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via Partition Separated Ordering

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Dependence Structure of Sojourn Times via Partition Separated Ordering

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Summary: We investigate the joint distribution of successive sojourn times of customers traversing a path in Jackson networks. It is shown in a general setting that sojourn times for a rather broad class of such paths exhibit positive dependence. This applies to paths which admit overtaking due to the network topology as well as due to the internal node structure. Our proofs utilize the concept of partition separated orders on multidimensional ordered spaces.

Key Words: Exponential multiserver, joint sojourn time vector, association, positive upper orthant dependence, partition separated ordering

AMS (1991) subject classification: 60K25, 60E15

Short title: Structure of sojourn time distributions.

1 Introduction

Stochastic orderings, dependence orderings, and related concepts have been proven to be a powerful tool to investigate qualitative behavior of stochastic models and to develop explicit quantitative bounds for performance measures of the systems described by these stochastic models, see [MS02], [Sze95], [SS94]. The key to many successful applications of the principles developed in this fields is often to find out suitable orderings on the state spaces of the models and, even more specific, to adapt stochastic orderings with respect to the state space order to the problem under consideration.

The present note is in the course of these developments: Using stochastic orderings and dependence orders we study space–time behaviour of stochastic networks which a customer experiences when traversing a path of this network.

We investigate open Jackson networks where customers arrive from an external source, circulate for some time through exponential multiserver nodes inside the network, and eventually leave the network. These networks belong to the class of networks where the steady state distribution is explicitly accessible and is of *product form.* The broadest subclasses of product form networks are networks of either exponential or symmetric nodes (see [Kel79], section 3.1 and 3.3,) and include especially the BCMP–networks [BCMP75] as special cases.

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Performance evaluation for these networks by analytical methods starts with determining explicit expressions for the steady state distributions of the Markov processes describing these systems. For Jackson networks these vector processes describe the joint behavior of all nodes and possess steady state distributions with independent coordinates. This indicates that in the long run the nodes behave in distribution as independent nodes, although they physically interact through circulating customers.

Therefore, in equilibrium we see that for a fixed time instant the correlation in space (across nodes) vanish. But it is well known that neither the correlation in time nor the space–time correlation vanishes. For detailed investigations of this phenomenon via simulation experiments see the works of Disney and Kiessler [DK87] and Kiessler [Kie80]. An investigation of correlation in space and time for stochastic network processes is included also in [DS95] in connection with generalized birth–death–processes and Markovian processes.

Clearly, the space–time correlation structure of the state process directly influences the sojourn time (delay time) distribution of customers visiting successive nodes on their itinerary through the network. But up till now this influence has not been well understood.

Realizing that explicit quantitative results are available only for a limited class of networks and for selected paths in the networks, we are faced with the problem to obtain qualitative information about the correlation structure of sojourn time vectors which yields useful performance bounds. We start from two observations concerning three–node networks, in [SF79], [Bur69]. In both cases it was proved that independence of sojourn times does not hold, either due to a node specific overtaking condition [Bur69], or due to a network topology which admits overtaking [SF79]. Further examples and contributions are described in Section 3. Especially, we focus on what is known about the joint distributions of successive sojourn times of a customer traversing in equilibrium overtake-free paths in Jackson networks.

We embed this three–node network problem into a general open exponential network setting and prove in section 4 a strong positive dependence (*positive upper orthant dependence*) for successive sojourn times of a customer on a path which may include up to four stations. This extends several previous results from the literature. Our analysis relies on the notion of partition separated ordering. This concept was developed in [DS95] as a versatile tool for tailoring order structures to complex multidimensional systems and processes.

We end the introduction with needed definitions on stochastic dependencies. For a more detailed introduction see, e.g., [Sze95].

Definition 1.1 *Let* $X_j : (\Omega,\mathcal{F},P) \longrightarrow (\mathbb{R}_+, \mathbb{B}_+), j = 1, \dots, J,$ *be random variables, and* $X = (X_j, j = 1, \dots, J)$.

- X *is associated, if for every pair of increasing functions (with respect to the coordinatewise ordering)* $F, G : (\mathbb{R}_+^J, \mathbb{B}_+^J) \longrightarrow (\mathbb{R}, \mathbb{B})$

$$E[F(X) \cdot G(X)] \geq E[F(X)] \cdot E[G(X)] \tag{1}$$

holds provided the integrals exist.

