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Another Remark on the Mysterious Theta Function

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The elementary classical Jacobi theta functions are given as in [EZ] p.58 for $\tau \in \mathfrak{H} = \{\tau = x + iy \in \mathbb{C}, y > 0\}$, $z \in \mathbb{C}$ and $e(z) := \exp(2\pi i z)$

$$\theta_{m,n}(\tau, z) := \sum_{\ell \in \mathbb{Z}} e^m((\ell + \mu/2m)^2\tau + 2(\ell + \mu/2m)z).$$

Here $\mu = 0, \ldots, 2m - 1$ and $2m \in \mathbb{N}$. Mumford in [M1] starts with the most elementary theta series

$$\vartheta(z, \tau) := \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z) \ ,$$

i.e. we can identify $\vartheta(z, \tau) = \theta_{1/2,0}(\tau, z)$. There is an enormous amount of literature on these functions and their generalizations. Besides Mumford's books "Tata Lectures on Theta I–III" ([M1-3]), for further references let us only mention Igusa's book [Ig] and the book by Lion and Vergne [LV]. The theta functions as functions in the complex variable z are quasiperiodic and for $z = 0$ (thetanullwerte) we get important modular forms. There is a recent note by Couwenberg [Cou] proving the modular property for the transformation $\tau \mapsto -1/\tau$ not via the usual Poisson sum procedure but basing on the fact that ϑ as a function of τ and z fulfills the heat equation. The thetas as functions in both variables play a fundamental role in the theory of Jacobi forms. The original ϑ can be understood as Jacobi form of weight $k = 1/2$ and index $m = 1/2$. We here only cite from [EZ] p.58 the following basic transformation properties

$$\theta_{m,\mu}(\tau + 1, z) = e\left(\mu^2/4m\right)\theta_{m,\mu}(\tau, z)$$

and

$$\theta_{m,\mu}(-1/\tau, z/\tau) = (\tau/(2mi))^{1/2} e^m(z^2/\tau) \sum_{\nu \bmod 2m} e(-\mu\nu/2m)\theta_{m,\nu}(\tau, z).$$

And from [M1] p.53 Proposition 11.1 we remind that $\vartheta(z, \tau)$ is characterized as the unique holomorphic function $f(z, \tau)$ on $\mathbb{C} \times \mathfrak{H}$ such that

a) $f(z + 1, \tau) = f(z, \tau)$,

b) $f(z + \tau, \tau) = \exp(-\pi i \tau - 2\pi i z) f(z, \tau)$,

c) $f(z+1/2, \tau+1) = f(z,\tau)$,

d) $f(z/\tau, -1/\tau) = (-i\tau)^{1/2} \exp(\pi i z^2/\tau) f(z,\tau)$,

e) $\lim_{\Im\tau\to\infty} f(z,\tau) = 1$ for all $z \in \mathbb{C}$.

As developed in the books cited above, there is an important representation theoretic background to the theta functions and their construction. I will comment briefly on this here because I am still mystified by the fact that so small a symmetrization as the summation over a single integer variable produces a function with such a high symmetry.
The procedure followed in [LV] and [M3] p.133ff consists in interpreting the thetas as appropriately chosen matrix coefficients for representations. We will indicate here the more elementary brute force method.

The Jacobi Group and its representations

The Jacobi group is the semidirect product of the symplectic group with an appropriate Heisenberg group. Here we look at

$$G^J(\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \ltimes \ \mathrm{Heis}(\mathbb{R})$$

where the multiplication law is fixed by the embedding into the symplectic group $\mathrm{Sp}(2,\mathbb{R})$ given by

$$\mathrm{Heis}\ (\mathbb{R}) \ni (\lambda,\mu,\kappa) \longmapsto \begin{pmatrix} 1 & & & \mu \\ \lambda & 1 & \mu & \kappa \\ & & 1 & -\lambda \\ & & & 1 \end{pmatrix},$$

$$\mathrm{SL}(2,\mathbb{R}) \ni M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & & b & \\ & 1 & & \\ c & & d & \\ & & & 1 \end{pmatrix}.$$

We will write

$$g = (p,q,\kappa)M \quad \text{or} \quad g = M(\lambda,\mu,\kappa) \in G^J(\mathbb{R}).$$

