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MacLane's planarity criterion
for locally finite graphs

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Abstract

MacLane's planarity criterion states that a finite graph is planar if and only if its cycle space has a basis $\mathcal{B}$ such that every edge is contained in at most two members of $\mathcal{B}$. Solving a problem of Wagner (1970), we show that the cycle space introduced recently by Diestel and Kühn allows a verbatim generalization of MacLane's criterion to locally finite graphs.

1 Introduction

For a (possibly infinite) graph G define the *finite-cycle space* to be the set of all symmetric differences of finitely many finite cycles (the finite 2-regular connected subgraphs). We denote this $\mathbb{Z}_2$-vector space by $\mathcal{C}_{\text{fin}}(G)$. A set (or family) $\mathcal{H}$ of subgraphs of G is called *simple*, if every edge of G lies in (the edge set of) at most two elements of $\mathcal{H}$. Then, MacLane's criterion states:

Theorem 1 (MacLane [9]). *A finite graph G is planar if and only if $\mathcal{C}_{\text{fin}}(G)$ has a simple generating set.*

Wagner [13] raised the question if MacLane's result could be extended so that it characterises also planar graphs which are infinite. Rather than modifying the planarity criterion, Thomassen [12] describes all infinite graphs that satisfy MacLane's condition. For this, recall that a *vertex accumulation point*, abbreviated VAP, of a plane graph Γ is a point p of the plane such that in every neighbourhood of p there are vertices of Γ.

Theorem 2 (Thomassen [12]). *Let G be an infinite 2-connected graph. Then G has a VAP-free drawing in the plane if and only if $\mathcal{C}_{\text{fin}}(G)$ has a simple generating set consisting of finite cycles.*

Bonnington and Richter [1] also provide a generalization of MacLane's theorem using the *even cycle space* $\mathcal{Z}(G)$, defined as the set of all subgraphs of G with all vertex degrees even. With this space they investigate which graphs have a drawing with k VAPs.

Our main result in this paper is a verbatim generalization of MacLane's theorem to locally finite graph:

Theorem 3. *Let G be a locally finite 2-connected graph. Then, G is planar if and only if $\mathcal{C}(G)$ has a simple generating set.*

Here, $\mathcal{C}(G)$ denotes the cycle space introduced for infinite graphs by Diestel and Kühn [5, 6]. Its basic constituents are not just finite cycles but arbitrary topological circles (i.e. homeomorphic images of the unit circle) in the space consisting of G together with its ends.

$\mathcal{C}(G)$ will be formally defined in Section 2. Let us remark here only that this notion of a cycle space has allowed a number of natural or verbatim generalizations of all the following basic results for finite graphs. Namely, that the cycle space is generated by the fundamental cycles of every spanning tree; that every element of the cycle space meets every cut in an even number of edges; that every element of the cycle space is the edge-disjoint union of cycles; and that the cycle space in a 3-connected graph is generated by the non-separating induced cycles (Tutte's theorem). See Diestel [3] for a gentle introduction and discussion of the cycle space of an infinite graph.

We discuss our main result in Section 3 after introducing the necessary definitions in the next section. In Section 4 we investigate some properties of simple generating sets. The main result will be proved in the course of Sections 5 and 6. In Section 7 we extend Tutte's planarity criterion to locally finite graphs.

2 Definitions and basic facts

The basic terminology we use can be found in Diestel [4]. All the graphs in this paper are simple and undirected. Let $G = (V, E)$ be a fixed graph.

A 1-way infinite path is called a *ray*, a 2-way infinite path is a *double ray*. The subrays of (double) rays are their *tails*. The *ends* of G are the equivalence classes of rays under the following equivalence relation: two rays in G are equivalent if no finite set of vertices separates them.

We define a topology on G together with its ends, i.e. our topological space consists of all vertices, all inner points of edges and all ends of G. On G the topology will be that of a 1-complex. Thus, the basic open neighbourhoods of an inner point on an edge are the open intervals on the edge containing that point, while the basic open neighbourhoods of a vertex x are the unions of half-open intervals containing x, one from every edge at x. For every end ω and any finite set $S \subseteq V$ there is exactly one component $C = C(S, \omega)$ of $G - S$ which contains a tail of every ray in ω. We say that ω *belongs* to C, and write $\overline{C}(S, \omega)$ for the component C together with all the ends belonging to it. Then the basic open neighbourhoods of an end ω are all sets of the form

$$\hat{C}(S, \omega) := \overline{C}(S, \omega) \cup E'(S, \omega)$$

where $E'(S, \omega)$ is any union of half-edges $(z, y] \subseteq e$, one for every edge $e = xy$ with $x \in S$, $y \in C$ (with $z \in \mathring{e}$). This topological space will be denoted by $|G|$. (For locally finite graphs G, $|G|$ is the Freudenthal compactification of G.) We shall freely view G either as an abstract graph or as a subspace of $|G|$, i.e. the union of all vertices and edges of G with the usual topology of a 1-complex. It is not difficult to see that if G is connected and locally finite, then $|G|$ is compact. We need the following result later on:

Theorem 4 (Diestel and Kühn [7]). *When a graph G is locally finite, every closed connected subset of $|G|$ is path-connected.*

A subset of $|G|$ which is homeomorphic to the unit interval $[0, 1]$ is called an *arc*. The images of 0 and of 1 are its *endpoints*. A set $C \subseteq |G|$ is a *circle* if it is a homeomorphic image of the unit circle. The next lemma will be of help when dealing with arcs and circles.

Lemma 5 (Diestel and Kühn [6]). *For every arc A and every circle C in $|G|$ the following is true:*

(i) The sets $A \cap G$ and $C \cap G$ are dense in A and C, respectively;

(ii) every arc in $|G|$ whose endpoints are vertices or ends, and every circle C in $|G|$, includes every edge of G of which it contains an inner point; and

(iii) if x is a vertex in $\mathring{A}$ (respectively in C), then A (respectively C) contains precisely two edges of G at x.

Therefore, given a circle $C \subseteq |G|$, the subset $C \cap G$ may be viewed as a subgraph of G, and will be called a *cycle* of G. Clearly, this definition includes traditional finite cycles but also allows infinite cycles. Such an infinite cycle D is the disjoint union of double rays whose ends fit together nicely (namely, so that the union of D and the set of these ends is a circle in $|G|$). Since, by Lemma 5, the closure of $C \cap G$ of the circle C coincides with C, we may speak of the (unique) defining circle of a cycle.

