Mathematisches Seminar

Hamburger Beiträge zur Mathematik
Heft 84

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Abstract

We consider a special class of Frobenius Groups, which generalizes the class of sharply 2-transitive groups in such a way that the construction of a neardomain can be generalized to the construction of a K-loop. The group then is shown to be a quasidirect product of that K-loop by a suitable automorphism group. The major advantage of this point of view is the existence of examples which are hoped to shed some light on the still open problem of the existence of proper neardomains.

1 Introduction

Let G be a group acting on a set P. The group G, or more precisely the pair (G, P), is called a *Frobenius group* if G acts transitively, but not regularly on P, and only the identity of G fixes more than one point, i.e., if Ω is the stabilizer of an arbitrary point $e \in P$, then $\Omega \neq \{1\}$ acts fixed-point-freely on $P \setminus \{e\}$.

The following theorem is the starting point of our investigations. Part (1) is a straightforward exercise, part (2) is FROBENIUS's famous theorem. It is proved in many books on group theory, e.g., [1, (35.24), p. 191].

Theorem 1.1 *Let G be a group. Consider the following conditions:*

(I) *$G = K\Omega$ contains subgroups K and $\Omega \neq \{1\}$ such that Ω acts fixed-point-freely on K by conjugation;*
(II) *G has a subgroup $\Omega \neq \{1\}$ such that $\Omega \cap g\Omega g^{-1} = \{1\}$ for all $g \in G \setminus \Omega$;*
(III) *G is a Frobenius group.*

We have

(1) (I) $\Longrightarrow$ (II) $\Longleftrightarrow$ (III).
(2) *If G is finite, then the above conditions are all equivalent.*

The implication "(I) $\implies$ (II)" cannot be reversed in general as we shall see later.

The point of FROBENIUS's theorem is the existence of the normal subgroup K, which has Ω as a complement. K is called the *Frobenius kernel* of G, and Ω the *Frobenius complement*. The group G is then the semidirect product of K by Ω.

An important class of Frobenius groups are the sharply 2-transitive groups, i.e., groups G which act on a set P in such a way that for all two pairs of distinct elements a_1, a_2 and b_1, b_2 in P there exists exactly one $\sigma \in G$ such that $\sigma(a_i) = b_i$. For this class of groups it is open, whether the implication "(I) $\implies$ (II)" can be reversed, or not. In fact, this question is equivalent with the question, whether proper neardomains exist, or not (see [22, V §1]).

Our aim is to establish a new class of Frobenius groups—Frobenius groups with many involutions—which generalizes the notion of a sharply 2-transitive group in a reasonable way. We'll develop the theory for these Frobenius groups, which to some extent parallels the construction of neardomains from sharply 2-transitive groups. The point of our generalization is that there do exist examples, which do not satisfy (I) in 1.1. It is hoped that studying such examples gives more insight into the problem mentioned in the preceding paragraph.

This approach is not completely new. In fact, GABRIEL has used similar methods in his dissertation [5]. We'll generalize GABRIEL's definition, and we'll establish the connection with K-loops. Indeed, we'll show that a generalization of (I) of 1.1 holds for our special class of Frobenius groups, where a K-loop plays the role of the Frobenius kernel K, and the semidirect product is replaced by the quasidirect product.

Therefore we'll need to recall some basic facts about K-loops and the construction of the quasidirect product in §2.

§3 is devoted to the connection between general Frobenius groups and certain loops, then, in §4, we introduce the notion which gives the title to this paper. It turns out that the theory splits in two cases.

§5 handles the case of characteristic 2. The remaining case is treated in §6. Here we also present our notion of specific groups.

The paper contains no examples. Most of the examples known can be obtained from a construction due to KOLB and KREUZER [14].

2 Preliminaries

A set L with a binary operation "$\cdot$" is called a *left loop* if there exists an (identity) element $1 \in L$ and for all $a, b \in L$ there exists a unique $x \in L$ with

$$a1 = 1a = a \quad \text{and} \quad ax = b$$

For every left loop the *left translations* $\lambda_a : L \to L; x \mapsto ax$ are bijections for all $a \in L$. We can therefore define $\delta_{a,b} := \lambda_{ab}^{-1}\lambda_a\lambda_b$. The group $\mathcal{D}(L)$ generated by all the $\delta_{a,b}$ is called the *left inner mapping group*.

Let G be a group with a subgroup Ω. A set L of representatives of the left cosets of Ω in G with $1 \in L$, will be called a *transversal* of G/Ω. More precisely, we require that

$$L \subseteq G \quad \text{such that} \quad \forall g \in G : |L \cap g\Omega| = 1 \quad \text{and} \quad 1 \in L.$$

This means in particular $G = L\Omega$.

For $a, b \in L$ let $a \circ b \in L$, $d_{a,b} \in \Omega$ be the unique elements such that

$$ab = (a \circ b)d_{a,b}\,.$$

Thus $(L, \circ)$ becomes a left loop (see [2] or [16, (3.2)]). Transversals are always assumed to carry this structure.

