metric. Then $\nabla^{\mathcal{E}^*}$ and $\nabla^{\mathcal{E}\otimes\mathcal{F}}$ are metric, and for $\alpha \in \Gamma(M, \mathcal{E}^*)$, $s \in \Gamma(M, \mathcal{E})$, $t \in \Gamma(M, \mathcal{F})$, $T \in \Gamma(M, \mathcal{E} \otimes \mathcal{F})$,*

$$\begin{aligned} (\nabla^{\mathcal{E}} s)^\flat &= \nabla^{\mathcal{E}^*}(s^\flat) & \qquad (\nabla^{\mathcal{E}^*}\alpha)^\sharp &= \nabla^{\mathcal{E}}(\alpha^\sharp) \\ g_{\mathcal{E}\otimes\mathcal{F}}(s \otimes t, T) &= g_{\mathcal{F}}(t, s^\flat \lrcorner T) & \qquad \|\alpha \lrcorner T\|_{\mathcal{F}} &\le \|\alpha\|_{\mathcal{E}^*} \|T\|_{\mathcal{E}\otimes\mathcal{F}} \end{aligned}$$

Proof. We first prove that $\sharp$ and $\flat$ commute with the covariant derivative:

$$\begin{aligned} (\nabla_X^{\mathcal{E}^*}\alpha)(s) &= X(\alpha(s)) - \alpha(\nabla_X^{\mathcal{E}} s) \\ &= X(g_{\mathcal{E}}(\alpha^\sharp, s)) - g_{\mathcal{E}}(\alpha^\sharp, \nabla_X^{\mathcal{E}} s) \\ &= g_{\mathcal{E}}(\nabla_X^{\mathcal{E}}(\alpha^\sharp), s) \end{aligned}$$

Hence $(\nabla^{\mathcal{E}^*}\alpha)^\sharp = \nabla^{\mathcal{E}}(\alpha^\sharp)$ (which implies also $(\nabla^{\mathcal{E}} s)^\flat = \nabla^{\mathcal{E}^*}(s^\flat)$).
Using this we have for $\beta \in \Gamma(M, \mathcal{E}^*)$:

$$\begin{aligned} X(g_{\mathcal{E}^*}(\alpha, \beta)) &= X(g_{\mathcal{E}}(\beta^\sharp, \alpha^\sharp)) \\ &= g_{\mathcal{E}}(\nabla_X^{\mathcal{E}}(\beta^\sharp), \alpha^\sharp) + g_{\mathcal{E}}(\beta^\sharp, \nabla_X^{\mathcal{E}}(\alpha^\sharp)) \\ &= g_{\mathcal{E}}((\nabla_X^{\mathcal{E}^*}\beta)^\sharp, \alpha^\sharp) + g_{\mathcal{E}}(\beta^\sharp, (\nabla_X^{\mathcal{E}^*}\alpha)^\sharp) \\ &= g_{\mathcal{E}^*}(\alpha, \nabla_X^{\mathcal{E}^*}\beta) + g_{\mathcal{E}^*}(\nabla_X^{\mathcal{E}^*}\alpha, \beta) \end{aligned}$$

A straightforward computation using the definitions gives the result for $\mathcal{E} \otimes \mathcal{F}$.

The formula $g_{\mathcal{E}\otimes\mathcal{F}}(s \otimes t, T) = g_{\mathcal{F}}(t, s^\flat \lrcorner T)$ is trivial if s is an element of an orthonormal basis, and we extend the result by (anti-)linearity. Then:

$$\begin{aligned} \|\alpha \lrcorner T\|_{\mathcal{F}}^2 &= g_{\mathcal{F}}(\alpha \lrcorner T, \alpha \lrcorner T) = g_{\mathcal{E}\otimes\mathcal{F}}(T, \alpha^\sharp \otimes (\alpha \lrcorner T)) \\ &\le \|T\|_{\mathcal{E}\otimes\mathcal{F}} \|\alpha^\sharp \otimes (\alpha \lrcorner T)\|_{\mathcal{E}\otimes\mathcal{F}} = \|T\|_{\mathcal{E}\otimes\mathcal{F}} \|\alpha\|_{\mathcal{E}^*} \|\alpha \lrcorner T\|_{\mathcal{F}} \end{aligned}$$

Thus

$$\|\alpha \lrcorner T\|_{\mathcal{F}} \le \|T\|_{\mathcal{E}\otimes\mathcal{F}} \|\alpha\|_{\mathcal{E}^*} \tag{3}$$

□

We said that there is no canonical way to choose a connection (i.e. a covariant derivative). In fact, as we will see, this statement is not always true (it is false at least for tensor bundles).

Definition 2.18. Let ∇ be a covariant derivative on $\mathfrak{X}(M)$. The *torsion* of ∇ is the section $T \in \Gamma(M, \wedge^2 T^*M \otimes TM)$ given by:

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]$$

If T vanishes for a covariant derivative ∇, we will say that ∇ is *torsion-free*.

Theorem 2.19. *Every Riemannian manifold M with metric g has a canonical covariant derivative called the* Levi-Civita connection*. It is the unique covariant derivative on M which is metric and torsion-free.*

Using the preceding results, unless explicitly stated, we will now consider that TM carries the Levi-Civita connection, and that the dual bundle or the tensor product of two bundles carry the natural covariant derivatives.

Definition 2.20. Let ∇ be a covariant derivative on $\mathcal{E}$. The *Riemannian curvature* is the element of $\mathcal{A}^2(M, \mathrm{End}(\mathcal{E}))$ defined by:

$$R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

for $X, Y \in \mathfrak{X}(M)$.

2.3 Formulas

For computations we use the following conventions: $dx^i, \frac{\partial}{\partial x^i}$ denote *any* coordinate system, and e^i, e_i local orthonormal frames obtained by radial parallel transport.

Proposition 2.21. *If ∇ is a torsion-free connection, the exterior differential on differential forms is given by:*

$$d = \varepsilon \circ \nabla$$

Proof. Let $\alpha, \beta \in \mathcal{A}(M)$ and $D := \varepsilon \circ \nabla$.

$$\begin{aligned} D(\alpha \wedge \beta) &= \sum_i dx^i \wedge \nabla_{\frac{\partial}{\partial x^i}}(\alpha \wedge \beta) \\ &= \sum_i dx^i \wedge (\nabla_{\frac{\partial}{\partial x^i}}\alpha) \wedge \beta + dx^i \wedge \alpha \wedge (\nabla_{\frac{\partial}{\partial x^i}}\beta) \\ &= (D\alpha) \wedge \beta + (-1)^{|\alpha|} \sum_i \alpha \wedge (dx^i \wedge \nabla_{\frac{\partial}{\partial x^i}}\beta) \\ &= (D\alpha) \wedge \beta + (-1)^{|\alpha|}\alpha \wedge (D\beta) \end{aligned}$$

So D satisfies the Leibniz's rule, and it is clear that D and d agree on functions. Hence it is sufficient to show that $D^2 = 0$. We recall that in local coordinates the Christoffel's symbols are defined by:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k} \Leftrightarrow \nabla_{\frac{\partial}{\partial x^i}} dx^j = -\sum_k \Gamma_{ik}^j dx^k$$

With these symbols we have:

$$T = \sum_k \sum_{i<j} T_{ij}^k dx^i \wedge dx^j \otimes \frac{\partial}{\partial x^k} = \sum_k \sum_{i<j} (\Gamma_{ij}^k - \Gamma_{ji}^k) dx^i \wedge dx^j \otimes \frac{\partial}{\partial x^k}$$

$$\begin{aligned} D^2 f &= \sum_i dx^i \wedge \nabla_{\frac{\partial}{\partial x^i}} df \\ &= \sum_{i,j} dx^i \wedge (\frac{\partial^2 f}{\partial x^i \partial x^j} dx^j) - \sum_{i,j,k} \frac{\partial f}{\partial x^j} dx^i \wedge \Gamma_{ik}^j dx^k \\ &= \sum_{i<j} (\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}) dx^i \wedge dx^j - \sum_j \frac{\partial f}{\partial x^j} \sum_{i<k} (\Gamma_{ik}^j - \Gamma_{ki}^j) dx^i \wedge dx^k \\ &= 0 - \sum_j \frac{\partial f}{\partial x^j} T_{ik}^j dx^i \wedge dx^k \\ &= -(df, T) \end{aligned}$$

Hence if T vanishes, $D^2 = 0$. ☐

We want now to compute some formal adjoints (see 4.3 for more details). For this we need to introduce the divergence of vector fields and one-forms. For any vector field X, ∇X defines a section in $\Gamma(M, \mathrm{End}(TM))$, so we can take the trace of the endomorphisms:

$$Tr(\nabla X) = \sum_i dx^i (\nabla_{\frac{\partial}{\partial x^i}} X)$$

For a one-form alpha we set $Tr(\nabla\alpha) := Tr(\nabla\alpha^\sharp)$ and in local coordinates, we have:

$$Tr(\nabla\alpha) = \sum_i (\nabla_{e_i}\alpha)(e_i) = \sum_i e_i(\alpha(e_i)) - \alpha(\nabla_{e_i} e_i)$$

Proposition 2.22. *Let M be a Riemannian manifold with canonical one-density $|dx|$. The divergence $\operatorname{div} X$ of a vector field is defined by $\mathcal{L}(X)|dx| =: (\operatorname{div} X)|dx|$. We have the formula $\operatorname{div} X = Tr(\nabla X)$.*

Proof. Locally, we can always consider a one-density as a n-form, since every manifold is locally orientable. In fact, given a simple connected domain $\mathcal{U}$ of M, there is an isomorphism between the one-densities and the n-forms with compact support in $\mathcal{U}$, which preserves the Lie and covariant derivatives. Informally, if we have a one-density $|dx|$ with associated n-form dx, we have $\mathcal{L}(X)|dx| = |\mathcal{L}(X)dx|$.

$$\begin{aligned}
\mathcal{L}(X)dx &= d(X \lrcorner\, dx) \\
&\underset{2.21}{=} \sum_{i=1}^{n} dx^i \wedge \nabla_{\frac{\partial}{\partial x^i}}(X \lrcorner\, dx) \\
&= \sum_{i=1}^{n} dx^i \wedge ((\nabla_{\frac{\partial}{\partial x^i}} X) \lrcorner\, dx) + dx^i \wedge (X \lrcorner\, \underbrace{\nabla_{\frac{\partial}{\partial x^i}} dx}_{=0}) \\
&\underset{(4)}{=} dx^i(\nabla_{\frac{\partial}{\partial x^i}} X)dx - (\nabla_{\frac{\partial}{\partial x^i}} X) \lrcorner\, (\underbrace{dx^i \wedge dx}_{=0}) \\
&= Tr(\nabla X)dx
\end{aligned}$$

Hence,
$$\mathcal{L}(X)|dx| = Tr(\nabla X)|dx|$$

□

Lemma 2.23. *For any compactly supported one-density $|\lambda|$, we have for any vector field X,*

$$\int_M \mathcal{L}(X)|\lambda| = 0$$

Proof. If $|\lambda|$ is a one-density with compact support, we can write it (using a partition of the unity) as a finite sum of compactly supported one-densities that can be interpreted as volume forms (see the discussion in the proof of proposition 2.22). So let us assume that $|\lambda|$ is the one-density associated to the n-form ω. For any vector field X we have $\mathcal{L}(X)|\lambda| = |d(X \lrcorner\, \omega)|$, hence $\int_M \mathcal{L}(X)(|\lambda|) = \int_M d(X \lrcorner\, \omega) = 0$ by Stokes' formula (where the "local orientation" used to compute the integral of the volume form is the same used to identify one-densities and volume forms). □

Proposition 2.24. *The divergence on one-forms is given by $d^*\alpha = -Tr(\nabla\alpha)$. So for any compactly supported C^1 one-form α,*

$$\int_M Tr(\nabla\alpha)|dx| = 0$$

Proof. Let $X := \alpha^\sharp$ and f a function with compact support. Then

$$0 \underset{2.23}{=} \int_M \mathcal{L}(X)(f|dx|) = \int_M X(f)|dx| + \int_M f\mathcal{L}(X)|dx|$$

So we have:

$$\int_M (\alpha, df)|dx| = \int_M X(f)|dx| = -\int_M f\mathcal{L}(X)|dx| \underset{2.22}{=} \int_M f(-Tr(\nabla X))|dx|$$

If α is compactly supported, let us take any cut-off function f identical to 1 on the support of α. We have:

$$\int_M d^*\alpha|dx| = \int_M \langle f, d^*\alpha\rangle|dx| = \int_M \langle df, \alpha\rangle|dx| = 0$$

□

Proposition 2.25. *Let ∇ be a torsion-free connection on M. Then the L^2 formal adjoint of the exterior derivative d is given by:*

$$d^* = -\iota \circ \nabla$$

where ι is the interior product (defined by means of the isomorphism $TM \simeq T^*M$ induced by the metric on M).

Proof. Let $\beta_p \in \mathcal{A}^p(M)$, $\beta_{p+1} \in \mathcal{A}^{p+1}(M)$ with compact support. We define a one-form α by $\alpha(X) := \langle \beta_p, X \lrcorner \beta_{p+1} \rangle$. We will use the fact that $\varepsilon^* = \iota \circ \sharp$. Then:

$$\begin{aligned}
\langle d\beta_p, \beta_{p+1} \rangle &= \sum_i \langle e^i \wedge \nabla_{e_i} \beta_p, \beta_{p+1} \rangle \\
&= \sum_i \langle \nabla_{e_i} \beta_p, e_i \lrcorner \beta_{p+1} \rangle \\
&= \sum_i e_i(\langle \beta_p, e_i \lrcorner \beta_{p+1} \rangle) - \langle \beta_p, \nabla_{e_i}(e_i \lrcorner \beta_{p+1}) \rangle \\
&= \sum_i e_i(\alpha(e_i)) - \langle \beta_p, (\nabla_{e_i} e_i) \lrcorner \beta_{p+1} \rangle - \langle \beta_p, e_i \lrcorner (\nabla_{e_i} \beta_{p+1}) \rangle \\
&= \sum_i e_i(\alpha(e_i)) - \alpha(\nabla_{e_i} e_i) - \langle \beta_p, e_i \lrcorner (\nabla_{e_i} \beta_{p+1}) \rangle \\
&= Tr(\nabla \alpha) + \langle \beta_p, -\iota \circ \nabla \beta_{p+1} \rangle
\end{aligned}$$

Integrating over M kills the divergence term, and yields the result. □

Proposition 2.26. *Let $\mathcal{E}$ be a vector bundle over M. The L^2 formal adjoint ∇^* of ∇ ($\nabla^* \colon \Gamma(M, T^*M \otimes \mathcal{E}) \to \Gamma(M, \mathcal{E})$) is given by:*

$$\nabla^* = -Tr(\nabla)$$

where "Tr" is the contraction on $\Gamma(M, T^*M \otimes T^*M \otimes \mathcal{E})$ defined by means of the metric.