We say a family $\mathcal{H}$ of subgraphs of G is *thin*, if no vertex lies on infinitely many of the elements of $\mathcal{H}$. Define the sum $\sum_{H\in\mathcal{H}} H$ of the thin family $\mathcal{H}$ to be the subgraph of G consisting of all edges which appear in an odd number of $H \in \mathcal{H}$ (together with the incident vertices). Note that in locally finite graphs a family is thin if and only if every edge appears in at most finitely many of its members. Thus, in locally finite graphs, thin families are exactly those for which we can decide whether an edge appears an even or an odd number of times in the involved subgraphs. We define the *cycle space* $\mathcal{C}(G)$ of G as the set of all sums over thin families of (finite or infinite) cycles. With the symmetric difference (denoted by $\triangle$) taken as addition, $\mathcal{C}(G)$ is a $\mathbb{Z}_2$-vector space. In a finite graph G, the cycle space $\mathcal{C}(G)$ and the finite-cycle space $\mathcal{C}_{\text{fin}}(G)$ coincide. A theorem in Diestel and Kühn [6] states that $\mathcal{C}(G)$ is closed under taking thin sums. Thus, thin sums seem to be completely natural in our context, and we will henceforth tacitly assume that every sum is thin.

3 Discussion of main result

First let us make the notion of a generating set more precise. A *generating set* of the cycle space will be a set $\mathcal{F} \subseteq \mathcal{C}(G)$ such that every element of $\mathcal{C}(G)$ can be written as a thin sum of elements of $\mathcal{F}$. Thus, in contrast to a generating set in the vector space sense we allow certain infinite sums. There are two reasons for this. First, thin sums are integral to the cycle space of an infinite graph, so it seems unnatural to forbid them. Second, MacLane's criterion is false if we insist that every $Z \in \mathcal{C}(G)$ is a finite sum of elements of a simple subset of $\mathcal{C}(G)$, as we shall see in Lemma 9.

To show that, in a certain sense, our main result, Theorem 3, is as strong as possible, we need the following theorem, which is of interest on its own. It will be proved in Section 4.

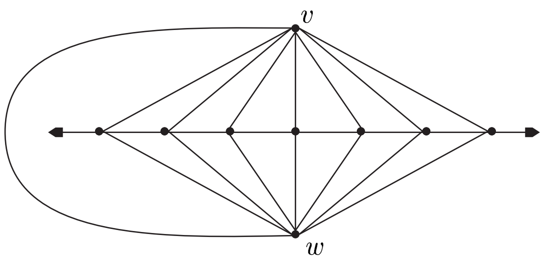

Figure 1: A planar graph without a 2-basis

Theorem 6. *Let G be a 3-connected graph, and let $\mathcal{F}$ be a simple generating set of $\mathcal{C}(G)$ consisting of cycles. Then every element of $\mathcal{F}$ is an induced non-separating cycle.*

First note that because of the following theorem, any cycle space that has a simple generating set also has a simple generating set consisting of cycles.

Theorem 7 (Diestel and Kühn [6]). *Every element of the cycle space of a graph is the edge-disjoint union of cycles.*

Theorem 3 is formulated for locally finite graphs, and indeed it is false for arbitrary infinite graphs. Indeed, consider the 3-connected graph G in Figure 1, which is not locally finite. By Theorem 6 and the remark following it, we may assume that a simple generating set $\mathcal{F}$ of $\mathcal{C}(G)$ consists of induced non-separating cycle (finite or infinite). In particular, no cycle which contains the edge vw is in $\mathcal{F}$. But then such a cycle cannot be generated by any sum of cycles of $\mathcal{F}$. Thus, there is no simple generating set of $\mathcal{C}(G)$, but G is clearly planar.

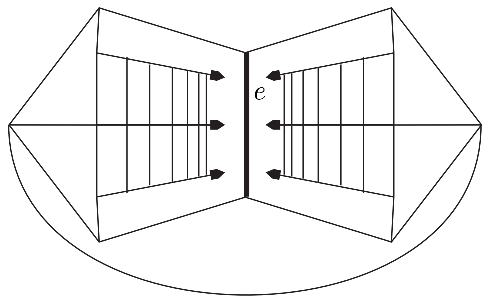

Figure 2: A locally finite graph without a 2-basis of finite cycles

Infinite cycles might seem, although topologically natural, combinatorially unwieldy. They are, however, inevitable in a certain sense: there is not always in a planar graph a simple generating set comprised of only finite cycles. Consider the graph G in Figure 2, and suppose there is a simple generating set $\mathcal{F}$ of $\mathcal{C}(G)$ consisting of finite cycles. Since G is 3-connected, every $C \in \mathcal{F}$ is, by Theorem 6, an induced non-separating cycle. Now, every finite cycle containing the edge e is clearly separating. Consequently, the cycles in $\mathcal{F}$ are not even sufficient to generate every finite cycle (namely those containing e).

There is another compelling reason for allowing infinite cycles in a generating set: for a locally finite graph G they are natural constituents of a reasonable notion of the cycle space. In Diestel [3] it is shown that, if one wants to generalize simultaneously all of the basic results for a cycle space we mentioned at the end of Section 1, one needs a cycle space which contains at least all of $\mathcal{C}(G)$.

4 Simple generating sets

As a tool, we introduce the notion of a 2-basis. For this, let $\mathcal{B} \subseteq \mathcal{C}(G)$ be a simple generating set of the cycle space of G. We call $\mathcal{B}$ a *2-basis* of $\mathcal{C}(G)$ if for every element $Z \in \mathcal{C}(G)$ there is a unique subset of $\mathcal{B}$, henceforth denoted by $\mathcal{B}_Z$, with $Z = \sum_{B\in\mathcal{B}_Z} B$. Observe that in a finite graph the 2-bases are exactly the simple bases of $\mathcal{C}(G)$, and thus conform with the traditional definition of a 2-basis in a finite graph.

Since we have left linear algebra with our definition of a 2-basis (allowing thin infinite sums), it is not clear if the properties usually expected of a basis are still retained. One of these, which we will need later on, is that a generating set always contains a basis. For simple sets this is true:

Lemma 8. *Let G be a locally finite 2-connected graph, and let $\mathcal{F}$ be a simple generating set of $\mathcal{C}(G)$. If $\mathcal{F}$ is not a 2-basis, then for any $Z \in \mathcal{F}$ the set $\mathcal{F}\setminus\{Z\}$ is a 2-basis of $\mathcal{C}(G)$.*

Proof. Observe first that it suffices to check the uniqueness required in the definition of a 2-basis for the empty set: a simple generating subset $\mathcal{B}$ of $\mathcal{C}(G)$ is a 2-basis if and only if every $\mathcal{B}' \subseteq \mathcal{B}$ with $\sum_{B\in\mathcal{B}'} B = \emptyset$ satisfies $\mathcal{B}' = \emptyset$.