For any set M denote the symmetric group on M by $\mathcal{S}_M$. The identity map is denoted by $\mathbf{1}$. For a left loop L define the *obstruction-map*

$$\chi : L \times \mathcal{S}_L \to \mathcal{S}_L\,;\ (a, \alpha) \mapsto \lambda_{\alpha(a)}^{-1}\alpha\lambda_a\alpha^{-1}.$$

Observe that $\alpha \in \mathcal{S}_L$ is an automorphism of L if and only if $\chi(a, \alpha) = \mathbf{1}$ for all $a \in L$. So χ "measures" the deviation of α from being an automorphism.

Following Sabinin (see [17]), we call a subgroup T of $\mathcal{S}_L$ with $\chi(L \times T) \subseteq T$, which fixes 1 and contains $\mathcal{D}(L)$ a *transassociant* of L. We emphasize that transassociants are groups by definition. All statements made for transassociants and the quasidirect product are proved in [20] and in [17, XII.6] (see also [12, §2.C]). $\mathcal{D}$ is always a transassociant, and $\operatorname{Aut} L$ is a transassociant if and only if $\mathcal{D} \subseteq \operatorname{Aut} L$.[1] A transassociant is called *fixed point free* if it acts fixed-point-freely on $L^{\#}$.[2] This is all we can get, since 1 is fixed by definition.

The main point of these definitions is the construction of the quasidirect product.

[1] Left loops with this property are called left A_ℓ-loops.

[2] $M^{\#} := M \setminus \{1\}$ for any subset of a left loop with identity 1.

Theorem 2.1 *Let $(L, \cdot)$ be a left loop, and let T be a transassociant of L. Then*

(1) *$L \times_Q T$, the set $L \times T$ with the multiplication*

$$(a, \alpha)(b, \beta) := (a \cdot \alpha(b), \delta_{a,\alpha(b)}\chi(b, \alpha)\alpha\beta), \quad \textit{for all } (a, \alpha), (b, \beta) \in L \times T,$$

is a group. The inverse of $(a, \alpha) \in L \times_Q T$ is given by

$$(a, \alpha)^{-1} = \left(\alpha^{-1}(a'),\ \alpha^{-1}\chi(\alpha^{-1}(a'), \alpha)^{-1}\delta_{a,a'}^{-1}\right), \quad \textit{where } a' = \lambda_a^{-1}(1)$$

is the right inverse of a in L.

(2) *$L \times_Q T$ acts faithfully and transitively on L by*

$$(a, \alpha)(x) := a \cdot \alpha(x) \quad \textit{for all } (a, \alpha) \in L \times_Q T,\ x \in L.$$

(3) *The map $T \to L \times_Q T$; $\alpha \mapsto (1, \alpha)$ is a monomorphism. The image will be denoted by $1 \times T$. It is the stabilizer of 1 for the above described action of $L \times_Q T$ on L.*

(4) *The set $L \times \mathbf{1} := \{(a, \mathbf{1}); a \in L\}$ is a transversal of $L \times_Q T/1 \times T$. The map $L \to L \times \mathbf{1}$; $a \mapsto (a, \mathbf{1})$ is an isomorphism of left loops.*

As a simple consequence we obtain

Theorem 2.2 *Let L be a left loop, and let $T \neq \{\mathbf{1}\}$ be a fixed point free transassociant of L. Then $(L \times_Q T, L)$ is a Frobenius group.*

PROOF. By (2.1.2) $L \times_Q T$ acts transitively on L. From (2.1.3) we know that $1 \times T$ is the stabilizer of 1. Since this group is non-trivial, the action is not regular.

Take an element $(a, \alpha) \in L \times_Q T$ which has two fixed points. Since the action is transitive there is no loss in generality, to assume that one of the fixed points is 1. Then $a = 1$ and $\alpha(x) = x$ for some $x \in L \setminus \{1\}$. By hypothesis, $\alpha = \mathbf{1}$, and the only element with more than one fixed point is $(1, \mathbf{1})$, the identity.

A left loop L is said to satisfy the *left inverse property* if $\lambda_a^{-1} \in \lambda(L) = \{\lambda_x\,;\, x \in L\}$ for all $a \in L$. In this case every element $a \in L$ has a unique (left and right) inverse a^{-1}, and $\lambda_a^{-1} = \lambda_{a^{-1}}$, as well as $\delta_{a,a^{-1}} = \mathbf{1}$ (see [19, I.4, p. 20,21]).

L is called a (left) *Bol loop* if it satisfies the (left) *Bol-identity*

$$a(b \cdot ac) = (a \cdot ba)c \quad \text{for all } a, b, c \in L.$$

Note that we use the *dot-convention* to save parenthesis, e.g., $a \cdot bc = a(bc)$. By [19, IV.6.3] Bol loops satisfy the left inverse property. Moreover, Sharma [21, Thm. 2] has shown that every Bol loop is in fact a loop, i.e., the equation $ya = b$ has the unique solution $y = a^{-1}(ab \cdot a^{-1})$.