Proof. Let $s \in \Gamma(M, \mathcal{E})$, $t \in \Gamma(M, T^*M \otimes \mathcal{E})$ with compact support. We define a one-form α by $\alpha(X) := \langle s, X \lrcorner t \rangle$. Then:

$$\begin{aligned}
\langle \nabla s, t \rangle &= \sum_i \langle \nabla_{e_i} s, e_i \lrcorner t \rangle \\
&= \sum_i e_i(\langle s, e_i \lrcorner t \rangle) - \langle s, \nabla_{e_i}(e_i \lrcorner t) \rangle \\
&= \sum_i e_i(\alpha(e_i)) - \langle s, (\nabla_{e_i} e_i) \lrcorner t) \rangle - \langle s, e_i \lrcorner (\nabla_{e_i} t) \rangle \\
&= \sum_i e_i(\alpha(e_i)) - \alpha(\nabla_{e_i} e_i) - \langle s, e_i \lrcorner (e_i \lrcorner \nabla t) \rangle \\
&= Tr(\nabla \alpha) + \langle s, -Tr(\nabla t) \rangle
\end{aligned}$$

Once again, integrating over M kills the divergence term, and yields the result. □

3 Spin geometry

In this section we describe briefly Clifford algebras and spin groups, giving the necessary elements required by Dirac operators. Our references for this section are [5, 6, 9].

3.1 Clifford algebras

Definition 3.1. Let V be a vector space over $\mathbb{K}$ of dimension n and q a non-degenerate symmetric bilinear form on V. The *Clifford algebra* $\mathcal{C}\ell(V, q)$ associated to q on V is the associative algebra with unit defined by

$$(\otimes V)/I(V, q)$$

where $\otimes V$ denote the tensor algebra of V, and $I(V, q)$ the two-sided ideal generated by the elements of the form $x \otimes x + q(x, x)1$, for $x \in V$.

By polarization, we can see that the ideal $I(V, q)$ is generated by the elements of the form $x \otimes y + y \otimes x + 2q(x, y)1$ for $x, y \in V$ (for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we have car$(\mathbb{K}) \neq 2$).

The Clifford algebra $\mathcal{C}\ell(V, q)$ is the solution of the following universal problem:

Proposition 3.2. *Let $\mathcal{A}$ be an associative $\mathbb{K}$-algebra with unit and $f\colon V \to \mathcal{A}$ a linear map such that*

$$\forall v \in V,\ f(v)^2 = -q(v,v)1$$

Then there exists a unique $\mathbb{K}$-algebra homomorphism F such that the following diagram is commutative:

$$\begin{tikzcd} V \arrow[d, "\llbracket\cdot\rrbracket"'] \arrow[dr, "f"] & \\ \mathcal{C}\ell(V,q) \arrow[r, "F"] & \mathcal{A} \end{tikzcd}$$

where $\llbracket . \rrbracket$ is the canonical map from V to $\mathcal{C}\ell(V,q)$ given by taking the equivalence class. Furthermore, $\mathcal{C}\ell(V,q)$ is the unique associative algebra with unit satisfying this property, up to an algebra isomorphism[3].

The non-degenerate bilinear form q gives us two isomorphisms between V and V^*, still denoted $\flat$ and $\sharp$, defined as follows:

$$\begin{tikzcd} V \arrow[r, shift left, "\flat"] & V^* \arrow[l, shift left, "\sharp"] \end{tikzcd}$$

For $v \in V$, $v^\flat$ is simply the linear form $v^\flat\colon u \mapsto q(v,u)$.

Consider the linear map τ from V to $\mathcal{C}\ell(V,q)$ defined by $\tau(v) := \llbracket -v \rrbracket$. This map satisfies $\tau(v)^2 = -q(v,v)$, so it admits a unique extension as an algebra endomorphism of $\mathcal{C}\ell(V,q)$, which we still denote τ. We see that $\tau^2 = Id$, so $\mathcal{C}\ell(V,q) = \ker(\tau - Id) \oplus \ker(\tau + Id)$. We set

$$\mathcal{C}\ell^+(V,q) := \ker(\tau - Id) \quad \text{and} \quad \mathcal{C}\ell^-(V,q) := \ker(\tau + Id)$$

This defines a $\mathbb{Z}_2$-grading of the algebra $\mathcal{C}\ell(V,q)$.

We now define the Clifford action on the exterior algebra $\wedge^* V$ of V. From the universal property of $\mathcal{C}\ell(V,q)$, we need only specify the action of V on $\wedge^* V$. From now on, the Clifford multiplication between two elements v and w will be denoted by "$v \cdot w$".

$$\forall v \in V,\ \alpha \in \wedge^* V, \quad v \cdot \alpha := v^\flat \wedge \alpha - v \lrcorner\, \alpha$$

where $v \lrcorner\, \alpha$ denotes the contraction of α with v. We check that this action defines a Clifford action as follows:

$$v \cdot v \cdot \alpha = \underbrace{v^\flat \wedge (v^\flat \wedge \alpha)}_{=0} - v^\flat \wedge (v \lrcorner\, \alpha) - v \lrcorner\, (v^\flat \wedge \alpha) + \underbrace{v \lrcorner\, (v \lrcorner\, \alpha)}_{=0}$$

but for two vectors v, w and a form α we have

$$w \lrcorner\, (v^\flat \wedge \alpha) = (w \lrcorner\, v^\flat) \wedge \alpha - v^\flat (w \lrcorner\, \alpha) \tag{4}$$

Hence, if $\iota(x)$ denotes the interior product with x and $\varepsilon(y^\flat)$ the exterior product with $y^\flat$, $\iota(w)\varepsilon(v^\flat) + \varepsilon(v^\flat)\iota(w) = q(v,w)$ and in particular $v^\flat \wedge v \lrcorner + v \lrcorner\, v^\flat \wedge = q(v,v)$, which gives the conclusion.

The *symbol map* $\sigma\colon \mathcal{C}\ell(V,q) \to \wedge^* V$ is defined by

$$\sigma(a) := a \cdot 1$$

Let e_i be an orthogonal basis of V, and denote by c_i the element in $\mathcal{C}\ell(V,q)$ corresponding to e_i. The symbol map has an inverse $\mathbf{c}$ given by

$$\mathbf{c}(e^{i_1} \wedge \cdots \wedge e^{i_k}) := c_{i_1} \cdot \ldots \cdot c_{i_k}$$

It follows that $\{e_{i_1} \cdot \ldots \cdot e_{i_k} \mid i_1 < \cdots < i_k,\ k \leq \dim V\}$ is a basis of the vector space $\mathcal{C}\ell(V,q)$, and so $\dim \mathcal{C}\ell(V,q) = 2^{\dim(V)}$. In particular, we have a natural inclusion of V in $\mathcal{C}\ell(V,q)$. As usual we simply identify V and its image under this inclusion, and consider that we have $V \subset \mathcal{C}\ell(V,q)$.

[3] This isomorphism is also unique if we consider a solution of the universal problem to be an algebra $\mathcal{C}\ell$ together with a map ι from V to $\mathcal{C}\ell$

3.2 *Pin* and *Spin* groups

Definition 3.3. The *Pin* group associated to V and q is the subgroup of $\mathcal{C}\ell^{\times}(V,q)$ (invertible elements of $\mathcal{C}\ell(V,q)$) generated by the (images of) vectors of V of norm ± 1:

$$Pin(V,q) := \left\{ e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_N} \mid \forall j = 1 \ldots N,\ e_{i_j} \in V,\ q(e_{i_j}, e_{i_j}) = \pm 1 \right\}$$

The *Spin* group associated to V and q is the evenly-graded part of $Pin(V,q)$, namely:

$$Spin(V,q) := Pin(V,q) \cap \mathcal{C}\ell^{+}(V,q)$$

Let us now consider the adjoint action of $\mathcal{C}\ell^{\times}(V,q)$ on V:

$$Ad_g v := g \cdot v \cdot g^{-1} \qquad \text{for } g \in \mathcal{C}\ell^{\times}(V,q),\ v \in V$$

Lemma 3.4. *We have the formula:*

$$-Ad_v w = w - 2\frac{q(v,w)}{q(v,v)} v$$

As a consequence, $-Ad_v$ is the reflection across $v^{\perp}$.

For $u = v_1 \cdot \ldots \cdot v_{2k} \in Spin(V,q)$ and $w \in V$ we get:

$$\begin{aligned} Ad_u w &= u \cdot w \cdot u^{-1} \\ &= v_1 \cdot \ldots \cdot v_{2k} \cdot w \cdot v_{2k}^{-1} \cdot \ldots \cdot v_1^{-1} \\ &= (-Ad_{v_1}) \circ \cdots \circ (-Ad_{v_{2k}})(w) \end{aligned}$$

So, this action is an even number of reflections of V. Thus, the adjoint action restricts to a map $Ad\colon Spin(V,q) \to SO(V,q)$. Conversely, we know by the Cartan-Dieudonné theorem that every element of $SO(V,q)$ can be written as an even number of reflections, which means that the map $Ad\colon Spin(V,q) \to SO(V,q)$ is surjective.

We now restrict ourselves to the case where $V = \mathbb{R}^n$ and $q = \langle .,. \rangle$ is the canonical scalar product. We use the following simplified notations:

$$\mathcal{C}\ell_n := \mathcal{C}\ell(\mathbb{R}^n, \langle .,. \rangle) \qquad Spin_n := Spin(\mathbb{R}^n, \langle .,. \rangle) \qquad \mathbb{C}\mathrm{l}_n := \mathcal{C}\ell_n \otimes_{\mathbb{R}} \mathbb{C}$$

Theorem 3.5. *The sequence:*

$$0 \to \mathbb{Z}_2 \to Spin_n \xrightarrow{Ad} SO_n \to 1$$

is a short exact sequence. Furthermore, if $n \geq 3$, $Spin_n$ is the universal covering of SO_n.

3.3 The complex spinor representation

For $n = 2m$ or $2m+1$ we set $\Sigma_n := \mathbb{C}^{2^m}$.

Theorem 3.6. *Let $\mathbb{C}(N)$ be the ring of $N \times N$ complex matrices. Then:*

- $\mathbb{C}\mathrm{l}_{2m} \simeq \mathbb{C}(2^m) \simeq \mathrm{End}_{\mathbb{C}}(\Sigma_n)$
- $\mathbb{C}\mathrm{l}_{2m+1} \simeq \mathbb{C}(2^m) \oplus \mathbb{C}(2^m) \simeq \mathrm{End}_{\mathbb{C}}(\Sigma_n) \oplus \mathrm{End}_{\mathbb{C}}(\Sigma_n)$

So we get a complex representation ρ of $\mathbb{C}\mathrm{l}_n$ on Σ_n by choosing in the odd case the projection onto one component. For $v \in \mathbb{C}\mathrm{l}_n$ and $\sigma \in \Sigma_n$ we write $v \cdot \sigma := \rho(v)(\sigma)$. The restriction of this representation to $Spin_n$ gives a complex spinor representation:

$$\rho\colon Spin_n \to \mathrm{Aut}(\Sigma_n)$$

Σ_n is called the space of *complex spinors*.

Let $(e_i)_{i=1\dots n}$ be an orthonormal basis of $\mathbb{R}^n$. The *complex volume element* $\omega_{\mathbb{C}}$ of $\mathbb{C}l_n$ for $n \in \{2m, 2m+1\}$ is:

$$\omega_{\mathbb{C}} := i^m e_1 \cdot \ldots \cdot e_n$$

It satisfies

$$\omega_{\mathbb{C}}^2 = 1 \qquad \text{and} \qquad x \cdot \omega_{\mathbb{C}} = (-1)^{n+1} \omega_{\mathbb{C}} \cdot x$$

for all $x \in \mathbb{R}^n \subset \mathbb{C}l_n$. From this follows the proposition:

Proposition 3.7. *If $n = 2m$ is even, the complex representation of $\mathbb{C}l_n$, restricted to the even part of the Clifford algebra: $\rho^+ \colon \mathbb{C}l_n^+ \to \mathrm{End}_{\mathbb{C}}(\Sigma_n)$ admits a decomposition $\Sigma_n = \Sigma_n^+ \oplus \Sigma_n^-$, where $\Sigma_n^\pm = \frac{1}{2}(1 \pm \omega_{\mathbb{C}}) \cdot \Sigma_n$. Moreover for $x \in \mathbb{R}^n \setminus \{0\}$, $\rho^+(x) \colon \Sigma_n^\pm \to \Sigma_n^\mp$ is an isomorphism of vector spaces.*

Theorem 3.8. *If $n = 2m$ is even, the complex spinor representation decomposes into*

$$\rho = \rho^+ \oplus \rho^- \text{ where } \rho^\pm \colon Spin_{2m} \to \mathrm{Aut}(\Sigma_{2m}^\pm)$$

The maps $\rho^\pm$ are the half-spinor representations*. They are irreducible and inequivalent.*

If $n = 2m+1$ is odd, the spinor representation is irreducible and does not depend on the choice of the component of $\mathrm{End}_{\mathbb{C}}(\Sigma_n) \oplus \mathrm{End}_{\mathbb{C}}(\Sigma_n)$ *(see above).*

Proposition 3.9. *Σ_n carries a natural Hermitian product such that:*

$$\forall x \in \mathbb{R}^n,\ \sigma_1, \sigma_2 \in \Sigma_n, \qquad \begin{cases} (x \cdot \sigma_1, \sigma_2) = -(\sigma_1, x \cdot \sigma_2) \\ \|x\|^2 (\sigma_1, \sigma_2) = (x \cdot \sigma_1, x \cdot \sigma_2) \end{cases}$$

Proof. Let $\langle .,. \rangle$ be an arbitrary Hermitian product on Σ_n, and Γ_n the multiplicative subgroup of $\mathcal{C}\ell_n^\times$ generated by a basis $(e_i)_i$ of $\mathbb{R}^n$. One sees easily that Γ_n is finite with $|\Gamma_n| = 2^{n+1}$. For $\sigma_1, \sigma_2 \in \Sigma_n$ we define:

$$(\sigma_1, \sigma_2) := \frac{1}{|\Gamma_n|} \sum_{x \in \Gamma_n} \langle x \cdot \sigma_1, x \cdot \sigma_2 \rangle$$

The map $x \mapsto e_i \cdot x$ is a bijection of Γ_n, and we see that

$$(e_i \cdot \sigma_1, e_i \cdot \sigma_2) = (\sigma_1, \sigma_2)$$

Thus, $$(e_i \cdot \sigma_1, \sigma_2) = (e_i \cdot e_i \cdot \sigma_1, e_i \cdot \sigma_2) = -(\sigma_1, e_i \cdot \sigma_2)$$

By bilinearity, $$\forall x \in \mathbb{R}^n,\ (x \cdot \sigma_1, \sigma_2) = -(\sigma_1, x \cdot \sigma_2)$$

Hence, $$\forall x \in \mathbb{R}^n,\ (x \cdot \sigma_1, x \cdot \sigma_2) = -(x \cdot x \cdot \sigma_1, \sigma_2) = \|x\|^2 (\sigma_1, \sigma_2)$$

□

3.4 Clifford, spinor and Dirac bundles

Definition 3.10. The Clifford bundle $\mathcal{C}\ell(M)$ of M is the bundle whose fiber at $x \in M$ is the Clifford algebra of $T_x M$. In terms of associated bundle, $\mathcal{C}\ell(M) := O(M) \times_{O_n} \mathcal{C}\ell(\mathbb{R})$.