Let us assume there is a non-empty set $\mathcal{D} \subseteq \mathcal{F}$ with $\sum_{B\in\mathcal{D}} B = \emptyset$, and suppose that $\mathcal{D} \neq \mathcal{F}$. Since G is 2-connected every edge of G appears in a cycle, and thus in at least one element of $\mathcal{F}$. But as $\mathcal{F}$ is simple and $\sum_{B\in\mathcal{D}} B = \emptyset$ no edge of G can lie in an element of $\mathcal{D}$ and at the same time in an element of $\mathcal{F} \setminus \mathcal{D}$.

So, $E_1 := \bigcup \mathcal{D}$ and $E_2 := \bigcup(\mathcal{F}\setminus\mathcal{D})$ define a partition of $E(G)$ (note that both sets are non-empty). Because G is 2-connected there is, by Menger's theorem, for any two edges a cycle through both of them. Therefore, there is a cycle D which shares an edge e_1 with E_1 and another edge e_2 with E_2. Let $\mathcal{D}' \subseteq \mathcal{F}$ be such that $D = \sum_{B\in\mathcal{D}'} B$. Then $D' := \sum_{B\in\mathcal{D}\cap\mathcal{D}'} B \subseteq D$, since for any edge $e \in E(D')\setminus E(D)$ there must be an element in $\mathcal{D}'\setminus\mathcal{D}$ which contains e; but then $e \in E_1 \cap E_2$. Since $D' \in \mathcal{C}(G)$ and since $e_1 \in E(D')$ we thus obtain $D' = D$, but D' cannot contain $e_2 \in E_2$, a contradiction.

We thus have shown:

$$\sum_{B\in\mathcal{D}} B = \emptyset \text{ for } \mathcal{D} \subseteq \mathcal{F} \text{ implies } \mathcal{D} = \emptyset \text{ or } \mathcal{D} = \mathcal{F}.$$

So, if $\mathcal{F}$ is not a 2-basis, then it (but none of its proper subsets) generates the empty set. For any $Z \in \mathcal{F}$,

$$Z = \sum_{B\in\mathcal{F}\setminus\{Z\}} B,$$

thus, the simple set $\mathcal{F} \setminus \{Z\}$ certainly generates the cycle space. It also is a 2-basis, as none of its non-empty subsets generates the empty set. $\square$

With our definition of a generating set, which allows infinite sums, we will show that MacLane's criterion holds for locally finite graphs. Since, in a vector space context, one usually allows only finite sums for a generating set, there is one obvious question: Does Theorem 3 remain true if we consider simple generating sets in the vector space sense? The answer is a strikingly clear no:

Lemma 9. *There is no locally finite* 2*-connected infinite graph in which the cycle space has a simple generating set in the vector space sense (i.e. allowing only finite sums).*

Proof. Suppose there is such a graph G so that $\mathcal{C}(G)$ has a simple set $\mathcal{A} \subseteq \mathcal{C}(G)$ which generates every $Z \in \mathcal{C}(G)$ through a finite sum. We determine the cardinality of $\mathcal{C}(G)$ in two ways.

First, since $\mathcal{A}$ is simple, every of the countably many edges of G lies in at most two elements of $\mathcal{A}$. Therefore, $\mathcal{A}$ is a countable set, and thus, $\mathcal{C}(G)$ also.

Second, there is, by Lemma 8, a 2-basis $\mathcal{B} \subseteq \mathcal{A}$. As $\mathcal{C}(G)$ is an infinite set (since G is infinite and 2-connected), so is $\mathcal{B}$. Hence, there are distinct $B_1, B_2, \ldots \in \mathcal{B}$. Also, as G is locally finite and $\mathcal{B}$ simple, all subsets of $\mathcal{B}$ are thin. Therefore, all the sums

$$\sum_{i \in I} B_i \text{ for } I \subseteq \mathbb{N}$$

are distinct elements of $\mathcal{C}(G)$. Since the power set of $\mathbb{N}$ has uncountable cardinality, it follows that $\mathcal{C}(G)$ is uncountable, a contradiction. $\square$

The rest of this section is devoted to the proof of Theorem 6, which we restate:

Theorem 6. *Let G be a* 3*-connected graph, and let $\mathcal{F}$ be a simple generating set of $\mathcal{C}(G)$ consisting of cycles. Then every element of $\mathcal{F}$ is an induced non-separating cycle.*

Let us first introduce some more concepts for a cycle C in a graph G. We define an equivalence relation on $E(G) \backslash E(C)$ by $e \sim f$ if there is a walk W such that the first edge of W is e and the last is f, and W meets C at most in its endsvertices. A connected non-trivial subgraph B of $G - E(C)$ whose edge set is closed under this equivalence relation is called a *bridge* of C. One sees easily that a bridge is either a chord of C or a subgraph of G consisting of a component K of $G - C$ with the edges $E(K, C)$ added. Note, that a cycle is induced and non-separating if and only if it has at most one bridge.

The vertices of B on C are called the *vertices of attachment* of B. If C' is the defining circle of C then there may be a point $x \in C'$ in which the vertices of attachment of B accumulate. Such a point is necessarily an end of G and called an *end of attachment* of B.

Should B have exactly one vertex of attachment then the whole circle C' is called the residual arc of B in C. Otherwise, a subarc L of C' is said to be a *residual arc* of B in C if L does not contain any vertices of attachment of B in its interior and is (inclusion-)maximal with that property.

All these concepts are straightforward generalizations of the terminology employed for finite graphs, and have, as the next lemma shows, similar properties.

Lemma 10. [2] *Let C be a cycle in a 2-connected graph G with a bridge B. Then the following statements are true:*

(i) The endpoints of a residual arc L of B in C are vertices of attachments or ends of attachment of B;

(ii) for a vertex $v \in C \setminus B$ there is exactly one residual arc L of B in C containing v; and

(iii) for a vertex $v \in C \cap B$ there are exactly two residual arcs of B in C with v as an endpoint.

Further, we say a bridge B of C *avoids* another bridge B' of C if there is a residual arc of B that contains all vertices of attachment of B'. Otherwise, they *overlap.* Two bridges B and B' of C are called *skew* if C' contains four vertices v, v', w, w' in that cyclic order such that v, w are vertices of attachment of B and v', w' vertices of attachment of B'. Clearly, if two bridges B, B' are skew they overlap.

