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Reconstructing the Number of Blocks of an Infinite Graph

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Abstract

It is shown that two hypomorphic infinite graphs always have the same number of blocks. This settles a problem of Nash–Williams (Reconstruction of infinite graphs, Discrete Mathematics 95 (1991) 221–229).

Keywords: Reconstruction; Infinite Graphs; Number of Blocks; Number of 2–connected components

1 Introduction

The graphs considered in this paper do not have loops or multiple edges and the letters G, H always denote graphs. Two graphs G and H are *hypomorphic* if there exists a bijection $\sigma : V(G) \to V(H)$ such that $G - v \simeq H - \sigma(v)$ for each $v \in V(G)$. We write $G \doteq H$ to indicate that G and H are hypomorphic. A graph G is *reconstructible* if $G \doteq H$ always implies $G \simeq H$. The well–known 'Reconstruction Conjecture' asserts that every finite graph with at least 3 vertices is reconstructible. For a survey on results concerning this conjecture and its variants, the reader may consult [5,6,12–14].

There are several partial results on the Reconstruction Conjecture asserting that hypomorphic finite graphs with at least 3 vertices must share a certain property; for example, the following two results are of this kind. (For proofs and for related results, see e.g. [3,5].)

Proposition 1. *Hypomorphic finite graphs with at least 3 vertices have the same number of components.*

Proposition 2. *Hypomorphic finite graphs with at least 3 vertices have the same number of blocks.*

The analogue for infinite graphs of the Reconstruction Conjecture is not in general true: simple counterexamples can be obtained as follows. Let T_α be a tree in which every vertex has degree α for some infinite cardinal α and let βT_α be a forest consisting of β components each isomorphic to T_α, where β is a cardinal with $2 \leq \beta \leq \alpha$. Then T_α and βT_α are hypomorphic but not isomorphic. (Examples of this kind were first presented in [9] and [16]. For a survey on the reconstruction of infinite graphs, see [14], and [2,7,10,15] for more recent results.)

By $c(G)$ we denote the number of components of G and, for each $x \in V(G)$, we denote by G_x the component of G containing x. The above examples also show that, for hypomorphic graphs G and H, the values of $c(G)$ and $c(H)$ may be quite different and thus, in particular, the analogue for infinite graphs of Proposition 1 is not in general true. Nevertheless, the following results of [1] show that, subject to an additional hypothesis, Proposition 1 can be extended to infinite graphs.

Theorem A. *Let G be an infinite graph such that $c(G_x - x)$ is finite for at least one $x \in V(G)$. Then $G \doteq H$ implies $c(G) = c(H)$.*

As an immediate consequence of Theorem A one obtains

Theorem B. *Let G be an infinite graph with at least one vertex of finite degree. Then $G \doteq H$ implies $c(G) = c(H)$.*

With the above Propositions 1 and 2, the described examples, and Theorem B in mind, it is natural to pose the following problem which is one of the unsolved problems presented by Nash–Williams in [14].

Problem (Nash–Williams [14]). *Must two hypomorphic infinite graphs have the same number of blocks? If not, can this conclusion be drawn subject to some additional hypothesis like that of Theorem B?*

It is the purpose of the present paper to solve this problem by showing that two hypomorphic infinite graphs always have the same number of blocks.

Our terminology is standard and essentially in accordance with most of the text–books in graph theory. For notions used but not defined here, we refer to Diestel [8]. By $V(G)$ and $E(G)$, we denote the set of vertices and edges of a graph G, respectively. We write $G \simeq H$ to indicate that G and H are isomorphic graphs. For $A \subseteq V(G)$, $G - A$ denotes the graph that results from G by deletion of A together with all edges which are incident with some vertex of A. For $v \in V(G)$, we write $G - v$ instead of $G - \{v\}$ and call $G - v$ a *vertex–deleted subgraph* of G; similarly, for a subgraph F of G, we write $G - F$ instead of $G - V(F)$. By $G + H$, we denote the disjoint union of G and H. For a cardinal α, we write αG for the union of α disjoint copies of G. A 2–way infinite path P consists of distinct vertices $a_i (i \in \mathbb{Z})$ with $a_i a_j \in E(P)$ if and only if $|i - j| = 1$. A *block* of G is a connected subgraph of G having no cutvertices and being maximal with these properties. By $bl(G)$ we denote the number of blocks of G. A graph is *trivial* if it consists of just one vertex.