Let us recall that a manifold M is called *orientable* when $GL(M)$ admits a reduction to a $GL_n^+(\mathbb{R})$-principal bundle where $GL_n^+(\mathbb{R})$ denotes the linear isomorphisms with positive determinant. It is equivalent to say that the volume form bundle $\Lambda = \wedge^n T^* M$ is trivial. In this case, the bundle $GL(M)$ has two connected components, and an *orientation* of M is the choice of one of the two components. An *oriented* manifold is a manifold together with an orientation.When M is an oriented Riemannian manifold, the orthonormal frame bundle $O(M)$ admits a reduction to a SO_n-principal bundle denoted by $SO(M)$.

Definition 3.11. Let (M, g) be a n-dimensional oriented Riemannian manifold. A *spin structure* on M is a pair $(Spin(M), \eta)$ where $Spin(M)$ is a $Spin_n$-principal bundle over M and η a two-fold covering such that the following diagram commutes:

$$\begin{tikzcd}
Spin(TM) \times Spin_n \arrow[r] \arrow[d, "\eta \times Ad"] & Spin(TM) \arrow[d, "\eta"] \arrow[dr, "\pi"] & \\
SO(M) \times SO_n \arrow[r] & SO(M) \arrow[r, "\pi"'] & M
\end{tikzcd}$$

where the maps in the rows are the right actions on the principal bundles.

Not every manifold admits a spin structure, and, conversely, it may admit more than one. In fact, the existence of a spin structure is equivalent to the fact that for the transition functions $\varphi_{\alpha\beta}$ of $SO(M)$, there is a choice of lifts to transition functions with values in $Spin_n$:

$$\begin{tikzcd} & Spin_n \arrow[d, "Ad"] \\ M \supset U_\alpha \cap U_\beta \arrow[ur, "\tilde{\varphi}_{\alpha\beta}"] \arrow[r, "\varphi_{\alpha\beta}"] & SO_n \end{tikzcd}$$

such that they still satisfy the cocycle conditions[4].

Definition 3.12. The *complex spinor bundle* associated to a spin structure is the complex vector bundle

$$\Sigma M := Spin(M) \times_\rho \Sigma_n$$

All associated vector bundles constructed with $SO(M)$ can be recovered as associated bundles of $Spin(M)$ using the adjoint action. In this way, we have:

$$TM = Spin(M) \times_{Ad} R^n \qquad SO(M) = Spin(M) \times_{Ad} SO_n \qquad C\ell(M) = Spin(M) \times_{Ad} C\ell_n$$

The spinorial Levi-Civita connection is given by lifting the Levi-Civita connection:

$$\begin{tikzcd} TSpin(TM) \arrow[r, "\tilde{\omega}"] \arrow[d, "\eta_*"] & \mathfrak{spin}_n \arrow[d, "Ad_*", "\simeq"'] \\ TSO(M) \arrow[r, "\omega"] & \mathfrak{so}_n \end{tikzcd}$$

Since $(Ad_g c)\cdot(g\cdot\sigma) = (g\cdot c\cdot g^{-1})\cdot(g\cdot\sigma) = g\cdot(c\cdot\sigma)$, the Clifford multiplication between $C\ell_n$ and Σ_n is $Spin_n$-equivariant, and so extends to a pointwise Clifford multiplication given by:

$$\begin{array}{ccc} \Gamma(M, C\ell(M)) \otimes \Gamma(M, \Sigma M) & \to & \Gamma(M, \Sigma M) \\ c \otimes \sigma & \mapsto & (x \mapsto c(x)\cdot\sigma(x)) \end{array}$$

which is compatible with the spinorial connection, in the sense that:

$$\forall X \in TM,\ c \in C\ell(M),\ \sigma \in \Sigma M, \quad \nabla_X(c\cdot\sigma) = (\nabla_X c)\cdot\sigma + c\cdot(\nabla_X \sigma)$$

The representation of $Spin_n$ on Σ_n is unitary for the natural Hermitian product. As a consequence, this Hermitian product extends to a Hermitian product $(.,.)$ on $\Gamma(M, \Sigma M)$ which is compatible with the spinorial connection:

$$\forall X \in TM,\ \sigma_1, \sigma_2 \in \Sigma M, \quad \nabla_X(\sigma_1, \sigma_2) = (\nabla_X \sigma_1, \sigma_2) + (\sigma_1, \nabla_X \sigma_2)$$

Definition 3.13. A *Dirac bundle* over a Riemannian manifold M is a bundle S of left modules over $C\ell(M)$ together with a Riemannian metric, such that both the Clifford multiplication and this metric are compatible with the covariant derivative.

[4]In terms of cohomology, this is equivalent to the nullity of the second Stiefel-Whitney class $\omega_2(M)$.

4 Laplace and Dirac operators

The aim is now to present Laplace and Dirac operators, and to prove some results, using the first three sections. From now on we assume M to be a complete Riemannian manifold. We will mostly use the fact that the closed balls are compact. The distance between x and y is denoted by $d(x, y)$. We will implicitly make use of the canonical isomorphism $TM \simeq T^*M$.

4.1 Differential operators

Proposition 4.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be two vector bundles over the manifold M. There is a one-to-one correspondence between $\mathbb{R}$-linear maps $\nu\colon \Gamma(M, \mathcal{E}) \to \Gamma(M, TM \otimes \mathcal{F})$ and $\mathbb{R}$-bilinear maps $\mathscr{D}\colon C^\infty(M, \mathbb{R}) \times \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$ satisfying:*

$$\mathscr{D}(fg, s)(x) = f(x)\mathscr{D}(g, s)(x) + g(x)\mathscr{D}(f, s)(x) \tag{5}$$

Proof. Given a map ν, we define $\mathscr{D}_\nu(f, s)(x) := df|_x (\nu(s)(x))$.

Given a map $\mathscr{D}$, we first establish some properties of $\mathscr{D}$. Taking $f = g \equiv 1$ in (5) and using linearity, we see that f constant $\Rightarrow \mathscr{D}(f, s) = 0$.

Let $f, g \in C^\infty(M, \mathbb{R})$, U an open set of M with $f|_U \equiv g|_U$, and $x \in U$. Choose a smooth function φ equal to 1 outside U and vanishing at x. Then:

$$\begin{aligned}\mathscr{D}(f, s)(x) - \mathscr{D}(g, s)(x) &= \mathscr{D}(f - g, s)(x) = \mathscr{D}(\varphi(f - g), s)(x)\\ &= \varphi(x)\mathscr{D}(f - g, s)(x) + (f - g)(x)\mathscr{D}(\varphi, s)(x) = 0\end{aligned}$$

This proves that the local value of $\mathscr{D}(f, s)$ depends only on the local value of f.

Suppose now that for $x_0 \in M$, $df_{x_0} = dg_{x_0}$. Choose (U, x^i) local coordinates around x_0 (with $x^i(x_0) = 0$). Then by a well-known lemma we can find smooth functions χ_i and κ_i such that for a smaller neighborhood $U' \subset \overline{U'} \subset U$ of x_0:

$$\forall x \in U',\ (f - g)(x) = (f - g)(x_0) + \sum_i \chi_i(x)\kappa_i(x)$$

with the conditions that $\chi_i|_{U'} \equiv x^i$ and $\kappa_i(x_0) = \frac{\partial(f-g)}{\partial x^i}(x_0)$. Using the properties of $\mathscr{D}$ and the fact that $\chi_i(x_0) = \kappa_i(x_0) = 0$ we obtain: $\mathscr{D}(f, s)(x_0) = \mathscr{D}(g, s)(x_0)$. Altogether we have proved that $\mathscr{D}(f, s)(x_0)$ depends only on s and df_{x_0}.

So for $x \in M$ and local coordinates x^i around x we set

$$\nu(\mathscr{D})(s)(x) := \sum_i \frac{\partial}{\partial x^i}(x) \otimes \mathscr{D}(x^i, s)(x)$$

This definition is consistent since it is enough to know the local value of x^i. It is obviously smooth and does not depend on the choice of local coordinates because for all $\xi \in T_x^*M$,

$$\xi(\nu(\mathscr{D})(s)(x)) = \sum_i \xi(\frac{\partial}{\partial x^i})\mathscr{D}(x^i, s)(x) = \mathscr{D}\left(\sum_i \xi(\frac{\partial}{\partial x^i})x^i, s\right)(x)$$

depends only on $\xi(\frac{\partial}{\partial x^i})\, dx^i\big|_x = \xi$ (if it were different in two local coordinate charts then we could find a ξ making the two contractions different).

$$\begin{aligned}\nu(\mathscr{D}_\nu)(s) &= \sum_i \frac{\partial}{\partial x^i} \otimes \mathscr{D}_\nu(x^i, s) = \sum_i \frac{\partial}{\partial x^i} \otimes dx^i(\nu(s))\\ &= \nu(s)\\ \mathscr{D}_{\nu(\mathscr{D})}(f, s) &= df(\nu(\mathscr{D})(s)) = df\left(\sum_i \frac{\partial}{\partial x^i} \otimes \mathscr{D}(x^i, s)\right) = \sum_i df(\frac{\partial}{\partial x^i})\mathscr{D}(x^i, s)\\ &= \mathscr{D}\left(\sum_i df(\frac{\partial}{\partial x^i})x^i, s\right) = \mathscr{D}(f, s)\end{aligned}$$

So we have an isomorphism. □

We use the classical notations

$$\partial^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} \qquad \xi^\alpha := \prod_{i=1}^n \xi_i^{\alpha_i} \qquad |\alpha| := \sum_{i=1}^n \alpha_i \qquad \binom{\alpha}{\beta} := \prod_{i=1}^n \binom{\alpha_i}{\beta_i}$$

for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^{\{1\ldots n\}}$, $\xi = \sum_{i=1}^n \xi_i dx^i$, and with the conventions that the derivative $\frac{\partial}{\partial x^i}$ does not appear when $\alpha_i = 0$.

Definition 4.2. Let M be a manifold, $\mathcal{E}$ and $\mathcal{F}$ two vector bundles over M with fibers E and F. A *linear smooth differential operator* of order d on M, from $\mathcal{E}$ to $\mathcal{F}$, is a linear smooth operator

$$\mathfrak{D} \colon \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{F})$$

such that given local trivializations of $\mathcal{E}$ and $\mathcal{F}$ over a coordinate patch U of M, $\mathfrak{D}$ has the local expression:

$$\mathfrak{D} = \sum_{\alpha, |\alpha| \le d} A_\alpha(x) \partial^\alpha$$

where $A_\alpha(x) \in \mathrm{End}(E, F)$ is smooth. The space of smooth linear differential operators of order d over M, from $\mathcal{E}$ to $\mathcal{F}$, will be denoted by $\mathcal{D}_d(M, \mathcal{E}, \mathcal{F})$. We set also:

$$\begin{aligned} \mathcal{D}(M, \mathcal{E}, \mathcal{F}) &:= \sum_{d \ge 0} \mathcal{D}_d(M, \mathcal{E}, \mathcal{F}) \\ \mathcal{D}_{=d}(M, \mathcal{E}, \mathcal{F}) &:= \mathcal{D}_d(M, \mathcal{E}, \mathcal{F}) \setminus \mathcal{D}_{d-1}(M, \mathcal{E}, \mathcal{F}) \end{aligned}$$

From now on we will simply say "differential operator" since all operators will be smooth and linear. An element $\mathfrak{D} \in \mathcal{D}_d(M, \mathcal{E}, \mathcal{E}) =: \mathcal{D}_d(M, \mathcal{E})$ will be called a differential operator of order d on $\mathcal{E}$. The space of all differential operators on $\mathcal{E}$ is clearly a graded algebra with respect to the vector subspaces $\mathcal{D}_d(M, \mathcal{E})$.

Differential operators are considered as linear operators in Hilbert spaces as follows. The domain of definition of a differential operator $\mathfrak{D} \in \mathcal{D}(M, \mathcal{E}, \mathcal{F})$ will be $\Gamma_c(M, \mathcal{E})$, the space of smooth sections in $\mathcal{E}$ with compact support. Given a scalar product $\langle .,. \rangle$ on $\mathcal{E}$ there is a natural scalar product $(.,.)$ on $\Gamma_c(M, \mathcal{E})$ given by:

$$(\varphi, \psi)_{L^2(M,\mathcal{E})} := \int_M \langle \varphi, \psi \rangle |dx|$$

with associated norm $\|.\|_{L^2(M,\mathcal{E})}$. Then $L^2(M, \mathcal{E})$ denotes the completion of $\Gamma_c(M, \mathcal{E})$ with respect to this norm. Doing the same with $\mathcal{F}$, differential operators appear to be densely defined linear operators in Hilbert spaces. From now on, the scalar product and the norm on the fibers are denoted by $\langle .,. \rangle$ and $|.|$, while the corresponding scalar product and norm on M are denoted by $(.,.)_{L^2(M,\mathcal{E})}$ and $\|.\|_{L^2(M,\mathcal{E})}$. We will often omit "$\mathcal{E}$" in these notations.

We want now to define the *formal adjoint* of a differential operator. We call it "formal" adjoint to avoid confusion with the strict L^2-adjoint.

So let $\mathfrak{D} \in \mathcal{D}(M, \mathcal{E})$ be a differential operator and φ_i a partition of the unity with $\mathrm{supp}\, \varphi_i \subset U_i$, where the U_i's are open sets on which normal coordinates are defined (i.e. $\exists x_i \in U_i, V_i \subset T_{x_i}M$ so that $\exp \colon V_i \to U_i$ is a diffeomorphism), and which form a locally finite covering of M. We define operators with compact support $\mathfrak{D}_i := \varphi_i \mathfrak{D}$. First we define formal adjoints for these operators.

$$\begin{aligned} (\mathfrak{D}_i f, g)_{L^2(M)} &= \int_M \langle \mathfrak{D}_i f, g \rangle |dx| \\ &= \int_{V_i} \langle \mathfrak{D}_i f, g \rangle \nu_g dx \\ &= \int_{V_i} \langle (\sum_\alpha \varphi_i A_{i,\alpha} \partial^\alpha f), g \nu_g \rangle dx \end{aligned}$$

$$
\begin{aligned}
(\mathfrak{D}_i f, g)_{L^2(M)} &= \int_{V_i} \langle \sum_\alpha \partial^\alpha f, \varphi_i A^*_{i,\alpha} g \nu_g \rangle dx \\
&= \int_{V_i} \langle f, \sum_\alpha (-1)^{|\alpha|} \varphi_i \partial^\alpha (A^*_{i,\alpha} g \nu_g) \rangle dx \\
&= \int_{V_i} \langle f, \mathfrak{D}^*_i g \rangle \nu_g dx
\end{aligned}
$$

where $(\mathfrak{D}^*_i g) := \frac{1}{\nu_g} \sum_\alpha (-1)^{|\alpha|} \partial^\alpha (\varphi_i A^*_{i,\alpha} g \nu_g)$. This expression makes sense only on U_i, but since it has compact support in U_i, we merely define $\mathfrak{D}^*_i$ to be zero outside U_i. This operator satisfies

$$
(\mathfrak{D}_i f, g)_{L^2(M)} = (f, \mathfrak{D}^*_i g)_{L^2(M)}
$$

for all $f, g \in C^\infty(M)$ (the requirement that they should have compact support is not necessary since $\mathfrak{D}$ and $\mathfrak{D}_i$ have compact support).