The following lemma is well-known for finite graphs:

Lemma 11. *Let G be a graph, and let C be a cycle in G. Then two overlapping bridges B and B' of C are skew or have three vertices of attachment in common.*

Proof. If B or B' has only two vertices of attachment it is easy to see that B and B' are skew. So let both have at least three vertices of attachment. Moreover, we may assume, that B' has a vertex of attachment u' which is not one of B as otherwise B and B' have at least three vertices of attachment in common, and we are done. By Lemma 10 (ii), u' lies in the interior of a residual arc L of B. Since B and B' overlap, we find a second vertex of attachment v' of B' with $v' \notin L$. Now, if the endpoints u, v of L are both vertices of attachment of B, then B and B' are clearly skew. So, assume that u, say, is an end of attachment. Then, there is a open interval $U \subseteq L$ around u such that $u', v' \notin U$. By definition of an end of attachment, there is a vertex of attachment of B in U. Using this vertex instead of u (and possibly a substitute vertex for v as well) we see that B and B' are skew. $\square$

We need that in a 3-connected graph there are always two overlapping bridges (if there is more than one bridge at all). For this, we define for a cycle C in the graph G the *overlap graph* of C in G as the graph on the bridges of C such that two bridges are adjacent if and only if they overlap. The next lemma ensures that there are overlapping bridges.

Lemma 12. [2] *For every cycle C in a 3-connected graph G the overlap graph of C in G is connected.*

The next simple lemma will be used repeatedly in the proof of Theorem 6.

Lemma 13. *Let G be a 3-connected graph, and let $\mathcal{B}$ be a 2-basis of $\mathcal{C}(G)$ consisting of cycles. Let C and D be cycles in G such that $C' \cap D'$ is an arc, where C' and D' are the defining circles of C and D. Suppose that $\mathcal{B}_C \cap \mathcal{B}_D \neq \emptyset$. Then, either $\mathcal{B}_C \subseteq \mathcal{B}_D$ or $\mathcal{B}_D \subseteq \mathcal{B}_C$.*

Proof. Put $K := \sum_{B\in\mathcal{B}_C\cap\mathcal{B}_D} B$ and consider an edge $e \notin E(C\cup D)$. Then both $\mathcal{B}_C$ and $\mathcal{B}_D$ contain either both or none of the at most two cycles $B \in \mathcal{B}$ with $e \in E(B)$. Thus, both or none of them is in $\mathcal{B}_C \cap \mathcal{B}_D$, and hence $e \notin E(K)$. Therefore, K is an element of the cycle space contained in $C \cup D$. These are precisely $\emptyset$, C, D and $C + D$ (since $C' \cap D'$ is an arc). Note that $K \neq \emptyset$ as $\mathcal{B}_C \cap \mathcal{B}_D \neq \emptyset$. Also, $K \neq C + D$, since otherwise

$$\mathcal{B}_C \cap \mathcal{B}_D = \mathcal{B}_K = \mathcal{B}_{C+D} = \mathcal{B}_C \triangle \mathcal{B}_D,$$

which is impossible. Consequently, we obtain either $K = C$ and thus, $\mathcal{B}_C \subseteq \mathcal{B}_D$, or $K = D$ and $\mathcal{B}_D \subseteq \mathcal{B}_C$. □

Proof of Theorem 6. Note that it suffices to prove the theorem for a 2-basis $\mathcal{B}$. Indeed, if $\mathcal{F}$ is not a 2-basis, consider two distinct elements Z_1 and Z_2 of $\mathcal{F}$. By Lemma 8, both $\mathcal{F} \setminus \{Z_1\}$ and $\mathcal{F} \setminus \{Z_2\}$ are a 2-basis of $\mathcal{C}(G)$, and, if Theorem 6 holds for these, it clearly also holds for $\mathcal{F}$.

Consider a cycle C of G which has a chord or is separating. Then, C has more than one bridge. Two of these, B and B' say, are, by Lemma 12, overlapping. By Lemma 11, they are skew or have three vertices of attachment in common. We show that $C \notin \mathcal{B}$ for each of the two cases.

(i) Suppose that B and B' are skew. Then, there are two C–paths $P = u\dots v$ and $P' = u'\dots v'$ such that u, u', v, v' appear in this order on C. Put $C_1 := uCv\cup P$, $C_2 := vCu\cup P$, $D_1 := u'Cv'\cup P'$ and $D_2 := v'Cu'\cup P'$. Observe that $C_1 + C_2 = C = D_1 + D_2$, and further, that C_i and D_j have an edge in common, for $i, j = 1, 2$.

Suppose $C \in \mathcal{B}$. Since

$$\mathcal{B}_{C_1} \triangle \mathcal{B}_{C_2} = \mathcal{B}_{C_1+C_2} = \mathcal{B}_C = \{C\},$$

not both of $\mathcal{B}_{C_1}$ and $\mathcal{B}_{C_2}$ may contain C. As the same holds for D_1 and D_2 we may assume that

$$C \notin \mathcal{B}_{C_1} \text{ and } C \notin \mathcal{B}_{D_1}. \tag{1}$$

Consider an edge $e \in E(C_1 \cap D_1) \subseteq E(C)$. Both of $\mathcal{B}_{C_1}$ and $\mathcal{B}_{D_1}$ must contain a cycle which contains e. By (1), this cannot be C. Therefore, and since $\mathcal{B}$ is simple, $\mathcal{B}_{C_1}$ and $\mathcal{B}_{D_1}$ contain the same cycle K with $e \in E(K)$. Consequently, $\mathcal{B}_{C_1} \cap \mathcal{B}_{D_1} \neq \emptyset$, and applying Lemma 13 we may assume that

$$\mathcal{B}_{C_1} \subseteq \mathcal{B}_{D_1}. \tag{2}$$

Now, consider an edge $e' \in E(C_1 \cap D_2) \subseteq E(C)$. There is a cycle $K' \in \mathcal{B}_{C_1}$ with $e' \in E(K')$. By (2), $K' \in \mathcal{B}_{D_1}$, but since e' lies in $C \cap D_2$ we have $e' \notin E(D_1)$. Thus, $\mathcal{B}_{D_1}$ also contains the other cycle in $\mathcal{B}$ that contains e', which is C, a contradiction to (1). Therefore, $C \notin \mathcal{B}$.