- X *is positive upper orthant dependent (PUOD) if for any vector* $(x_1, \dots, x_J) \in \mathbb{R}^J$

$$P(X_j > x_j, j = 1, \dots, J) \geq \prod_{j=1}^{J} P(X_j > x_j). \tag{2}$$

Definition 1.2 (Partial sum ordering) *The partial sum ordering $\leq_*$ on $\mathbb{R}^n$ is defined by:*

$$(x_1, \ldots, x_n) \leq_* (y_1, \ldots, y_n) \qquad iff \qquad \sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} y_i, \qquad j = 1, \ldots, n.$$

In [DS95] partition separated orders on product spaces were introduced, and were shown to provide a versatile tool to construct orderings on these product spaces which describe interactions of different character in space or time jointly. While in general this notion is defined for normally ordered Polish spaces we restrict our attention here to the case of $\mathbb{R} \times \mathbb{R} \ldots \times \mathbb{R}$.

Consider $\mathbb{R}^n$ and a partition $\mathcal{P}$ of the index set $\{1, 2, \ldots, n\}$, which is a class of disjoint subsets (segments) of $\{1, 2, \ldots, n\}$, which in common contain all of its elements. For each segment J of partition $\mathcal{P}$ we consider a closed partial order $\prec_J$ on $\times_{i \in J} \mathbb{R} = \mathbb{R}^J$. Usually these orders are thought to be of different nature on each segment; say, on one segment we have coordinatewise ordering, on some other segment we have partial sum ordering, and so on.

Definition 1.3 (Partition separated order) *For a partition $\mathcal{P}$ of the index set $\{1, 2, \ldots, n\}$ for $\mathbb{R}^n$ and a set of closed partial orders $\prec_J$ on $\times_{i \in J} \mathbb{R} = \mathbb{R}^J$, $J \in \mathcal{P}$, the set $\{\prec_J : J \in \mathcal{P}\}$ of orders is called a set of partition separated orders. This set defines a partition separated ordering $\prec_{(\mathcal{P})}$ on the space $\times_{J \in \mathcal{P}} \mathbb{R}^J$ by*

$$x \prec_{(\mathcal{P})} y \quad iff \quad x_J \prec_J y_J \quad \forall J \in \mathcal{P}, \qquad x, y \in \mathbb{R}^n, \tag{3}$$

where $x_J \in \mathbb{R}^J$ is the projection of $x \in \mathbb{R}^n$ onto $\mathbb{R}^J$.

Note, that from an abstract point of view partition separated ordering is a formal product order structure the components of which are not homogeneous. It is obvious how to apply this principle to component orderings on different coordinate spaces. If on the other hand for each $J \in \mathcal{P}$ $\prec_J$ is coordinatewise ordering, then $\prec_{(\mathcal{P})}$ reduces to the coordinatewise ordering on $\mathbb{R}^n$.

In the light of Definition 1.3 the following generalization of Definition 1.1 (2), is natural.

Definition 1.4 (Partition separated upper orthant dependence) *(a) Consider state space $\mathbb{R}^n$ for a random vector $X = (X_j, j = 1, \ldots, n)$, $X_j : (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}, \mathbb{B}), j = 1, \ldots, n$, a partition $\mathcal{P}$ of the index set $\{1, 2, \ldots, n\}$, a set of closed partial orders $\prec_J$ on $\times_{i \in J} \mathbb{R} = \mathbb{R}^J$, $J \in \mathcal{P}$, and the associated partition separated ordering $\prec_{(\mathcal{P})}$ on the space $\times_{J \in \mathcal{P}} \mathbb{R}^J$. X is partition separated positive upper orthant dependent (with respect to $\prec_{(\mathcal{P})}$) if for any vector $x = (x_J : J \in \mathcal{P}) \in \times_{J \in \mathcal{P}} \mathbb{R}^J$*

$$P(X_J \succ_J x_J, J \in \mathcal{P}) \geq \prod_{J \in \mathcal{P}} P(X_J \succ_J x_J). \tag{4}$$

where $x_J \in \mathbb{R}^J$ is the respective marginal of x.

(b) For vectors X with only non negative values partion separated positive upper orthant depnedence is defined similarly.

Here is an example which will be of use later.