The formal adjoint $\mathfrak{D}^*$ of $\mathfrak{D}$ is defined by:

$$
\mathfrak{D}^* := \sum_i \mathfrak{D}^*_i
$$

This definition is valid, because this sum is locally finite. With this definition, we have for any $f \in \Gamma_c(M, \mathcal{E})$, $g \in \Gamma_c(M, \mathcal{F})$:

$$
\begin{aligned}
(\mathfrak{D} f, g)_{L^2(M)} &= \sum_i \int_{U_i} \langle \mathfrak{D}_i f, g \rangle |dx| = \sum_i \int_{U_i} \langle f, \mathfrak{D}^*_i g \rangle |dx| \\
&= \int_M \langle f, \sum_i \mathfrak{D}^*_i g \rangle |dx| = \int_M \langle f, \mathfrak{D}^* g \rangle |dx| \\
&= (f, \mathfrak{D}^* g)_{L^2(M)}
\end{aligned}
$$

(we have the right to exchange the sum and the integral because we work on a compact, so the sum is finite). It is clear that it is the only operator defined on $\Gamma_c(M, \mathcal{E})$ which is formal adjoint to $\mathfrak{D}$. So we have proved:

Proposition 4.3. *Let $\mathcal{E}, \mathcal{F}$ be two finite dimensional vector bundles over M with scalar products $\langle .,. \rangle_{\mathcal{E}}$ and $\langle .,. \rangle_{\mathcal{F}}$, $\mathfrak{D} \in \mathcal{D}_d(M, \mathcal{E}, \mathcal{F})$ be a differential operator. Then there exists a differential operator $\mathfrak{D}^* \in \mathcal{D}_d(M, \mathcal{F}, \mathcal{E})$ which satisfies:*

$$
\forall \varphi \in \Gamma_c(M, \mathcal{E}), \psi \in \Gamma_c(M, \mathcal{F}), \qquad \int_M \langle \mathfrak{D} \varphi, \psi \rangle_{\mathcal{F}} = \int_M \langle \varphi, \mathfrak{D}^* \psi \rangle_{\mathcal{E}}
$$

For $\mathfrak{D} \in \mathcal{D}(M, \mathcal{E}, \mathcal{F})$, $f \in C^\infty(M)$ and $s \in \Gamma(M, \mathcal{E})$ we set

$$
((\operatorname{ad} f)\mathfrak{D})s := [\mathfrak{D}, f]s = \mathfrak{D}(fs) - f(\mathfrak{D}s)
$$

Lemma 4.4. *Let $\mathfrak{D}$ be a differential operator. Then for all $f \in C^\infty(M)$:*

(i) $\mathfrak{D} \in \mathcal{D}_0(M, \mathcal{E}, \mathcal{F}) \Rightarrow (\operatorname{ad} f)\mathfrak{D} = 0$

(ii) $\mathfrak{D} \in \mathcal{D}_{d+1}(M, \mathcal{E}, \mathcal{F}) \Rightarrow (\operatorname{ad} f)\mathfrak{D} \in \mathcal{D}_d(M, \mathcal{E}, \mathcal{F})$

Proof. (i) This is clear because $\mathcal{D}_0(M, \mathcal{E}, \mathcal{F})$ is the space of sections in the vector bundle $\operatorname{End}(\mathcal{E}, \mathcal{F})$ over M.

(ii) For two multi-indices α and β, we note $\beta \leq \alpha$ when $\forall i, \beta_i \leq \alpha_i$, and in this situation, $\alpha - \beta$ is the multi-index $(\alpha_i - \beta_i)$. By recurrence, it is easy to show that

$$
(\operatorname{ad} f)\partial^\alpha = \sum_{\substack{\beta \leq \alpha \\ |\beta| \geq 1}} \binom{\alpha}{\beta} (\partial^\beta f) \partial^{\alpha - \beta}
$$

So it remains only derivatives with multi-indices of length $|\alpha - \beta| = |\alpha| - |\beta| < |\alpha| \leq d + 1$. The fact that endomorphisms commute with pointwise multiplication by a function yields the result.

□

Proposition 4.5. *Let $\mathfrak{D}$ be a differential operator.*

$$\mathfrak{D} \in \mathcal{D}_d(M,\mathcal{E},\mathcal{F}) \Leftrightarrow \forall f \in C^\infty(M),\ (\operatorname{ad} f)^{d+1}\mathfrak{D} = 0 \tag{6}$$
$$\Leftrightarrow \forall f_1,\dots,f_{d+1} \in C^\infty(M),\ (\operatorname{ad} f_1)\circ\dots\circ(\operatorname{ad} f_{d+1})\mathfrak{D} = 0 \tag{7}$$

Proof. Note that the right-hand side of (7) implies the right-hand side of (6). We use this fact in the proof. If the local expression of $\mathfrak{D}$ is given by $\mathfrak{D} = \sum_{\alpha,|\alpha|\le d} A_\alpha(x)\partial^\alpha$, we have

$$(\operatorname{ad} f_1)\circ\dots\circ(\operatorname{ad} f_{d+1})\mathfrak{D} =$$
$$\sum_{|\alpha|\le d}\ \sum_{\substack{|\beta_k|\ge 1\\ \sum_{k=1}^{d+1}\beta_k\le\alpha}} A_\alpha(x)\left(\left(\prod_i \frac{\alpha_i!}{\beta_{1,i}!\dots\beta_{d+1,i}!(\alpha_i-\sum_k(\beta_k)_i)!}\right)\prod_k \partial^{\beta_k} f_k\right)\partial^{\alpha-\sum\beta_k}$$

"$\Rightarrow$" This is a direct consequence of lemma 4.4.

"$\Leftarrow$" Using the remark above, we suppose here that $f_1 = \dots = f_{d+1}$. If $\mathfrak{D}$ is *not* in $\mathcal{D}_d(M,\mathcal{E},\mathcal{F})$, then there exists a patch of local coordinates and a non-zero A_γ of maximal order, with $|\gamma| \ge d+1$. We consider the terms of highest order in $(\operatorname{ad} f)^{d+1}\mathfrak{D}$. They come from $d+1$-tuples (β_k) where each β_k has length 1. Hence the coefficient of a term of highest order is a homogeneous polynomial of degree $d+1$ in the $\frac{\partial f}{\partial x^i}$'s, and each coefficient of the monomials in these polynomials is *one* A_α, up to a non-zero integer coefficient, where α is of maximal length. Since one of these does not vanish, and because $f \mapsto df|_x$ is surjective on T_x^*M, we can always find a smooth function f such that at least one term of highest order in $(\operatorname{ad} f)^{d+1}\mathfrak{D}$ does *not* vanish.

□

Corollary 4.6. *So, if $\mathfrak{D} \in \mathcal{D}_{=d}(M,\mathcal{E},\mathcal{F})$, d is given by:*

$$d = \inf\left\{k\in\mathbb{N} \mid \forall f\in C^\infty(M), (\operatorname{ad} f)^{k+1}\mathfrak{D} = 0\right\}$$
$$= \inf\left\{k\in\mathbb{N} \mid \forall f_1,\dots,f_{k+1}\in C^\infty(M), (\operatorname{ad} f_1)\circ\dots\circ(\operatorname{ad} f_{k+1})\mathfrak{D} = 0\right\}$$

We see from this that for $\mathfrak{D} \in \mathcal{D}_{=d}(M,\mathcal{E},\mathcal{F})$, $(\operatorname{ad} f)^d\mathfrak{D} \in \mathcal{D}_0(M,\mathcal{E},\mathcal{F})$. Furthermore, the computation made above shows that the associated endomorphism $((\operatorname{ad} f)^d\mathfrak{D})(x)\colon \mathcal{E}_x \to \mathcal{F}_x$ depends on f only by the first-order derivatives $\left.\frac{\partial f}{\partial x^i}\right|_x$, i.e. it depends only on df_x.

Definition 4.7. The *d-symbol* $\sigma_d(\mathfrak{D})$ of the differential operator $\mathfrak{D} \in \mathcal{D}_d(M,\mathcal{E},\mathcal{F})$ is the section in $\Gamma(T^*M, \operatorname{End}(\mathcal{E},\mathcal{F}))$ given by[5]:

$$\sigma_d(\mathfrak{D})(x,\xi) := \frac{i^d}{d!}(\operatorname{ad} f)^d\mathfrak{D}$$

where f is any smooth function with $df|_x = \xi$. If $\mathfrak{D} \in \mathcal{D}_{=d}(M,\mathcal{E},\mathcal{F})$, $\sigma_d(\mathfrak{D})$ will be simply called *symbol* of $\mathfrak{D}$.

Example. The symbol of the covariant derivative is given by:

$$\sigma_1(\nabla)(x,\xi)s = i\xi\otimes s \tag{8}$$

If the local expression of $\mathfrak{D}$ is:

$$\sum_{|\alpha|\le d} A_\alpha(x)\partial^\alpha$$

then the computations made above show that:

$$\sigma_d(\mathfrak{D})(x,\xi) = i^d\sum_{|\alpha|=d} A_\alpha(x)\xi^\alpha$$

Hence the symbol is polynomial along the fibers.

[5]Here we assume that we have complex bundles. If not, one should delete the term "i^d" and modify the formulas according to this change.

Definition 4.8. A differential operator is called *elliptic* when its symbol is an isomorphism for all $\xi \in T^*M$ with $\|\xi\| \neq 0$.

Proposition 4.9. *For two differential operators $\mathfrak{D}_1 \in \mathcal{D}(M, \mathcal{E}, \mathcal{F})$, $\mathfrak{D}_2 \in \mathcal{D}(M, \mathcal{F}, \mathcal{G})$ and a smooth function f, we have:*

$$(\operatorname{ad} f)^d(\mathfrak{D}_2\mathfrak{D}_1) = \sum_{k=0}^{d} \binom{d}{k} ((\operatorname{ad} f)^k\mathfrak{D}_2)((\operatorname{ad} f)^{d-k}\mathfrak{D}_1) \tag{9}$$

Proof. We are sure the reader will be able to do the recurrence himself. □

Proposition 4.10. *Let $\mathfrak{D}_1 \in \mathcal{D}_{d_1}(M, \mathcal{E}, \mathcal{F})$, $\mathfrak{D}_2 \in \mathcal{D}_{d_2}(M, \mathcal{F}, \mathcal{G})$. Then:*

$$\sigma_{d_1+d_2}(\mathfrak{D}_2\mathfrak{D}_1) = \sigma_{d_2}(\mathfrak{D}_2)\sigma_{d_1}(\mathfrak{D}_1)$$

Proof. Since $\mathfrak{D}_1 \in \mathcal{D}_{d_1}(M, \mathcal{E}, \mathcal{F})$ and $\mathfrak{D}_2 \in \mathcal{D}_{d_2}(M, \mathcal{F}, \mathcal{G})$, $(\operatorname{ad} f)^k\mathfrak{D}_2$ vanishes for $k > d_2$ and $(\operatorname{ad} f)^{d-k}\mathfrak{D}_1$ vanishes for $d - k > d_1$. Hence, taking $d = d_1 + d_2$ in (9), the only term remaining is with $k = d_2$, i.e.:

$$\begin{aligned}(\operatorname{ad} f)^{d_1+d_2}(\mathfrak{D}_2\mathfrak{D}_1) &= \binom{d_1+d_2}{d_2} ((\operatorname{ad} f)^{d_2}\mathfrak{D}_2)((\operatorname{ad} f)^{d_1+d_2-d_2}\mathfrak{D}_1) \\ &= \frac{(d_1+d_2)!}{d_1!d_2!}((\operatorname{ad} f)^{d_2}\mathfrak{D}_2)((\operatorname{ad} f)^{d_1}\mathfrak{D}_1)\end{aligned}$$

□

Proposition 4.11. *Let $\mathfrak{D} \in \mathcal{D}_d(M, \mathcal{E})$. Then $\sigma_d(\mathfrak{D}^*) = \sigma_d(\mathfrak{D})^*$. In particular, the formal adjoint of an elliptic operator is elliptic.*

Proof. It is not difficult to see that for $f \in C^\infty(M, \mathbb{R})$, $[\mathfrak{D}, f]^* = [f, \mathfrak{D}^*] = -[\mathfrak{D}^*, f]$, and by recurrence, $((\operatorname{ad} f)^d\mathfrak{D})^* = (-1)^d(\operatorname{ad} f)^d\mathfrak{D}^*$. Since $i^d(-1)^d = (-i)^d = \overline{i^d}$ we obtain the result. We can also see this with the local expression of $\mathfrak{D}^*$ computed above. □

Proposition 4.12. *Let $\mathfrak{D} \in \mathcal{D}_{d+1}(M, \mathcal{E})$ be an elliptic differential operator. Then for any $\mathfrak{d} \in \mathcal{D}_d(M, \mathcal{E})$, $\mathfrak{D} + \mathfrak{d}$ is elliptic.*

Proof. We have $\sigma_{d+1}(\mathfrak{D} + \mathfrak{d}) = \sigma_{d+1}(\mathfrak{D}) + \underbrace{\sigma_{d+1}(\mathfrak{d})}_{=0} = \sigma_{d+1}(\mathfrak{D})$. □

4.2 Laplace operators

Definition 4.13. A generalized Laplace operator (or Laplacian) Δ on $\mathcal{E}$ is a second-order differential operator Δ such that

$$\sigma_2(\Delta)(\xi) = \|\xi\|^2$$

It is equivalent to require that the expression of Δ in local coordinates is of the form:

$$\Delta = -\sum_{i,j} g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j} + \text{lower order terms}$$

Using the expression for the symbol, we also get that Δ is a generalized Laplacian when it satisfies

$$[[\Delta, f], f] = -2\|df\|^2$$

Given a covariant derivative $\nabla^{\mathcal{E}}$ on $\mathcal{E}$, we can define a canonical Laplace operator, using $\nabla^{\mathcal{E}}$ and the Levi-Civita connection ∇. We have seen that having a covariant derivative on TM and $\mathcal{E}$ enables us to define a canonical associated covariant derivative $\nabla^{T^*M\otimes\mathcal{E}}$ on $T^*M \otimes \mathcal{E}$.

Definition 4.14. Let $\mathcal{E}$ be a vector bundle over M with covariant derivative $\nabla^{\mathcal{E}}$. We can define a generalized Laplace operator $\Delta^{\mathcal{E}}$ by the composition:

$$\Gamma(M, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} \Gamma(M, T^*M \otimes \mathcal{E}) \xrightarrow{\nabla^{T^*M\otimes\mathcal{E}}} \Gamma(M, T^*M \otimes T^*M \otimes \mathcal{E}) \xrightarrow{-Tr} \Gamma(M, \mathcal{E}),$$

where "Tr" denotes the contraction using the metric on M.