(ii) Suppose that B and B' have three vertices of attachment, v_1, v_2, v_3, in common. Then there is a vertex $x \in V(B - C)$ and three x–C paths $P_i = x\dots v_i \subseteq B$, $i = 1, 2, 3$ whose interiors are pairwise disjoint. Let $Q_i = y\dots v_i$ be analogous paths in B'. Define $C_1 := v_1Cv_2 \cup P_1 \cup P_2$, $C_2 := v_2Cv_3 \cup P_2 \cup P_3$, $C_3 := v_3Cv_1 \cup P_3 \cup P_1$, and analogously D_i with the paths Q_i such that $C_i \cap C = D_i \cap C$ $(i = 1, 2, 3)$. Note that $C_1 + C_2 + C_3 = C = D_1 + D_2 + D_3$.

Now suppose $C \in \mathcal{B}$. As

$$\mathcal{B}_{C_1} \triangle \mathcal{B}_{C_2} \triangle \mathcal{B}_{C_3} = \mathcal{B}_{C_1+C_2+C_3} = \mathcal{B}_C = \{C\}$$

either C lies in all of the $\mathcal{B}_{C_i}$ or in only one of them, in $\mathcal{B}_{C_3}$, say. In both cases, we have

$$C \notin \mathcal{B}_{C_1+C_2} \text{ and } C \in \mathcal{B}_{C_3}.$$

We obtain the same result for the D_i: either C lies in all of the $\mathcal{B}_{D_i}$ or in only one of them. In any case, we can define D as either D_1 or $D_2 + D_3$ such that $C \notin \mathcal{B}_D$. Put $D' := C + D$, and note that $\mathcal{B}_{D'} = \mathcal{B}_D \cup \{C\}$.

Then, since $C_1 + C_2$ shares an edge of C with D, and neither $\mathcal{B}_{C_1+C_2}$ nor $\mathcal{B}_D$ contains C, we have $\mathcal{B}_{C_1+C_2} \cap \mathcal{B}_D \neq \emptyset$. Applying Lemma 13, we obtain that one of the two sets $\mathcal{B}_{C_1+C_2}, \mathcal{B}_D$ is contained in the other.

First assume that $\mathcal{B}_{C_1+C_2} \subseteq \mathcal{B}_D$, and consider an edge e that lies on both C_1+C_2 and D'. Then e lies on a cycle $K \neq C$ in $\mathcal{B}_{C_1+C_2}$, and thus also $K \in \mathcal{B}_D$. But as $e \notin E(D)$, there is another cycle in $\mathcal{B}_D$ that contains e, which can only be C, a contradiction.

So, we may assume that $\mathcal{B}_D \subseteq \mathcal{B}_{C_1+C_2}$. Because $\mathcal{B}_{D'} = \mathcal{B}_D \cup \{C\}$ we even have $\mathcal{B}_D \subseteq \mathcal{B}_{C_1+C_2} \cap \mathcal{B}_{D'}$. Thus, by Lemma 13, either $\mathcal{B}_{C_1+C_2} \subseteq \mathcal{B}_{D'}$ or $\mathcal{B}_{D'} \subseteq \mathcal{B}_{C_1+C_2}$. The latter is impossible as $C \notin \mathcal{B}_{C_1+C_2}$. Therefore, we obtain

$$\mathcal{B}_D \subseteq \mathcal{B}_{C_1+C_2} \subseteq \mathcal{B}_{D'} = \mathcal{B}_D \cup \{C\}.$$

Now, from $C \notin \mathcal{B}_{C_1+C_2}$ follows that $\mathcal{B}_{C_1+C_2} = \mathcal{B}_D$, contradicting $C_1 + C_2 \neq D$. Thus, $C \notin \mathcal{B}$. $\square$

5 The backward implication

In this section, we show the backward implication of our main result, starting with the following lemma.

Lemma 14. *Let G be a 2-connected graph such that $\mathcal{C}(G)$ has a 2-basis, and let $H \subseteq G$ be a finite 2-connected subgraph. Then $\mathcal{C}(H)$ has a 2-basis.*

Proof. Let $\mathcal{B}$ be the 2-basis of $\mathcal{C}(G)$. Since H is finite, there are $Z \in \mathcal{C}(H)$ with a non-empty generating set $\mathcal{B}_Z \subseteq \mathcal{B}$ which is $\subseteq$-minimal among all B_Z with $Z \in \mathcal{C}(H)$. Let us denote these by $Z_1, \ldots, Z_k$.

Consider a $D \in \mathcal{C}(H)$ with $\mathcal{B}_D \cap \mathcal{B}_{Z_i} \neq \emptyset$ for some i. We claim that $\mathcal{B}_{Z_i} \subseteq \mathcal{B}_D$. First, note that

$$C := \sum_{B \in \mathcal{B}_D \cap \mathcal{B}_{Z_i}} B \subseteq H.$$

Indeed, consider an edge $e \notin E(H)$. Since Z_i and D are subgraphs of H and since $\mathcal{B}$ is simple, e either lies on exactly two or on none of the cycles of $\mathcal{B}_{Z_i}$, and the same holds for $\mathcal{B}_D$. Furthermore, if e lies on two cycles of $\mathcal{B}_{Z_i}$ and on two of $\mathcal{B}_D$, these must be the same. So, $e \notin E(C)$.

Therefore, C is contained in H, and thus $C \in \mathcal{C}(H)$. As $\mathcal{B}_C \subseteq \mathcal{B}_{Z_i}$ we obtain, by the minimality of $\mathcal{B}_{Z_i}$, that $C = Z_i$. Consequently, $\mathcal{B}_{Z_i} = \mathcal{B}_C \subseteq \mathcal{B}_D$, as claimed.

This result also implies $\mathcal{B}_{Z_i} \cap \mathcal{B}_{Z_j} = \emptyset$ for all $1 \leq i < j \leq k$. Thus, every edge of H appears in at most two of the Z_i. Further, we claim that $\{Z_1, \ldots, Z_k\}$ is a generating set for $\mathcal{C}(H)$. Then, $\{Z_1, \ldots, Z_k\}$ contains a 2-basis of $\mathcal{C}(H)$, and we are done.