Example 1.5 *Consider state space $\mathbb{R}^4_+$ with partition $\mathcal{P}_4 = \{\{1\}, \{2\}, \{3, 4\}\}$ of the index set $\{1, 2, 3, 4\}$, assume on the first and second coordinate of $\times_{J \in \mathcal{P}_4} \mathbb{R}^J_+$ be given the natural order $\leq$*

and on the third coordinate the (closed) partial sum order $\leq_*$ *. Denote the associated partition separated ordering by* $\prec_{(\mathcal{P}_4)}$ *.*
The random vector $X = (X_j, j = 1, \ldots, 4)$, $X_j : (\Omega, \mathcal{F}, P) \longrightarrow (\mathbb{R}_+, \mathbb{B}_+), j = 1, \ldots, 4,$ *is partition separated positive upper orthant dependent with respect to* $\prec_{(\mathcal{P}_4)}$ *if for any vector* $x = (x_J : J \in \mathcal{P}) \in \times_{J \in \mathcal{P}} \mathbb{R}_+^J$

$$\begin{aligned} & P(X_1 > x_1, X_2 > x_2, X_3 > x_3, X_3 + X_4 > x_3 + x_4,) \\ \geq \;& P(X_1 > x_1) \cdot P(X_2 > x_2) \cdot P(X_3 > x_3, X_3 + X_4 > x_3 + x_4). \end{aligned} \tag{5}$$

2 Description of the system

We consider exponential networks of multiserver queues with random routing (Jackson networks [Jac57]) consisting of J nodes numbered $1, \ldots, J$. Customers arrive from the external source at node j according to a Poisson stream with finite intensity $\lambda_j \geq 0$. The arrival streams are mutually independent. The customers in the network are indistinguishable and follow the same rules.

Node j has $m_j \geq 1$ service channels, an infinite waiting room which is organized according to the first–come–first–served (FCFS) regime, and customers finding at their arrivals a number of idle channels select one of them at random. All customers arriving at node j request for an amount of service time which is node specific and $\exp(\mu_j)$–distributed. The set of all service times is an independent set of variables and is independent of the arrival streams.

A customer departing from node j immediately proceeds to node i with probability $r(j,i) \geq 0$, and departs from the network with probability $r(j,0)$. The routing decisions are independent of the past of the system given the node where the customer departs. We assume that every node can be reached from any other node in a finite number of steps with positive probability and that all customers eventually leave the network with probability 1. This ensures that the traffic equations

$$\eta(j) = \lambda_j + \sum_{i=1}^{J} \eta(i) r(i,j), \qquad j = 1, \ldots, J, \tag{6}$$

have a unique solution which we denote by $\eta = (\eta(j) : j = 1, \ldots, J)$.

Let $X = (X(t) : t \geq 0)$ denote the vector process recording the joint queue lengths in the network for time variable t. $X(t) = (X_1(t), \ldots, X_J(t)) \in \mathbb{N}^J$ reads: at time t there are $X_j(t)$ customers present at node j, either in service or waiting. The assumptions put on the system imply that X is a strong Markov process with the state space $\mathbb{N}^J$. The following theorem is classical.

Theorem 2.1 *X is ergodic if and only if for all j, $\eta(j) < \mu_j m_j$. If X is ergodic, its unique steady–state and limiting distribution* $\pi = (\pi(n_1, \ldots, n_J) : n_j \in \mathbb{N}, j = 1, \ldots, J)$ *is given by*

$$\pi(n_1, \ldots, n_J) = \prod_{j=1}^{J} \pi_j(n_j), \tag{7}$$

where

$$\pi_j(n_j) = \begin{cases} \left(\frac{\eta_j}{\mu_j}\right)^{n_j} \frac{1}{n_j!} C(j)^{-1} & if \quad n_j \leq m_j, \\ \left(\frac{\eta_j}{\mu_j}\right)^{n_j} \frac{1}{m_j!} \left(\frac{1}{m_j}\right)^{n_j - m_j} C(j)^{-1} & if \quad n_j \geq m_j, \end{cases} \tag{8}$$

and the $C(j)$ are norming constants.

¿From now on we assume that X is ergodic.

3 Review of previous results on customers' sojourn time distributions in product form networks

We consider a network as described in section 2 in equilibrium. A path of length N in the network is a finite sequence of nodes $P = (j_1, j_2, \ldots, j_N)$, not necessarily distinct, which a customer can visit consecutively, i.e., $r(j_k, j_{k+1}) > 0, k = 1, \ldots, N-1$. For a customer traversing path P we denote by $(S_1, S_2, \ldots, S_N)$ the vector of his successive sojourn times at the nodes of the path. Our interest is focused on determining the joint distribution of the vector $(S_1, S_2, \ldots, S_N)$ in equilibrium. In general this is an unsolved problem, explicit expressions are rare.