Proof. From proposition 2.26, we have $\Delta^{\mathcal{E}} = \nabla^*\nabla$. So:

$$\begin{aligned}\sigma_2(\nabla)(df) &\underset{4.10}{=} \sigma_1(\nabla^*)(df)\sigma_1(\nabla)(df) \underset{4.11}{=} \sigma_1(\nabla)(df)^*\sigma_1(\nabla)(df) \underset{(8)}{=} (i.df\otimes)^*(idf\otimes)\\ &\underset{2.17}{=} df\lrcorner\, df\otimes = \|df\|^2\end{aligned}$$

Since we will need also the expression of $[\Delta^{\mathcal{E}}, f]$, we give another proof. For $f \in C^\infty(M)$ we note "Δ" this operator associated to the Levi-Civita connection. If (e_i) is a local orthonormal frame and $t \in \Gamma(M, T^*M \otimes T^*M \otimes \mathcal{E})$, then Tr is given by the formula:

$$Tr(t) = \sum_i e_i\lrcorner\,(e_i\lrcorner\, t)$$

We have

$$\begin{aligned}\Delta^{\mathcal{E}}(fs) &= -Tr(\nabla^{T^*M\otimes\mathcal{E}}\nabla^{\mathcal{E}}(fs))\\ &= -Tr(\nabla^{T^*M\otimes\mathcal{E}}(df\otimes s + f\nabla^{\mathcal{E}}s))\\ &= -Tr((\nabla df)\otimes s + df\otimes(\nabla^{\mathcal{E}}s) + df\otimes(\nabla^{\mathcal{E}}s) + f\nabla^{T^*M\otimes\mathcal{E}}\nabla^{\mathcal{E}}s)\end{aligned}$$

Hence

$$\begin{aligned}[\Delta^{\mathcal{E}}, f]s &= (\Delta f)s - 2Tr(df\otimes(\nabla^{\mathcal{E}}s))\\ &= (\Delta f)s - 2\sum_i df(e_i)(\nabla_{e_i}{}^{\mathcal{E}}s)\\ &= (\Delta f)s - 2(\nabla^{\mathcal{E}}_{\sum_i df(e_i)e_i}s)\\ &= (\Delta f)s - 2(\nabla^{\mathcal{E}}_{df^\sharp}s)\end{aligned} \tag{10}$$

Thus

$$\begin{aligned}[[\Delta^{\mathcal{E}}, f], g] &= -2dg(df^\sharp)\\ &= -2\langle df, dg\rangle\end{aligned}$$

This shows that $\Delta^{\mathcal{E}}$ is a generalized Laplacian (in particular on functions). □

Remark. On $\mathbb{R}^n$, this Laplacian is given by

$$\Delta^{\mathbb{R}^n} f = \sum_{i=1}^n -\frac{\partial^2 f}{(\partial x^i)^2}$$

which is nothing but the opposite of the "classical" Laplacian.

Proposition 4.15. *The canonical Laplacian $\Delta^{\mathcal{E}}$ associated to $\nabla^{\mathcal{E}}$ is symmetric (as operator defined on $\Gamma_c(M, \mathcal{E})$).*

Proof. This is a direct consequence of proposition 2.26 because it implies that $\Delta^{\mathcal{E}} = (\nabla^{\mathcal{E}})^*\nabla^{\mathcal{E}}$, so

$$(\Delta^{\mathcal{E}}s, t)_{L^2(M)} = (\nabla^{\mathcal{E}}s, \nabla^{\mathcal{E}}t)_{L^2(M)} = (s, \Delta^{\mathcal{E}}t)_{L^2(M)}$$

□

Proposition 4.16. *If H is a generalized Laplacian on a vector bundle $\mathcal{E}$, there exists a covariant derivative $\nabla^{\mathcal{E}}$ on $\mathcal{E}$ such that $H - \Delta^{\mathcal{E}}$ is a zeroth-order operator.*

Proof. For $f \in C^\infty(M)$ and $s \in \Gamma(M, \mathcal{E})$ we set

$$D_f s := \frac{1}{2}([f, H]s + (\Delta f)s)$$

From $[[H, f+g], f+g] = -2\|d(f+g)\|^2$ we deduce that $[[H, f], g] = -2\langle df, dg\rangle$, and a simple computation shows that $[[H, f], g] = [H, fg] + f[g, H] + g[f, H]$. We have also $\Delta(fg) = f(\Delta g) + g(\Delta f) - 2\langle df, dg\rangle$. Altogether,

$$\begin{aligned}D_{fg}s &= \frac{1}{2}([fg, H]s + (\Delta(fg))s)\\ &= \frac{1}{2}(f[g, H]s + g[f, H]s - [[H, f], g]s + f(\Delta g)s + g(\Delta f)s - 2\langle df, dg\rangle s)\\ &= fD_g s + gD_f s\end{aligned}$$

Hence, from proposition 4.1, this defines a linear map

$$D\colon \Gamma(M,\mathcal{E}) \to \Gamma(M,T^*M\otimes\mathcal{E}) \simeq \Gamma(M,TM\otimes\mathcal{E})$$

Furthermore, this map satisfies the following commutation relation:

$$[D_f,g] = \frac{1}{2}([[f,H],g] + \underbrace{[\Delta f,g]}_{=0}) = \langle df,dg\rangle$$

which proves that it is a covariant derivative, from now on denoted by $\nabla^{\mathcal{E}}$. So, by the very definition of $\nabla^{\mathcal{E}}$:

$$[H,f] = -2\nabla^{\mathcal{E}}_{df^\sharp} + (\Delta f) \underset{(10)}{=} [\Delta^{\mathcal{E}},f]$$

thus $H - \Delta^{\mathcal{E}}$ is a first-order differential operator. □

4.3 Dirac operators

Definition 4.17. Let $\mathcal{E}$ be a bundle of left Clifford modules, i.e. $\mathcal{E}$ is a left module over $C\ell(M)$. A generalized Dirac operator on $\mathcal{E}$ is a first-order differential operator $\not{D}$ such that

$$\sigma_1(\not{D})(\xi) = i\xi\cdot$$

From $\sigma(\mathfrak{D}_1\circ\mathfrak{D}_2) = \sigma(\mathfrak{D}_1)\circ\sigma(\mathfrak{D}_2)$, we see that the square of a generalized Dirac operator is a generalized Laplacian:

$$\sigma_2(\not{D}^2)(\xi) = \sigma_1(\not{D})(\xi)\circ\sigma_1(\not{D})(\xi) = i^2\xi\cdot\xi\cdot \;\; = \|\xi\|^2$$

Remark. This fact explains to us where the Clifford algebras come from. Suppose we want a first order differential operator $D = \sum_i a^i(x)\frac{\partial}{\partial x^i}$ to be the square root of a Laplacian. We see that (after symmetrization):

$$D^2 = \sum_{i,j}\frac{1}{2}(a^i(x)a^j(x) + a^j(x)a^i(x))\frac{\partial^2}{\partial x^i\partial x^j} + \text{first order operator}$$

So the coefficients must satisfy

$$a^ia^j + a^ja^i = -2g^{ij}$$

Conversely, we have the following proposition:

Proposition 4.18. *Let $\not{D}$ be a first-order differential operator such that $\not{D}^2$ is a generalized Laplacian. Then we can define a Clifford action by means of:*

$$\xi\cdot\varphi := [\not{D},f]\varphi$$

where $x\in M$, $\xi\in T^*_xM$ and f in any smooth function such that $df_x = \xi$.

Proof. First, we must prove that this definition is consistent. $\not{D}$ is a first-order differential operator, so we have:

$$[\not{D},fg] = f[\not{D},g] + g[\not{D},f] + \underbrace{[[\not{D},f],g]}_{=0} = f[\not{D},g] + g[\not{D},f]$$

Hence, by proposition 4.1, $f\mapsto[\not{D},f]$ defines a section in $\Gamma(M,TM\otimes\mathrm{End}(\mathcal{E}))$, i.e. linear maps $T^*_xM \to \mathrm{End}(\mathcal{E}_x)$. Let $c(df) := [\not{D},f]$. Since $\not{D}$ is a first-order differential operator we have $[[\not{D},f],f] = 0$ hence $[\not{D},f]f = f[\not{D},f]$.

$$\begin{aligned}
2c(df)^2 &= 2[\not{D},f][\not{D},f]\\
&= ([\not{D},f](\not{D}f - f\not{D}) + (\not{D}f - f\not{D})[\not{D},f])\\
&= (([\not{D},f]\not{D}f - f[\not{D},f]\not{D}) + (\not{D}[\not{D},f]f - f\not{D}[\not{D},f]))\\
&= \big((\not{D}f\not{D}f - f\not{D}^2f - f\not{D}f\not{D} + f^2\not{D}^2) + (\not{D}^2f^2 - \not{D}f\not{D}f - f\not{D}^2f + f\not{D}f\not{D})\big)\\
&= \big(\not{D}^2f^2 - 2f\not{D}^2f + f^2\not{D}^2\big)\\
&= [[\not{D}^2,f],f]\\
&= -2\|df\|^2
\end{aligned}$$

So the action of T^*M extends to a Clifford action. □

Proposition 4.19. *Given a Dirac bundle $\mathcal{E}$, we can define a canonical Dirac operator, by composing the covariant derivative ∇ with the Clifford multiplication c:*

$$\not{D} : \Gamma(M,\mathcal{E}) \xrightarrow{\nabla} \Gamma(M, T^*M \otimes \mathcal{E}) \xrightarrow{c} \Gamma(M,\mathcal{E})$$

Proof. If (e_i) is a local orthonormal frame, we have $\not{D}\sigma := \sum_i e_i \cdot \nabla_{e_i}\sigma$. Hence:

$$[\not{D}, f]\sigma = \sum_i e_i \cdot \nabla_{e_i}(f\sigma) - f e_i \cdot \nabla_{e_i}\sigma = \sum_i e_i \cdot (\nabla_{e_i} f)\sigma = df \cdot \sigma$$

□

Examples.

- For $\mathbb{R}$ we have $\mathcal{C}\ell_1 \simeq \mathbb{C}$. With this identification the Dirac operator on $C^\infty(\mathbb{R},\mathbb{C})$ is given by:

$$\not{D} f = i\frac{df}{dx}$$

- For $\mathbb{R}^2$ we have $\mathcal{C}\ell_2 = \text{span}\,\{1, e_1, e_2, e_1 \cdot e_2\}$ with the product relations:

$$e_1^2 = e_2^2 = (e_1 \cdot e_2)^2 = -1 \qquad e_1 \cdot e_2 = e_1 \cdot e_2 \qquad e_2 \cdot (e_1 \cdot e_2) = e_1 \qquad (e_1 \cdot e_2) \cdot e_1 = e_2$$

  so we see that $\mathcal{C}\ell_2 \simeq \mathbb{H} = \text{span}\,\{1, \mathcal{I}, \mathcal{J}, \mathcal{K}\}$. On $C^\infty(\mathbb{R}^2, \mathbb{H})$ the Dirac operator is given by:

$$\not{D} f = \mathcal{I}\frac{\partial f}{\partial x^1} + \mathcal{J}\frac{\partial f}{\partial x^2}$$

- We have seen that $\wedge T^*M$ is a left module over $\mathcal{C}\ell(M)$ under the action defined by $v \cdot \alpha := v^\flat \wedge \alpha - v \lrcorner\, \alpha$ for $v \in TM$. The Dirac operator is given by:

$$\not{D}\alpha = \sum_i e^i \wedge \nabla\alpha - e_i \lrcorner\, \nabla\alpha = (\varepsilon \circ \nabla - \iota \circ \nabla)\alpha$$

  So by the formulas shown previously:

$$\not{D} = d + d^*$$

Proposition 4.20. *Let $\not{D}$ be the Dirac operator on a Dirac bundle. Then $\not{D}$ is symmetric.*

Proof. We set $\alpha(X) := \langle \sigma_1, X \cdot \sigma_2 \rangle$

$$\begin{aligned}
\langle \not{D}\sigma_1, \sigma_2 \rangle &= \sum_i \langle e_i \cdot \nabla_{e_i}\sigma_1, \sigma_2 \rangle \\
&= -\sum_i \langle \nabla_{e_i}\sigma_1, e_i \cdot \sigma_2 \rangle \\
&= \sum_i -e_i(\langle \sigma_1, e_i \cdot \sigma_2 \rangle) + \langle \sigma_1, \nabla_{e_i}(e_i \cdot \sigma_2) \rangle \\
&= \sum_i -e_i(\alpha(e_i)) + \langle \sigma_1, (\nabla_{e_i} e_i) \cdot \sigma_2 \rangle + \langle \sigma_1, e_i \cdot \nabla_{e_i}\sigma_2 \rangle \\
&= \left(\sum_i -e_i(\alpha(e_i)) + \alpha(\nabla_{e_i} e_i) \right) + \langle \sigma_1, \not{D}\sigma_2 \rangle \\
&= -Tr(\nabla\alpha) + \langle \sigma_1, \not{D}\sigma_2 \rangle
\end{aligned}$$

Integrating over M kills the divergence. □

4.4 Essentially self-adjointness

4.4.1 Preliminaries

Proposition 4.21. *Let M be a complete Riemannian manifold, $p \in M$. Let B_r denote the closed ball of radius r and center p. Then $\forall \varepsilon, r > 0$, $\exists \rho_{\varepsilon,r} \in C^\infty(M,\mathbb{R})$ satisfying:*

(i) $\forall x \in B_r,\ |\rho_{\varepsilon,r}(x) - 1| < \varepsilon,$

(ii) $\forall x \in M \setminus B_{2r},\ \rho_{\varepsilon,r}(x) = 0,$

(iii) $\forall x \in M,\ 0 \le \rho_{\varepsilon,r}(x) \le 1 + \varepsilon,$

(iv) $\forall x \in M,\ \|d\rho_{\varepsilon,r}(x)\| \le \frac{1+\varepsilon}{r}$

Proof. So let M, p as above, $r_1, r_2 > 0$ with $r < r_1 < r_2 < 2r$. r denotes the distance between p and x:

$$r(x) := d(p,x)$$

We define $\rho \in C^\infty(M)$ by:

$$\rho(x) = \begin{cases} 1 & \text{if } r(x) \le r_1, \\ \frac{r_2 - r(x)}{r_2 - r_1} & \text{if } r_1 < r(x) < r_2 \\ 0 & \text{if } r(x) \ge r_2 \end{cases}$$

Now let $\theta \in C_c^\infty(\mathbb{R}, \mathbb{R})$ satisfying:

- $\theta \ge 0,$
- θ is locally constant at 0,
- $\int_{\mathbb{R}^n} \theta(\|x\|)dx = 1$

With these definitions, we set: $$\theta_t(x) := \frac{1}{t^n}\theta(\frac{x}{t}) \quad \text{and} \quad \rho_t(x) := \int_M \rho(y)\theta_t(d(x,y))|dx|$$

Of course the interesting case is when $t \to 0$, so we will assume $t < 1$. Let us establish the properties of this new function. Choose $x \in M$. Using normal coordinates at x, the expression of ρ_t on T_xM is given by:

$$\rho_t(\exp X) = \int_{\mathbb{R}^n} \rho(\exp Y)\theta_t(d(\exp X, \exp Y))\nu_g(Y)dY$$

where $\nu_g := \sqrt{|\det g|}$ denotes the volume element on T_xM, and where we suppose that the support of θ is small enough so that exp is a diffeomorphism where θ does not vanish. We know that $x \mapsto d(x,y)$ is smooth if $0 < d(x,y) < \mathrm{injrad}(y)$, where "injrad" denotes the injectivity radius. Since θ_t is locally constant at 0, and using the fact that the injectivity radius is continuous, we see that if t is small enough, say smaller than $\mathrm{injrad}(x)/2$, then ρ_t is smooth in a neighborhood of x.

Let $R_t := \sup\{|x| \in \mathbb{R} \mid \theta_t(x) \neq 0\}$. It is obvious that $0 \le \rho_t$, and that ρ_t vanishes outside $B_{r_2+R_t}$, so we can restrict ourselves to a compact set, and this fact enables us to have uniform bounds for smooth functions and derivatives. It is not difficult to see that ρ_t converges uniformly to ρ when $t \to 0$. Hence the properties (i), (ii) and (iii) are satisfied for t small enough.

