On Self–Immersions of Infinite Graphs

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Abstract

The existence of an infinite graph which is not isomorphic to a proper minor of itself was proved by Oporowski. In the present note, it is shown that an analogous result holds when immersions are considered instead of minors. The question whether or not the same is true for weak immersions remains open.

Keywords: immersion, infinite graphs, Seymour's self–minor conjecture, well–quasi–ordering, antichains

1 Introduction

Our graph theoretic terminology is standard and widely in accordance with most of the textbooks. Graphs are understood to be simple, i.e. without loops or multiple edges. For the reader's convenience, some definitions and notational conventions are explained at the end of this introduction; for graph theoretic notions used but not explained in this note, we refer to Diestel [2].

The notion of an immersion of a graph into another one was introduced by Nash–Williams in his paper on well–quasi–ordering infinite trees [7]. For a graph G, let $\mathcal{P}(G)$ denote the set of nontrivial paths of G. For graphs G, H, an *immersion* $f : H \to G$ is a mapping $f : V(H) \cup E(H) \to V(G) \cup \mathcal{P}(G)$ such that

(i) if $v \in V(H)$, then $f(v) \in V(G)$

(ii) if $v, v' \in V(H), v \neq v'$, then $f(v) \neq f(v')$

(iii) if $e = vv' \in E(H)$, then $f(e) \in \mathcal{P}(G)$ and the path $f(e)$ connects $f(v)$ with $f(v')$

(iv) if $e, e' \in E(H)$ are distinct, then $f(e)$ and $f(e')$ are edge–disjoint

(v) if $e = vv' \in E(H)$ and $v'' \in V(H)$, then $f(v'') \notin V(f(e)) \backslash \{f(v), f(v')\}$.

We write $H \leq_1 G$ to indicate that there exists an immersion $f : H \to G$. Note that if in (iv) the word "edge–disjoint" is changed to "internally disjoint", then instead of the *immersion relation* $\leq_1$ one obtains the well–known subdivision relation. Hence, beside the extensively studied minor relation, the immersion relation is another natural refinement of the subdivision relation, and thus one may ask which of the known results on minors have analogues for immersions. For example, a problem of this type has been settled by Robertson and Seymour who (in [10]) announced a proof of a theorem showing that there does not exist an infinite $\leq_1$–antichain of finite graphs, thus providing an analogue to their graph minor theorem. Further, by modifying some of the ideas of Thomas [12], it has been shown in [1] that infinite $\leq_1$–antichains of uncountable graphs exist, which is an analogue to a result on minors obtained in [12]. For other results on the minor versus immersion theme, see [6].

In his paper [9], Oporowski established the existence of an infinite graph which is not isomorphic to a proper minor of itself. It is the purpose of the present note to derive an analogous result for immersions (Theorem 1). The proof of this result is based on a series of lemmas some of which are interesting in their own right since they establish the existence of certain infinite $\leq_1$–antichains. Some of our arguments make use of methods of Oporowski [9] and Komjáth [4]; in particular , the proof of Lemma 3 is based on a technique of Komjáth [4] involving disjoint stationary subsets of regular cardinals.

We also discuss a modified version of the immersion relation which was introduced by Nash–Williams in [8] and which we refer to as the weak immersion relation $\leq_2$. The question whether the main result of the present note still holds when weak immersions are considered instead of immersions is posed as an open problem.

As announced, we collect some definitions and notational conventions. For a graph G, if $A, B \subseteq V(G)$, then a path connecting a vertex of A with a vertex of B is refered to as an *A, B–path*; for $a, b \in V(G)$, the term *a, b–path* is defined similarly. A path is *nontrivial* if it contains at least two vertices. A *one–way infinite path* P consists of an infinite sequence $v_0, v_1, \dots$ of distinct vertices together with the edges $v_i v_{i+1} (i = 0, 1, \dots)$; v_0 is called the *initial vertex* of P. A *two–way infinite path* is defined similarly. For $v \in V(G)$, the degree of v in G is denoted by $d_G(v)$. An immersion $f : G \to G$ is called a *self–immersion* of G. A self–immersion of G is *trivial* if $f(v) = v$ for all $v \in V(G)$. (Of course, a trivial self–immersion of G maps each edge of G onto the corresponding path of length one, and so, for each graph G, there is just one trivial self–immersion.) For an immersion $f : H \to G$ and a subgraph H' of H, we denote by $f(H')$ the subgraph of G consisting of all vertices $f(v)$ with $v \in V(H')$ together with all vertices and edges lying on a path $f(e)$ with $e \in E(H')$. Let $\leq$ be a relation defined on the class of graphs. Then a set $\mathcal{H}$ of graphs is called an *antichain with respect to* $\leq$ (or a *$\leq$–antichain* for short) if $H \not\leq G$ for each pair of distinct members H, G of $\mathcal{H}$. For an $\leq$–antichain $\mathcal{H}$, $|\mathcal{H}|$ is called its *length*.