Elements of the representation theory of $G^J(\mathbb{R})$ can be found in the work of Pyatetski–Shapiro, Satake, Kirillov, Howe, Guillemin, Sternberg, Igusa, Mumford and many others. They are to a large part summed up in [BeS],

the essential outcome being that the representations π of $G^J(\mathbb{R})$ with non-trivial central character are essentially products of the representations of the metaplectic group $\mathrm{Mp}(2,\mathbb{R})$ with a fundamental representation π^m_{SW} of the metaplectic cover $\tilde{G}^J$ of G^J called the "Schrödinger–Weil representation" in [BeS], as it is in turn the product of the Schrödinger representation π_S of the Heisenberg group Heis($\mathbb{R}$) and the (Segal–Shale–) Weil or oszillator representation π_W of Mp
This representation π^m_{SW} is a representation of "lowest weight" $k = 1/2$ and index m. It is characterized by the fact that all its K^J–types are onedimensional ($K^J = SO(2) \times \mathbb{R}$), which in turn is characterized by the fact that the lowest weight vector for π^m_{SW} satisfies the heat equation. To be a bit more explicit, the complexified Liealgebra of the Jacobi group

$$\mathfrak{g}^J_c =< X_\pm, Z, Y_\pm, Z_0 >$$

is realized (see [BeS] p.12) by the left invariant differential operators

$$\begin{aligned}
\mathcal{L}_{Z_0} &= i\partial_\kappa \\
\mathcal{L}_{Y_\pm} &= (1/2)y^{-1/2}e^{\pm i\theta}(\partial_p - (x \pm iy)\partial_q - (p(x+iy)+q)\partial_\kappa) \\
\mathcal{L}_{X_\pm} &= \pm(i/2)e^{\pm 2i\theta}(2y(\partial_x \mp i\partial_y) - \partial_\theta) \\
\mathcal{L}_Z &= -i\partial_\theta
\end{aligned}$$

acting on differentiable functions $\phi = \phi(g)$ with the coordinates coming from

$$g = (p,q,\kappa)n(x)t(y^{1/2})r(\theta).$$

where

$$n(x) = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}, t(a) = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix},\ r(\theta) = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},\ \alpha = \cos\theta,\ \beta = \sin\theta.$$

and

$$g(i,0) = (\tau, z) = (x+iy, p\tau + q).$$

Now, as to be seen for instance in [BeS] p.28ff, a "lowest weight" vector ϕ_0 for a discrete series representation $\pi^+_{m,k}$ of $G^J(\mathbb{R})$ realized by right translation in a space $\mathcal{H}^+_{m,k}$ of smooth functions $\phi = \phi(g)$ on $G^J(\mathbb{R})$ is characterized by the equations

$$\mathcal{L}_{X_-}\phi_0 = \mathcal{L}_{Y_-}\phi_0 = 0,\ \mathcal{L}_{Z_0}\phi_0 = 2\pi m\phi_0,\ \mathcal{L}_Z\phi_0 = k\phi_0\ .$$

It is well known that $\pi^+_{m,k}$ decomposes under the subgroup $K^J = SO(2) \times \mathbb{R}$ as

$$\begin{aligned}
\mathrm{mult}\left(\zeta_{m,\tilde{k}},\ \pi^+_{m,k}\right) &= 1 \text{ for } \tilde{k} = k \\
&= 1 + \ell \text{ for } \tilde{k} = k + 2\ell \text{ or} \\
&= 0 \text{ for } \tilde{k} < k.
\end{aligned}$$

Here the representation $\zeta_{m,k}$ of K^J is given by

$$\zeta_{m,k}((0,0,\kappa)r(\theta)) = e^{ik\theta+2\pi im\kappa} \quad \kappa\in\mathbb{R},\ \theta\in(0,2\pi).$$

As is well known and shown in [Be], these multiplicities are all zero or one exactly if ϕ_0 is annihilated by the heat operator

$$4\pi m\mathcal{L}_{X_+} + \mathcal{L}_{Y_+^2} = \mathcal{L}_{D_+}\ ,$$

and this is exactly the case for $\pi^+_{m,k}$ with $k = 1/2$.
By some general considerations on induced representations (see[BeS] p.49) or brute force calculation, one verifies that

$$\phi_0(g) = y^{1/4}\zeta^m e^{i\theta/2}e^m(pz)$$

is such a lowest weight vector of type $k = 1/2, m$.

