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Abstract

Samaga [6] proved that there are at least two classes of Minkowski planes with respect to the validity of special configurations of Miquel's Theorem. The only configuration with 6 to 8 points holding in the Moulton-Minkowski plane $\mathcal{M}_\gamma$ for $\gamma \neq 1$ is the 6-point proposition (BD_1M3)[1].
In this paper we study a similar statement for Laguerre planes. Here it is an 8-point theorem $(M2)$ that holds in a non-Miquelian Laguerre plane of shear type[2] and even in a non-embeddable Laguerre plane. For proving this, we establish a connection to a much simpler problem of linear algebra. Furthermore we prove in an incidence-geometrical way that even three other configurations[3] hold in these Laguerre planes.

Keywords nonlinear incidence geometry, chain geometries, Benz planes, Laguerre planes, Miquel's Theorem[4]

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[1]For the notation of Miquelian propositions see Samaga [6].
[2]Laguerre planes of shear type have been introduced by Löwen and Pfüller [3].
[3]These are the configurations (BD_2M2), (CC_2M1) and $(BBDDM1)$.
[4]AMS classification 51B05, 51B15.

1 Introduction

Van der Waerden and Smid formulated in [7] the so-called *full theorem of Miquel* (VM). A more general Miquelian theorem (VM^a) has been proved by Samaga [4] for non-minimal chain geometries $\Sigma(\mathbb{K},\mathbb{L})^5$.

(VM^a). Let $k_0, \ldots, k_4$ be circles and let $A, \ldots, H$ be points such that $E \neq F$. If the assumption

$$k_0 \cap k_i = \begin{cases} \{A\ ,\ B\} \\ \{B\ ,\ C\} \\ \{C\ ,\ D\} \\ \{D\ ,\ A\} \end{cases},\ k_i \cap k_{i+1} = \begin{cases} \{B\ ,\ F\} \\ \{C\ ,\ G\} \\ \{D\ ,\ H\} \\ \{A\ ,\ E\} \end{cases} \text{ for } \ i = \begin{cases} 1 \\ 2 \\ 3 \\ 4\ (5 \equiv 1) \end{cases}$$

holds, then there is a unique circle k such that

$$k \cap k_i = \begin{cases} \{E\ ,\ F\} \\ \{F\ ,\ G\} \\ \{G\ ,\ H\} \\ \{H\ ,\ E\} \end{cases} \text{ for } \ i = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \end{cases}.$$

Axiomatic Laguerre planes are defined as follows:
Let $\mathbb{P}$ be the set of *points* and $\mathbb{K} \subseteq \mathfrak{P}(\mathbb{P})$ be the set of *regular circles*. Furthermore let $\|$ be an equivalence relation on the point set and let $\mathbb{G}$ be the set of all equivalence classes of $\|$, called *generators*. Two points P,Q are called *parallel*, P$\|$Q, if they belong to the same generator. We call ($\mathbb{P}$, $\mathbb{K}$, $\mathbb{G}$) a *Laguerre plane* if the following five axioms are satisfied:

(L1) For every point P there exists exactly one generator g with P$\in g$.

(L2) For every pair $(k, g) \in \mathbb{K}\times\mathbb{G}$ there exists exactly one point P with P$\in k \cap g$.

(L3) Any three pairwise non-parallel points determine a regular circle uniquely.

(L4) There exists a regular circle k with $|k| \geq 3$. $\mathbb{P}\backslash k \neq \emptyset$, $\emptyset \notin \mathbb{K}$.

(L5) For non-parallel points P,Q and a regular circle k with P$\in k$ and Q $\notin k$ there exists exactly one regular circle l with Q $\in l$ and $k \cap l = \{$P$\}$.

