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Marsden-Weinstein Reduction and Representations of the Jacobi Group

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Here we follow ideas of Kirillov, Kostant, Duflo, Guillemin, Sternberg, Vogan and others ([Ki], [Du], [GS], [Vo])to look for geometric constructions of unitary representations of the Jacobi group and to try to give a first still very rough outline of geometric interpretations of their structure.

1.

Let G be a Lie group and $\mathfrak{g} = \text{Lie}\, G$ identified with its dual $\mathfrak{g}^*$ via

$$\mathfrak{g}^* \ni \eta \longmapsto X_\eta \in \mathfrak{g},\ \text{ defined by } \eta(Y) = \text{tr}\, X_\eta Y =: < X_\eta, Y > \ \text{ for all } Y \in \mathfrak{g}.$$

Moreover let $\mathcal{O} = G \cdot \eta$ be the coadjoint orbit through $\eta \in \mathfrak{g}^*$, i.e.

$$\mathcal{O} = \{gX_\eta g^{-1}, g \in G\} \simeq G/G_\eta \subset \mathbf{R}^n,$$

if G has a realization as a subgroup of $GL(n, \mathbf{R})$ and G_η is the stabilizing group of η in G. $\mathcal{O}$ is a symplectic manifold with Kirillov–Kostant form ω given by

$$\omega(X, Y) = \eta([X, Y]), \text{for } X, Y \in \mathfrak{g}.$$

If $\mathcal{O}$ is prequantizable, i.e. $[\omega] \in \mathrm{H}^2(\mathcal{O}, \mathbf{Z})$, one has a metrized hermitian line bundle $L \longrightarrow \mathcal{O}$ with connection ω and the space of sections $\Gamma(L)$ is the space of a unitary representation of G which in general is not irreducible. If moreover one has an appropriate polarization $\mathcal{P}$, the space $\Gamma_{\mathcal{P}}(L) = \mathcal{H}_\pi$ of polarized sections is the space of an irreducible unitary representation π of G.

Example 1: $G = G_1 := SL(2, \mathbb{R})$
We have

$$\mathfrak{g}_1 := \mathfrak{s}l_2 =< X, Y, Z >,\ [X, Y] = 2Y,\ [X, Z] = -2Z,\ [Y, Z] = X,$$

where

$$X = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\ Y = \begin{pmatrix} & 1 \\ & \end{pmatrix},\ Z = \begin{pmatrix} & \\ 1 & \end{pmatrix},$$

and

$$\mathfrak{g}_1{}^* :=< X_*, Y_*, Z_* >, X_* = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\ Y_* = \begin{pmatrix} & \\ 1 & \end{pmatrix},\ Z_* = \begin{pmatrix} & 1 \\ & \end{pmatrix},$$

such that

$$< X_*, X >= 2, < Y_*, Y >= 1, < Z_*, Z >= 1,$$

and all other brackets are zero.
Then the orbit belonging to η with $X_\eta = k'(Y_* - Z_*), k' \neq 0$, is given by

$$M = G_1 \cdot X_\eta = \{(x, \check{y}, \check{z}) \in \mathbf{R}^3;\ x^2 + \check{y}^2 - \check{z}^2 + k'^2 = 0, z > 0\,\}.$$

Here we use the coordinates

$$xX_* + yY_* + zZ_* \in \mathfrak{s}l_2, y := \check{y} + \check{z}, z := \check{y} - \check{z}.$$

The stabilizing group G_η of X_η is $K_1 = \mathrm{SO}(2)$ and by the general procedure sketched above the orbit $M = G_1/K_1, K_1 = SO(2)$ is integral iff

$$ik' < Y_* - Z_*, \vartheta(Y - Z) >= 2ik'\vartheta$$

integrates to a character ρ of $G_\eta = \mathrm{SO}(2)$. The characters ρ of $\mathrm{SO}(2)$ are given by

$$\rho_k(r(\vartheta)) = e^{ik\vartheta}, r(\vartheta) = (\begin{matrix} \alpha & \beta \\ -\beta & \alpha \end{matrix}), \alpha = \cos(\vartheta),\ \beta = \sin(\vartheta),$$

i.e. M is integral iff

$$2k' = k \in \mathbf{Z}.$$

Then for $k \in \mathbf{N}$ M carries the discrete series representation π_k^+ of G_1. This representation is also a representation of K_1 and we have

$$\mathrm{mult}(\rho_{\tilde{k}}, \pi_k^+) = 1 \text{ for } \tilde{k} = k + 2l, l \in \mathbf{N}_0 \text{ and } = 0 \text{ else.}$$