Given hypomorphic graphs G, H, we always assume without further mention that we have arbitrarily picked some bijection $\sigma : V(G) \to V(H)$ such that $G - x \simeq H - \sigma(x)$ for all $x \in V(G)$ and, for simplicity, we always write x' instead of $\sigma(x)$.

2 Proof of the main result

By $c^*(G)$, we denote the number of 2–connected components of G. We start with two lemmas the first of which being of some interest in its own right. It states that, for hypomorphic graphs G and H, $c^*(G)$ and $c^*(H)$ may differ by at most 1. At the end of this paper we show that this is best possible in the following sense. For each nonnegative integer n, we give examples of infinite graphs G, H with $G \doteq H$ and $c^*(G) = n = c^*(H) - 1$.

Lemma 1. *If $G \doteq H$, then either $c^*(G) = c^*(H)$ or $c^*(G),\ c^*(H) < \aleph_0$ and $|c^*(G) - c^*(H)| = 1$.*

Proof. Suppose the contrary. Then $c^*(G) + 1 < c^*(H)$ or $c^*(H) + 1 < c^*(G)$ and we may assume that the former holds. Clearly, for each $x \in V(G)$, we have

$$c^*(H) \leq c^*(H - x') + 1 = c^*(G - x) + 1$$

which, when taken together with $c^*(G) + 1 < c^*(H)$, yields $c^*(G) < c^*(G - x)$ for each $x \in V(G)$. Consequently, no component of G can be 2–connected and, for each $x \in V(G)$, there exists a 2–connected component of $G_x - x$. For each $x \in V(G)$, let $C(x)$ denote some 2–connected component of $G_x - x$ and let $v \in V(G)$ be arbitrary. Then, clearly, we can find a vertex $w \in C(v)$ such that $G_w - w$ is connected. Hence $C(w) = G_w - w$ and thus $G_w - w$ is 2–connected. Since w is a vertex of the 2–connected graph $C(v)$, we conclude that w has at least two neighbors in $G_w - w$ and thus the 2–connectedness of $G_w - w$ implies that G_w is 2–connected. This contradicts the above stated fact that G has no 2–connected components. ∎

Lemma 2. *If G is a graph with $bl(G) \geq \aleph_0$, then $bl(G - x) \geq bl(G)$ for all $x \in V(G)$.*

Proof. Let $x \in V(G)$. If x is isolated, then obviously $bl(G - x) = bl(G)$ since $bl(G)$ is infinite. Hence assume that x is nonisolated. Let $\mathcal{B}$ be the set of bridges of G which are incident with x. One easily finds that

$$bl(G - x) + |\mathcal{B}| \geq bl(G). \tag{1}$$

Moreover

$$|\mathcal{B}| \leq bl(G - x) \tag{2}$$

since, obviously, there exists an injective mapping from $\mathcal{B}$ into the set of blocks of $G - x$. From (1), together with $bl(G) \geq \aleph_0$, one obtains $bl(G - x) + |\mathcal{B}| \geq \aleph_0$ and thus, by (2), $bl(G - x) + |\mathcal{B}| = bl(G - x)$. Hence $bl(G - x) \geq bl(G)$ by (1). ∎