A left loop with unique inverses is said to satisfy the *automorphic inverse property* if

$$(ab)^{-1} = a^{-1}b^{-1}.$$

A Bol loop with automorphic inverse property is called a *K-loop* (or sometimes a *Bruck loop*). Notice that an associative K-loop is an abelian group.

From [19, IV.6.5 p. 114] one can derive (see also [12, (6.1)]).

2.3 *In every Bol loop we have $\lambda_{a^k} = \lambda_a^k$ for all $k \in \mathbf{Z}$. Therefore $|a| = |\lambda_a|$.*

For later use we record

2.4 *Let L be a left loop such that $\mathcal{D}(L)$ acts fixed-point-freely on $L^{\#}$. If every element of L has unique inverses, i.e., $ab = 1 \implies ba = 1$, then L satisfies the left inverse property.*

PROOF. Let $a \in L$. We have $a = a \cdot a^{-1}a = aa^{-1} \cdot \delta_{a,a^{-1}}(a) = \delta_{a,a^{-1}}(a)$. The hypothesis enforces $\delta_{a,a^{-1}} = \mathbf{1}$. Thus L satisfies the left inverse property.

One can also show that if L satisfies the hypothesis of the preceding theorem, and is automorphic inverse, then L is also left alternative.

Now, we give a construction for left loops using certain sets of involutions in transitive permutation groups. This construction occurs in [10, §6] in a completely different context. It is used to coordinatize absolute spaces with K-loops. This approach has been further generalized by Gabrieli, Karzel in [6–8], and by Im, Ko in [9]. The following theorem and its proof are due to Karzel. It is also worked out in [12, (7.1)].

Theorem 2.5 *Let G be a group acting on a set P with $J = J(G) := \{\alpha \in G; \alpha^2 = \mathbf{1}\}$. For the fixed element $1 \in P$, denote the stabilizer by Ω. For a map $\mu : P \to J$ with $\mu_x(1) = x$ for all $x \in P$ define*

$$\lambda : P \to G;\ x \mapsto \lambda_x := \mu_x\mu_1 .$$

Then we have

(1) *$L := \lambda(P)$ is a transversal of G/Ω. The left loop L has unique inverses. The bijection λ allows to carry the left loop structure over to P. Then, the*

λ_x *are the left translations of* P*. Moreover,* $\mu_1(x) = x^{-1}$ *for all* $x \in P$*, where* x^{-1} *denotes the inverse of* x *in* P*.*

(2) L *is a loop if and only if the set* $\mu(P)$ *acts regularly on* P*.*

(3) *The following are equivalent*

  (I) $\mu_1\mu(P)\mu_1 \subseteq \mu(P)$*;*

  (II) $\mu_1\mu_x\mu_1 = \mu_{x^{-1}}$ *for all* $x \in P$*;*

  (III) L *has the automorphic inverse property;*

  (IV) L *has the left inverse property.*

(4) L *is a K-loop if and only if* $\mu_x\mu(P)\mu_x \subseteq \mu(P)$ *for all* $x \in P$*.*

3 Frobenius Groups

In the case we are most interested in, the Frobenius group G has no kernel, and is therefore not a semidirect product. However, transversals, and the quasidirect product serve as a substitute:

Theorem 3.1 *Let* G *be a Frobenius group acting on a set* P*. For a fixed* $e \in P$ *let* Ω *be the stabilizer of* e *and let* L *be a transversal. Then*

(1) Ω *acts faithfully on* L *by*

$$\begin{cases} \Omega \times L \to L; \\ (\omega, a) \mapsto \theta_\omega(a), \quad \textit{where } \omega a\Omega \cap L = \{\theta_\omega(a)\}. \end{cases}$$

Therefore Ω *can be viewed as a subgroup of* $\mathcal{S}_L$*. In this sense,* Ω *is a transassociant of* L*.*

(2) *The map* $\phi : L \to P$*;* $\alpha \mapsto \alpha(e)$ *is bijective, and the map*

$$G \to L \times_Q \Omega;\ g \mapsto (a, a^{-1}g), \quad \textit{where } a = \phi^{-1}(g(e))$$

is an isomorphism. In fact, this map induces an equivalence of the permutation representations of G *on* P *and of* $L \times_Q \Omega$ *on* L *via* ϕ*.*

PROOF. (1) Clearly, θ defines an action. If $\theta_\omega = \mathbf{1}$, then $a^{-1}\omega a \in \Omega$ for all $a \in L$, hence (II) of (1.1) shows $\omega = 1$. Therefore, the action is faithful.

(2) ϕ is bijective, because G acts transitively, and L is a transversal.