We use axiomatic set theory notation. Cardinals are identified with their initial ordinals, ω denotes the set $\{0, 1, \dots\}$ of natural numbers, and $\mathrm{cf}(\alpha)$ denotes the cofinality of an ordinal α. For set theoretic basics, for the notion of a closed unbounded set, and for the notion of a stationary set, see [5].

2 Infinite antichains of bridgeless connected graphs and graphs without nontrivial self–immersions

The following Lemmas 1 and 2 show that an infinite $\leq_1$–antichain $\mathcal{H}$ of bridgeless connected graphs can be used to obtain an infinite graph admitting no nontrivial self–immersion, provided that the length of $\mathcal{H}$ is at least the number of vertices of each of its members. For a graph G, we denote by $\mathcal{M}(G)$ the set of its maximal bridgeless connected subgraphs.

Lemma 1. Let $\mathcal{H}$ be an infinite set of pairwise disjoint bridgeless connected nonempty graphs and assume that

$$|H| \leq |\mathcal{H}| \quad \text{for all} \quad H \in \mathcal{H}. \tag{1}$$

Then there exists a graph G with $\mathcal{M}(G) = \mathcal{H}$ and with the property that

$$\text{each } v \in V(G) \text{ is contained in } |\mathcal{H}| \text{ bridges.} \tag{2}$$

Proof. Put $\kappa = |\mathcal{H}|$ and denote the members of $\mathcal{H}$ by H_ν with ν ranging over the ordinals $< \kappa$ and $H_\nu \neq H_\mu$ for $\nu \neq \mu$. Let V be the union of the vertex sets of the graphs $H_\nu (\nu < \kappa)$. By (1), we have $|V| = \kappa$ and thus it is possible to assign to each $v \in V$ a set $I_v \subseteq \kappa$ such that $|I_v| = \kappa$ and such that $I_v \cap I_{v'} = \emptyset$ for each pair of distinct $v, v' \in V$. Clearly, for each $\nu < \kappa$ and each $v \in V(H_\nu)$, we may assume that $\nu < \mu$ for all $\mu \in I_v$ (because, if necessary, we could delete from I_v all elements $\leq \nu$). Now, for each $v \in V$ and each $\mu \in I_v$, draw an edge connecting v with an arbitrary vertex of H_μ and let G be the resulting graph. One easily finds that all edges of G connecting distinct graphs H_ν are bridges of G. Hence $\mathcal{M}(G) = \mathcal{H}$ and G possesses property (2). $\square$

Lemma 2. Let $\mathcal{H}$ and G be as in Lemma 1 and, in addition, assume that $\mathcal{H}$ is a $\leq_1$–antichain. Then no self–immersion of G exists, except for the trivial one.

Proof. As in the proof of Lemma 1, let $\kappa = |\mathcal{H}|$ and $\mathcal{H} = \{H_\nu : \nu < \kappa\}$. Let f be a self–immersion of G. From $\mathcal{M}(G) = \mathcal{H}$, together with the assumption that $\mathcal{H}$ is a $\leq_1$–antichain, one readily obtains $f(H_\nu) \subseteq H_\nu$ for all $\nu < \kappa$. Pick $\nu < \kappa$. We are done if we can show that the restriction of f to $V(H_\nu)$ is the identity map on $V(H_\nu)$. Thus, for a contradiction, suppose that $f(u) = v \neq u$ for some $u \in V(H_\nu)$. By condition (ii) in the definition of "immersion", we must have $f(v) \neq v$. By (2), there exists a bridge $e = vv'$ with $v' \in V(H_\mu)$ for some $\mu \neq \nu$, and thus $f(e)$ is a path connecting $f(v) \in V(H_\nu)$ with $f(v') \in V(H_\mu)$. Consequently, e is an edge of $f(e)$ and we have $f(v) \neq v \neq f(v')$. Thus v is an inner vertex of $f(e)$, in contradiction to the fact that $f(u) = v$. $\square$

For the proof of the preceding lemma, a condition weaker than (2) stating that each vertex of G is contained in at least one bridge would have been sufficient. Condition (2) will be used in full when, in Section 4, we apply Lemma 1 a second time.