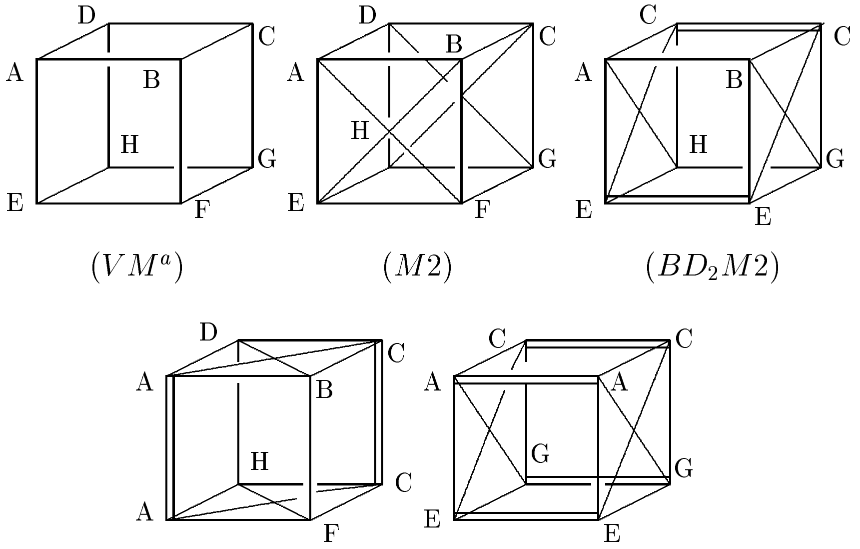
From now on we investigate axiomatic Laguerre planes and ask for all configurations that meet the conditions of (VM^a). We are free to let coincide any pair of points that forms an edge of our cube model (see Figure 1 (VM^a)). This will be denoted by a double-line. We are even allowed to assume any two points to be parallel ($\|$) if they are not lying on a common edge of the cube. Parallel points will be denoted by a diagonal line in the cube model. We have to fulfil the following two conditions

[5] For the definition of chain geometries see Benz [1].

- not any three or more vertices of the cube model represent the same point
- not any three or more points are pairwise parallel.

Working out all possible configurations, we get 39 Miquelian propositions[6]. As we are interested in some special cases, we figure out four of them:

($M2$). For eight different points A,...,H let $(\mathrm{ABCD})_r$, $(\mathrm{BCFG})_r$, $(\mathrm{ADEH})_r$ be regular circles[7] and let A$\|$F, B$\|$E, C$\|$H and D$\|$G. Then there exists a unique regular circle k=$(\mathrm{EFGH})_r$ (see Figure 1 ($M2$)).



(VM^a) ($M2$) (BD_2M2)

(CC_2M1) ($BBDDM1$)

(Figure 1)

(BD_2M2). Let A,B,C,E,G,H be six different points and let $(\mathrm{ABC})_r$, $(\mathrm{CGH})_r$, $(\mathrm{ABE})_r$ be regular circles such that $(\mathrm{ABC})_r$ touches $(\mathrm{CGH})_r$ in C and let A$\|$H, B$\|$G and C$\|$E. Then there exists a unique regular circle $(\mathrm{EGH})_r$ with $(\mathrm{ABE})_r\mathrm{E}(\mathrm{EGH})_r$ (see Figure 1 (BD_2M2)).

(CC_2M1). Let A,B,C,D,F,H be six different points and let A$\|$C and B$\|$D and $(\mathrm{ABF})_r\mathrm{A}(\mathrm{ADH})_r$ and $(\mathrm{BCF})_r\mathrm{C}(\mathrm{CDH})_r$. Then F$\|$H holds (see Figure 1 ($CC_2M1$)), i.e. $(\mathrm{ACFH})_i$.

[6] See Bröcker [2].

[7] $(\mathrm{ABCD})_r$ means that the points A,B,C and D lie on a regular circle. $(\mathrm{ABCD})_i$ means that the points A,B,C,D lie on an irregular circle, i.e. $(\mathrm{ABCD})_i \in \{g_1 \cup g_2 \mid g_1 \neq g_2;\ g_1, g_2 \in \mathbb{G}\}$. Note that an irregular circle is not uniquely determined by three points. Touching will be denoted by $k\mathrm{A}l$, for regular circles k, l and a point A such that k touches l in A.

($BBDDM1$). Let A,C,E,G be four different points and let A$\|$G and C$\|$E. Let k_0 be a regular circle containing the points A and C. Let k_1 be a regular circle through A and E touching k_0 in A. Let k_2 be a regular circle through C and G touching k_0 in C and let k_3 be a regular circle through E and G touching k_2 in G. Then k_1Ek_3 holds (see Figure 1 ($BBDDM1$)).

All 39 propositions of Miquel's Theorem hold in the above mentioned chain geometries $\Sigma(\mathbb{K}, \mathbb{L})$[8]. Van der Waerden and Smid proved in [7] that Laguerre planes are Miquelian iff they can be described as chain geometries.
For axiomatic Laguerre planes Samaga [5] showed that 8 of the 39 Miquelian propositions lead to (VM) with eight different points. Chen proved that (VM) with eight different points implies (VM)[9]. Thus, Laguere planes that fulfill one of the eight propositions can be coordinized by chain geometries.
We prove for a well-known non-Miquelian Laguerre plane that the four above mentioned propositions hold (see Figure 1).
A consequence of this fact is that we have at least two classes of Laguerre planes with respect to Miquelian propositions. One class – the *Miquelian class* – can be characterized by any of the nine propositions worked out in [5]. A second class is characterized by ($M2$). We start by constructing the fitting Laguerre plane.