So consider a $D \in \mathcal{C}(H)$, and let I denote the set of those indices i with $\mathcal{B}_{Z_i} \cap \mathcal{B}_D \neq \emptyset$. We may assume $I = \{1, \ldots, k'\}$ for a $k' \leq k$. Then, by $\mathcal{B}_{Z_i} \subseteq \mathcal{B}_D$ and $\mathcal{B}_{Z_i} \cap \mathcal{B}_{Z_j} = \emptyset$ for $i, j \in I$, it follows that $\mathcal{B}_D$ is the disjoint union of the sets $\mathcal{B}_{Z_1}, \mathcal{B}_{Z_2}, \ldots, \mathcal{B}_{Z_{k'}}$ and of

$$\mathcal{B}' := \mathcal{B}_D \setminus \bigcup_{i=1}^{k'} \mathcal{B}_{Z_i}.$$

Consequently,

$$\sum_{B \in \mathcal{B}'} B = \sum_{B \in \mathcal{B}_D} B + \sum_{B \in \mathcal{B}_{Z_1}} B + \ldots + \sum_{B \in \mathcal{B}_{Z_{k'}}} B = D + Z_1 + \ldots + Z_{k'} \subseteq H$$

as all the summands lie in H. Now, if $\mathcal{B}' \neq \emptyset$ then there is a $Z \in \mathcal{C}(H)$ with a non-empty and minimal $\mathcal{B}_Z \subseteq \mathcal{B}'$ which then must be one of the Z_i, a contradiction. Thus, $\mathcal{B}'$ is empty and we have $D = \sum_{i=1}^{k'} Z_i$. $\square$

We need the Kuratowski planarity criterion for countable graphs.

Theorem 15 (Dirac and Schuster [8]). *Let G be a countable graph. Then, G is planar if and only if G contains neither a subdivision of K^5 nor a subdivision of $K_{3,3}$.*

For the backward implication of Theorem 3, we use the well-known fact that the cycle space of every subdivision of K^5 or of $K_{3,3}$ fails to have a 2-basis (see, for instance, Diestel [4]).

Lemma 16. *Let G be a locally finite 2-connected graph such that $\mathcal{C}(G)$ has a simple generating set. Then G is planar.*

Proof. Suppose not. Then G contains, by Theorem 15, a subdivision H of K^5 or of $K_{3,3}$ as subgraph. By Lemma 8, $\mathcal{C}(G)$ has a 2-basis. Then, by Lemma 14, $\mathcal{C}(H)$ also has a 2-basis, which is impossible. $\square$

6 The forward implication

To show the forward implication of Theorem 3 we proceed as in the finite case: we embed our graph G in the sphere and then show that the set of face boundaries is a simple generating set. So, our first priority is to ensure that every face boundary is indeed a member of the cycle space, namely a cycle.

This, however, is certainly not the case when a VAP of the embedded graph coincides with a vertex or an inner point of an edge. To avoid this problem we consider topological embeddings of the space $|G|$ in the sphere (rather than graph embeddings of G), which, in our context, is no restriction:

Theorem 17 (Richter and Thomassen [11]). *Let G be a locally finite 2-connected planar graph. Then $|G|$ embeds in the sphere.*

Let $\varphi : |G| \to S^2$ be an embedding (i.e. a homeomorphism of $|G|$ onto $\varphi(|G|)$) of a connected locally finite graph G in the unit sphere S^2 Putting $\Gamma := \varphi(|G|)$, we note that Γ is compact. Call the connected components of $S^2 \setminus \Gamma$ the *faces* of φ. We also say that f is a face of Γ. Then, it can be shown in a similar way as for finite plane graphs (see for instance Diestel [4] and note Theorem 20) that if an inner point of an edge lies on the frontier of a face f then the whole edge lies on it. Thus, if $F \subseteq \Gamma$ is the frontier of f then $\varphi^{-1}(F) \cap G$ is a subgraph of G, which we will denote by $G[f]$.

When G is locally finite and connected, then every face of an embedding $\varphi : |G| \to S^2$ is a region (open and connected), and hence path-connected. As we will now see, its frontier, too, is path-connected.

Lemma 18. *Let G be a connected locally finite graph with an embedding $\varphi : |G| \to S^2$ in the sphere. Then for every face f of φ its frontier $\partial f \subseteq S^2$ is path-connected.*

Proof. Let us first suppose that ∂f is not even connected. Then, there are two disjoint open sets $U_1, U_2 \subseteq S^2$ such that $\partial f \subseteq U_1 \cup U_2$ and neither of $U_i \cap \partial f$ for $i = 1, 2$ is empty. Note that together with ∂f also f must meet both U_1 and U_2. But f, being connected, cannot be contained in $U_1 \cup U_2$, and so meets $S^2 \setminus (U_1 \cup U_2)$. As $\partial f \subseteq U_1 \cup U_2$ it follows that $S^2 \setminus (U_1 \cup U_2) \subseteq f$. Now, since $|G|$ is path-connected by Theorem 4, and as φ is a homeomorphism, also $\varphi(|G|)$ is path-connected. Thus, there is an arc in $\varphi(|G|)$ from $U_1 \cap \partial f$ to $U_2 \cap \partial f$. This arc clearly meets $S^2 \setminus (U_1 \cup U_2)$, which is contained in f, a contradiction.

Therefore, ∂f is a connected and closed subset of $\varphi(|G|)$. Since φ is a homeomorphism, this also holds for the preimage, which then is, by Theorem 4, path-connected. Hence, ∂f is also path-connected. $\square$

This enables us to prove the following:

Lemma 19. *Let G be a connected locally finite graph with an embedding $\varphi : |G| \to S^2$ in the sphere, and let f be face of φ. Then $\varphi(G[f])$ is dense in the frontier ∂f of f.*

Proof. Consider a point $p \in \partial f$. If $\partial f = \{p\}$ then $\partial f = \varphi(|G|)$, and, thus $\partial f = \varphi(G[f])$. So, let there be a second point $q \in \partial f$. By Lemma 18, there is an arc $A \subseteq \partial f$ connecting p to q. Since, by Lemma 5, $\varphi(G) \cap A$ is dense in A, every neighbourhood of p meets $\varphi(G[f])$. $\square$

We need the following theorem:

Theorem 20 (Mohar and Thomassen [10, p.29]). *Let $\varphi : A \to B$ be a homeomorphism of arcs A and B in the plane. Then φ can be extended to a homeomorphism of the plane.*

Consider an arc A in the plane. By definition, A is homeomorphic to the unit intervall, which we may see as a subset of the plane. By Theorem 20, this homeomorphism can be extended to the plane. Clearly, the unit intervall leaves exactly one region in the plane, which, thus, holds also for A. Compactifying the plane we have proved:

Lemma 21. *Let A be an arc in the sphere S^2. Then $S^2 \setminus A$ has exactly one region.*

With this result and the following topological one we can finally show that the face boundaries are cycles.