By $\mathcal{F}(G)$ we denote the *block–cutvertex graph* of G which is defined in the usual manner: if $\mathcal{C}$ and $\mathcal{B}$ are the sets of cutvertices and blocks of G, respectively, then $\mathcal{F}(G)$ is the bipartite graph with vertex set $\mathcal{C} \cup \mathcal{B}$ and edge set $E(\mathcal{F}(G)) = \{cB : c \in \mathcal{C},\ B \in \mathcal{B},\ c \in V(B)\}$. Then, clearly, $\mathcal{F}(G)$ is a forest and the trivial components of $\mathcal{F}(G)$ (isolated vertices of $\mathcal{F}(G)$) are precisely those blocks of G which are either isolated vertices, or isolated edges (with ends), or 2–connected components of G. The following facts will frequently be used without further mention. Let $x \in V(G)$. Then

- x is an isolated vertex of G if and only if $c(G_x - x) = 0$
- if x is not an isolated vertex of G, then $c(G_x - x)$ is equal to the number of blocks of G containing x
- x is a cutvertex of G if and only if $c(G_x - x) \geq 2$
- for a cutvertex x of G, $c(G_x - x)$ is the degree of x in $\mathcal{F}(G)$.

Theorem. *If G, H are hypomorphic infinite graphs, then $bl(G) = bl(H)$.*

Proof. Let G, H be infinite with $G \doteq H$. For each nontrivial component T of $\mathcal{F}(G)$, pick $r \in V(T)$ such that r is a cutvertex of G and call r the *root* of T. For each cutvertex x of G, let $A(x)$ be the set of blocks of G which are direct successors of x in the rooted tree (T, r), where T is the component of $\mathcal{F}(G)$ to which x belongs. Further, let A_0 be the set of blocks of G corresponding to the trivial components of $\mathcal{F}(G)$. This way, we obtain a partition of the set of blocks of G into classes, where – contrary to the usual terminology – the class A_0 may be empty. In a similar manner, we define a partition of the set of blocks of H into classes $B(y)$ and B_0 with y running over all cutvertices of H and B_0 possibly being empty.

Put $a := bl(G)$, $b := bl(H)$ and suppose $a < b$, Then (clearly) $|V(G)| = |V(H)| \geq b$ and, because $|A(x)| \geq 1$ for all cutvertices x of G,

(3) $\qquad$ G has at most a cutvertices.

Hence there exist vertices of G which are no cutvertices of G and thus we conclude from Theorem A that

(4) $\qquad$ $c(G) = c(H)$.

In the following, we seperately discuss the cases $c(G) < \aleph_0$ and $c(G) \geq \aleph_0$.

(I) Let $c(G) < \aleph_0$. Then (4) implies

(5) $\qquad$ $c(G_x - x) = c(H_{x'} - x')$ for all $x \in V(G)$.

In particular, one obtains from (5) that x is a cutvertex of G if and only if x' is a cutvertex of H. Hence, writing X for the set of cutvertices of G, we have

(6) $$a = |A_0| + \sum_{x \in X} |A(x)|, \quad b = |B_0| + \sum_{x \in X} |B(x')|.$$

By a result of Bondy and Hemminger [4], the number of components which are isomorphic to some given finite graph is the same for G and H , and thus G and H must have the same number of isolated vertices and the same number of isolated edges. From this, together with Lemma 1, one obtains that

(7) $\qquad$ either $|A_0| = |B_0|$ or $|A_0|,\ |B_0| < \aleph_0$ and $||A_0| - |B_0|| = 1$.

Further, one easily concludes from (5) that

(8) for each $x \in X$, either $|A(x)| = |B(x')|$ or

$$|A(x)|,\ |B(x')| < \aleph_0 \text{ and } ||A(x)| - |B(x')|| = 1.$$

¿From (6–8) one obtains $a + |X| + 1 \geq b$ and thus

(9) $a < b \leq a + |X| + 1.$

We claim that

(10) $a \geq |X| + 1.$

For a proof suppose $a < |X| + 1$. By the first equality of (6), this can only happen if $|A_0| = 0$ and $|A(x)| = 1$ for all $x \in X$. But then there is no nontrivial component of $\mathcal{F}(G)$ because $|A(x)| \geq 2$ if x is the root of such a component. Consequently, $\mathcal{F}(G)$ must have a trivial component, in contradiction to $|A_0| = 0$. Hence (10).