Example 2: The Jacobi group

$$G = G^J := G_1 \times H,\ G_1 = SL(2,\mathbf{R}),\ H = \mathrm{Heis}(\mathbf{R})$$

is coordinatized in the following ways. As in [BeS] we write

$$G^J \ni g = (M, v, \kappa)$$

with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbf{R}),\ v = (\lambda, \mu) \in \mathbf{R^2},\ \kappa \in \mathbf{R},$$

and the multiplication law

$$gg' = (MM', vM' + v', \kappa + \kappa' + \omega(vM', v')),\ \omega(w, v) = wJ^tv,\ J = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

or, as in [EZ] and [Ya1] for its realization as a subgroup of the symplectic group $G_2 = \mathrm{Sp}(2, \mathbf{R})$

$$g^{-1} = \begin{pmatrix} a & 0 & b & a\mu - b\lambda \\ \lambda & 1 & \mu & \kappa \\ c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As in [BeS] or [Ya1,2], we describe the Lie algebra $\mathfrak{g}^J$ as a subalgebra of $\mathfrak{g}_2 = \mathfrak{sp}(2, \mathbf{R})$ by

$$G(x, y, z, p, q, r) = \begin{pmatrix} x & 0 & y & q \\ p & 0 & q & r \\ z & 0 & -x & -p \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and denote

$$X = G(1, 0, \ldots, 0), \ldots, R = G(0, \ldots, 0, 1).$$

We get the commutators

$$\begin{array}{lll} [X, Y] = \ 2Y, & [X, Z] = -2Z, & [Y, Z] = X, \\ [X, P] = -P, & [X, Q] = \ Q, & [P, Q] = 2R, \\ [Y, P] = -Q, & [Z, Q] = -P, & \end{array}$$

all others are zero. As usual, we identify the dual $\mathfrak{sp}^*$ of $\mathfrak{sp}$ by the $\mathrm{Sp}(2, \mathbf{R})$ invariant isomorphism $\eta \mapsto X_\eta$ given by

$$\eta(\check{Y}) = \mathrm{tr}(X_\mu \check{Y}) =: < X_\mu, \check{Y} > \ \text{ for all } \check{Y} \in \mathfrak{sp}$$

$\eta_1, \ldots, \mu_6$ denotes a basis of $(\mathfrak{g}^J)^*$ dual to

$$(X_1, \ldots, X_6) := (X, \ldots, R).$$

Following closely Yang, we realize $(\mathfrak{g}^J)^*$ as a subspace of $\mathfrak{sp}$ by the matrices

$$M(x, y, z, p, q, r) = \begin{pmatrix} x & p & z & 0 \\ 0 & 0 & 0 & 0 \\ y & q & -x & 0 \\ q & r & -p & 0 \end{pmatrix}, x, \ldots, r \in \mathbf{R},$$

and put

$$X_* = M(1, 0, \ldots, 0), \ldots, R_* = M(0, \ldots, 0, 1).$$

Then

$$X_{\eta_1} = X_*, \ldots, X_{\eta_6} = R_*$$

is a basis of $(\mathfrak{g}^J)^*$ with

$$\begin{array}{l} < X_*, X >= 2, \; < Y_*, Y >= 1, \; < Z_*, Z >= 1 \\ < P_*, P >= 2, \; < Q_*, Q >= 2, \; < R_*, R >= 1 \end{array}$$

If g^{-1} is denoted as above, the coadjoint action of $\mathfrak{g}$ on $M(x, \ldots, r)$ is given by

$$\mathrm{Ad}^*(g)M(x, \ldots, r) = M(\tilde{x}, \ldots, \tilde{r})$$

with

$$(*) \qquad \begin{aligned} \tilde{x} &= (ad + bc)x + bdy - acz + (2ac\mu - (ad + bc)\lambda)p \\ &\qquad + ((ad + bc)\mu - 2bd\lambda)q + r(a\mu - b\lambda)(c\mu - d\lambda), \\ \tilde{y} &= 2dcx + d^2 y - c^2 z + 2(c\mu - d\lambda)(cp + dq) + r(c\mu - d\lambda)^2, \\ \tilde{z} &= -2abx - b^2 y + a^2 z - 2(a\mu - b\lambda)(ap + bq) - r(a\mu - b\lambda)^2, \\ \tilde{p} &= ap + bq + r(a\mu - b\lambda), \\ \tilde{q} &= cp + dq + r(c\mu - d\lambda), \\ \tilde{r} &= r. \end{aligned}$$

resp.