The first results were obtained by Reich [Rei57], [Rei63], and Burke [Bur56], [Bur68]. They proved that for a J–stage tandem $(r(j, j+1) = 1, j = 1, \ldots, J-1, r(J, 0) = 1, \lambda_j = 0, j \geq 2)$, where the nodes $2, 3, \ldots, J-1$ are single servers, and nodes $1, J$ may be multiservers, in equilibrium the successive sojourn times of a customer traversing the tandem are independent. For closed cycles the parallel results were developed by Chow [Cho80], Schassberger and Daduna [SD83], and Boxma, Kelly, and Konheim [BKK84]. Independence was not found due to the negative correlation of the queue lengths in the network, but a product form structure emerged as well.

The research that followed the mentioned results was concentrated on proving that similar results hold for *overtake–free* paths. Extensions to single server overtake–free paths for networks with general topology were obtained for the open network case by Walrand and Varaiya [WV80] and Melamed [Mel82], and for closed networks by Kelly and Pollett [KP83]. The result for overtake–free paths with multiserver stations at the beginning and the end of the path was proved by Schassberger and Daduna [SD87]. (For a more detailed review see [BD90].)

It turned out that defining overtake–free property was a major task when customers of different classes where included in systems' description. In the present context with indistinguishable customers for single server networks overtaking has to be understood physically.

The most prominent example where overtaking appears is the Simon–Foley network [SF79] of single server queues, see figure 1. A customer traversing path $(1, 2, 3)$ can be overtaken by customers proceeding directly to node 3 when departing from node 1. Simon and Foley proved that the vector (S_1, S_2, S_3) of the successive sojourn times for a customer traversing path $(1, 2, 3)$ has dependent components S_1 and S_3, [SF79]. On the other hand it follows from Theorem 3.1 below that the pairs (S_1, S_2) and (S_2, S_3) both have independent coordinates. Moreover, the same holds for customers traversing the network without entering node 2.

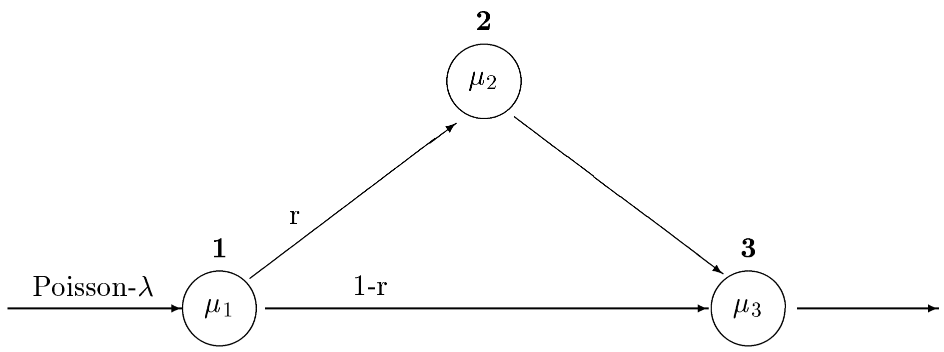

Figure 1: The Simon–Foley network with overtaking due to the network topology

While the Simon–Foley network provides us with an example where overtaking is due to the topological structure of the network, Burke [Bur69] (see figure 2) showed that overtaking due to the internal node structure prevents sojourn times on a linear path from independence as well: a three–station path $(1,2,3)$ with a multiserver node 2 ($m_2 > 1$) has dependent components S_1 and S_3.

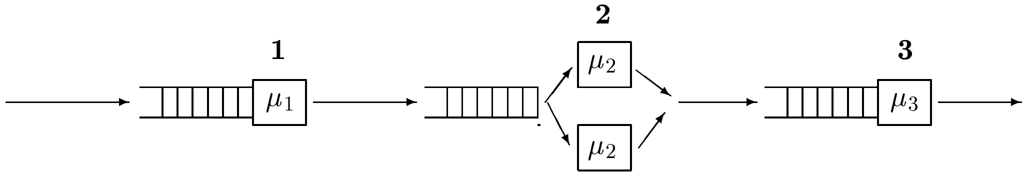

Figure 2: The tandem network with overtaking due to the internal node structure

Two–station paths with multiserver nodes admit physical overtaking – nevertheless the following general theorem holds (see for the case of single server nodes [WV80], and for the general case [SD87], Corollary 2.1 (b).)