Let $g \in G$, and $p \in P$. We'll show that $\phi(a, a^{-1}g)\phi^{-1}(p) = g(p)$, where $a = \phi^{-1}(g(e))$. Notice that only this choice of a puts $a^{-1}g$ inside Ω. Take $\alpha \in L$ with $\alpha(e) = p$, then we an compute

$$\phi(a, a^{-1}g)\phi^{-1}(p) = \phi(a, a^{-1}g)\alpha = \phi(a \circ \theta_{a^{-1}g}(\alpha)).$$

Now $(a \circ \theta_{a^{-1}g}(\alpha))\,\Omega = aa^{-1}g\alpha\Omega = g\alpha\Omega$. Therefore,

$$\phi(a \circ \theta_{a^{-1}g}(\alpha)) = g\alpha(e) = g(p).$$

This shows that ϕ induces the stated equivalence, which in turn implies the remaining statements.

Remarks 3.2 **1.** *This is a straightforward generalization of the construction [16, (7.2)] for loops. It also generalizes* (I) $\Longrightarrow$ (II) *in (1.1).*

2. *If L is a loop, then clearly all elements of $L \times \mathbf{1}$ act fixed-point-freely on L. This need not be the case, when L is only a left loop. Indeed, put $L := \mathbf{Z}_3$ and define*

$$a \circ b := \begin{cases} b & \text{if } a = 0 \\ a - b & \text{if } a \neq 0, \end{cases}$$

then $(L, \circ)$ is a left loop such that $\mathcal{D}(L) = \{\pm\mathbf{1}\}$ is a fixed point free transassociant. Thus $L \times_Q \mathcal{D}$ is a Frobenius group. It is easy to see that the elements of $L \times \mathbf{1} \setminus \{(0, \mathbf{1})\} = \{(1, \mathbf{1}), (2, \mathbf{1})\}$ are involutions, which necessarily have fixed points. In fact, $L \times_Q \mathcal{D}$ is isomorphic to $\mathcal{S}_3$. Of course, $\mathcal{S}_3$ does have a kernel, namely $\langle(1, 2, 3)\rangle$, so in a sense we only chose the transversal in a silly way.

3. *There are examples where one cannot make a better choice. Specifically, according to [4] (see also [3, p. 205]), there exists a Frobenius group G with $G = \bigcup_{g \in G} g\Omega g^{-1}$ (Ω a one point stabilizer). Therefore G contains no fixed point free elements at all. Here no transversal of G/Ω can be a loop.*

4 Involutions

We are now ready to introduce our new class of Frobenius groups. We also present the connection with other approaches, in particular with [5]. First a simple lemma.

4.1 *Let (G, P) be a Frobenius group such that the set $J = J(G)$ acts transitively on P. Then for all $x, y \in P$ with $x \neq y$ there exists a unique element $\alpha \in J$ with $\alpha(x) = y$.*

PROOF. The existence of α is an assumption. If $\beta \in J$ also has the property $\beta(x) = y$, then

$$\alpha\beta(x) = \alpha(y) = x \quad \text{and} \quad \alpha\beta(y) = y, \quad \text{hence} \quad \alpha\beta = 1,$$

since G is a Frobenius group. Therefore $\alpha = \beta$ is unique.

A Frobenius group (G, P) is said to have *many involutions* if $J = J(G)$ acts transitively on P, and if the one point stabilizers contain at most one involution. Note that since all one point stabilizers are conjugate, they all contain the same number of involutions.

Theorem 4.2 *Let (G, P) be a Frobenius group with many involutions. For a fixed $e \in P$ let Ω be the stabilizer of e.*

(1) *There exists a unique map $\mu : P \to J$ with $\mu_x(e) = x$ for all $x \in P$ such that $\mu(P) = J \setminus \{1\}$ or $\mu(P) = J$. The two cases are mutually exclusive.*
(2) *$\mu(P)$ acts regularly on P.*
(3) *$L := \mu(P)\mu_e$ is a transversal of G/Ω, which is a K-loop.*
(4) *$\mu(P) = J \setminus \{1\}$ if and only if every involution in G has exactly one fixed point. In this case we have $\mathcal{C}_G(\mu_e) = \Omega$, and L contains no elements of order 2.*
(5) *If $\mu(P) = J$, then $L = J$ is of exponent two, and every involution is fixed point free.*

PROOF. (1) Let $x \in P$. If $x \neq e$, there exists $\mu_x \in J$ with $\mu_x(e) = x$, which is unique by (4.1). If $e = x$, then we consider two cases: Firstly, assume that there is no involution in Ω, i.e., no involution has a fixed point, then $\mu_e = 1$, and $\mu(P) = J$. Secondly, assume that there exists a (unique!) involution $\alpha \in \Omega$, then put $\mu_e := \alpha$. Through this choice, the map μ has the desired properties. Note that there is no other way to do it — μ is unique.