$$\begin{aligned} \tilde{x} &= (ad + bc)x + (bd - ac)\check{y} + (bd + ac)\check{z} + (2ac\mu - (ad + bc)\lambda)p \\ &\qquad + ((ad + bc)\mu - 2bd\lambda)q + r(a\mu - b\lambda)(c\mu - d\lambda), \\ \tilde{\check{y}} &= (cd - ab)x + (1/2)(a^2 - b^2 - c^2 + d^2)\check{y} + (1/2)(c^2 + d^2 - a^2 - b^2)\check{z} + \; \ldots \;, \\ \tilde{\check{z}} &= (ab + cd)x + (1/2)(b^2 + d^2 - a^2 - c^2)\check{y} + (1/2)(c^2 + d^2 + a^2 + b^2)\check{z} + \; \ldots \;. \end{aligned}$$

Here we look at the elliptic coadjoint orbit

$$M^J = G^J \cdot \eta = G^J / K^J$$

with $\eta = k'(Y_* - Z_*) + mR_*$ and its stabilizing group $G_\eta = K_1 \times C =: K^J$, C the center of G^J. It is the four-dimensional manifold given by $(x, \check{y}, \check{z}, p, q, r) \in \mathbf{R}^6$ with

$$\begin{aligned} x^2 + \check{y}^2 - \check{z}^2 + k'^2 &= (1/m)(2pqx - (p^2 - q^2)\check{y} + (p^2 + q^2)\check{z}), \\ r &= m. \end{aligned}$$

This orbit carries the (lowest weight) discrete series representation $\pi^+_{m,k}$ of G^J which is characterized by its K^J-types

$$\begin{aligned}
\text{mult}\,(\rho_{m,\tilde{k}}, \pi^+_{m,k}) &= 1 & &\text{for } \tilde{k} = k,\\
&= 1+l & &\text{for } \tilde{k} = k+2l \text{ or } = k+2l+1, l \in \mathbf{N}_0\\
&= 0 & &\text{for } \tilde{k} < k,
\end{aligned}$$

.

Here the representation $\rho_{m,\tilde{k}}$ of K^J is given by

$$\rho_{m,\tilde{k}}(r(\vartheta), (0,0,\kappa)) = e^{(i\tilde{k}\vartheta)+2\pi i m\kappa}, \vartheta \in [0, 2\pi], \kappa \in \mathbf{R}.$$

It belongs to the coadjoint orbit of K^J consisting of the point

$$\mathcal{O} = K^J \cdot \eta = \{(0,0,k',0,0,m)\}.$$

2.

Following an idea of Heckman, Guillemin and Sternberg propose in [GS2] a geometric interpretation of the formulae for the multiplicities and more generally of the spaces $\mathrm{Hom}_G(\rho, \pi)$ of intertwining operators between different representations ρ and π of a group G using the Marsden-Weinstein reduction in the following way.

Given a prequantized symplectic manifold (M, ω) with an Hamiltonian G-operation and a moment map $\Phi \to \mathfrak{g}^*$ under nice conditions the *Marsden-Weinstein reduction*

$$M_{\text{red}} = \Phi^{-1}(0)/\,G$$

is again a symplectic manifold (or at least an orbifold). If $\mathcal{Q}$ denotes an appropriate quantization procedure associating a representation π to the given situation (like the one sketched above in 1. or as in more recent publications (see for instance Vergne [Ve1]) using an equivariant Dirac operator), we have the *Guillemin-Sternberg Conjecture* stating that quantization commutes with reduction

$$\mathcal{Q}(M_{\text{red}}) = \mathcal{Q}(M)^G.$$

If $\mathcal{O}$ is a coadjoint G-orbit belonging to the irreducible unitary representation ρ of G the space $M \times \mathcal{O}$ is symplectic with moment map

$$\Psi : M \times \mathcal{O} \longrightarrow \mathfrak{g}^*\,,\ (m, \eta) \mapsto \Phi(m) - \eta.$$

If everything is nice, we then have the *Marsden-Weinstein reduction of M with respect to $\mathcal{O}$*

$$M_{\mathcal{O}} = \Psi^{-1}(0)/G \simeq \Phi^{-1}(\eta)/G_\eta$$

and $\mathcal{Q}(M_{\mathcal{O}})$ is the geometric analogue of $\mathrm{Hom}_G(\rho, \pi)$ (see Theorem 6.2 in [GS2]). There is already a lot of work precising this (see for instance section 3 in [GLS]). In particular, the multiplicity of ρ in π is zero if is $M_{\mathcal{O}}$ empty and the multiplicity is one if $M_{\mathcal{O}}$ is just a point. And in certain compact cases the multiplicity is related to the Riemann-Roch number respective the symplectic volume of $M_{\mathcal{O}}$. We will now look at what comes out in the two examples given above.