In order to simplify the notations, we will drop "exp", and consider functions as if defined on the tangent space. By the coordinate change $Y = X + Z$ we get:

$$\rho_t(X) = \int_{\mathbb{R}^n} \rho(X+Z)\theta_t(d(X, X+Z))\nu_g(X+Z)dZ$$

From the properties of ρ we have almost everywhere:

$$\begin{aligned}
\sum_{i=1}^n \frac{\partial \rho_t}{\partial x^i}(0)dx^i &= \sum_{i=1}^n \left(\int_{\mathbb{R}^n} \frac{\partial}{\partial x^i}\Big(\rho(X+Z)\theta_t(d(X,X+Z))\nu_g(X+Z)\Big)\Big|_{X=0} dZ \right) dx^i \\
&= \int_{\mathbb{R}^n} \left(\sum_{i=1}^n \frac{\partial \rho(X+Z)}{\partial x^i}\Big|_{X=0} dx^i \right) \theta_t(\|Z\|)\nu_g(Z)dZ && \text{(T1)} \\
&\quad + \sum_{i=1}^n \left(\int_{\mathbb{R}^n} \rho(Z)\ \frac{\partial \theta_t(d(X,X+Z))}{\partial x^i}\Big|_{X=0} \nu_g(Z)dZ \right) dx^i && \text{(T2)} \\
&\quad + \sum_{i=1}^n \left(\int_{\mathbb{R}^n} \rho(Z)\theta_t(\|Z\|)\ \frac{\partial \nu_g(X+Z)}{\partial x^i}\Big|_{X=0} dZ \right) dx^i && \text{(T3)}
\end{aligned}$$

Using the boundedness of ρ and $d\rho$ (we have $\|d\rho\| \le \frac{1}{r_2 - r_1}$ where it is smooth), and the uniform convergences for the metric coefficients $g^{ij}(Z) \to \delta^{ij}$, $\frac{\partial g^{ij}}{\partial x^k}(Z) \to 0$ when $Z \to 0$, we get that[6]:

$$\begin{aligned}
\|(\mathrm{T1})\| &\le \|d\rho\|_\infty + o(1)\\
\|(\mathrm{T2})\| &= O\left(\int_{B_{R_t}} \|d_X\theta_t(d(X, X+Z))\| dZ\right)\\
\|(\mathrm{T3})\| &= O(t)
\end{aligned}$$

But we have $|d(X, X+Z) - \|Z\|| = O(\|X\|.\|Z\|.\|dg\|)$, so

$$(\mathrm{T2}) = o\left(\int_{B_{R_t}} \|\theta_t'\|_\infty \|Z\| dZ\right)$$

It is clear that $\int_{B_{R_t}} \|Z\| dZ = O(t^{n+1})$, and from $\theta_t(x) = \frac{1}{t^n}\theta(\frac{x}{t})$ we get $\|\theta_t'\|_\infty = O(\frac{1}{t^{n+1}})$. Hence

$$(\mathrm{T2}) = o(1)$$

All "big-O's" and "small-o's" used are uniform on the compact set B_{2r+R_1}. Altogether we obtain that

$$\|d\rho_t\|_\infty \le \|d\rho\|_\infty + \underset{t\to 0}{o}(1)$$

Since ρ_t is smooth, this inequality holding almost everywhere holds in fact everywhere. So for r_1, r_2 close enough to r and $2r$, then t small enough, the property (iv) is also satisfied. ☐

Proposition 4.22. *Let M be a complete Riemannian manifold, $p \in M$. Let B_r denote the closed ball of radius r and center p. Then $\forall \varepsilon, r > 0$, $\exists \rho_{\varepsilon,r} \in C^\infty(M, \mathbb{R})$ satisfying:*

(i) $\forall x \in B_r,\ \rho_{\varepsilon,r}(x) = 1,$
(ii) $\forall x \in M \setminus B_{2r},\ \rho_{\varepsilon,r}(x) = 0,$
(iii) $\forall x \in M,\ 0 \le \rho_{\varepsilon,r}(x) \le 1,$
(iv) $\forall x \in M,\ \|d\rho_{\varepsilon,r}(x)\| \le \frac{1+\varepsilon}{r}$

Proof. Choose r_1, r_2 with $r < r_1 < r_2 < 2r$ and ρ as above.

$$P_t(x) := \frac{\int_M \rho(y)\theta_t(d(x,y))|dx|}{\int_M \theta_t(d(x,y))|dx|} = \frac{\rho_t(x)}{u_t(x)}$$

It is obvious that the denominator does not vanish since θ does not vanish identically. Using the discussion made above, we see that for t small enough, both ρ_t and u_t are smooth, hence P_t is smooth for t small enough. Now:

- $\rho = 1$ on B_r implies that $P_t = 1$ on B_r for t small enough.
- $\rho = 0$ outside B_{r_2} implies that $P_t = 0$ outside B_{2r} for t small enough.
- $0 \le \rho \le 1 \Rightarrow 0 \le P_t \le 1$ everywhere.
- Since $u - t$ is obtained from the constant function 1 by the same method as ρ_t, and because $\|d1\|_\infty = 0$, we obtain that $\|u_t(x) - 1\|_{B_{2r},\infty} = o(1)$ and $\|du_t\|_{B_{2r},\infty} = o(1)$ for $t \to 0$ (here $\|.\|_{B_{2r},\infty}$ denotes the sup on the compact set B_{2r}). Now,

$$\begin{aligned}
\|dP_t\| &= \left\|\frac{d\rho_t}{u_t} - \frac{\rho_t}{u_t^2}du_t\right\|\\
&\le \underbrace{\|d\rho_t\|}_{\le \frac{1}{r_2-r_1}+o(1)} \cdot \underbrace{\left\|\frac{1}{u_t}\right\|}_{1+o(1)} + \underbrace{\|\rho_t\|}_{=1+o(1)} \cdot \underbrace{\left\|\frac{1}{u_t^2}\right\|}_{1+o(1)} \cdot \underbrace{\|du_t\|}_{=o(1)}\\
&\le \frac{1}{r_2 - r_1} + o(1)
\end{aligned}$$

Hence for $r_1 \approx r$, $r_2 \approx 2r$ and t small enough, the bound on $\|dP_t\|$ is satisfied.

[6] "big-O's" and "small-o's" used are in $t \to 0$.

□

Remark. In the particular case where $\exp\colon T_xM \to M$ is a diffeomorphism, this proposition is easy to prove. Indeed, we can work in polar coordinates and restrict ourselves to functions depending only on the distance. Hence solving the problem on $\mathbb{R}$ (which is not difficult) yields the result. The problem here is that, in general, the cut-locus is *not* empty.

Let us now recall some fundamental results of the theory of elliptic operators. We refer to [11] (p. 379 Th. 11.1) for the result. It is also stated in [9] (p. 193 Th. 5.2) in the case of compact manifolds (but that does not make a great difference since everything is considered locally).

Theorem 4.23 (Elliptic regularity). *Let $\mathfrak{D} \in \mathcal{D}(M,\mathcal{E})$ be an elliptic differential operator. Then for any distribution φ,*

$$\mathfrak{D}\varphi = 0 \Rightarrow \varphi \in C^\infty(M)$$

Corollary 4.24. *Let $\mathfrak{D} \in \mathcal{D}(M,\mathcal{E}) \setminus \mathcal{D}_0(M,\mathcal{E})$ be an elliptic differential operator of order at least 1. Then for all $\lambda \in \mathbb{C}$ and any distribution φ,*

$$\mathfrak{D}\varphi = \lambda\varphi \Rightarrow \varphi \in C^\infty(M)$$

Proof. From proposition 4.12, $\mathfrak{D} - \lambda$ is elliptic, so it is a direct consequence of theorem 4.23. □

Theorem 4.25. *Let M be a* compact *manifold, $\mathfrak{D} \in \mathcal{D}_d(M,\mathcal{E},\mathcal{F})$ an elliptic symmetric differential operator. Then $\mathfrak{D}$ is essentially self-adjoint.*

Proof. Since M is compact, elliptic regularity tells us that $\forall z \in \mathbb{C}$, $\ker(z - \mathfrak{D}^*) \subset \mathrm{dom}(\mathfrak{D})$, hence $\ker(z - \mathfrak{D}^*) = \ker(z - \mathfrak{D})$. Eigenvalues of $\mathfrak{D}$ are real, so:

$$\ker(\pm i - \mathfrak{D}^*) = \ker(\pm i - \mathfrak{D}) = \{0\}$$

Corollary 1.28 yields the result. □

Thanks to the previous theorem, we may assume for the following proofs that M is non-compact.

4.4.2 Essentially self-adjointness of Laplace operators

Proposition 4.26. *Let $\mathcal{E}$ be a vector bundle over M with covariant derivative $\nabla^\mathcal{E}$ and associated Laplacian $\Delta^\mathcal{E}$. Let $\varphi \in \Gamma(M,\mathcal{E}) \cap L^2(M,\mathcal{E})$ with $\Delta^\mathcal{E}\varphi \in L^2(M,\mathcal{E})$. Then $\nabla^\mathcal{E}\varphi \in L^2(M, T^*M \otimes \mathcal{E})$.*

Proof. Let $\rho_{\varepsilon,r}$ be as in proposition 4.22. For $f \in C^\infty(M,\mathbb{R})$ we have

$$[(\nabla^\mathcal{E})^*, f] = [f, \nabla^\mathcal{E}]^* = (-df\otimes)^* = -df^\sharp\lrcorner$$

Then

$$\begin{aligned}
\|\rho_{\varepsilon,r}\nabla^\mathcal{E}\varphi\|^2_{L^2(M)} &= ((\nabla^\mathcal{E})^*(\rho^2_{\varepsilon,r}\nabla^\mathcal{E}\varphi),\varphi)_{L^2(M)} \\
&= (\rho^2_{\varepsilon,r}(\nabla^\mathcal{E})^*\nabla^\mathcal{E}\varphi,\varphi)_{L^2(M)} + ([(\nabla^\mathcal{E})^*,\rho^2_{\varepsilon,r}]\nabla^\mathcal{E}\varphi,\varphi)_{L^2(M)} \\
&= (\rho^2_{\varepsilon,r}\Delta^\mathcal{E}\varphi,\varphi)_{L^2(M)} - (\nabla^\mathcal{E}_{(d\rho^2_{\varepsilon,r})^\sharp}\varphi,\varphi)_{L^2(M)} \\
&= (\rho^2_{\varepsilon,r}\Delta^\mathcal{E}\varphi,\varphi)_{L^2(M)} - (2\rho_{\varepsilon,r}\nabla^\mathcal{E}_{d\rho^\sharp_{\varepsilon,r}}\varphi,\varphi)_{L^2(M)}
\end{aligned}$$

Hence

$$\begin{aligned}
\|\rho_{\varepsilon,r}\nabla^\mathcal{E}\varphi\|^2_{L^2(M)} &\underset{(3)}{\leq} \|\Delta^\mathcal{E}\varphi\|_{L^2(M)}.\|\varphi\|_{L^2(M)} + 2\|d\rho_{\varepsilon,r}\|_\infty.\|\rho_{\varepsilon,r}\nabla^\mathcal{E}\varphi\|_{L^2(M)}.\|\varphi\|_{L^2(M)} \\
&\leq \|\Delta^\mathcal{E}\varphi\|_{L^2(M)}.\|\varphi\|_{L^2(M)} + \frac{2(1+\varepsilon)}{r}\|\rho_{\varepsilon,r}\nabla^\mathcal{E}\varphi\|_{L^2(M)}.\|\varphi\|_{L^2(M)} \\
&\leq \|\Delta^\mathcal{E}\varphi\|_{L^2(M)}.\|\varphi\|_{L^2(M)} + \frac{1+\varepsilon}{r}(\|\rho_{\varepsilon,r}\nabla^\mathcal{E}\varphi\|^2_{L^2(M)} + \|\varphi\|^2_{L^2(M)})
\end{aligned}$$

Thus

$$(1 - \frac{1+\varepsilon}{r})\|\rho_{\varepsilon,r}\nabla^\mathcal{E}\varphi\|^2_{L^2(M)} \leq \|\Delta^\mathcal{E}\varphi\|_{L^2(M)}.\|\varphi\|_{L^2(M)} + \frac{1+\varepsilon}{r}\|\varphi\|^2_{L^2(M)} < \infty$$

Letting $\varepsilon \to 0$ then $r \to \infty$, we obtain that

$$\nabla^{\mathcal{E}}\varphi \in L^2(M, T^*M \otimes \mathcal{E})$$

Using this fact in the computation above gives also:

$$(\nabla^{\mathcal{E}}\varphi, \nabla^{\mathcal{E}}\varphi)_{L^2(M)} = (\Delta^{\mathcal{E}}\varphi, \varphi)_{L^2(M)}$$

□

Theorem 4.27. *Let M be a complete manifold, $\mathcal{E}$ a vector bundle over M with covariant derivative $\nabla^{\mathcal{E}}$. The canonical Laplace operator $\Delta^{\mathcal{E}}$ is essentially self-adjoint.*

Proof. Here we want to use the criterion stated in corollary 1.28. So let $(\Delta^{\mathcal{E}})^*$ denote the L^2-adjoint[7] of $\Delta^{\mathcal{E}}$, and suppose $\exists\varphi \in \ker(i - (\Delta^{\mathcal{E}})^*) \cap L^2(M, \mathcal{E})$. A L^2 solution is in particular a solution in distributional sense, and $(\Delta^{\mathcal{E}})^*$ is elliptic (considered as operator on distributions). So, by elliptic regularity (corollary 4.24), we know that $\varphi \in C^\infty(M)$. Moreover, since φ is an eigenfunction for the L^2 adjoint, we have $\varphi \in L^2(M, \mathcal{E})$. We choose $\rho_{\varepsilon,r}$ as in proposition 4.22, with any point p as center of the balls, and χ is any cut-off function identical to 1 on B_{2r} (we have to introduce it because strictly speaking $\Delta^{\mathcal{E}}\varphi$ does not make sense since φ is not compactly supported). Then:

$$\begin{aligned}
((\Delta^{\mathcal{E}})^*\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)} &= (\varphi, \Delta^{\mathcal{E}}(\rho_{\varepsilon,r}\varphi))_{L^2(M)} \\
&= (\varphi, [\Delta^{\mathcal{E}}, \rho_{\varepsilon,r}](\chi\varphi))_{L^2(M)} + (\varphi, \rho_{\varepsilon,r}\Delta^{\mathcal{E}}(\chi\varphi))_{L^2(M)} \\
&= (\varphi, -2\nabla^{\mathcal{E}}_{d\rho^\sharp_{\varepsilon,r}}(\chi\varphi))_{L^2(M)} + (\varphi, (\Delta\rho_{\varepsilon,r})(\chi\varphi))_{L^2(M)} + (\Delta^{\mathcal{E}}(\rho_{\varepsilon,r}\varphi), \chi\varphi))_{L^2(M)} \\
&= (\varphi, -2\nabla^{\mathcal{E}}_{d\rho^\sharp_{\varepsilon,r}}\varphi)_{L^2(M)} + (\varphi, (\Delta\rho_{\varepsilon,r})\varphi)_{L^2(M)} + (\Delta^{\mathcal{E}}(\rho_{\varepsilon,r}\varphi), \varphi))_{L^2(M)}
\end{aligned}$$

because $\chi \equiv 1$ where $\rho_{\varepsilon,r} \neq 0$. Hence:

$$\begin{aligned}
((\Delta^{\mathcal{E}})^*\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)} &= (\varphi, -2\nabla^{\mathcal{E}}_{d\rho^\sharp_{\varepsilon,r}}\varphi)_{L^2(M)} + (\varphi, (\Delta\rho_{\varepsilon,r})\varphi)_{L^2(M)} + (\varphi, \rho_{\varepsilon,r}(\Delta^{\mathcal{E}})^*\varphi)_{L^2(M)} \\
&= (\varphi, -2\nabla^{\mathcal{E}}_{d\rho^\sharp_{\varepsilon,r}}\varphi)_{L^2(M)} + (\varphi, (\Delta\rho_{\varepsilon,r})\varphi)_{L^2(M)} + i(\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)}
\end{aligned}$$