A special theta transform

There are general procedures to construct modular and automorphic forms by the formation of Eisenstein series, Poincaré series or the "theta transform". In our case, let's look at the maximal unipotent subgroup of G^J

$$N^J = \{\tilde{n}(x,q,\kappa) = (0,q,\kappa)n(x),\ q,\kappa,x\in\mathbb{R}\}$$

and the discrete group

$$\Gamma^J = SL(2,\mathbb{Z})\ltimes\mathbb{Z}^3\ .$$

Then we have (at least formally) the theta transform associating to a N^J–leftinvariant function ϕ the Γ^J invariant function

$$\vartheta\phi(g) = \sum_{\gamma\in(\Gamma^J\cap N^J)\backslash\Gamma^J}\phi(\gamma g).$$

The "vacuum vector" ϕ_0 is N^J–leftinvariant and one can use it to construct the sum

$$\Theta(g) := \sum_{\ell\in\mathbb{Z}}\phi_0((\ell,0,0)g).$$

Via

$$(\ell,0,0)(p,q,\kappa)M = (p,+\ell,q,\kappa+\ell q)M$$

we get

$$\begin{aligned}\Theta(g) &= \sum_{\ell} y^{1/4} e^m(\kappa) e^{i\vartheta/2} e^m(((p+\ell)((p+\ell)\tau+q)) e^m(q^\ell) \\ &= \phi_0(g) \underbrace{\sum_{\ell} e^m(\ell^2\tau + 2\ell z)}_{\Theta_{m,0}(\tau,z).}\end{aligned}$$

The usual way to lift automorphic forms f from a homogeneous space $\mathcal{H} = G/K$ to the group G is via the appropriate automorphic factor. In our case a Jacobi form f of type (k,m) on $\mathfrak{H} \times \mathbb{C}$ is lifted to a Γ^J–invariant function on $G^J(\mathbb{R})$ by

$$f \longmapsto \phi_f \text{ with } \phi_f(g) = j_{k,m}(g(i,0) f(g(i,0)).$$

Here the automorphic factor $j_{k,m}$ is given in the coordinates $(p,q,\kappa,x,y,\vartheta)$ by

$$j_{k,m}(g,(i,0)) = y^{k/2} e^{ik\vartheta} e^m(\kappa + pz).$$

This shows that the function Θ constructed above is just the $(1/2,m)$–lift of the theta function $\Theta_{m,0}(\tau,z)$. Obviously, by a slight generalization, we can do as well for the $\Theta_{m,\mu}$.

The transformation formula d) for $\vartheta = \Theta_{1/2,0}$ translates into the invariance of Θ under the left multiplication by $w = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ and thus under Γ^J. This invariance comes here as a gift offered by the construction. There are several ways to see this. The most refined one probably is by interpreting the theta function as matrix elements as already mentioned. Another one comes out by pursuing material assembled in [BeS] or [M3]. The basic realization of the "Schrödinger–Weil representation" π^m_{SW} of the metaplectic cover $\tilde{G}^J(\mathbb{R}$ of the Jacobi group $G^J(\mathbb{R})$ is given for $m = 1/2$ on the space $\mathcal{H} = L^2(\mathbb{R})$ by the formulae (see [BeS] p.27 or [M3] p.135)

$$\left(\pi_W \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} f\right)(\xi) = e\left(b\xi^2/2\right) f(\xi), \quad b \in \mathbb{R}$$

$$\left(\pi_W \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix} f\right)(\xi) = a^{1/2} f(a\xi), \qquad a > 0$$

$$\left(\pi_W \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} f\right)(\xi) = i^{1/2}\hat{f}(\xi) = i^{1/2} \int f(\eta) e^{-\pi i \mu \xi d\eta}.$$

$$(\pi_S(\lambda,\mu,\kappa)f)(\xi) = e((\kappa + (2\xi+\lambda)\mu)/2) f(\xi+\lambda).$$