In **Example 1** we take

$$G = K_1, M = G_1 \cdot \eta, \text{and } \mathcal{O} = K_1 \cdot \tilde{\eta}$$

for η corresponding to $k'(Y - Z)$ resp. $\tilde{\eta}$ corresponding to $\tilde{k}'(Y - Z)$ with stabilizing group $G_{\tilde{\eta}} = SO(2)$. Then we have

$$\begin{aligned} M_{\mathcal{O}} &= \{(x, \breve{y}) \in \mathbf{R}^2;\, x^2 + \breve{y}^2 - \tilde{k'}^2 + k'^2 = 0\,\}/\,SO(2), \\ &= \emptyset \qquad \text{for } \tilde{k}' < k', \\ &= \{\text{point}\} \ \text{ for } \tilde{k}' \geq k'. \end{aligned}$$

This reflects exactly the multiplicity formula given above.

In **Example 2** we take

$$G = K^J, M = G^J \cdot \eta, \text{and } \mathcal{O} = K^J \cdot \tilde{\eta}$$

for $\eta = k'(Y_* - Z_*) + mR_*$ resp. $\tilde{\eta} = \tilde{k}'(Y_* - Z_*) + mR_*$ with stabilizing group $G_{\tilde{\eta}} = \mathrm{SO}(2) \times C = K^J$. Then the reduced space is here $M_{\mathcal{O}} = \Phi^{-1}(\tilde{\eta})/K^J$ where $\Phi^{-1}(\tilde{\eta})$ is given by

$$\{(x, \breve{y}, p, q) \in \mathbf{R}^4;\; x^2 + \breve{y}^2 - \tilde{k'}^2 + k'^2 = (1/m)(2pqx - (p^2 - q^2)\breve{y} + (p^2 + q^2)\tilde{k}')\}.$$

The group K^J acts here by

$$\begin{pmatrix} x \\ \breve{y} \end{pmatrix} \longmapsto r(2\vartheta)\begin{pmatrix} x \\ \breve{y} \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \longmapsto r(\vartheta)\begin{pmatrix} p \\ q \end{pmatrix}$$

for $r(\vartheta) \in SO(2)$. So, the reduced space $M_{\mathcal{O}}$ is in bijection to (for instance)

$$\{(x, \breve{y}, q) \in \mathbf{R}^3;\; x^2 + \breve{y}^2 - \tilde{k'}^2 + k'^2 = (1/m)q^2(\breve{y} + \tilde{k}')\}.$$

Things look a bit nicer if we introduce the complex variables

$$\tau = x + i\check{y},\ w = p + iq.$$

Then we have

$$\Phi^{-1}(\tilde{\eta}) = \{(\tau, w)\,;\, |\tau|^2 - \tilde{k'}^2 + k'^2 = (1/m)(\mathrm{Re}(i\bar{w}^2\tau) + |w|^2\tilde{k}')\}$$

and the action of SO(2) is given by

$$w \longmapsto e^{i\vartheta}w,\ \tau \longmapsto e^{i2\vartheta}\tau.$$

It would be nice to find a geometric prescription deducing the multiplicity formula given above from this situation.
Similar questions arise if one reduces the coadjoint G^J-orbit from example 2 as well as the other G^J-orbits with respect to the orbits of K^J and other subgroups of G^J, in particular $\mathrm{SL}(2, \mathbf{R})$ and $\mathrm{Heis}(\mathbf{R})$. This can easily be done starting from the descriptions of the coadjoint G^J-orbits in [Be]. For instance, we have the following situation.
In **Example 3** we take

$$G = G_1, M = G^J \cdot \eta,\ \text{and}\ \mathcal{O} = G_1 \cdot \tilde{\eta},$$

where η is as above and $\tilde{\eta} = \tilde{k}'(Y_* - Z_*)$. Then the reduced space is here $M_{\mathcal{O}} = \Phi^{-1}(\tilde{\eta})/K_1$ where $\Phi^{-1}(\tilde{\eta})$ is given by

$$\{(p, q) \in \mathbf{R}^2;\ -\tilde{k'}^2 + k'^2 = (1/m)(p^2 + q^2)\tilde{k}'\},$$

i. e. the reduced space is a point for $\tilde{k}' \geq k'$ and empty otherwise. This is in relation to the fact that a discrete series representation of G^J is the tensor product of the corresponding discrete sreies representation of the metaplectic group with the Schrödinger Weil representation.

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