(2) We have seen that Ω contains the involution μ_e if $\mu(P) \neq J$. This involution has the fixed point e. Then $\mu_x\mu_e\mu_x$ has fixed point $x \in P$. Thus for every $x, y \in P$ there exists $\alpha \in \mu(P)$ with $\alpha(x) = y$, even if $x = y$ and $1 \notin \mu(P)$.

The uniqueness of α comes from (4.1).

(3) is a direct consequence of (1), (2) and (2.5), since $\mu(P)$ is obviously invariant.

(4) The first statement is clear from the proof of (1). It follows immediately from (1.1), (II), that $\mathcal{C}_G(\mu_e) \subseteq \Omega$. For $\omega \in \Omega$ the element $\omega\mu_e\omega^{-1}$ is an involution with fixed point e. Therefore $\omega\mu_e\omega^{-1} = \mu_e$, and $\omega \in \mathcal{C}_G(\mu_e)$.

(5) is clear from the previously proved statements, and from (2.3).

In the case where L is of exponent 2 (see (5) in the preceding theorem), the Frobenius group G (with many involutions) is said to have *characteristic* 2, in symbols $\operatorname{char} G = 2$. Otherwise, we write $\operatorname{char} G \neq 2$. The latter is the situation where involutions have fixed points.

Theorem 4.3 *Let G be a sharply 2-transitive group, then G is a Frobenius group with many involutions. The K-loop constructed in (4.2) is isomorphic to the additive loop of the corresponding neardomain F.*[3] *Moreover,* $\operatorname{char} G = 2 \iff \operatorname{char} F = 2$.

PROOF. Clearly, G is a Frobenius group. [22, V.2, (i), (ii), (iv), pp. 229f] says that G has many involutions.

5 Characteristic 2

In the case of characteristic 2 we obtain a satisfactory converse. At the same time, we get a considerable strengthening of (2.4) for loops of exponent 2.

Theorem 5.1 *Let L be a loop of exponent 2, and let T be a fixed point free transassociant. Then T contains no involutions, $G := L \times_Q T$ is a Frobenius group with many involutions and* $\operatorname{char} G = 2$. *Moreover, L is a K-loop, and $T \subseteq \operatorname{Aut} L$.*

PROOF. G is a Frobenius group by (2.2), and the set $L \times \mathbf{1}$ acts transitively. By (2.4) L satisfies the left inverse property. Thus for all $a \in L$

$$\delta_{a,a} = \mathbf{1} \quad \text{and so} \quad (a, \mathbf{1})(a, \mathbf{1}) = (a^2, \delta_{a,a}) = (1, \mathbf{1}).$$

Therefore, $L \times \mathbf{1}$ consists of involutions and the identity.

We identify T and $1 \times T$ according to (2.1). Assume, there exists an involution $\alpha \in T$. Then for $x \in L \setminus \{1\}$ we have $\alpha(x) = y \neq x$. Take $a \in L$ with $ax = y$. Then $g := (a, \mathbf{1}) \in G$ is an involution, and $g(x) = y$. By (4.1) we arrive at the contradiction $g = \alpha$. Therefore T contains no involutions. This also implies that G is a Frobenius group with many involutions, and $\operatorname{char} G = 2$.

By (4.2.3) $L \times \mathbf{1}$ is a K-loop, which is isomorphic to L (2.1.4). For $a \in L$, $\gamma \in T$ we have

$$(1, \gamma)(a, \mathbf{1})(1, \gamma)^{-1} = (\gamma(a), \chi(a, \gamma)) \in L \times \mathbf{1},$$

since the conjugate of an involution is an involution. Therefore, $\chi(a, \gamma) = \mathbf{1}$ for all $a \in L$, and γ is an automorphism. Hence $T \subseteq \operatorname{Aut} L$.

[3] For this correspondence, and the definition of the characteristic of a neardomain the reader is referred to [22, V §1,2].

It would be desirable to relax the definition of "Frobenius group with many involutions" to the sole requirement that the set J acts transitively. This is possible in the previously discussed special case.

Theorem 5.2 *Let (G, P) be a Frobenius group, and take Ω to be the stabilizer of a fixed $e \in P$. If there exists a subset F of $J(G)$, $1 \in F$, which acts fixed-point-freely and transitively, then G is a Frobenius group with many involutions and* $\operatorname{char} G = 2$. *Furthermore, $F = J(G)$, and F is a transversal of G/Ω, which is a K-loop.*

PROOF. From (4.1) and the assumption we conclude that F acts regularly. Therefore, F is a transversal of G/Ω. We'll show that F is a loop: Let $\alpha, \beta \in F$, then there exists exactly one $\gamma \in F$ with $\gamma(\alpha(e)) = \beta(e)$. Hence $\gamma\alpha \in \beta\Omega$, and $\gamma \circ \alpha = \beta$. Thus the equation $\xi \circ \alpha = \beta$ has the unique solution $\xi = \gamma$, and F is a loop.