If $|X| \geq \aleph_0$, then (10) implies $a + |X| + 1 = a$ which, together with (9), yields a contradiction. Hence $|X| < \aleph_0$ and thus $a, b < \aleph_0$ by (9). Further, $c(\mathcal{F}(G)) = c(\mathcal{F}(H))$ by (4). Hence $\mathcal{F}(G)$ and $\mathcal{F}(H)$ are finite forests having the same number of components. Moreover, one easily obtains from (5) that $\mathcal{F}(G)$ and $\mathcal{F}(H)$ have the same number of edges. Hence

$$\begin{gathered} a = |V(\mathcal{F}(G))| - |X| = |E(\mathcal{F}(G))| + c(\mathcal{F}(G)) - |X| = \\ |E(\mathcal{F}(H))| + c(\mathcal{F}(H)) - |X| = |V(\mathcal{F}(H))| - |X| = b, \end{gathered}$$

in contradiction to $a < b$. This settles the case $c(G) < \aleph_0$.

(II) Let $c(G) \geq \aleph_0$. Then, by (4), $c(H) \geq \aleph_0$ and (of course) $a, b \geq \aleph_0$. In the sequel, the fact that $c(G),\ c(H),\ a, b$ are infinite cardinals will several times be used without further mention. As an immediate consequence of (3), one obtains the following.

(11) There are at most a vertices x of G such that $G - x$ has two or more components C with $bl(C) > a$.

We claim that

(12) H has exactly one component with b blocks while all other components of H have at most a blocks.

For the proof of (12), we first show that H has at least one component with more than a blocks. Suppose the contrary. Then, because $bl(H) > a$, we must have $c(H) > a$ and thus, by (4), $bl(G) \geq c(G) = c(H) > a$, in contradiction to the definition of a.

Next we show that H has at most one component with more than a blocks. For a contradiction, suppose that H has distinct components C_1, C_2 with $bl(C_i) > a (i = 1, 2)$. One concludes from (11) that $|V(H - (C_1 \cup C_2))| \leq a$ and, consequently, $|V(C_1 \cup C_2)| \geq b$. Further, again by (11), there are at most a vertices $x' \in C_1 \cup C_2$ such that $bl(C) > a$ for some component C of $H_{x'} - x'$. Hence, we can pick $x' \in V(C_1 \cup C_2)$ such that $bl(C) \leq a$ for each component C of $H_{x'} - x'$. Then, by Lemma 2 and because $H_{x'} = C_i$ for $i = 1$ or 2, we have $bl(H_{x'} - x') \geq bl(H_{x'}) > a$ which is compatible with the choice of x' only if $c(H_{x'} - x') > a$. Hence, for the corresponding $x \in V(G)$, we have $c(G - x) > a$ and thus $bl(G) > a$, which is a contradiction.

So far, we have shown that there exists a uniquely determined component C of H with $bl(C) > a$. Hence, for the proof of (12), it remains to show that $bl(C) = b$. Suppose $bl(C) < b$. Then, because $bl(H) = b$ and since $bl(C') \leq a$ for all components C' of H with $C' \neq C$, we must have $c(H) = b$. Hence, by (4), $c(G) = b$, in contradiction to $bl(G) < b$. This proves (12).

Denote by C the uniquely determined component of H with $bl(C) = b$. We show that

(13) there are at most a vertices $x' \in H - C$ such that $bl(H_{x'} - x') > a$.

For a proof, let $x' \in V(H - C)$ with $bl(H_{x'} - x') > a$. By (12), we have $bl(H_{x'}) \leq a$ and, consequently, $c(H_{x'} - x') \leq a$. Hence, because $bl(H_{x'} - x') > a$, there must be a component of $H_{x'} - x'$ with more than a blocks and thus there are two components of $H - x'$ with more than a blocks. Hence assertion (13) follows from (11).