On the other hand, we have:

$$((\Delta^{\mathcal{E}})^*\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)} = -i(\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)}$$

so we get:

$$-2i(\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)} = (\varphi, -2\nabla^{\mathcal{E}}_{d\rho^\sharp_{\varepsilon,r}}\varphi)_{L^2(M)} + (\varphi, (\Delta\rho_{\varepsilon,r})\varphi)_{L^2(M)}$$

taking the imaginary part, and using the fact that $(\varphi, (\Delta\rho_{\varepsilon,r})\varphi)_{L^2(M)}$ is real, we obtain:

$$(\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)} = \operatorname{Im}(\varphi, \nabla^{\mathcal{E}}_{d\rho^\sharp_{\varepsilon,r}}\varphi)_{L^2(M)}$$

Then:

$$\begin{aligned}
\|\varphi\|^2_{L^2(B_r)} &\leq \int_{B_{2r}} \langle\varphi, \rho_{\varepsilon,r}\varphi\rangle \\
&\leq \operatorname{Im}\int_{B_{2r}} \langle\varphi, \nabla^{\mathcal{E}}_{d\rho^\sharp_{\varepsilon,r}}\varphi\rangle \\
&\leq \left|\int_{B_{2r}} \langle\varphi, \nabla^{\mathcal{E}}_{d\rho^\sharp_{\varepsilon,r}}\varphi\rangle\right| \\
&\leq \int_{B_{2r}} |\varphi|.|\nabla^{\mathcal{E}}_{d\rho^\sharp_{\varepsilon,r}}\varphi| \\
&\underset{(3)}{\leq} \int_{B_{2r}} |\varphi|.|d\rho_{\varepsilon,r}|.|\nabla^{\mathcal{E}}\varphi| \\
&\leq \|d\rho_{\varepsilon,r}\|_\infty \sqrt{\int_{B_{2r}} |\varphi|^2}.\sqrt{\int_{B_{2r}} |\nabla^{\mathcal{E}}\varphi|^2} \\
&\leq \|d\rho_{\varepsilon,r}\|_\infty \|\varphi\|_{L^2(M)} \|\nabla^{\mathcal{E}}\varphi\|_{L^2(M)}
\end{aligned}$$

[7]Let us here stress the importance of paying great attention to domains of definition of objects and domain of validity of formulas. Otherwise one comes very quickly to the surprising -and false!- result that each densely defined symmetric operator is essentially self-adjoint . . .

Thus, taking the limit $\varepsilon \to 0$,

$$\|\varphi\|^2_{L^2(B_r)} \leq r^{-1} \|\nabla^{\mathcal{E}}\varphi\|_{L^2(M)} \|\varphi\|_{L^2(M)}$$

so by letting $r \to \infty$,

$$\|\varphi\|_{L^2(M)} = 0 \Rightarrow \varphi = 0$$

The proof that $\ker(-i - (\Delta^{\mathcal{E}})^*) = \{0\}$ is exactly the same. $\square$

Remark. The only properties of $\Delta^{\mathcal{E}}$ that we used are:

- $\Delta^{\mathcal{E}}$ is a smooth second-order linear differential operator,
- $\Delta^{\mathcal{E}}$ is symmetric, elliptic,
- $[\Delta^{\mathcal{E}}, f] = -2\nabla^{\mathcal{E}}_{df^\sharp} + (\Delta f)$

We call *potential* a zeroth-order differential operator. We can easily see that adding a smooth symmetric potential (for example a smooth real function) to $\Delta^{\mathcal{E}}$ does not affect these properties, hence the result is still valid for such operators.

4.4.3 Essentially self-adjointness of Dirac operators

Theorem 4.28. *Let M be a complete manifold. Any symmetric generalized Dirac operator is essentially self-adjoint. In particular, if $\mathcal{E}$ is a Dirac bundle over M, the canonical Dirac operator is essentially self-adjoint.*

Proof. The method is the same as that used to prove proposition 4.27. So let $\varphi \in \ker(i - \not{D}^*)$ and $\rho_{\varepsilon,r}$, χ be as above. Let us recall that we have $\varphi \in C^\infty(M) \cap L^2(M, \mathcal{E})$.

$$\begin{aligned}
(\not{D}^*\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)} &= (\varphi, \not{D}(\rho_{\varepsilon,r}\varphi))_{L^2(M)} \\
&= (\varphi, [\not{D}, \rho_{\varepsilon,r}](\chi\varphi))_{L^2(M)} + (\varphi, \rho_{\varepsilon,r}\not{D}(\chi\varphi))_{L^2(M)} \\
&= (\varphi, d\rho_{\varepsilon,r} \cdot (\chi\varphi))_{L^2(M)} + (\not{D}(\rho_{\varepsilon,r}\varphi), \chi\varphi)_{L^2(M)} \\
&= (\varphi, d\rho_{\varepsilon,r} \cdot \varphi)_{L^2(M)} + (\not{D}(\rho_{\varepsilon,r}\varphi), \varphi)_{L^2(M)}
\end{aligned}$$

because $\chi|_{\operatorname{supp}\rho_{\varepsilon,r}} \equiv 1$, so $\not{D}(\rho_{\varepsilon,r}\varphi) \neq 0 \Rightarrow \chi = 1$. Hence:

$$\begin{aligned}
(\not{D}^*\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)} &= (\varphi, d\rho_{\varepsilon,r} \cdot \varphi)_{L^2(M)} + (\varphi, \rho_{\varepsilon,r}\not{D}^*\varphi)_{L^2(M)} \\
&= (\varphi, d\rho_{\varepsilon,r} \cdot \varphi)_{L^2(M)} + i(\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)}
\end{aligned}$$

On the other hand, we have:

$$(\not{D}^*\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)} = -i(\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)}$$

so we get:

$$-2i(\varphi, \rho_{\varepsilon,r}\varphi)_{L^2(M)} = (\varphi, d\rho_{\varepsilon,r} \cdot \varphi)_{L^2(M)}$$

Then:

$$\begin{aligned}
2\|\varphi\|^2_{L^2(B_r)} &\leq 2\int_{B_{2r}} \langle \varphi, \rho_{\varepsilon,r}\varphi \rangle \\
&\leq -\operatorname{Im} \int_{B_{2r}} \langle \varphi, d\rho_{\varepsilon,r} \cdot \varphi \rangle \\
&\leq \left| \int_{B_{2r}} \langle \varphi, d\rho_{\varepsilon,r} \cdot \varphi \rangle \right| \\
&\leq \int_{B_{2r}} |\varphi|.|d\rho_{\varepsilon,r} \cdot \varphi| \\
&\leq \int_{B_{2r}} |\varphi|^2.|d\rho_{\varepsilon,r}| \\
&\leq \|d\rho_{\varepsilon,r}\|_\infty \|\varphi\|^2_{L^2(M)}
\end{aligned}$$

Thus, taking the limit $\varepsilon \to 0$,

$$2\|\varphi\|^2_{L^2(B_r)} \leq r^{-1} \|\varphi\|^2_{L^2(M)}$$

thus, by letting $r \to \infty$,

$$\|\varphi\|_{L^2(M)} = 0 \Rightarrow \varphi = 0$$

The proof that $\ker(-i - D\!\!\!/^*) = \{0\}$ is exactly the same. Once again, adding a smooth symmetric potential to $D\!\!\!/$ does not affect the result. $\square$

Remark. Let us give a much more direct proof for the canonical Dirac operator associated to the spinor bundle. For this Dirac operator, we have the *Schrödinger-Lichnerowicz formula*:

$$D\!\!\!/^2 = \nabla^*\nabla + \frac{s}{4} \tag{11}$$

where s denotes the scalar curvature of M. So $D\!\!\!/^2$ differs from $\nabla^*\nabla$ only by a symmetric smooth potential, hence is essentially self-adjoint. By corollary 1.29, $D\!\!\!/$ is essentially self-adjoint.

4.4.4 Dirac operator with Coulomb potential

For physical reasons, the Coulomb potential $V = \frac{1}{r}$ is particularly interesting. Unfortunately, this potential is *not* smooth, so we cannot simply use the proof given in theorem 4.28. In this case, theorems like 1.30 or 1.31 are very useful, but obviously we have to take the geometry of M into account. We will prove in a simple case (where computations are straightforward), that adding a not too big Coulomb potential to the spinorial Dirac operator does not affect its essentially self-adjointness.

if $n \geq 3$, $\frac{1}{r} \in L^2_{loc}(M)$ because $\frac{1}{r^2}$ is integrable at zero. So $f \mapsto \frac{f}{r}$ defines an operator in $L^2(M,\mathcal{E})$, with domain $\Gamma_c(M,\mathcal{E})$. For the following discussion, we choose $p \in M$, normal coordinates on T_pM and $r(x)$ denotes the distance between p and x:

$$r(x) := d(p,x)$$

Sphere and balls are taken with center p. Polar coordinates on T_pM are denoted by (r,ω).

Lemma 4.29. *Let $\nu_g := \sqrt{|\det g|}$ be the volume element on the tangent space T_pM, $\frac{\partial}{\partial r}$ the radial vector field, i.e. $\frac{\partial}{\partial r}\big|_{(x^1\dots x^n)} := \sum_{i=1}^n \frac{x^i}{r}\frac{\partial}{\partial x^i}$ ($\frac{\partial}{\partial r}$ is not defined at 0). Then*

$$\frac{\partial \nu_g}{\partial r} = \left(Tr(\nabla\frac{\partial}{\partial r}) - \frac{n-1}{r}\right)\nu_g$$

Proof. Let $dx = dx^1 \wedge \dots \wedge dx^n$ be the Euclidian n-form on T_pM. On n-forms we have $\mathcal{L} = d \circ \iota$ hence $\mathcal{L}(fY)dx = \mathcal{L}(Y)(fdx)$. Then:

$$\mathcal{L}(r\frac{\partial}{\partial r})dx \begin{cases} = \operatorname{div}(r\frac{\partial}{\partial r})dx = n.dx \\ = \mathcal{L}(\frac{\partial}{\partial r})(rdx) = dx + r\mathcal{L}(\frac{\partial}{\partial r})dx \end{cases}$$

so

$$\mathcal{L}(\frac{\partial}{\partial r})dx = \frac{n-1}{r}dx$$

Then

$$Tr(\nabla\frac{\partial}{\partial r})(\nu_g dx) = \mathcal{L}(\frac{\partial}{\partial r})(\nu_g dx) = \frac{\partial \nu_g}{\partial r}dx + \nu_g\mathcal{L}(\frac{\partial}{\partial r})dx = \frac{\partial \nu_g}{\partial r}dx + \nu_g\frac{n-1}{r}dx$$

Altogether,

$$\frac{\partial \nu_g}{\partial r} = \left(Tr(\nabla\frac{\partial}{\partial r}) - \frac{n-1}{r}\right)\nu_g$$

$\square$

We want now to describe briefly the endomorphism $\nabla\frac{\partial}{\partial r}$. The following results can be found in [2] or [7]. Let $\mathcal{S}_r$ denote the sphere of radius r and center p in M.

Proposition 4.30. *At (r,ω), $\nabla\frac{\partial}{\partial r}$ defines a symmetric endomorphism of $T\mathcal{S}_r$. $S = -\nabla\frac{\partial}{\partial r}$ is the* Weingarten map. *The eigenvalues λ_i of S are called* principal curvatures, *and $H = \frac{1}{n-1}Tr(S)$ is the* mean curvature.

Proof. Since radial rays are geodesics, we have $\nabla_{\frac{\partial}{\partial r}}\frac{\partial}{\partial r}=0$. Given (r,ω), let X and Y be two vector fields tangent to $\mathcal{S}_r$. From $\|\frac{\partial}{\partial r}\|=1$, we deduce that

$$\langle\nabla_X\frac{\partial}{\partial r},\frac{\partial}{\partial r}\rangle=\underbrace{\frac{\partial}{\partial r}\left\|\frac{\partial}{\partial r}\right\|^2}_{=0}-\langle\frac{\partial}{\partial r},\nabla_X\frac{\partial}{\partial r}\rangle\Rightarrow\nabla_X\frac{\partial}{\partial r}\perp\frac{\partial}{\partial r}$$

Hence S is an endomorphism of $T\mathcal{S}_r$. Furthermore, since X, Y and $[X,Y]$ are tangent to $\mathcal{S}_r$, and ∇ is torsion-free, we have:

$$\langle\nabla_X\frac{\partial}{\partial r},Y\rangle-\langle\nabla_Y\frac{\partial}{\partial r},X\rangle=X(\underbrace{\langle\frac{\partial}{\partial r},Y\rangle}_{=0})-Y(\underbrace{\langle\frac{\partial}{\partial r},X\rangle}_{=0})-\underbrace{\langle\frac{\partial}{\partial r},[X,Y]\rangle}_{=0}=0$$

hence S is symmetric. ☐

Definition 4.31. The *generalized cotangent function* is defined as follows:

$$\mathfrak{ct}_\kappa(x):=\begin{cases}\frac{\sqrt{\kappa}\cos(\sqrt{\kappa}x)}{\sin(\sqrt{\kappa}x)} & \text{if } \kappa>0,\ \sqrt{\kappa}x\not\equiv 0 \bmod \pi\\ \frac{1}{x} & \text{if } \kappa=0,\ x\neq 0\\ \frac{\sqrt{|\kappa|}\cosh(\sqrt{|\kappa|}x)}{\sinh(\sqrt{|\kappa|}x)} & \text{if } \kappa<0,\ x\neq 0\end{cases}$$

Lemma 4.32. *For $\kappa<0$, $h(x):=\frac{1}{x}-\mathfrak{ct}_\kappa(x)$ is a decreasing function with*

$$\lim_{x\to-\infty}h(x)=\sqrt{|\kappa|}\qquad\lim_{x\to 0}h(x)=0\qquad\lim_{x\to+\infty}h(x)=-\sqrt{|\kappa|}$$