2 A non-Miquelian Laguerre plane $\mathcal{E}$ of shear type

Let

$$\mathbb{P} = (\mathbb{R} \cup \{\infty\}) \times \mathbb{R}$$

be the set of *points* and

$$\mathbb{G} = \{\{x\} \times \mathbb{R} \mid x \in \mathbb{R} \cup \{\infty\}\}$$

be the set of *generators* (or *parallel classes*). For $a, b, c \in \mathbb{R}$,

$$k(a, b, c) = \{(x, ax^4 + bx + c) \mid x \in \mathbb{R}\} \cup \{(\infty, a)\}$$

we define the set of *regular circles* as

$$\mathbb{K} = \{k(a, b, c) \mid a, b, c \in \mathbb{R}\}.$$

Proposition 1. The incidence structure $\mathcal{E} = (\mathbb{P}, \mathbb{K}, \mathbb{G})$ is an ovoidal Laguerre plane. Furthermore $\mathcal{E}$ is not Miquelian.

Proof. We define the function

[8] This fact has been proved by using cross-ratios, see Benz [1], Samaga [4].
[9] See Benz [1].

$$f : \begin{cases} \mathbb{R} \cup \{\infty\} & \to & \mathbb{R} \\ x & \mapsto & x^4, \quad x \in \mathbb{R} \\ \infty & \mapsto & 1 \end{cases}$$

which is differentiable on $\mathbb{R}$ and satisfies the normalization conditions[10]

$$f(0) = f'(0) = 0, \quad f(1) = f(\infty) = 1.$$

f is strictly convex on $\mathbb{R}$ and its graph is

$$K = \{(x, f(x)) \mid x \in \mathbb{R} \cup \{\infty\}\}.$$

Define Ξ as the group of permutations of $\mathbb{P}$ with

$$\Xi = \{\sigma(a, s, v) \mid a, s, v \in \mathbb{R}, a \neq 0\} \text{ and}$$

$$(x, y)^{\sigma(a,s,v)} = (x, ay + sx + v), \quad a, s, v, x, y \in \mathbb{R}, a \neq 0,$$

$$(\infty, y)^{\sigma(a,s,v)} = (\infty, ay), \quad a, s, v, y \in \mathbb{R}, a \neq 0.$$

Then $(\mathcal{L} \backslash \mathbb{G}) \cup K^{\Xi}$ is the set of all regular circles where $\mathcal{L}$ is the set of all ordinary lines of the affine plane augmented by the point $(\infty, 0)$. By Löwen and Pfüller [3] we know that the incidence structure $\mathcal{E}$ – induced by f – is a Laguerre plane. Furthermore, by theorem 4 of [3] we know that the Laguerre plane is ovoidal but not Miquelian[11]. □

3 $(M2)$ holds in $\mathcal{E}$

We call

$$I = \{(\infty, r) \mid r \in \mathbb{R}\}$$

the *set of infinite points*. For the central theorem of this paper we need the following

Proposition 2. For three pairwise non-parallel points $X = (x_1, x_2)$, $Y = (y_1, y_2)$, $Z = (z_1, z_2) \in \mathbb{P}$ there exists a unique regular circle $k(m_1, m_2, m_3) \in \mathbb{K}$ such that $X, Y, Z \in k(m_1, m_2, m_3)$ holds[12]. m_1, m_2, m_3 are determined by

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[10]See Löwen and Pfüller [3].

[11]$\mathcal{E}$ is Miquelian iff $f(x) = x^2$ for $x \in \mathbb{R}$ and $f(\infty) = 1$.