By (3.1) Ω is a fixed point free transassociant of F, and $G = F \times_Q \Omega$. By (2.4) F fulfills the left inverse property. Therefore, F is of exponent 2. Now (5.1) applies, and gives all the assertions.

An analogous theorem is true for all finite Frobenius groups (see [18, 18.1, p. 193]). No decisive results have been achieved in cases not mentioned in the theorem. So we do not know if there exist Frobenius groups with "too many" involutions, i.e., such that the set J acts transitively, and a one point stabilizer contains more than one involution.

6 Characteristic not 2 and Specific Groups

We now turn to the case of characteristic $\neq 2$. Here we get only a partial converse. A left loop is called *uniquely 2-divisible* if the map $x \mapsto x^2$ is a bijection. In this case, we denote the unique solution of $y^2 = x$ by $y = x^{1/2}$. Notice that the map $x \mapsto x^{1/2}$ is the inverse of the square map. By (2.3) the usual laws for powers (including ${}^{1/2}$) hold in uniquely 2-divisible K-loops. We'll make use of this in the proof of the following

Theorem 6.1 *Let L be a uniquely 2-divisible K-loop, and let Ω be a fixed point free subgroup of* $\operatorname{Aut} L$ *which contains $\mathcal{D}(L)$ and an involution ι. Then*

(1) *$\iota(x) = x^{-1}$ for all $x \in L$.*
(2) *$L \times_Q \Omega$ is a Frobenius group with many involutions of characteristic $\neq 2$.*
(3) *(a, ι) is an involution, and $\lambda_a(x) = (a, \iota)(1, \iota)(x)$ for all $a, x \in L$.*

PROOF. (1) First, we claim

$$a(a^{-1}b)^{1/2} = b(b^{-1}a)^{1/2} \quad \text{for all} \quad a, b \in L.$$

By [16, (2.6)] we have $\delta_{b^{-1},a}(a^{-1}b) = (b^{-1}a)^{-1}$, hence we can compute

$$b^{-1} \cdot a(a^{-1}b)^{1/2} = b^{-1}a \cdot \delta_{b^{-1},a}(a^{-1}b)^{1/2} = b^{-1}a \cdot (b^{-1}a)^{-1/2} = (b^{-1}a)^{1/2}.$$

Multiplying with b on both sides gives the claim. Using this, we find

$$\iota\Big(x(x^{-1}\iota(x))^{1/2}\Big) = \iota(x)(\iota(x)^{-1}x)^{1/2} = x(x^{-1}\iota(x))^{1/2}.$$

Therefore the assumption implies $x(x^{-1}\iota(x))^{1/2} = 1$. By the left inverse property we get $x^{-1}\iota(x) = x^{-2}$, and then $\iota(x) = x^{-1}$.

(2) and (3): Ω, sitting inside $\operatorname{Aut} L$, is a transassociant. Therefore, (2.2) shows that $L \times_Q \Omega$ is a Frobenius group. By (1) Ω contains exactly one involution.

A simple calculation shows that (a, ι) is an involution in $L \times_Q \Omega$. Therefore, the involutions act transitively. More specific: Let $x, y \in L$ and choose $a \in L$ such that $ax^{-1} = y$. Then $(a, \iota)(x) = y$.

Finally, $(a, \iota)(1, \iota)(x) = a(x^{-1})^{-1} = ax = \lambda_a(x)$.

The following lemma shows that the presence of an involution in Ω is not a strong assumption. Indeed, it can always be achieved.

6.2 *Let L be a K-loop, and let Ω be a subgroup of $\operatorname{Aut} L$ with $\iota \notin \Omega$. Then*

$$\langle \Omega \cup \{\iota\} \rangle = \Omega \times \langle \iota \rangle \, .$$

If L is uniquely 2-divisible, and Ω acts fixed-point-freely, then so does $\Omega \times \langle \iota \rangle$.

PROOF. Since ι centralizes every automorphism, the first statement is clear.

Now, assume that L is uniquely 2-divisible, and Ω acts fixed-point-freely. Every element of $\Omega \times \langle \iota \rangle$ is of the form ω or $\omega\iota$ for some $\omega \in \Omega$. If this element is not the identity, and has a fixed point $x \in L \setminus \{1\}$, then it must be of the form $\omega\iota$, because of the assumption. Now $x = \omega\iota(x) = \omega(x^{-1})$ implies $\omega^2(x) = x$. Therefore $\omega^2 = \mathbf{1}$. By (6.1.1) we necessarily have $\omega = \mathbf{1}$, since $\omega = \iota$ contradicts our hypothesis. However, $x = \iota(x) = x^{-1}$ is a contradiction, too, because a uniquely 2-divisible loop cannot have elements of order 2.