Theorem 4.33. *Let M be a Riemannian manifold with sectional curvature K bounded by $\kappa_1\le K\le\kappa_2$. Choose $(r,\omega)\in T_pM$, and suppose that the geodesic $\gamma(t)=\exp(t\omega)$ has no point conjugate to p for $t\in(0,r]$, and that $d(p,x)$ is smooth at $x=\exp(t\omega)$ for all $t\in(0,r]$. Denote by $(\lambda_i(t))_{i=1\dots n-1}$ the principal curvatures at $\exp(t\omega)$. Then:*

$$\forall t\in[0,r],\ \forall i=1\dots n-1,\ -\mathfrak{ct}_{\kappa_1}(t)\le\lambda_i(t)\le-\mathfrak{ct}_{\kappa_2}(t)$$

Corollary 4.34. *Let M be a simply connected Riemannian manifold with non-positive sectional curvature K bounded by $\kappa\le K\le 0$. Then $H(r,\omega)+\frac{1}{r}$ is bounded by $\sqrt{|\kappa|}$.*

Proof. By the theorem of Cartan-Hadamard, we know that exp is a diffeomorphism from T_pM to M. Hence there are no conjugate points, and $d(p,x)$ is smooth for $x\neq p$. The result follows from the preceding propositions. ☐

Theorem 4.35. *Let M be a simply connected manifold of dimension $n\ge 3$, with sectional curvature K satisfying $-\kappa\le K\le 0$ for a constant $\kappa>0$, and $D\!\!\!/$ be the spinorial Dirac operator (M is spin because diffeomorphic to $\mathbb{R}^n$). Then $D\!\!\!/+\frac{n-2}{2r}$ is essentially self-adjoint.*

Proof. As previously, we know that exp is a diffeomorphism from T_pM to M. Hence we have *global* normal coordinates. Let S_1 denotes the unit sphere in T_pM, and ν_g the volume element. For all $\psi\in\Gamma_c(M,\Sigma M)$ (in order to make the computation clearer, we omit the argument "$(r\omega)$"):

$$\begin{aligned}\int_{r>\varepsilon}\frac{|\psi|^2}{r^2}\nu_g dx&=\int_{S_1}\int_\varepsilon^\infty r^{n-3}|\psi|^2\nu_g dr d\omega\\&=\int_{S_1}\left[\frac{r^{n-2}}{n-2}|\psi|^2\nu_g\right]_\varepsilon^\infty d\omega-\int_{S_1}\int_\varepsilon^\infty\frac{r^{n-2}}{n-2}\frac{\partial}{\partial r}(|\psi|^2\nu_g)drd\omega\\&=\int_{S_1}\frac{\varepsilon^{n-2}}{n-2}|\psi(\varepsilon\omega)|^2\nu_g(\varepsilon\omega)d\omega-\int_{S_1}\int_\varepsilon^\infty\frac{r^{n-2}}{n-2}2\operatorname{Re}\langle\psi,\nabla_{\frac{\partial}{\partial r}}\psi\rangle\nu_g drd\omega\\&\quad-\int_{S_1}\int_\varepsilon^\infty\frac{r^{n-2}}{n-2}|\psi|^2\frac{\partial\nu_g}{\partial r}drd\omega\\&=O(\varepsilon)-\frac{2}{n-2}\int_{r>\varepsilon}\frac{1}{r}\operatorname{Re}\langle\psi,\nabla_{\frac{\partial}{\partial r}}\psi\rangle\nu_g dx+\frac{n-1}{n-2}\int_{r>\varepsilon}\frac{1}{r}|\psi|^2(H+\frac{1}{r})\nu_g dx\end{aligned}$$

Letting $\varepsilon\to 0$,

$$
\begin{aligned}
\left\|\frac{\psi}{r}\right\|^2_{L^2(M)} &= -\frac{2}{n-2}\int_M \frac{1}{r}\operatorname{Re}\langle\psi,\nabla_{\frac{\partial}{\partial r}}\psi\rangle|dx| + \frac{n-1}{n-2}\int_M \frac{1}{r}|\psi|^2(H+\frac{1}{r})|dx| \\
&\leq \frac{2}{n-2}\left\|\frac{\psi}{r}\right\|_{L^2(M)}\left\|\nabla_{\frac{\partial}{\partial r}}\psi\right\|_{L^2(M)} + \sqrt{\kappa}\frac{n-1}{n-2}\left\|\frac{\psi}{r}\right\|_{L^2(M)}\|\psi\|_{L^2(M)} \\
\left\|\frac{\psi}{r}\right\|_{L^2(M)} &\leq \frac{2}{n-2}\|\nabla\psi\|_{L^2(M)} + \sqrt{\kappa}\frac{n-1}{n-2}\|\psi\|_{L^2(M)}
\end{aligned}
$$

The bounds $-\kappa \leq K \leq 0$ imply that the scalar curvature s is bounded by:

$$-\kappa n(n-1) \leq s \leq 0$$

Using Schrödinger-Lichnerowicz formula (11) we have:

$$
\begin{aligned}
\|D\!\!\!/\psi\|^2_{L^2(M)} &= (D\!\!\!/^2\psi,\psi)_{L^2(M)} \\
&= ((\nabla^*\nabla + \frac{s}{4})\psi,\psi)_{L^2(M)} \\
&= \|\nabla\psi\|^2_{L^2(M)} + (\frac{s}{4}\psi,\psi)_{L^2(M)}
\end{aligned}
$$

So,

$$
\begin{aligned}
\|\nabla\psi\|^2_{L^2(M)} &= \|D\!\!\!/\psi\|^2_{L^2(M)} - (\frac{s}{4}\psi,\psi)_{L^2(M)} \\
&\leq \|D\!\!\!/\psi\|^2_{L^2(M)} + \frac{\kappa n(n-1)}{4}\|\psi\|^2_{L^2(M)}
\end{aligned}
$$

Hence,

$$\|\nabla\psi\|_{L^2(M)} \leq \|D\!\!\!/\psi\|_{L^2(M)} + \frac{\sqrt{\kappa n(n-1)}}{2}\|\psi\|_{L^2(M)}$$

Using these results, we obtain:

$$\left\|\frac{\psi}{r}\right\|_{L^2(M)} \leq \frac{2}{n-2}\|D\!\!\!/\psi\|_{L^2(M)} + C\|\psi\|_{L^2(M)}$$

with

$$C := \frac{\sqrt{\kappa(n-1)}}{n-2}\left(\sqrt{n}+\sqrt{n-1}\right)$$

So $\exists C'$, $\forall\psi\in\Gamma_c(M,\Sigma M)$,

$$\left\|\frac{n-2}{2r}\psi\right\|_{L^2(M)} \leq \|D\!\!\!/\psi\|_{L^2(M)} + C'\|\psi\|_{L^2(M)}$$

By theorem 1.31, $D\!\!\!/ + \frac{n-2}{2r}$ is essentially self-adjoint. $\square$

Let us discuss the preceding theorem. We used above all two facts. Firstly, the exponential map is a diffeomorphism, which allows us to do computation without boundary terms, because the cut-locus is empty. Secondly, we have an explicit Weitzenböck formula for the spinorial Dirac operator, so that we gain a uniform elliptic estimate (see below for the notion of elliptic estimate).

In a more general case, we would face the two corresponding problems, namely the (possible) existence of a non-empty cut-locus, and maybe no explicit Weitzenböck formula.

However, the Laplace-Beltrami operator $(d+d^*)^2 = dd^* + d^*d$ on differential forms has an explicit Weitzenböck formula on one-forms given by:

$$(d+d^*)^2 = \Delta^{\wedge T^*M} + \operatorname{Ric}$$

where the Ricci curvature is considered as section in the bundle $\operatorname{End}(T^*M)$. Since bounds for the sectional curvature imply bounds for the Ricci curvature, we could do a similar proof in this case.

4.5 Decomposition principle

The *Sobolev space* of sections in $\mathcal{E}$ whose covariant derivatives up to order k are L^2 is denoted by $H^{k,2}(M,\mathcal{E})$. The *Sobolev norm* is defined by

$$\|\varphi\|^2_{H^{k,2}(M,\mathcal{E})} := \sum_{j=0}^{k} \|\underbrace{\nabla \ldots \nabla}_{j \text{ times}} \varphi\|^2_{L^2(M)}$$

The references for the next proposition are the same as that given for elliptic regularity (the results are most of the time stated together).

Proposition 4.36 (Elliptic estimate). *Let $\mathfrak{D} \in \mathcal{D}_d(M,\mathcal{E})$ be an elliptic differential operator. Then for any compact subset $K \subset M$, there exists a positive constant $C(K)$ such that*

$$\forall \varphi \in \operatorname{dom}(\mathfrak{D}), \qquad \|\varphi\|_{H^{d,2}(K,\mathcal{E})} \leq C(K).\left(\|\varphi\|_{L^2(M,\mathcal{E})} + \|\mathfrak{D}\varphi\|_{L^2(M,\mathcal{E})}\right)$$

Theorem 4.37. *Let M be a complete manifold, $\mathfrak{D} \in \mathcal{D}_d(M,\mathcal{E})$ an essentially self-adjoint elliptic differential operator of order $d \geq 1$, with domain* $\operatorname{dom}(\mathfrak{D}) = \Gamma_c(M,\mathcal{E})$*, $\lambda \in \sigma_e(\overline{\mathfrak{D}})$. Then for all compact subset $K \subset M$ we can find a Weyl sequence of smooth compactly supported sections φ_i associated to $\overline{\mathfrak{D}}, \lambda$ with supports outside K, i.e.*

$$\forall i \in \mathbb{N},\ \operatorname{supp}(\varphi_i) \cap K = \emptyset$$

In particular if M is compact we have $\sigma_e(\overline{\mathfrak{D}}) = \emptyset$.

Proof. We will assume that M is non-compact. The proof given here is easily adaptable to the case M compact to show that $\overline{\mathfrak{D}}$ has a pure discrete spectrum.

Take any Weyl sequence of smooth compacty supported sections φ_i associated to $\overline{\mathfrak{D}}, \lambda$ and K_1, K_2 two other compact subsets with $K \subset \mathring{K}_1 \subset K_1 \subset \mathring{K}_2 \subset M$. Since for a Weyl sequence $\liminf \|\varphi_i\| > 0$, without loss of generality we can assume that $\forall i, \|\varphi_i\| = 1$. By the elliptic estimate (proposition 4.36), we have

$$\begin{aligned}
\|\varphi_i\|_{H^{d,2}(K_2,\mathcal{E})} &\leq C(K_2).\left(\|\varphi_i\|_{L^2(M,\mathcal{E})} + \|\mathfrak{D}\varphi_i\|_{L^2(M,\mathcal{E})}\right)\\
&\leq C(K_2).\left(1 + |\lambda|.\|\varphi_i\|_{L^2(M,\mathcal{E})} + \|(\mathfrak{D}-\lambda)\varphi_i\|_{L^2(M,\mathcal{E})}\right)\\
&\leq C_1
\end{aligned}$$

Since $(\varphi_i)_i$ is bounded in the $H^{d,2}$ norm and K_2 is compact, we can find, by the Rellich's lemma, a subsequence which converges in $H^{d-1,2}(K_2,\mathcal{E})$ to φ_∞. Since any subsequence of a Weyl sequence is a Weyl sequence, we simply assume that $\varphi_i \to \varphi_\infty$.

Let κ be a cut-off function identical to 1 on K_1 and vanishing outside K_2. On the one hand,

$$(\kappa\varphi_i, \kappa\varphi_\infty)_{L^2(M,\mathcal{E})} = (\kappa\varphi_i, \kappa\varphi_\infty)_{L^2(K_2,\mathcal{E})} \xrightarrow{(\varphi_i \to \varphi_\infty)} (\kappa\varphi_\infty, \kappa\varphi_\infty)_{L^2(K_2,\mathcal{E})}$$

On the other hand,

$$(\kappa\varphi_i, \kappa\varphi_\infty)_{L^2(M,\mathcal{E})} = (\varphi_i, \kappa^2\varphi_\infty)_{L^2(M,\mathcal{E})} \xrightarrow{(\varphi_i \rightharpoonup 0)} 0$$

So $\|\kappa\varphi_\infty\|_{L^2(K_2,\mathcal{E})} = 0$ and $\varphi_\infty|_{K_1} = 0$. Therefore

$$\|\varphi_i\|_{H^{d-1,2}(K_1,\mathcal{E})} \to 0 \tag{12}$$

Choose now a smooth function ψ vanishing on K, identical to 1 outside K_1, with $0 \leq \psi \leq 1$ everywhere. We set $\psi_i := \psi\varphi_i$.

$$\|\psi_i\|_{L^2(M,\mathcal{E})} \geq \|\varphi_i\|_{L^2(M\setminus K_1,\mathcal{E})}$$

Hence, by (12) $\qquad \liminf \|\psi_i\|_{L^2(M,\mathcal{E})} \geq \liminf \|\varphi_i\|_{L^2(M\setminus K_1,\mathcal{E})} = 1$

Furthermore, for any $f \in L^2(M, \mathcal{E})$,

$$(\psi_i, f)_{L^2(M,\mathcal{E})} = (\varphi_i, \psi f)_{L^2(M,\mathcal{E})} \xrightarrow{(\varphi_i \rightharpoonup 0)} 0$$

$[\mathfrak{D}, \psi]$ is a differential operator of order $d-1$, vanishing outside K_1. Thus:

$$\exists C_2 > 0, \qquad \|[\mathfrak{D}, \psi]\varphi_i\|_{L^2(M,\mathcal{E})} \leq C_2.\|\varphi_i\|_{H^{d-1,2}(K_1,\mathcal{E})} \to 0$$

We conclude

$$\begin{aligned} \|\mathfrak{D}\psi_i - \lambda\psi_i\|_{L^2(M,\mathcal{E})} &\leq \|\psi(\mathfrak{D}\varphi_i - \lambda\varphi_i)\|_{L^2(M,\mathcal{E})} + \|[\mathfrak{D}, \psi]\varphi_i\|_{L^2(M,\mathcal{E})} \\ &\leq \|(\mathfrak{D} - \lambda)\varphi_i\|_{L^2(M,\mathcal{E})} + \|[\mathfrak{D}, \psi]\varphi_i\|_{L^2(M,\mathcal{E})} \to 0 \end{aligned}$$

Hence the sequence ψ_i is a Weyl sequence of $\overline{\mathfrak{D}}$ associated to λ, and $\forall i, \operatorname{supp}(\psi_i) \cap K = \emptyset$. $\square$

Corollary 4.38. *Let M be a complete manifold, $\mathfrak{D} \in \mathcal{D}_d(M, \mathcal{E})$ a self-adjoint elliptic differential operator of order $d \geq 1$, with domain $\operatorname{dom}(\mathfrak{D}) = \Gamma_c(M, \mathcal{E})$. Let $\lambda \in \sigma_e(\overline{\mathfrak{D}})$. Then there exists a Weyl sequence of smooth compactly supported sections φ_i associated to $\overline{\mathfrak{D}}, \lambda$ with pairwise disjoint supports and leaving any compact, i.e.:*

(i) $i \neq j \Rightarrow \operatorname{supp}(\varphi_i) \cap \operatorname{supp}(\varphi_j) = \emptyset$.

(ii) $\forall K \subset M$ *compact,* $\operatorname{supp}(\varphi_i) \cap K = \emptyset$ *for i large enough.*

This corollary yields the so-called *decomposition principle*: the essential spectrum of $\overline{\mathfrak{D}}$ depends only on $\mathfrak{D}$ and the geometry of M by the values at "infinity". Thus, if we change M and $\mathfrak{D}$ on a compact domain, the essential spectrum remains the same.

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