[12]See chapter 2 for the definition of $k(a, b, c)$.

$$m_1 = \frac{(z_2 - x_2)(y_1 - x_1) - (y_2 - x_2)(z_1 - x_1)}{(z_1^4 - x_1^4)(y_1 - x_1) - (y_1^4 - x_1^4)(z_1 - x_1)},$$

$$m_2 = \frac{(z_1^4 - x_1^4)(y_2 - x_2) - (y_1^4 - x_1^4)(z_2 - x_2)}{(z_1^4 - x_1^4)(y_1 - x_1) - (y_1^4 - x_1^4)(z_1 - x_1)},$$

$$m_3 = \frac{x_1^4(y_2 z_1 - y_1 z_2) + y_1^4(x_1 z_2 - x_2 z_1) + z_1^4(x_2 y_1 - x_1 y_2)}{(z_1^4 - x_1^4)(y_1 - x_1) - (y_1^4 - x_1^4)(z_1 - x_1)}$$

if $X, Y, Z \notin I$ and

$$m_1 = x_2,$$

$$m_2 = \frac{z_2 - y_2 - x_2(z_1^4 - y_1^4)}{z_1 - y_1},$$

$$m_3 = \frac{y_2 z_1 - y_1 z_2 - x_2(y_1^4 z_1 - y_1 z_1^4)}{z_1 - y_1}$$

if $X = (\infty, x_2) \in I$ and $Y, Z \notin I$.

Proof. For X,Y,Z$\notin I$ consider the system of linear equations

$$\begin{pmatrix} x_1^4 & x_1 & 1 \\ y_1^4 & y_1 & 1 \\ z_1^4 & z_1 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}.$$

The numbers x_1, y_1, z_1 are pairwise distinct. Thus, this system has, after a few easy steps, the unique solution as mentioned above. X$\in I$ and Y,Z$\notin I$: By the definition of $k(m_1, m_2, m_3)$ we get $m_1 = x_2$. The remaining equation system is

$$\begin{pmatrix} y_1 & 1 \\ z_1 & 1 \end{pmatrix} \begin{pmatrix} m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} y_2 - x_2 y_1^4 \\ z_2 - x_2 z_1^4 \end{pmatrix}.$$

□

Theorem 1. (M2) holds in the Laguerre-plane $\mathcal{E}$, for $\mathcal{E}$ as defined in 2.

Proof. Let A=:(a_1, a_2),...,H=:(h_1, h_2) be eight different points with circles given as in the hypothesis of $(M2)$ (see Figure 1 $(M2)$). Note that (a_1, a_2)=A$\|$F=(f_1, f_2) implies $a_1 = f_1$. Analogously, we get $b_1 = e_1, c_1 = h_1$ and $d_1 = g_1$. Since E,F,G are pairwise non-parallel points, we get (EFG)$_r$. We have to show H$\in$(EFG)$_r$. This can be done in seven steps:

1. calculate the circle $k_0 := k(m_1, m_2, m_3) := (\mathrm{ABC})_r$ by using Proposition 2,
2. for given d_1 calculate d_2,
3. calculate the circle $k_1 := k(m_4, m_5, m_6) := (\mathrm{BCF})_r$,
4. for given g_1 calculate g_2,
5. calculate the circle $k_2 := k(m_7, m_8, m_9) := (\mathrm{ADE})_r$,
6. calculate the circle $k_3 := k(m_{10}, m_{11}, m_{12}) := (\mathrm{EFG})_r$,
7. calculate the point set $(\mathrm{ADE})_r \cap (\mathrm{EFG})_r$.

We distinguish two cases:

1. none of the points A,...,H are elements of I,
2. $\mathrm{A} \in I$.

All other cases run as case 2 by symmetry. Starting at case 1, step 1, and using Proposition 2 we determine the regular circle $(\mathrm{ABC})_r$ by

$$m_1 = \frac{(c_2 - a_2)(b_1 - a_1) - (b_2 - a_2)(c_1 - a_1)}{(c_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(c_1 - a_1)},$$

$$m_2 = \frac{(c_1^4 - a_1^4)(b_2 - a_2) - (b_1^4 - a_1^4)(c_2 - a_2)}{(c_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(c_1 - a_1)},$$

$$m_3 = \frac{a_1^4(b_2 c_1 - b_1 c_2) + b_1^4(a_1 c_2 - a_2 c_1) + c_1^4(a_2 b_1 - a_1 b_2)}{(c_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(c_1 - a_1)}.$$

Step 2. Let $d_1 \in \mathbb{R}$. D is a point on $(\mathrm{ABC})_r \Rightarrow d_2 = m_1 d_1^4 + m_2 d_1 + m_3$.
Steps 3 and 4 are similar to steps 1 and 2.

$$m_4 = \frac{(c_2 - f_2)(b_1 - a_1) - (b_2 - f_2)(c_1 - a_1)}{(c_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(c_1 - a_1)},$$

$$m_5 = \frac{(c_1^4 - a_1^4)(b_2 - f_2) - (b_1^4 - a_1^4)(c_2 - f_2)}{(c_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(c_1 - a_1)},$$

$$m_6 = \frac{a_1^4(b_2 c_1 - b_1 c_2) + b_1^4(a_1 c_2 - f_2 c_1) + c_1^4(f_2 b_1 - a_1 b_2)}{(c_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(c_1 - a_1)}.$$