Theorem 3.1 *For any two–station path of distinct nodes the successive sojourn times are independent and their joint distribution is given by the Laplace–Stieltjes transform*

$$E\left[e^{-\theta_1 S_1 - \theta_2 S_2}\right] = \prod_{j=1}^{2} \left(\frac{\mu_j}{\mu_j + \theta_j} \sum_{n_j=0}^{\infty} \pi_j(n_j) \left(\frac{\mu_j m_j}{\mu_j m_j + \theta_j} \right)^{(n_j+1-m_j)_+} \right), \quad Re(\theta_j) \geq 0, j = 1, 2, \tag{9}$$

where for $a \in I\!R$, $(a)_+ = \max(a, 0)$.

For arbitrary three–station paths an independence results does not hold, as the examples in figures 1 and 2 show. The detailed correlation structure of the sojourn time vector (S_1, S_2, S_3) on these paths seems to be complicated. The following results [DS00] hold for three–station walks in exponential networks. They are in accordance with simulation results which have been obtained in several studies in the literature, see [Kie80], [DK87]. For further complementary results and remarks see [DS00], Section 5.

Theorem 3.2 *In equilibrium, the successive sojourn times of the test customer on a three node path of distinct nodes are positive upper orthant dependent according to (2) and hence positively correlated.*

Corollary 3.3 *In equilibrium the test customer's successive sojourn times on a three–node path of two distinct nodes, where the second node is visited twice, are positive upper orthant dependent and hence positively correlated.*

Foley and Kiessler [FK89] proved that in the Simon–Foley network (Figure 1) the vector (S_1, S_3) is associated. A careful inspection of their proof shows that a similar property holds in the single server network of Figure 3:

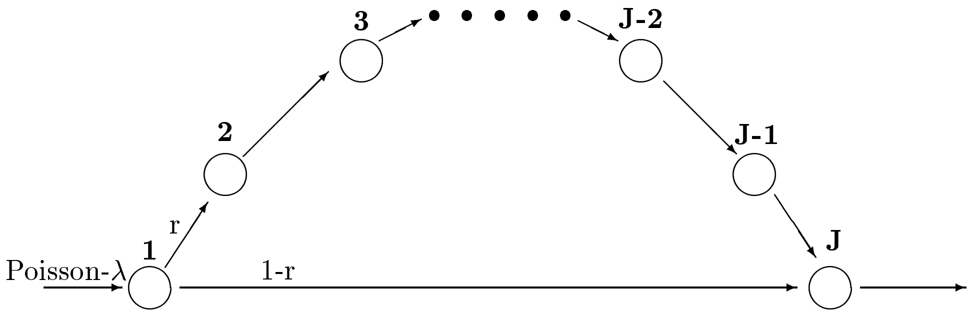

Figure 3: The generalized Simon–Foley network with overtaking due to the network topology

For the path $(1, 2, \ldots, J-1, J)$ with sojourn time vector $(S_1, S_2, \ldots, S_{J-1}, S_J)$ of the successive holding times the vector (S_1, S_J) is associated. Here the paths $(1, 2, \ldots, J-2, J-1)$ and $(2, 3, \ldots, J-1, J)$ are overtake–free and therefore $(S_1, S_2, \ldots, S_{J-2}, S_{J-1})$ and $(S_2, S_3, \ldots, S_{J-1}, S_J)$ have independent coordinates, see [WV80].

But the question whether on the three–station path of the Simon–Foley network the complete sojourn time vector (S_1, S_2, S_3) is associated remains unanswered. The methods provided by the proof of Foley and Kiessler [FK89] seemingly do not apply to that problem.

It should be mentioned that Kiessler and Disney [KL82] presented a set of equations the solution of which in principle would provide the passage time distribution of a customer in the Simon–Foley network. Unfortunately however it seems not possible to conclude from these equations statements about the qualitative behavior of passage times. The same comments apply to the results presented by Fayolle, Iasnogorodski, and Mitrani [FIM83] who reduced the passage time problem for this network to a boundary value problem. Knessl and Morrison [KM91] studied the effect of overtaking in this network using heavy traffic analysis.

4 On the correlation structure of the sojourn time vector for a 4–station–walk

We consider a network as described in section 2 and some path P consisting of four distinct nodes which for ease of notation we assume to be numbered $P = (1, 2, 3, 4)$.