Remarks 6.3 **1.** *If L is a uniquely 2-divisible K-loop, then Ω (as in (6.1)) cannot contain more than one involution by (6.1.1). If Ω happens to have* *no involution, then $L \times_Q \Omega$ can be embedded into the Frobenius group $L \times_Q (\Omega \times \langle \iota \rangle)$.*

2. *One can relax the hypothesis of (6.1) to a left power alternative, left A_ℓ-loop. Then it follows that L is a K-loop (see [12, (7.10)].) This and (1) of the theorem are due to* Kist *[13, (1.2.e), (1.4.b)].*

3. *There exist examples of K-loops with $\iota \notin \mathcal{D}(L)$. It seems to be open whether ι is always an element of $\mathcal{D}(L)$ when L is uniquely 2-divisible. Of course, this would make the hypothesis about ι in (6.1) redundant.*

4. *The converse in theorem (6.1) is only partial, because we need the assumption that L be uniquely 2-divisible. (4.2.4) together with [15, (1.4)] only implies that the map $x \mapsto x^2$ is injective. Surjectivity is missing. There are in fact counterexamples arising from [11].*

Theorem 6.4 *Let G be a group with set of involutions $J^\#$, and let Ω be a subgroup of G. The following are equivalent*

(I) *$(G, G/\Omega)$ is a Frobenius group with many involutions, and* $\operatorname{char} G \neq 2$*;*

(II) *$\Omega = \mathcal{C}_G(\mu)$ for some $\mu \in J^\#$, and for all $\alpha, \beta \in J^\#, \alpha \neq \beta$, there exists $\gamma \in J^\#$ with $\gamma\alpha\gamma = \beta$, and $\mathcal{C}_G(\alpha) \cap \mathcal{C}_G(\beta) = \{1\}$.*

If (II) *holds, then γ is uniquely determined given α and β.*

PROOF. (I) $\Longrightarrow$ (II): From (4.2.4) we have that $\Omega = \mathcal{C}_G(\mu)$ for some $\mu \in J^\#$.

By assumption, α and β each have exactly one fixed point, denote them by x, y, respectively, i.e., $\alpha x = x$, $\beta y = y$. Since the involutions act transitively, there exists $\gamma \in J^\#$ with $\gamma x = y$. Now, $\gamma\alpha\gamma$ is an involution with fixed point y. Since there is only one such involution, we must have $\gamma\alpha\gamma = \beta$.

Finally, there exist $g, h \in G$ with $g\mu g^{-1} = \alpha$, $h\mu h^{-1} = \beta$, therefore, using (II) from (1.1), we find

$$\mathcal{C}_G(\alpha) \cap \mathcal{C}_G(\beta) = \mathcal{C}_G(g\mu g^{-1}) \cap \mathcal{C}_G(h\mu h^{-1}) = g\Omega g^{-1} \cap h\Omega h^{-1} = \{1\}.$$

Indeed, the case $h^{-1}g \in \Omega$ leads to $\alpha = \beta$, which is excluded by the hypothesis.

(II) $\Longrightarrow$ (I): All the conjugates of Ω are centralizers of involutions, thus (II) of (1.1) is satisfied, and $(G, G/\Omega)$ is a Frobenius group.

To see that $J^\#$ acts transitively, take $X, Y \in G/\Omega$. The stabilizer of $\Omega \in G/\Omega$ is Ω itself, and contains the involution μ. Hence every one point stabilizer contains an involution, because they are conjugate to Ω. In particular, there exists an involution α with fixed point X. Also, there exists an element $g \in G$ with $gX = Y$. Now $g\alpha g^{-1}$ is an involution, therefore there exists $\gamma \in J^\#$ with $\gamma(g\alpha g^{-1})\gamma = \alpha$. Hence $\gamma g\alpha = \alpha\gamma g$, and $\gamma g \in \mathcal{C}_G(\alpha)$, the stabilizer of X. This implies $\gamma \in g\,\mathcal{C}_G(\alpha)$, and so $\gamma X = Y$. Thus $J^\#$ acts transitively.

The assumption clearly implies that there is at most one involution in $\mathcal{C}_G(\alpha)$. Thus the Frobenius group G has many involutions. Clearly, $\operatorname{char} G \neq 2$.

For the final statement, let $\gamma\alpha\gamma = \beta$. If $\alpha = \beta$, then $\gamma\alpha\gamma\alpha = 1$. Hence γ and α commute, i.e., $\gamma \in \mathcal{C}_G(\alpha)$, therefore $\gamma = \alpha$. This shows uniqueness in the case of equality.

If $\alpha \neq \beta$, let $\gamma' \in J^\#$ also have the property $\gamma'\alpha\gamma' = \beta$. Then

$$\gamma\alpha\gamma = \gamma'\alpha\gamma' \implies \gamma'\gamma\alpha = \alpha\gamma'\gamma \quad \text{and, similarly} \quad \gamma'\gamma\beta = \beta\gamma'\gamma.$$

Therefore, $\gamma'\gamma \in \mathcal{C}_G(\alpha) \cap \mathcal{C}_G(\beta) = \{1\}$, and $\gamma = \gamma'$ is unique.