This determines the circle $(\mathrm{BCF})_r$ and for g_2 we get $g_2 = m_4 d_1^4 + m_5 d_1 + m_6$.
Steps 5 and 6. The circles $(\mathrm{ADE})_r$ and $(\mathrm{EFG})_r$ are determined by

$$m_7 = \frac{(d_2 - a_2)(b_1 - a_1) - (e_2 - a_2)(d_1 - a_1)}{(d_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(d_1 - a_1)},$$

$$m_8 = \frac{(d_1^4 - a_1^4)(e_2 - a_2) - (b_1^4 - a_1^4)(d_2 - a_2)}{(d_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(d_1 - a_1)},$$

$$m_9 = \frac{a_1^4(e_2 d_1 - b_1 d_2) + b_1^4(a_1 d_2 - a_2 d_1) + d_1^4(a_2 b_1 - a_1 e_2)}{(d_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(d_1 - a_1)}$$

and

$$m_{10} = \frac{(g_2 - f_2)(b_1 - a_1) - (e_2 - f_2)(d_1 - a_1)}{(d_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(d_1 - a_1)},$$

$$m_{11} = \frac{(d_1^4 - a_1^4)(e_2 - f_2) - (b_1^4 - a_1^4)(g_2 - f_2)}{(d_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(d_1 - a_1)},$$

$$m_{12} = \frac{a_1^4(e_2 d_1 - b_1 g_2) + b_1^4(a_1 g_2 - f_2 d_1) + d_1^4(f_2 b_1 - a_1 e_2)}{(d_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(d_1 - a_1)},$$

respectively. Now we prove that the two circles $(\mathrm{ADE})_r$ and $(\mathrm{EFG})_r$ have H in common. We ask for the solutions of the equation

$$(m_{10} - m_7)x^4 + (m_{11} - m_8)x + (m_{12} - m_9) = 0. \tag{1}$$

One solution is b_1 because E is a common point of $(\mathrm{ADE})_r$ and $(\mathrm{EFG})_r$, i.e.

$$(m_{10} - m_7)b_1^4 + (m_{11} - m_8)b_1 + (m_{12} - m_9) = 0. \tag{2}$$

By using the above mentioned equations we get

$$m_{11} - m_8 = \frac{(d_1^4 - a_1^4)(a_2 - f_2) - (b_1^4 - a_1^4)(g_2 - d_2 + a_2 - f_2)}{(d_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(d_1 - a_1)},$$

$$m_{10} - m_7 = \frac{(b_1 - a_1)(g_2 - d_2 + a_2 - f_2) - (d_1 - a_1)(a_2 - f_2)}{(d_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(d_1 - a_1)},$$

$$g_2 - d_2 = \frac{(f_2 - a_2)[d_1(b_1^4 - c_1^4) + c_1(d_1^4 - b_1^4) + b_1(c_1^4 - d_1^4)]}{(c_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(c_1 - a_1)}.$$

It is easy to verify that

$$g_2 - d_2 = \frac{(f_2 - a_2)[b_1(c_1^4 - b_1^4) - d_1(c_1^4 - b_1^4) + d_1^4(c_1 - b_1) - b_1^4(c_1 - b_1)]}{(b_1 - a_1)(c_1^4 - b_1^4) - (b_1^4 - a_1^4)(c_1 - b_1)}$$

holds. We multiply this equation by the denominator and get

$$(g_2 - d_2)[(b_1 - a_1)(c_1^4 - b_1^4) - (b_1^4 - a_1^4)(c_1 - b_1)] =$$

$$= (f_2 - a_2)[d_1(b_1^4 - c_1^4) + c_1(d_1^4 - b_1^4) + b_1(c_1^4 - d_1^4)].$$

This equation leads to

$$(b_1 - a_1)(g_2 - d_2 + a_2 - f_2)(c_1^4 - b_1^4) - (d_1 - a_1)(a_2 - f_2)(c_1^4 - b_1^4) +$$

$$+(d_1^4 - a_1^4)(a_2 - f_2)(c_1 - b_1) - (b_1^4 - a_1^4)(g_2 - d_2 + a_2 - f_2)(c_1 - b_1) = 0.$$