It follows from the Lemmas 1 and 2 that, in order to obtain an infinite graph possessing no nontrivial self–immersion, we have to construct an infinite $\leq_1$–antichain $\mathcal{H}$ of bridgeless connected graphs such that condition (1) holds. This will be carried out in the Lemmas 3 and 4. The proof of Lemma 3 is postponed to Section 3.

Lemma 3. Let λ, κ bei cardinals with $\omega < \lambda < \kappa$, λ regular, and $\mathrm{cf}(\kappa) = \omega$. Then there exists a $\leq_1$–antichain $\mathcal{H}$ of length λ such that for all $H \in \mathcal{H}$

(a) $|H| = \kappa$

(b) $|\{v \in V(H) : d_H(v) \geq 3\}| = \lambda$

(c) H is bridgeless and connected.

Lemma 4. Let κ be a cardinal with $\omega < \kappa$ and $\mathrm{cf}(\kappa) = \omega$. Then there exists a $\leq_1$–antichain $\mathcal{H}$ of bridgeless connected graphs such that $|\mathcal{H}| = \kappa$ and $|H| = \kappa$ for all $H \in \mathcal{H}$.

Proof. Because $\mathrm{cf}(\kappa) = \omega < \kappa$ there exists a sequence $\lambda_n (n \in \omega)$ of regular cardinals with $\omega < \lambda_0 < \lambda_1 < \ldots$ and $\kappa = \cup\{\lambda_n : n \in \omega\}$. For each $n \in \omega$, application of Lemma 3 yields a $\leq_1$–antichain $\mathcal{H}_n$ of bridgeless connected graphs such that $|\mathcal{H}_n| = \lambda_n$, $|H| = \kappa$, and $|\{v \in V(H) : d_H(v) \geq 3\}| = \lambda_n$ for all $H \in \mathcal{H}_n$.

Clearly, we may assume $\mathcal{H}_n \cap \mathcal{H}_m = \emptyset$ for all $n, m \in \omega$ with $n \neq m$ (since, otherwise, we could replace the involved graphs by appropriate copies). Let

$\{F_1, F_2, \dots\}$ be a countable subset of $\mathcal{H}_0$. For each $n \in \omega$ with $n \geq 1$, let κF_n denote a graph consisting of κ components each isomorphic to F_n and each disjoint to every member of $\mathcal{H}_n$. For each $n \in \omega$ with $n \geq 1$ and each $H \in \mathcal{H}_n$, let H^+ be a graph that results from $H \cup \kappa F_n$ by adding a new vertex h to $H \cup \kappa F_n$ and joining h to every component of $H \cup \kappa F_n$ by exactly two edges.

Let $\mathcal{H}$ be the set of the graphs H^+. Then, clearly, the members of $\mathcal{H}$ are bridgeless connected graphs and $|H^+| = \kappa$ for all $H^+ \in \mathcal{H}$. Thus it remains to show that $\mathcal{H}$ is a $\leq_1$–antichain of length κ. For this purpose, let $H \in \mathcal{H}_{n_1}$, $G \in \mathcal{H}_{n_2}$ with $n_1, n_2 \geq 1$ and assume $H \neq G$. We are done if we can show that this implies $H^+ \not\leq_1 G^+$. (Note that, in particular, this means that $H \neq G$ implies $H^+ \neq G^+$ and thus we have $|\mathcal{H}| = |\cup_{n \geq 1} \mathcal{H}_n| = \kappa$.).

Hence, for a contradiction, suppose that there exists an immersion $f : H^+ \to G^+$. Let h and g be the vertices which have been added to $H \cup \kappa F_{n_1}$ and $G \cup \kappa F_{n_2}$ in order to obtain H^+ and G^+, respectively.