The "fundamental lowest weight vector" or "vacuum vector" is given in this realization by

$$f_0(\xi) = e^{-\pi\xi^2} = e(i\xi^2/2) \ .$$

We intertwine this representation with the subrepresentation of the induced representation

$$\mathrm{ind}_{B^J}^{G^J}\chi_{1/2,-1}$$

which is discussed on p.48f in [BeS]. This representation is given by right translation on the space of smooth functions ϕ on $\tilde{G}^J(\mathbb{R})$ of K^J–type $(k = 1/2,\ m = 1/2)$ which come as images from $\mathcal{S}(\mathbb{R})$ by the mapping

$$f \longmapsto \phi_f \text{ with } \phi_f(g) := (\pi_{SW}(g)f)(0).$$

A small computation using the formulae above leads to

$$(*) \qquad (\pi_{SW}((p,q,\kappa)n(x)t(y))f_0)(\xi) = y^{1/4}e^{1/2}(\kappa + pz + \xi^2\tau + 2\xi z),$$

i.e. the vacuum vector f_0 goes to the same vector as above

$$\phi_0(g) = \phi_{f_0}(g) = y^{1/4}e^{i\theta/2}e((\kappa + pz)/2).$$

The formulae (*) shows that we have

$$\begin{aligned}\Theta(g) &= \sum_{\ell\in\mathbb{Z}} \phi_0((\ell,0,0)g), \\ &= \sum_{\ell\in\mathbb{Z}} (\pi_{SW}(g)f_0)(\ell).\end{aligned}$$

Now, we can use the most elementary Poisson summation formula to see the invariance of our Θ:

$$\begin{aligned}\Theta(wg) &= \sum_{\ell\in\mathbb{Z}} (\pi_{SW}(wg)f_0)(\ell), \\ &= \sum_{\ell\in\mathbb{Z}} (\pi_W(w)\pi_{SW}(g)f_0)(\ell), \\ &= \gamma(1)\sum_{\ell\in\mathbb{Z}} (\pi_{SW}(g)f_0)(\ell), \\ &= \gamma(1)\Theta(g).\end{aligned}$$

This invariance translates into the theta–transformation formula by the standard procedure: We have

$$w(p,q,\kappa) = (q,-p,\kappa)w$$

and

$$wn(x)t(y)r(\vartheta) = n(\check{x})t(\check{y})r(\check{\vartheta})$$

with

$$\check{x} = \frac{-x}{x^2+y^2},\ \check{y} = \frac{y}{x^2+y^2},\ e^{i\check{\vartheta}} = e^{i\vartheta}\left(\frac{\overline{\tau}}{\tau}\right)^{1/2}$$

i.e. $\check{\tau} = -1/\tau$. $\check{z} = \check{p}\check{\tau}.\check{q} = -qz/\tau$. This leads to

$$\begin{aligned}\Theta(wg) &= \sum_{\ell\in\mathbb{Z}} \phi_0((\ell,0,0)wg)\\ &= \sum \phi_0((q+\ell,-p,\kappa-\ell p)n(\check{x})t(\check{y})r(\check{\vartheta}))\\ &= y^{1/4}e^{i\vartheta/2}\ell^{1/2}(\kappa-qz/\tau)\tau^{-1/2}\sum_{\ell}(-\ell^2/\tau-2\ell z/\tau)\end{aligned}$$

Comparing this with $(*)$, we have the usual Jacobi form relation

$$\vartheta(-1/\tau,-z/\tau) = (\tau/i)^{-1/2}e^{1/2}(z^2/\tau)\vartheta(\tau,z)$$

References

[Be] Berndt, R.: *Die Jacobigruppe und die Wärmeleitungsgleichung.* Math. Z. **191** (1986) 351–361

[BeS] Berndt, R., Schmidt, R.: *Elements of the Representation Theory of the Jacobi Group.* PM 163, Birkhäuser, Basel 1998.

[Ig] Igusa, J.I.: *Theta Functions.* Springer, Berlin 1972.

[LV] Lion, G., Vergne, M.: *The Weil representation, Maslov index and Theta series.* PM 6, Birkhäuser, Boston 1980.

[Cou] Couwenberg, W.: *A simple proof of the modular identity for theta functions.* Proc. AMS **131** (2003) 3305–3307.

[EZ] Eichler, M., Zagier, D.: *The Theory of Jacobi Forms.* PM 55, Birkhäuser, Boston 1985.

[M1–3] Mumford, D.: *Tata Lectures on Theta I–III.* PM 28, 43, 97, Birkhäuser Boston 1983, 1984, 1991.