Dividing the equation by $(d_1^4 - a_1^4)(b_1 - a_1) - (b_1^4 - a_1^4)(d_1 - a_1)$ yields to

$$(m_{10} - m_7)(c_1^4 - b_1^4) + (m_{11} - m_8)(c_1 - b_1) = 0$$

which is equivalent to

$$(m_{10} - m_7)c_1^4 + (m_{11} - m_8)c_1 + (m_{12} - m_9) = (m_{10} - m_7)b_1^4 + (m_{11} - m_8)b_1 + (m_{12} - m_9).$$

By using (2) we get $(m_{10} - m_7)c_1^4 + (m_{11} - m_8)c_1 + (m_{12} - m_9) = 0$.
This means that c_1 is a solution of (1). Together, there is a point H$'\|$H with H$' \in$(ADE)$_r\cap$(EFG)$_r$. Since H$\in$(ADE)$_r$ holds, we get H=H$'$.

Case 2. Let A $\in I$ be the infinite point $(\infty, m), m \in \mathbb{R}$. Since A$\|$F and A$\neq$F, we have a second infinite point F=(∞, n) with $m \neq n$. Note that every regular circle in our Laguerre plane has exactly one point of this kind. Therefore all other points on the circles (ABCD)$_r$, (ADEH)$_r$, (BCFG)$_r$ and (EFG)$_r$ have to be points not in I. For (ABCD)$_r = k(m_1, m_2, m_3)$ the number m_1 is determined by A. Analogously, for (ADEH)$_r = k(m_4, m_5, m_6)$, (BCFG)$_r = k(m_7, m_8, m_9)$ and (EFG)$_r = k(m_{10}, m_{11}, m_{12})$ the numbers m_4, m_7 and m_{10} are determined by A and F, respectively. We have $m_1 = m_7 = m$ and $m_4 = m_{10} = n$. By using Proposition 2 we calculate the circles (ABCD)$_r$ and (BCFG)$_r$:

$$m_2 = \frac{c_2 - b_2 - m_1(c_1^4 - b_1^4)}{c_1 - b_1},$$

$$m_3 = \frac{b_2 c_1 - b_1 c_2 - m_1(b_1^4 c_1 - b_1 c_1^4)}{c_1 - b_1},$$

$$d_2 = m_1 d_1^4 + m_2 d_1 + m_3,$$

$$m_5 = \frac{c_2 - b_2 - m_4(c_1^4 - b_1^4)}{c_1 - b_1},$$

$$m_6 = \frac{b_2 c_1 - b_1 c_2 - m_4(b_1^4 c_1 - b_1 c_1^4)}{c_1 - b_1},$$

$$g_2 = m_4 d_1^4 + m_5 d_1 + m_6.$$

The circles $(\mathrm{ADEH})_r$ and $(\mathrm{EFG})_r$ are determined by:

$$m_8 = \frac{d_2 - e_2 - m_7(d_1^4 - b_1^4)}{d_1 - b_1},$$

$$m_9 = \frac{d_1 e_2 - d_2 b_1 - m_7(b_1^4 d_1 - b_1 d_1^4)}{d_1 - b_1}$$

and

$$m_{11} = \frac{g_2 - e_2 - m_{10}(d_1^4 - b_1^4)}{d_1 - b_1},$$

$$m_{12} = \frac{d_1 e_2 - g_2 b_1 - m_{10}(b_1^4 d_1 - b_1 d_1^4)}{d_1 - b_1}.$$

As in case 1 we consider the equation

$$(m_{10} - m_7)x^4 + (m_{11} - m_8)x + (m_{12} - m_9) = 0.$$

A solution of this equation is b_1 because $\mathrm{E} \in (\mathrm{ADE})_r \cap (\mathrm{EFG})_r$, i.e.

$$(m_{10} - m_7)b_1^4 + (m_{11} - m_8)b_1 + (m_{12} - m_9) = 0.$$

We start with

$$(g_2 - d_2)(c_1 - b_1) = (m_4 - m_1)(c_1 - b_1)d_1^4 + (m_5 - m_2)(c_1 - b_1)d_1 + (m_6 - m_3)(c_1 - b_1) =$$

$$(m_4 - m_1)(c_1 - b_1)d_1^4 - (m_4 - m_1)(c_1^4 - b_1^4)d_1 - (m_4 - m_1)(b_1^4 c_1 - b_1 c_1^4) =$$

$$(m_4 - m_1)(c_1 d_1^4 - b_1 d_1^4 - c_1^4 d_1 + b_1^4 d_1 - b_1^4 c_1 + b_1 c_1^4) =$$

$$(m_{10} - m_7)((d_1^4 - b_1^4)(c_1 - b_1) - (c_1^4 - b_1^4)(d_1 - b_1)).$$

Together, we get

$$(m_{10} - m_7)(c_1^4 - b_1^4)(d_1 - b_1) + (g_2 - d_2)(c_1 - b_1) - (m_{10} - m_7)(d_1^4 - b_1^4)(c_1 - b_1) = 0.$$