At time 0 a customer, henceforth called the test customer, enters node 1 starting an itinerary through P with sojourn time vector (S_1, S_2, S_3, S_4). We do not distinguish between customers entering P from the outside or entering P from some other node inside the network. We assume that he finds a network population of customers distributed according to π, he himself is not counted in this description. This initial distribution of the system is suggested by the celebrated *arrival theorem*, see [Mel82], [SD87]. For customers arriving from the external source this is the PASTA property (Poisson–Arrivals–See–Time–Averages). For open networks these arrival theorems state that the limiting distribution for the *"disposition" of the other customers* seen by jumping customers on a specific jump is exactly the steady state of the network – the jumping (arriving) customer is not counted in the disposition. On the other hand this limiting distribution is the *steady state distribution of the embedded Markov chain* recording the system's state at time instances where the specific jump happens. Formally this is a result on Palm distributions conditioned on the occurrence of the specific jump at time 0, see [KS93] for a more detailed discussion.

The test customer entering path P at time 0, starting his journey according to this situation is from now on said the customer observing sojourn times in equilibrium. It follows that the distribution of the vector (S_1, S_2, S_3, S_4) is the limiting and stationary distribution of the sequence of these vectors indexed by the successive customers traversing P.

Note that following the preceding discussion all the probabilities in Theorem 4.1 and afterwards are Palm probabilities conditioned that at time 0 an arbitrary customer enters node 1 and then proceeds to nodes 2, 3, and 4, in that order.

We equip the state space $\mathbb{R}_+^4$ of (S_1, S_2, S_3, S_4) with the partition separated ordering $\prec_{(\mathcal{P}_4)}$ of Example 1.5, where the partition of the index set $\{1,2,3,4\}$ is $\mathcal{P}_4 = \{\{1\},\{2\},\{3,4\}\}$. All statements in the present section refer to that ordering.

Theorem 4.1 *In customer stationary equilibrium, the successive sojourn times of the test customer on a four node path of distinct nodes are partition separated positive upper orthant dependent with respect to* $\prec_{(\mathcal{P}_4)}$.

Proof: We describe here the main lines of the proof and postpone details to the appendix 6. We denote by $\tilde{X}(t)$ the vector of numbers of customers other then the test customer present at the nodes of the network at time t. So in jump instants of the test customer $\tilde{X}$ describes the *disposition* of the other customers seen by the jumping test customer. We have

$$
\begin{aligned}
& P(S_j > x_j : j = 1,2,3, S_3 + S_4 > x_3 + x_4) \\
= & \sum_{(n_1,\ldots,n_J)\in \mathbb{N}^J} P^{\tilde{X}(S_1)}(n_1,\ldots,n_J) P(S_1 > x_1 | \tilde{X}(S_1) = (n_1,\ldots,n_J)) \\
& \quad \cdot P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4 | S_1 > x_1, \tilde{X}(S_1) = (n_1,\ldots,n_J)) \\
\overset{(1)}{=} & \sum_{n_1\in\mathbb{N}} \pi_1(n_1) \Big((1 - (\exp(\mu_1) * \Gamma_{\mu_1 m_1, (n_1+1-m_1)_+})(x_1)) \\
& \cdot \sum_{(n_2,\ldots,n_J)\in\mathbb{N}^{J-1}} \prod_{j=2}^{J} \pi_j(n_j) P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4 | \tilde{X}(S_1) = (n_1,\ldots,n_J)) \Big) \\
\overset{(2)}{\geq} & \left(\sum_{n_1\in\mathbb{N}} \pi_1(n_1)(1 - (\exp(\mu_1) * \Gamma_{\mu_1 m_1, (n_1+1-m_1)_+})(x_1)) \right) \\
\cdot & \left(\sum_{n_1\in\mathbb{N}} \pi_1(n_1) \right. \\
& \left. \cdot \sum_{(n_2,\ldots,n_J)\in\mathbb{N}^{J-1}} \prod_{j=2}^{J} \pi_j(n_j) P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4 | \tilde{X}(S_1) = (n_1,\ldots,n_J)) \right) \\
\overset{(3)}{=} & P(S_1 > x_1) \cdot P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4) \\
\overset{(4)}{\geq} & P(S_1 > x_1) \cdot P(S_2 > x_2) \cdot P(S_3 > x_3, S_3 + S_4 > x_3 + x_4).
\end{aligned}
$$