Case 1: $f(h) = g$. Then f maps each component of $H \cup \kappa F_{n_1}$ into a component of $G \cup \kappa F_{n_2}$. Note that the graph κF_{n_1} contains κ vertices of degree ≥ 3 while G contains less than κ such vertices. Hence $\kappa F_{n_1} \not\leq_1 G$ and thus f maps at least one component of κF_{n_1} into a component of κF_{n_2}. Hence $F_{n_1} \leq_1 F_{n_2}$ and, consequently, $n_1 = n_2$. Further, $H \not\leq_1 F_{n_2}$ since H contains λ_{n_1} vertices of degree ≥ 3, while F_{n_2} contains only λ_0 such vertices. Hence $H \leq_1 G$ which, together with $n_1 = n_2$, implies $H = G$, in contradiction to our assumption on H and G.

Case 2: $f(h) \neq g$. Then $f(h) \in D$ for some component D of $G \cup \kappa F_{n_2}$. It follows that $f(C) \not\subseteq D$ can hold for at most finitely many components C of $H \cup \kappa F_{n_1}$ since there are just finitely many edges of G connecting a vertex of D with a vertex not in D. Hence $\kappa F_{n_1} \leq_1 D$, which is impossible since κF_{n_1} contains too many vertices of degree ≥ 3. $\square$

As an immediate consequence of the Lemmas 2 and 4, one obtains the following.

Theorem 1. There exist infinite graphs possessing no nontrivial self–immersion.

More specifically, it follows from the construction that there exists a graph of order $\aleph_\omega$ possessing no nontrivial self–immersion. The question whether or not there exists such a graph of order $\aleph_n$ for some $n \in \omega$ remains open.

We shortly discuss a related problem which comes to mind in the context of Komjáth's paper [4]. For an infinite cardinal κ, let $\mathcal{H}$ be a $\leq_1$–antichain such that $|H| = \kappa$ for all members H of $\mathcal{H}$. Then, clearly, the length of $\mathcal{H}$ is at most 2^κ and thus the question arises whether or not $|\mathcal{H}| = 2^\kappa$ is possible. In this context, the following lemma is useful. (See also [4], where a similar lemma was proved for minors.)

Lemma 5. For an infinite cardinal κ, if there exists a $\leq_1$–antichain $\mathcal{H}'$ such that, for each $H \in \mathcal{H}'$, H is connected and $|H| = |\mathcal{H}'| = \kappa$, then there also exists a $\leq_1$–antichain $\mathcal{H}$ with $|H| = \kappa$ for all $H \in \mathcal{H}$ and $|\mathcal{H}| = 2^\kappa$.

Proof. It is well–known that, for each infinite set of cardinality λ, there exist 2^λ *independent subsets*, i.e., none of these subsets is contained in another one (cf. Hausdorff [3]). Let Γ be a set of 2^κ independent subsets of $\mathcal{H}'$. For each $\mathcal{G} \in \Gamma$, consider a graph which is the disjoint union of the members of $\mathcal{G}$ and let $\mathcal{H}$ be the set of all these graphs. One easily finds that $\mathcal{H}$ has the desired properties. $\square$

As an immediate consequence of the Lemmas 4 and 5, one obtains a first result on the above question.

Theorem 2. Let κ be a cardinal with $\omega < \kappa$ and $\mathrm{cf}(\kappa) = \omega$. Then there exists a $\leq_1$–antichain $\mathcal{H}$ with $|H| = \kappa$ for all $H \in \mathcal{H}$ and $|\mathcal{H}| = 2^\kappa$.

Thus, in particular, there exists a $\leq_1$–antichain $\mathcal{H}$ of length $2^{\aleph_\omega}$ with $|H| = \aleph_\omega$ for all $H \in \mathcal{H}$. On the other hand, it remains open whether there exists an infinite $\leq_1$–antichain of graphs of order $< \aleph_\omega$.

3 A proof of Lemma 3

Our proof of Lemma 3 is based on a method of Komjáth which was used in [4] to prove that, for each cardinal $\lambda > \omega$, there exists an antichain $\mathcal{H}$ with respect to the minor relation such that $|\mathcal{H}| = 2^\lambda$ and $|H| = \lambda$ for each $H \in \mathcal{H}$. In particular, we need the following result of Komjáth ([4, Lemma 3]).