Lemma 22 (Mohar and Thomassen [10, p.28]). *Let P_1, P_2, P_3 be three arcs in the sphere S^2, between the same two endpoints but otherwise disjoint. Then $S^2 \setminus (P_1 \cup P_2 \cup P_3)$ has exactly three regions with frontiers $P_1 \cup P_2$, $P_1 \cup P_3$ and $P_2 \cup P_3$.*

Lemma 23. *Let G be a locally finite 2-connected graph with an embedding $\varphi : |G| \to S^2$. Then the face boundaries of φ are cycles.*

Proof. Consider a face f of $\Gamma := \varphi(|G|)$. First, we claim:

There is no point $p \in \partial f$ such that there are three arcs $A_1, A_2, A_3 \subseteq \partial f$ with p as an endpoint but otherwise disjoint. (3)

Suppose there is such a p and arcs A_1, A_2, A_3. Obviously, p cannot be an inner point of an edge. For $i = 1, 2, 3$ let C_i denote the component of $\Gamma - p$ that meets A_i. Now, if two of these components are not the same, $C_1 \neq C_2$ say, then p separates C_1 from C_2 in Γ. But then in G, each of $\varphi^{-1}(C_1)$ and $\varphi^{-1}(C_2)$ contains a vertex v_1, v_2, and every v_1–v_2 path meets $\varphi^{-1}(p)$, which thus is a vertex and separates v_1 and v_2. This leads to a contradiction, as G is 2-connected.

So, $C_1 = C_2 = C_3$, and we find a point $q \in S^2$ and three arcs P_1, P_2, P_3 between p and q but otherwise disjoint such that P_i contains a subarc of A_i. Applying Lemma 22, we see that $S^2 \setminus (P_1 \cup P_2 \cup P_3)$ has three regions with respective frontiers $P_1 \cup P_2$, $P_2 \cup P_3$ and $P_1 \cup P_3$. One of these regions contains f, say the one with frontier $P_1 \cup P_2$. But then, $A_3 \cap P_3 \not\subseteq \partial f$, a contradiction. Thus, (3) is proved.

Next, we show that

$$\textit{if } \partial f \textit{ contains a circle } C \textit{ then } G[f] \textit{ is a cycle.} \tag{4}$$

Observe that $\varphi^{-1}(C) \cap G$ is a cycle since φ is a homeomorphism. Suppose $C \neq \partial f$, and consider a point $p \in \partial f \setminus C$. By Lemma 18, there is an arc $A \subseteq \partial f$ from C to p. As C is compact, there is a last point $q \in C$ on A. Then, there are three arcs in ∂f with q as an endpoint but otherwise disjoint, in contradiction to (3). Hence, $C = \partial f$ and (4) is established.

So, from now on, suppose that ∂f does not contain a circle. Then,

$$\textit{there are arcs } A_1 \subseteq A_2 \subseteq \ldots \subseteq \partial f \textit{ such that } \varphi(G[f]) \subseteq \bigcup_{i=1}^{\infty} A_i. \tag{5}$$

Let $v_0, v_1, \ldots$ be an enumeration of $V(G[f])$. By Lemma 18, there is an arc $A_1 \subseteq \partial f$ with endpoints $\varphi(v_0)$ and $\varphi(v_1)$. Now, for $i > 1$ assume the arc $A_{i-1} \subseteq \partial f$ to be constructed such that $\varphi(\{v_0, \ldots, v_{i-1}\}) \subseteq A_{i-1}$. If also $\varphi(v_i) \in A_{i-1}$, we put $A_i := A_{i-1}$, else by Lemma 18 there exists an arc $B \subseteq \partial f$ between a point $p \in A_{i-1}$ and $\varphi(v_i)$. As A_{i-1} is closed, B has a last point in A_{i-1} and we may suppose B to be such that $A_{i-1} \cap B = \{p\}$. We then put $A_i := A_{i-1} \cup B$. Because of (3), p is one of the endpoints of A_{i-1}. Thus, $A_i \subseteq \partial f$ clearly is an arc and we proceed with $i + 1$. This proves (5).

Put $A := \bigcup_{i=1}^{\infty} A_i$. As can be easily seen, by piecing together the homeomorphisms $[0, 1] \to A_i$ we obtain a continuous bijection $\sigma : I \to A$, where I is one of $[0,1]$, $[0,1)$, $(0,1]$ or $(0,1)$. For $p, q \in A$, let $p \le q$ if $\sigma^{-1}(p) \le \sigma^{-1}(q)$. This defines a linear ordering $\le$ on A such that if $p \le q$ are contained in A_n, for some n, then also $z \in A_n$ whenever $p \le z \le q$. We claim:

> *There is an $x \in \partial f$ with the following property: every sequence $x_1 < x_2 < \ldots$ in A for which there is no $n \in \mathbb{N}$ with $(x_i)_{i\in\mathbb{N}} \subseteq A_n$ converges against x. For strictly decreasing sequences a similar result is true with a possibly different limit.* (6)

As every sequence in S^2 has a converging subsequence, it is sufficient to suppose, by way of contradiction, that there are two sequences $p_1 < p_2 < \ldots$ and $q_1 < q_2 < \ldots$ in A such that

$$\lim_{i\to\infty} p_i =: p \neq q := \lim_{i\to\infty} q_i,$$

and such that there is no $n \in \mathbb{N}$ with $(p_i)_{i\in\mathbb{N}} \subseteq A_n$ or $(q_i)_{i\in\mathbb{N}} \subseteq A_n$. Note that $p, q \in \partial f$, and that $\omega_p := \varphi^{-1}(p)$ and $\omega_q := \varphi^{-1}(q)$ are distinct ends. Choose a finite set $S \subseteq V(G)$ that separates these ends in $|G|$, i.e.

$$\overline{C_p} \cap \overline{C_q} = \emptyset \text{ where } C_p = C(S, \omega_p) \text{ and } C_q = C(S, \omega_q).$$

By substituting for a subsequence we may assume that $(p_i)_{i\in\mathbb{N}} \subseteq \varphi(\overline{C_p})$, $(q_i)_{i\in\mathbb{N}} \subseteq \varphi(\overline{C_q})$, and also that

$$p_1 < q_1 < p_2 < q_2 < \ldots .$$

Indeed, otherwise we may suppose that there is a $j \in \mathbb{N}$ with $q_i < p_j$ for all $i \in \mathbb{N}$. Let $n \in \mathbb{N}$ be such that $q_1 \in A_n$ and $p_j \in A_n$. Then, $(q_i)_{i\in\mathbb{N}} \subseteq A_n$, which is impossible.