Dividing this equation by $(d_1 - b_1)$ and using

$$m_{11} - m_8 = \frac{g_2 - d_2 - (m_{10} - m_7)(d_1^4 - b_1^4)}{d_1 - b_1}$$

we get $(m_{10} - m_7)(c_1^4 - b_1^4) + (m_{11} - m_8)(c_1 - b_1) = 0$ which is equivalent to

$$(m_{10} - m_7)c_1^4 + (m_{11} - m_8)c_1 + (m_{12} - m_9) = (m_{10} - m_7)b_1^4 + (m_{11} - m_8)b_1 + (m_{12} - m_9).$$

By using the equation $(m_{10} - m_7)b_1^4 + (m_{11} - m_8)b_1 + (m_{12} - m_9) = 0$ we get

$$(m_{10} - m_7)c_1^4 + (m_{11} - m_8)c_1 + (m_{12} - m_9) = 0.$$

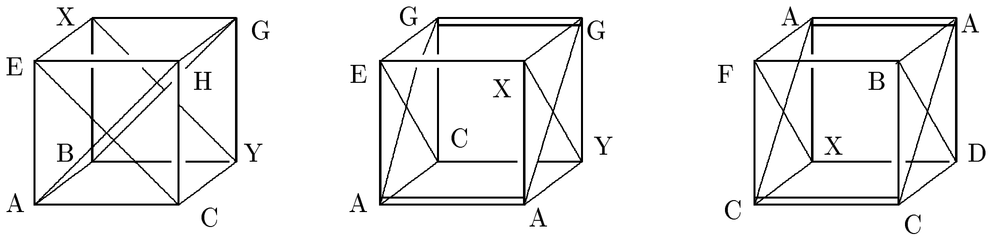
Thus c_1 is a solution of $(m_{10} - m_7)x_1^4 + (m_{11} - m_8)x_1 + (m_{12} - m_9) = 0$. With the same argument as in case 1 we get H$\in$(EFG)$_r$. □

4 Incidence-geometrical implications

The following propositions have been proved in an incidence-geometrical way. This means that they hold in every axiomatic Laguerre plane.

Proposition 3. $(M2) \Rightarrow (BD_2M2)$.

Proof. Let $(M2)$ be true and let A,...,H be eight different points as in the hypothesis of (BD_2M2). E,G,H are pairwise non-parallel $\Rightarrow$(EGH)$_r$. We have to prove (ABE)$_r$E(EGH)$_r$. Assume (ABE)$_r$ and (EGH)$_r$ to be equal. Then (ABEGH)$_r$ is a regular circle. But this is a contradiction to B$\|$G.
Assume (ABE)$_r \cap$(EGH)$_r$ = {E,X} with X$\neq$E. X=A $\Rightarrow$(AEGH)$_r$, contradiciton to A$\|$H. X=B $\Rightarrow$(BEGH)$_r$, contradiction to B$\|$G. X=C $\Rightarrow$(CEGH)$_r$, contradiction to C$\|$E. X=G $\Rightarrow$(ABEG)$_r$, contradiction to B$\|$G. X=H $\Rightarrow$(ABEH)$_r$, contradiction to A$\|$H. Hence X$\notin$ {A,B,C,G,H}. There exists a point Y$\in$(CGH)$_r$ parallel to X. Y=E is not possible because Y$\nparallel$E. X is not a point on (CGH)$_r$ otherwise (CGHX)$_r$ would lead to (CEGHX)$_r$, contradiction to C$\|E$. This means X$\neq$Y. Furthermore we have X$\nparallel$A and X$\nparallel$B. Therefore we can apply $(M2)$ to the points



(a) (b) (c)

(FIGURE 2)

E,H,G,X,A,C,Y,B (see Figure 2a). $(M2)$ yields $(\mathrm{ABCY})_r$. This is a contradiction to $(\mathrm{ABC})_r\mathrm{C}(\mathrm{CGHY})_r$. Since A$\|$H holds, $(\mathrm{ABC})_r{=}(\mathrm{CGHY})_r$ is not possible. Thus, $(\mathrm{ABE})_r\mathrm{E}(\mathrm{EGH})_r$ holds. $\square$

Proposition 4. $(BD_2M2) \Rightarrow (BBDDM1)$.