Lemma 6. Let $\lambda > \omega$ be a regular cardinal. Then, for each successor ordinal $x < \lambda$, there exists a natural number $r(x)$ and a finite sequence $s(i,x)(i = 0,\dots,r(x))$ of ordinals satisfying:

(i) $s(0,x) < \cdots < s(r(x),x) = x$

(ii) if $s(i,x) = s(j,y)$ then $i = j$ and $s(k,x) = s(k,y)$ for all $k \leq i$

(iii) there exists a closed unbounded set $C \subseteq \lambda$ such that if $\alpha \in S := \{\beta \in C : \mathrm{cf}(\beta) = \omega\}$ then an increasing sequence of successor ordinals $\gamma_n(\alpha)(n = 0,1,\dots)$ with $\gamma_n(\alpha) \to \alpha$ can be found such that, for all $n \in \omega$, $r(\gamma_n(\alpha)) = n$ and $s(i,\gamma_n(\alpha)) = \gamma_i(\alpha)$ for all $i \leq n$.

For the proof of Lemma 3, let λ, κ be cardinals with $\omega < \lambda < \kappa$, λ regular, and $\mathrm{cf}(\kappa) = \omega$. Before proving the existence of a $\leq_1$–antichain $\mathcal{H}$ as is claimed in Lemma 3, we first show that there exists a $\leq_1$–antichain $\mathcal{H}'$ of length λ such that for all $H \in \mathcal{H}'$ condition (a) of Lemma 3 holds, together with the conditions

(b') $|\{v \in V(H) : d_H(v) \geq 3 \text{ or } d_H(v) = 0\}| = \lambda$

(c') H is bridgeless and has at most λ components

(d') each component of H is either infinite or an isolated vertex.

Let $r(x), s(i,x), S, \gamma_n(\alpha)$ be as in Lemma 6. Let further $\kappa_n (n \in \omega)$ be a sequence of cardinals with $\lambda \leq \kappa_0 < \kappa_1 < \dots$ and $\kappa = \cup\{\kappa_n : n \in \omega\}$.

For all $\alpha \in S$ and $n \in \omega$, let $A_{\alpha,n}$ be a set of cardinality κ_n such that $A_{\alpha,n} \cap \lambda = \emptyset$ and $A_{\alpha,n} \cap A_{\beta,m} = \emptyset$ whenever $(\alpha,n) \neq (\beta,m)$. For each nonempty $X \subseteq S$, we define a graph H_X by

$$\begin{aligned} V(H_X) &:= \lambda \cup \bigcup_{\substack{\alpha \in X \\ n \in \omega}} A_{\alpha,n} \\ E(H_X) &:= \bigcup_{\substack{\alpha \in X \\ n \in \omega}} \{av : a \in A_{\alpha,n}, v \in \{\gamma_n(\alpha), \alpha\}\}. \end{aligned}$$

It follows from the definitions that the graphs $H = H_X$ meet the conditions (a), (b'), (c'), (d'). Note also that it follows from the definition of S that S is a stationary set (cf.[5]) and thus, by a result of Solovay [11], there exist λ

disjoint stationary subsets of S. Hence, in order to obtain $\mathcal{H}'$ as desired, it suffices to prove the following.

$$\text{For} \quad X, Y \subseteq S, \text{ if } X \backslash Y \text{ is stationary, then } H_X \not\leq_1 H_Y. \tag{3}$$

For a proof by contradiction, let $X, Y \subseteq S$ such that $X \backslash Y$ is stationary and suppose that there exists an immersion $f : H_X \to H_Y$. From [5, II 6.13] one readily obtains that there exists a closed unbounded set $D \subseteq \lambda$ such that, for each $\alpha \in D$, the following holds.

$$\text{Let } \beta \in \lambda \text{ with } f(\beta) \in \lambda. \text{Then } \beta < \alpha \text{ if and only if } f(\beta) < \alpha. \tag{4}$$

Since $X \backslash Y$ is stationary, we can pick $\alpha \in (X \backslash Y) \cap D$. Then $f(\alpha) \in \lambda$ since α has infinite degree in H_X and thus cannot be mapped by f onto a vertex of degree two. For a similar reason, we have $f(\gamma_i(\alpha)) \in \lambda$ for $i = 0, 1, \ldots$ .Application of (4) to the case $\beta = \alpha$ yields $\alpha \leq f(\alpha)$. Note that α is an isolated vertex in H_Y since neither is α a member of Y nor is it a successor ordinal. Hence $f(\alpha) = \alpha$ is impossible and thus we have $\alpha < f(\alpha)$. Since $\gamma_i(\alpha) < \alpha (i = 0, 1, \ldots)$, we can apply (4) to obtain $f(\gamma_i(\alpha)) < \alpha$ for $i = 0, 1, \ldots$. Since $\gamma_i(\alpha)$ and α are joined by κ_i edge–disjoint paths of H_X the same must be true for $f(\gamma_i(\alpha))$ and $f(\alpha)$ in H_Y. Thus, summarizing, we have obtained the following.