Now, if for $i \in \mathbb{N}$, $B_i \subseteq A$ is the arc with endpoints p_i and q_i then the B_i are pairwise disjoint. Now, as $p_i \in \varphi(\overline{C_p})$ and $q_i \in \varphi(\overline{C_q})$ every B_i has to pass through the finite set $\varphi(S)$, a contradiction. Therefore, $(p_i)_{i\in\mathbb{N}}$ and $(q_i)_{i\in\mathbb{N}}$ have the same limit, proving (6).

Finally we show that

$$\partial f \textit{ is an arc.} \tag{7}$$

Then, Lemma 21 implies that $S^2 \backslash \partial f$ has only one region. Consequently, $\partial f = \Gamma$, which is impossible since G, as a 2-connected graph, contains a finite cycle. So, we arrive at a contradiction, showing that ∂f contains a circle. Hence, from (4) it follows that $G[f]$ is cycle, as desired.

To show (7), consider any converging sequence $(z_i)_{i\in\mathbb{N}}$ in A with limit z. If there is an $n \in \mathbb{N}$ such that $(z_i)_{i\in\mathbb{N}} \subseteq A_n$ then $z \in A_n$, since A_n is closed. If there is no such n, we may assume, by substituting $(z_i)_{i\in\mathbb{N}}$ for a subsequence, that $(z_i)_{i\in\mathbb{N}}$ is strictly increasing or strictly decreasing. Then, $(z_i)_{i\in\mathbb{N}}$ is a sequence as in (6), and z is the respective limit. Now, if A has a maximum, denote it by x, if not, then there is an increasing sequence as in (6) and we denote by x its limit. Analogously, we denote the minimum or the lower limit of A by y. Thus, we have shown that the closure $\overline{A}$ of A is $A \cup \{x, y\}$. As, by assumption, $\overline{A}$ does not contain a circle, it follows that $x \neq y$ and $x, y \notin \mathring{A}$. So, we may extend $\sigma : I \to A$ to a continuous bijection $[0,1] \to \overline{A}$. Consequently, $\overline{A}$ is an arc. Finally, since $\varphi(G[f])$ is dense in ∂f, by Lemma 19, and since $\varphi(G[f]) \subseteq A$, by (5), we obtain $\overline{A} = \partial f$. $\square$

Showing the forward implication, we now complete the proof of Theorem 3:

Lemma 24. *Let G be a locally finite 2-connected planar graph. Then, $\mathcal{C}(G)$ has a simple generating set.*

Proof. By Theorem 17, $|G|$ has an embedding $\varphi : |G| \to S^2$ in the sphere. We show that the set of face boundaries of φ is a simple generating set of $\mathcal{C}(G)$. Certainly, the set of face boundaries is simple, and, by Lemma 23, a subset of $\mathcal{C}(G)$. So, we only have to prove that every element of the cycle space is the sum of certain face boundaries.

Put $\Gamma := \varphi(|G|)$, and fix a face f^* of Γ. First, consider a cycle C in G with defining circle C'. By definition, $\varphi(C')$ is homeomorphic to the unit circle and, thus, bounds two faces (by the Jordan-curve theorem). Let f_C be the face not containing f^*. As G is 2-connected every edge e lies on a finite cycle, and therefore on the boundaries of exactly two faces of Γ, which we denote by f_e and f'_e. Hence, the set

$$\mathcal{B}^C := \{G[f] : f \subseteq f_C \text{ is a face of } \Gamma\}$$

is thin. Moreover, as we have $f_e, f'_e \subseteq f_C$ or $f_e, f'_e \nsubseteq f_C$ if and only if $e \notin E(C)$, it follows that

$$\sum_{B \in \mathcal{B}^C} B = C. \tag{8}$$

Now, consider an arbitrary element Z of the cycle space. By definition, there is a thin family $\mathcal{D}$ of cycles with $Z = \sum_{C \in \mathcal{D}} C$. If none of the face boundaries of Γ appears in $\mathcal{B}^C$ for infinitely many C, then the family $\mathcal{B}$, which we define to be the (disjoint) union of all $\mathcal{B}^C$ with $C \in \mathcal{D}$, is thin (since every edge lies on exactly two face boundaries). Then, $Z = \sum_{B \in \mathcal{B}} B$, and we are done. Therefore, if $F(\Gamma)$ is the set of faces of Γ it suffices to show that the set

$$F := \{f \in F(\Gamma) : f \subseteq f_C \text{ for infinitely many } C \in \mathcal{D}\}$$

is empty.

So suppose $F \neq \emptyset$. By definition of f_C, we have $f^* \nsubseteq f_C$ for all $C \in \mathcal{D}$, and thus also $F \neq F(\Gamma)$. Hence, there is an edge e such that one of its adjacent faces, say f_e, lies in F and the other, f'_e, in $F(\Gamma) \setminus F$. Then, $G[f_e]$ appears in infinitely many $\mathcal{B}^C$ while e lies on only finitely many $C \in \mathcal{D}$. Thus, the boundary $G[f'_e]$ of the other adjacent face also lies in infinitely many $\mathcal{B}^C$, which implies $f'_e \in F$, a contradiction. □

7 Tutte's planarity criterion

For finite 3-connected graphs there is another well-known planarity criterion, namely Tutte's criterion. It follows from MacLane's criterion together with Tutte's theorem. The latter is known to be true for locally finite graphs:

Theorem 25. [**2**] *Let G be a locally finite 3-connected graph. Then the induced non-separating cycles generate the cycle space.*

We know from Lemma 23 that the face boundaries of a locally finite 2-connected planar graph are cycles. When G is 3-connected then, as for finite graphs (see Diestel [4]), the Jordan curve theorem implies that these cycles are precisely the non-separating and induced cycles:

Lemma 26. *Let G be a locally finite 3-connected graph with an embedding $\varphi : |G| \to S^2$ into the sphere. Then, the face boundaries are precisely the induced non-separating cycles.* □

Now, we easily obtain a verbatim generalization of Tutte's criterion for locally finite graphs.

Theorem 27. *Let G be a locally finite 3-connected graph. Then G is planar if and only if every edge lies on at most two induced non-separating cycles.*

Proof. If G is planar then there is, by Theorem 17, also an embedding of $|G|$, in which, by Lemma 26, every induced non-separating cycle is a face boundary. Thus, every edge lies on at most two induced non-separating cycles.

For the backward implication let $\mathcal{F}$ be the set of all induced non-separating cycles, which then is simple. Thus, $\mathcal{F}$ is, by Theorem 25, a simple generating set, and hence G planar, by Theorem 3. □

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