The proof of the preceding theorem has been inspired by [5], in particular § 3.2, p. 20f. Besides its intrinsic interest, this theorem will serve to establish the connection with Gabriel's thesis.

In [5], Gabriel has introduced *specific groups* in order to axiomatize the subgroup of a sharply 2-transitive group generated by the set of involutions. We give a generalized notion, which includes the whole sharply 2-transitive group, and more. A group G is called *specific* if the set $J^\#$ of involutions in G has at least 2 elements, i.e., $|J^\#| \geq 2$, and there exists $p \in \mathbf{N} \cup \{\infty\}$ such that for all $\alpha, \beta \in J^\#$ with $\alpha \neq \beta$, there is a $\gamma \in J^\#$ with

$$|\alpha\beta| = p, \quad \gamma\alpha\gamma = \beta \quad \text{and} \quad \mathcal{C}_G(\alpha) \cap \mathcal{C}_G(\beta) = \{1\}.$$

Notice that G cannot be abelian, since two distinct involutions do not commute. As mentioned already, our definition slightly deviates from Gabriel's. In fact, in [5, p. 6] it is required that G is non-abelian, and that $J^\#$ generates G. It is then derived that there are at least 2 involutions. This is used to show that p is either an odd prime, or ∞, [5, 2.4, p. 6]. Therefore, this statement holds in our more general situation. Call

$$\operatorname{char} G := \begin{cases} 0 & \text{if } p = \infty \\ p & \text{otherwise} \end{cases}$$

the *characteristic* of G. We therefore have

6.5 *The characteristic of a specific group is* 0 *or an odd prime.*

We now hook up the notion of a specific group to the main subject of this section.

Theorem 6.6 *Let G be a specific group of characteristic p. If μ is an involution, then $(G, G/\mathcal{C}_G(\mu))$ is a Frobenius group with many involutions, and* $\operatorname{char} G \neq 2$. *Moreover, the K-loop L coming with G is of exponent p, or in case $p = 0$, every element in $L^\#$ has infinite order.*

PROOF. Let $J^{\#}$ be the set of involutions. By (6.4) G is a Frobenius group with many involutions. For $a \in L$ we have $\lambda_a = \alpha\beta$ with $\alpha, \beta \in J^{\#}$ from (4.2.3). Therefore $|\lambda_a| = p$, or $|\lambda_a| = \infty$, according to the cases in the hypothesis. But $|\lambda_a|$ is the order of a in L, by (2.3).

This theorem has the following converse

Theorem 6.7 *Let L be a K-loop, and let Ω be a fixed point free subgroup of* $\mathrm{Aut}\, L$ *which contains $\mathcal{D}(L)$ and ι. Put $G := L \times_Q \Omega$.*

(1) *If L is of exponent p, where p is an odd prime, then G is a specific group with* $\mathrm{char}\, G = p$.
(2) *If every element in $L^{\#}$ has infinite order, then G is a specific group with* $\mathrm{char}\, G = 0$.
(3) *In both cases (1) and (2), $\Omega = \mathcal{C}_G(\iota)$.*

PROOF. The assumptions about the orders of elements in L imply that L is uniquely 2-divisible. Thus by (6.1) $(G, G/\Omega)$ is a Frobenius group with many involutions. The statement about $|\alpha\beta|$, also, comes directly from (6.1) and the hypothesis in both cases. From (4.2.4) we infer that $\Omega = \mathcal{C}_G(\iota)$. Now (6.4) gives all the remaining conditions for specific groups.

Remarks 6.8 **1.** *In the preceding theorem, the involutions generate G if and only if $\Omega = \langle \mathcal{D} \cup \{\iota\} \rangle$, the minimal possible choice for Ω satisfying the hypothesis if Ω acts fixed-point-freely.*

2. *The fact that specific groups are Frobenius groups has been observed by* Gabriel *in [5, 4.2, p. 27]. He has used the phrase "generalized Frobenius group" to denote what we call a Frobenius group.*

3. *Let G be a sharply 2-transitive group with the corresponding neardomain $(F, +, \cdot)$ of characteristic $p \neq 2$, and let K be the subgroup of G generated by the involutions in G. Then every subgroup of G containing K is a specific group of characteristic p. In* Gabriel*'s setting only K qualifies, see [5, 2.4, p. 20] and remark 1 above. This shows the extent to which* Gabriel*'s notion has been generalized.*

4. *Examples of specific groups of characteristic 0 and 2 abound. They can be obtained from a construction of* Kolb *and* Kreuzer *[14]. If one uses a field F of characteristic 2, replacing the complex numbers, then this construction gives K-loops of exponent 2, to which (5.1) applies. If* $\mathrm{char}\, F \neq 2$, *then one can show that the resulting K-loops give specific groups of characteristic 0.*

5. *It does not seem to be easy to come by specific groups in characteristic $p > 2$. The only examples knows to the author arise from infinite Burnside groups as described in [5, 5.18, p. 50].*

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