In $\overset{(1)}{=}$ we used that a customer departing from node 1 with destination node 2 observes the other customers distributed according to π, (this is the arrival theorem for open exponential networks which holds independently of whether the stream of customers from node 1 to node 2 is Poissonian,) and

$$
E\left[e^{-\theta_1 S_1} | \tilde{X}(S_1) = (n_1,\ldots,n_J)\right] = \frac{\mu_1}{\mu_1 + \theta_1} \left(\frac{\mu_1 m_1}{\mu_1 m_1 + \theta_1} \right)^{(n_1+1-m_1)_+},
$$

which follows from the key Lemma 2.1 in [SD87], and, using the strong Markov property of X, that

$$\begin{aligned} & P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4 | S_1 > x_1, \tilde{X}(S_1) = (n_1, \dots, n_J)) \\ = \ & P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4 | \tilde{X}(S_1) = (n_1, \dots, n_J)). \end{aligned}$$

For $\stackrel{(2)}{\geq}$ we have to show that

$$g(n_1) = (1 - (\exp(\mu_1) * \Gamma_{\mu_1 m_1, (n_1+1-m_1)_+})(x_1)),$$

and

$$h(n_1) = \sum_{(n_2,\dots,n_J) \in I\!N^{J-1}} \prod_{j=2}^{J} \pi_j(n_j) P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4 | \tilde{X}(S_1) = (n_1, \dots, n_J))$$

are increasing functions in n_1. Because any probability measure on $I\!N$ is associated (see [Sze95], p. 136), we then conclude

$$\sum_{n_1 \in I\!N} \pi_1(n_1) g(n_1) h(n_1) \geq \left(\sum_{n_1 \in I\!N} \pi_1(n_1) g(n_1) \right) \left(\sum_{n_1 \in I\!N} \pi_1(n_1) h(n_1) \right).$$

That $g(\cdot)$ is increasing is obvious, because the argument $(n_1 + 1 - m_1)_+$ counts exponential phases.

To prove that $h(\cdot)$ is increasing we show that

$$P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4 | \tilde{X}(S_1) = (n_1, \dots, n_J))$$

is increasing in n_1 for any fixed $(n_2, \dots, n_J)$ by a coupling argument of two parallel systems; this is performed in the Appendix 6.

Applying then the arrival theorem again and interpreting now π as the arrival distribution for the test customer just before entering node 1 and π_1 as the distribution of the queue length at node 1, which due to the FCFS regime completely determines the sojourn time distribution for the test customer at node 1, we obtain for the equality $\stackrel{(3)}{=}$

$$\sum_{n_1 \in I\!N} \pi_1(n_1)(1 - (\exp(\mu_1) * \Gamma_{\mu_1 m_1, (n_1+1-m_1)_+})(x_1)) = P(S_1 > x_1).$$

The second factor of $\stackrel{(3)}{=}$ is obtained simply by deconditioning. The validity of equality $\stackrel{(4)}{\geq}$ can be seen as follows:
Consider the three–station walk of nodes $(2, 3, 4)$ which the test customer passes in this order. Insert an additional (virtual) node into this path between node 2 and node 3 with service rate ∞, i.e service time 0. Apply to that passage the arguments which lead to the previous equality $\stackrel{(3)}{=}$, we obtain

$$P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4) \geq P(S_2 > x_2) \cdot P(S_3 > x_3, S_3 + S_4 > x_3 + x_4),$$

which finally yields the inequality $\stackrel{(4)}{\geq}$. $\odot$

Remark 4.2 *The results in Theorem 4.1 on positive correlations for customers' successive sojourn times on 4–stations paths provide us with exact lower bounds for joint sojourn time quantiles in cases where the exact probabilities are still unknown.*

Theorem 3.2 is a direct consequence of Theorem 4.1. This is seen by setting $x_4 = 0$ *in*

$$P(S_j > x_j : j = 1, 2, 3, S_3 + S_4 > x_3 + x_4) = P(S_1 > x_1) \cdot P(S_2 > x_2) \cdot P(S_3 > x_3, S_3 + S_4 > x_3 + x_4).$$

How to generalize the result of Theorem 4.1 to more complex paths seems to be a difficult task. It is at a first glance tempting to expect that the proof presented in the Appendix can be used to state a similar monotonicity for past conditional sojourn time vector distributions and then to split, say, a six station path using time reversing arguments. Indeed the first observation is true. But the result can not be incorporated into a true sequence of arguments similar to that in the proof of Theorem 4.1. ⊙