$$\begin{gathered}\text{For all } i = 0, 1, \ldots, f(\gamma_i(\alpha)) < \alpha < f(\alpha) \text{ and there are } \kappa_i \\ \text{edge–disjoint paths of } H_Y \text{ connecting } f(\gamma_i(\alpha)) \text{ with } f(\alpha).\end{gathered} \tag{5}$$

Let $A = \{\gamma \in \lambda : \gamma < \alpha\}$ and $B = \{\beta \in \lambda : \alpha < \beta\}$. Call a path of H_Y a *link* if it is a path of length two of the form $(\gamma_j(\beta), a, \beta)$ with $\beta \in Y, j \in \omega$, and $a \in A_{\beta,j}$. A link $(\gamma_j(\beta), a, \beta)$ is an *A, B–link* if $\gamma_j(\beta) \in A$ and $\beta \in B$. Note that each A, B–path of H_Y contains an A, B–link as a subpath. (This immediately follows from the definition of H_Y, together with the fact that α is isolated in H_Y.) From this, together with (5), one easily concludes that there exists a connected subgraph F of H_Y such that $V(F) \cap \lambda \subseteq B$ and such that, for each $i \in \omega$, there exists a set $\mathcal{L}_i$ of κ_i A, B–links terminating in F.(For each of the A, B–paths P mentioned in (5), consider the "last" A, B–link $(\gamma_j(\beta), a, \beta)$ contained in P, where "last" means that the $\beta, f(\alpha)$–subpath P' of P does not contain a vertex of A; define F to be the union of all such paths P'.)

Now, we can proceed similar as in [4] to obtain a contradiction. Let $\bar{x} = \{x_0, \ldots, x_n\} \subseteq A$ satisfy $x_0 < \cdots < x_n$. Define $\Gamma(\bar{x}) \subseteq B$ as follows. If $y \in B$ is a successor ordinal, then $y \in \Gamma(\bar{x})$ if $r(y) > n$, $s(i, y) = x_i$ for $i \leq n$, and $s(n+1, y) \in B$. If $\beta \in B \cap S$, then $\beta \in \Gamma(\bar{x})$ if $\gamma_i(\beta) = x_i$ for $i \leq n$ and $\gamma_{n+1}(\beta) \in B$. In this way, we clearly partition the vertices of B which are non–isolated in H_Y and it follows from the definition of H_Y that, if two vertices of $H_Y \cap B$ are joined by a link and one of these vertices is in $\Gamma(\bar{x})$ then so is the other (cf. Lemma 6(iii)). Applying this to the graph F, one obtains from the connectedness of F that there exists an $\bar{x} = \{x_0, \ldots, x_n\} \subseteq A$ with $x_0 < \cdots < x_n$ such that all vertices of $V(F) \cap \lambda$ are in $\Gamma(\bar{x})$. Consequently, if $(\gamma_j(\beta), a, \beta)$ is an A, B–link with $\beta \in V(F)$, then $\gamma_j(\beta) \in \bar{x}$ and thus there are at most $\sum_{j=0}^{n} \lambda \cdot \kappa_j = \kappa_n$ A, B–links terminating in F. This contradicts the existence of the sets $\mathcal{L}_i$ for $i > n$. Hence we have proved (3) and thus can pick a $\leq_1$–antichain $\mathcal{H}'$ having the claimed properties. We now modify each $H \in \mathcal{H}'$ as follows.

(I) For $H \in \mathcal{H}'$, let H_1 be a graph that results from H by replacing each isolated vertex by a component which is isomorphic to a triangle.

(II) Let H_2 result from H_1 by doubling each component of H_1.

(III) Add a new vertex h to H_2 and, for each component C of H_2, draw exactly two edges joining h with C.

For each $H \in \mathcal{H}'$, denote the thus resulting graph by H^+ and let $\mathcal{H}$ be the set of graphs H^+ with $H \in \mathcal{H}'$. From the fact that (a), (b'), (c') hold for each $H \in \mathcal{H}'$ one easily concludes that each $H^+ \in \mathcal{H}$ meets the conditions (a), (b), (c). Thus it remains to prove that $\mathcal{H}$ is a $\leq_1$–antichain of length λ. For this purpose, let $H, G \in \mathcal{H}', H \neq G$. Let H_1, H_2, h be as in (I)–(III) and use the notations G_1, G_2, g similarly. We have to show that $H^+ \not\leq_1 G^+$. Thus, for a contradiction, assume that there exists an immersion $f : H^+ \to G^+$.