Proof. Let (BD_2M2) be true. Let A,C,E,G be four different points as in the hypothesis of $(BBDDM1)$. Let k_0 be a regular circle through A and C, k_1 a regular circle through A and E with $k_0\mathrm{A}k_1$. Furthermore let k_2 be a regular circle containing the points C and G, $k_0\mathrm{C}k_2$. Let k_3 be a regular circle through the points E and G, $k_2\mathrm{G}k_3$. Let A$\|$G and C$\|$E. We have to show $k_1\mathrm{E}k_3$. Assume $k_1 = k_3$. Then $(\mathrm{AEG})_r$ is a regular circle, contradiction to A$\|$G. Assume $k_1 \cap k_3 = \{E, X\}$ with E$\neq$X. Then X$\nparallel$G holds, otherwise we get X=A. This means A$\in k_3$, contradiction to (A$\|$G and k_3 regular). Since two regular circles cannot intersect in two parallel points, X$\nparallel$E holds. There exists a point Y$\|$X with Y$\in k_2$. Assume X=Y. Then X $\in k_2$ leads to $k_2 = (\mathrm{CGX})_r$, contradiction to $k_2\mathrm{G}k_3$ (it is $k_2 \neq k_3$). We get X$\neq$Y. The point A is not an element of the circle k_3 because A$\|$G and G $\in k_3$ hold. X=C $\Rightarrow$ A$\|$E, contradiction to A,E $\in k_1$ (k_1 is regular). Together, X$\notin$ {A,C,E,G,Y}. Furthermore it is Y$\notin \{A, C, E, G\}$. Apply (BD_2M2) to the points E,X,G,A,Y,C (see Figure 2b). We get $k_1 = (\mathrm{AEX})_r\mathrm{A}(\mathrm{ACY})_r$.
Hence Y$\in k_0 \cap k_2$. Since $k_0\mathrm{C}k_2$ and $k_0 \neq k_2$, we get the contradiction Y=C. Thus, $k_1\mathrm{E}k_3$ is true. $\square$

Proposition 5. $(BD_2M2) \Rightarrow (CC_2M1)$.

Proof. Let (BD_2M2) be true. Let A,B,C,D,F,H be six different points as in the hypothesis of (CC_2M1). We have to prove F$\|$H. A$\|$C $\Rightarrow(\mathrm{ACF})_i$. Let X$\in(\mathrm{ADH})_r$ with X$\|$F. X=A,B,C,D implies (in this order) A$\|$F, B$\|$F, C$\|$F, B$\|$F, which contradicts the assumptions. X=F means $(\mathrm{ABF})_r{=}(\mathrm{ADH})_r$, contradiction. Together, X$\notin${A,B,C,D,F}. We have to show X=H. Assume X$\neq$H. Apply (BD_2M2) to the points F,B,A,C,D,X (see Figure 2c) $\Rightarrow(\mathrm{BCF})_r\mathrm{C}(\mathrm{CDX})_r$. $(\mathrm{BCF})_r\mathrm{C}(\mathrm{CDH})_r$ $\Rightarrow(\mathrm{CDX})_r{=}(\mathrm{CDH})_r \Rightarrow(\mathrm{CDHX})_r \Rightarrow(\mathrm{ACDHX})_r$, contradiction to the assumption A$\|$C. Hence, X=H. $\square$

Together, 4 of 39 Miquelian propositions hold in the above mentioned Laguerre plane. It is not known if the other direction of the implications given in the propositions 3,4 and 5 are valid. Note that these four propositions even hold in the non-ovoidal Laguerre plane of shear type constructed by the function[13]

$$f_{2,3} : \begin{cases} \mathbb{R} \cup \{\infty\} & \to & \mathbb{R} \\ x & \mapsto & x^2, \quad x \in \mathbb{R}_{\geq 0} \\ x & \mapsto & (-x)^3, \quad x \in \mathbb{R}_{<0} \\ \infty & \mapsto & 1. \end{cases}$$

The proof runs analogously to the proof of Theorem 1.
The class of Laguerre planes characterized by the proposition $(M2)$ contains all Miquelian Laguerre planes. It is an open question whether $(M2)$ holds in every ovoidal Laguerre plane.

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[13] See Löwen and Pfüller [3]