(14) For all $x \in V(G)$ there exists a component $C(x)$ of $G_x - x$ such that $bl(C(x)) > a$.

For a proof , let $x \in V(G)$. By Lemma 2, $bl(G - x) = bl(H - x') \geq bl(H) = b > bl(G)$ and thus $bl(G_x - x) \geq b > a$. Further, since $bl(G) = a$, we have $c(G_x - x) \leq a$ which, together with $bl(G_x - x) > a$, implies (14).

Now we obtain a contradiction as follows. Let $x \in V(G)$ such that $x' \in V(C)$ and such that x is not a cutvertex of G. (This choice of x is possible by (3) and because $|V(C)| \geq bl(C) = b$.) Let $\phi : G - x \to H - x'$ be an isomorphism and let $C(x)$ be as in (14). By (12) and because $bl(C(x)) > a$, $\phi(C(x))$ is a component of $C - x'$. Since x is not a cutvertex, we have $C(x) = G_x - x$ and thus each component of $H - C$ is mapped by ϕ^{-1} onto a component of G. Consequently, one obtains from (14) that for each $y' \in V(H - C)$ there exists a component $C(y')$ of $H_{y'} - y'$ such that $bl(C(y')) > a$. Hence, by (13), we have $|V(H - C)| \leq a$. On the other hand, note that $V(H - C) \neq \emptyset$

since $c(H) \geq \aleph_0$ and thus , picking an arbitrary $y' \in V(H-C)$, one obtains $|V(H-C)| \geq |C(y')| \geq bl(C(y')) > a$, which is a contradiction. ∎

3 Examples

As announced we present examples showing that the statement of Lemma 1 is, in a sense, best possible. Essential parts of the following constructions are based on ideas of Harary, Schwenk and Scott [11].

Example. For each $n = 0, 1, \ldots$, we intend to construct infinite graphs G_n, H_n such that $G_n \doteq H_n$ and $c^*(G_n) = n = c^*(H_n) - 1$. For this purpose, let P be a 2–way infinite path and let S be a graph resulting from P by adding a new vertex which is adjacent to each vertex of P. Then, clearly, S is 2–connected and

(15) for each finite nonempty subset F of $V(S)$,
there is no 2–connected component of $S - F$.

For each positive integer k, let R_k be a graph consisting of k disjoint copies of S together with one additional vertex of degree k which is adjacent to the k vertices of infinite degree occuring in the copies of S. We assume that the so–defined graphs $R_1, R_2 \ldots$ are disjoint.

Let $\mathcal{A}$ be the set of graphs A such that $A \not\simeq S$ and such that, for some positive integer k and some finite (possibly empty) $F \subseteq V(R_k)$, A is a component of $R_k - F$. Note that it follows from the definitions that each $A \in \mathcal{A}$ occurs in $\mathcal{A}$ with multiplicity $\aleph_0$ (i.e., there are exactly $\aleph_0$ members of $\mathcal{A}$ which are isomorphic to A). Further, one easily obtains from (15) that no $A \in \mathcal{A}$ is 2–connected.

Let B be the disjoint union of the members of $\mathcal{A}$ and put

$$A_n := B + \; nS \quad (n = 0, 1, \ldots).$$

Note that, as a consequence of the constructions, the following holds.

(16) For both A_0 and A_1, the vertex–deleted subgraphs are exactly
the graphs $A_j (j = 0, 1, \ldots)$ each occuring with multiplicity $\aleph_0$.

Hence $A_0 \doteq A_1$ and, by construction, $c^*(A_0) = 0, \; c^*(A_1) = 1$. Denoting by K the complete graph with $\aleph_0$ vertices, we put

$$G_n := A_0 + \; nK, \; H_n := A_1 + \; nK \quad (n = 0, 1, \ldots).$$

Then, by construction and in particular by (16), we have $G_n \doteq H_n$ and $c^*(G_n) = n = c^*(H_n) - 1$.

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