Case 1: $f(h) = g$. For a graph F, denote by $\text{comp}F$ the set of its components. By the hypothesis of Case 1, there exists an injective mapping $\hat{f} : \text{comp}H_2 \to \text{comp}G_2$ such that $f(C) \subseteq \hat{f}(C)$ for all $C \in \text{comp}H_2$. (The injectivity of $\hat{f}$ follows from the fact that, for each $D \in \text{comp}G_2$, there are just two edges of G^+ joining D with g.)

Put $A := \text{comp}H_1$ and $B := \text{comp}G_1$. We may assume that $A \cap B = \emptyset$. We define a bipartite auxiliary graph $G(A, B)$ on $A \cup B$ as follows. Let $C_1 \in A, D_1 \in B$. Denote by $C_2 \in \text{comp}H_2$ the copy of C_1 which has been added in Step (II) in order to double C_1; similarly, define $D_2 \in \text{comp}G_2$. We draw an edge of $G(A, B)$ between C_1 and D_1 if there exist indices $i, j \in \{1, 2\}$ with $\hat{f}(C_i) = D_j$. Then, in $G(A, B)$, each vertex of A has degree one or two. Further, it follows from the injectivity of $\hat{f}$ that each vertex of B has degree at most two and, if a vertex of A has degree one, then its neighbor in B also has degree one. From this one concludes that each component of $G(A, B)$ either is an isolated vertex contained in B, or it is a complete graph K_2, or it is an even cycle, or it is a path having its ends in B, or it is a one–way infinite path having its initial vertex in B, or it is a two–way infinite path. Consequently, $G(A, B)$ has a matching M such that every vertex of A is incident with an edge in M. Hence $H_1 \leq_1 G_1$ and thus (because of (d')) we have $H \leq_1 G$. This implies $H = G$ since H, G are members of the $\leq_1$–antichain $\mathcal{H}'$. This contradicts our assumption on H, G and thus Case 1 is settled.

Case 2: $f(h) \neq g$. Then $f(h) \in D$ for some component D of G_2. One readily concludes from the constructions that $f(C) \not\subseteq D$ can hold for at most one component C of H_2. It follows that $H_1 \leq_1 D$ since H_2 has been obtained from H_1 by doubling each component of H_1. From $H_1 \leq_1 D$ one easily obtains $H \leq_1 G$, which (as in Case 1) contradicts the choice of H and G. $\square$

4 Weak immersion

In his paper [8], Nash–Williams proposed a modified version of the immersion relation which we call "weak immersion" and for which we use the notation $H \leq_2 G$. The weak immersion relation is obtained by redefining $H \leq_1 G$ so that distinct vertices of H can be mapped into the same vertex of G, while the edges of H are mapped into edge–disjoint paths and cycles of G.

More formally, this definition can be restated as follows. Let $\mathcal{C}(G)$ denote the set of cycles of a graph G. For graphs G, H, a *weak immersion* $f : H \to G$ is a mapping $f : V(H) \cup E(H) \to V(G) \cup \mathcal{P}(G) \cup \mathcal{C}(G)$ such that (i), (ii'), (iii'), (iv), (v) hold, where (i), (iv) and (v) are as in the definition of "immersion" and the statements (ii'), (iii') are as follows.

(ii') if $e = vv' \in E(H)$ and $f(v) \neq f(v')$, then $f(e) \in \mathcal{P}(G)$ and the path $f(e)$ connects $f(v)$ with $f(v')$

(iii') if $e = vv' \in E(H)$ and $f(v) = f(v')$, then $f(e) \in \mathcal{C}(G)$ and $f(v)$ is a vertex of the cycle $f(e)$.

A weak immersion $f : G \to G$ is called a *weak self–immersion* of G and a weak self–immersion of G is *trivial* if on $V(G)$ it is the identity map. For a weak immersion $f : H \to G$ and a subgraph H' of H, we use the notation $f(H')$ similarly as for immersions. The following Problem 1 immediately comes to mind in the context of the present note.

Problem 1. Does there exist an infinite graph not possessing a nontrivial weak self–immersion?

It has been shown in [1] that infinite $\leq_2$–antichains of (uncountable) graphs exist and thus one may wish to use such an antichain to obtain an affirmative answer to Problem 1. This idea results into the next problem, and the subsequent Lemma 7 shows that an affirmative answer to Problem 2 would yield an affirmative answer to Problem 1.

Problem 2. Does there exist an infinite $\leq_2$–antichain $\mathcal{H}$ such that the members of $\mathcal{H}$ are bridgeless connected graphs and such that $|H| \leq |\mathcal{H}|$ for all $H \in \mathcal{H}$?

Lemma 7. Let $\mathcal{H}$ and G be as in Lemma 1 and, in addition, assume that $\mathcal{H}$ is a $\leq_2$–antichain. Then no weak self–immersion of G exists, except for the trivial one.

Proof. As in the proofs of the Lemmas 1,2, let $\kappa = |\mathcal{H}|$ and $\mathcal{H} = \{H_\nu : \nu < \kappa\}$. Let f be a weak self–immersion of G. From $\mathcal{M}(G) = \mathcal{H}$, together with the assumption that $\mathcal{H}$ is a $\leq_2$–antichain, one obtains $f(H_\nu) \subseteq H_\nu$ for all $\nu < \kappa$. Let $\nu < \kappa$. It remains to show that the restriction of f to $V(H_\nu)$ is the identity map on $V(H_\nu)$. For this purpose, suppose that $f(u) \neq u$ for some $u \in V(H_\nu)$. Let $e = uu'$ be a bridge of G. Then $f(e)$ is a $f(u), f(u')$–path containing e. (This follows from the fact that $u, f(u) \in H_\nu$ and $u', f(u') \in H_\mu$ for some $\mu \neq \nu$.) In particular, we have $u \in V(f(e))$, and the subpath of $f(e)$ connecting $f(u)$ with u is contained in H_ν. As this holds for each of the κ

bridges of G containing u (cf.(2)), we have found that there exist κ edge–disjoint $u, f(u)$–paths of H_ν. It follows that $H_\mu \leq_2 H_\nu$ for all μ since a weak immersion $H_\mu \to H_\nu$ can be obtained by mapping all vertices of H_μ onto u and representing the edges of H_μ by edge–disjoint cycles through u. (This is possible since, by (1), H_μ has at most κ edges and because κ edge–disjoint $u, f(u)$–paths of H_ν can be used to construct κ edge–disjoint cycles of H_ν passing through u.) Thus we have derived a contradiction to the assumption that $\mathcal{H}$ is a $\leq_2$–antichain. $\square$

References

[1] T. Andreae, On immersions of uncountable graphs, *J. Combin. Theory*, Ser. B, **87** (2003) 130-137.

[2] R. Diestel, *Graph Theory.* Springer, New York, 2000.

[3] F. Hausdorff, Über zwei Sätze von G. Fichtenholz und L. Kantorovitch, *Studia Math.* **6** (1936) 18–19.

[4] P. Komjáth, A note on minors of uncountable graphs, *Math.Proc.Camb.Phil.Soc.* **117** (1995) 7–9.

[5] K. Kunen, *Set Theory,* Studies in Logic **102**. North–Holland, Amsterdam, 1980.

[6] F. Lescure and H. Meyniel, On a problem upon configurations contained in graphs with given chromatic number. *Graph theory in memory of G.A. Dirac* (Papers presented at the meeting held at Sandbjerg, Denmark from 2 to 7 June 1985), Ann. Discrete Math. **41** (1989) 325–331.

[7] C.St.J.A. Nash-Williams, On well–quasi–ordering infinite trees, *Proc.Cambridge Philos.Soc.* **61** (1965) 697–720.

[8] C.St.J.A. Nash–Williams, A glance at graph theory – part II, *Bull. London Math.Soc.* **14** (1982) 294–328.

[9] B. Oporowski, A counterexample to Seymour's self–minor conjecture, *J. Graph Theory* **14** (1990) 521–524.

[10] N. Robertson and P.D. Seymour, Graph Minors XX. Wagner's conjecture (Preprint).

[11] R. Solovay, Real–valued measurable cardinals. *Axiomatic set theory*, Proc.Sympos.Pure Math. **13**, Part I (1971) 397–428.

[12] R. Thomas, A counter–example to 'Wagner's conjecture' for infinite graphs, *Math.Proc.Camb.Phil.Soc.* **103** (1988) 55–57.