5 Complements and extensions

We formulated our problems and stated our theorems in the setting of networks with classical multiserver nodes, random routing, and only exponential distributions for service and interarrival times. It is well known [Kel79] that in open product form networks with nodes having general service disciplines and exponentially distributed service times or having symmetric service disciplines with generally distributed service times, multiserver nodes as considered here behave in equilibrium as if they work in isolation with suitably adjusted arrival rates $\eta(\cdot)$ according to (6). It follows that Theorem 4.1 holds in such networks as well. This is true even for networks with customers of different types entering the network and possibly changing their types during their passage through the network. Here we may allow additionally that the service time distributions at symmetric nodes are type dependent.

A more general situation is as follows: We may allow type dependent fixed routing and embed our multiserver paths of interest into a network where other subnetworks show no explicit product structure with respect to the different nodes of the subnetwork, as is considered in [KS93] and [Dad97]. However it is not clear whether our observations and theorems for 4–station paths of multiserver nodes embedded in such networks hold true in these settings. This is due to the fact that in such networks the coupling proof is not directly applicable because the extra customers in network II may delay other regular customers at nodes outside of the path of interest, who may therefore enter in network I node 3 or node 4 before they will do that in network II.

6 Appendix

Coupling for the proof of Theorem 4.1:
To prove that $h(\cdot)$ is increasing we show that

$$P(S_2 > x_2, S_3 > x_3, S_3 + S_4 > x_3 + x_4 | \tilde{X}(S_1) = (n_1, \ldots, n_J))$$

is increasing in n_1 for any fixed $(n_2, \ldots, n_J)$. We use a coupling argument, see e.g. [FK89] for a similar construction in case of natural coordinatewise ordering on $\mathbb{R}^3$.

We consider two copies of our network: network I starts at time 0 with a customer population $n = (n_1, n_2, \ldots, n_J)$ and the test customer just on the arrival at node 2, network II starts at time 0 with a customer population $n' = (n_1 + k, n_2, \ldots, n_J), k > 0$, and the test customer just on the arrival at node 2. In both state descriptions n, n' the test customer is not counted. Let us call henceforth the k customers additionally present at node 1 in network II the *extra customers*, while the other customers, present at time 0 in the network or arriving later on, are called *regular customers.* So in network I we shall find only regular customers. The test customer is not regular.

While network I develops as described above, we give additional rules for network II: all regular customers have absolute preemptive priority over the extra customers, only the test customer may be delayed by the extra customers. Because the customers are indistinguishable and the service and interarrival times are memory-less this handling of the extra customers does not change the test customers' sojourn times in network II, compared with his sojourn times in the (original) network II without the additional preemption rules and with initial configuration n'.

The stochastic evolution of both networks is driven by the same random sequences: Poisson–λ_j processes for the external arrivals at node j in both networks, and for each service channel at node j a Poisson–μ_j process generating the service times at similar channels of similar nodes in both networks, and sequences of routing decisions for customers moving in both networks. If an event is generated by some service process, i.e., a *service time* should expire if possible, and if there is no customer in service at the associated node in one or both networks that event is lost at the idle nodes. Similarly, a routing decision made in a situation where in some network no customer is available for this movement, is skipped for that network.
To generate jointly the stochastic sequences of inter event times in both systems associated with the nodes is the usual procedure to construct pathwise couplings. We shall need additional data associated with the test customers' sojourns at the successive nodes: We generate independent of anything else in the system a random vector on $\mathbb{R}^4$ with independent coordinates and the i–th coordinate distributed according to an exponential distribution with paprameter μ_i. This vector serves in both networks to determine the test customers successive service times on the path.

We denote quantities in network II with an additional prime, if necessary for better reading. The first consequence of the regulation of customer movements and their priority rules is that in both networks the regular customers and the test customer will be served and move completely in parallel until at least the time when the test customer departs from node 2 in both systems at time $S_2 = S_2'$.

Our first observation is: Both test customers depart from node 2 at the same time and arrive at node 3 at the same time. Up to that moment the regular customers at nodes 3 and 4 in both systems are served synchronously, and regular customers arrived in parallel there. So at their arrival the test customers find exactly the same number of regular customers at nodes 3 and 4 before them in service or in the queue. Only in network II the test customer may find further extra customers at nodes 3 and 4 (or somewhere else in the network) who have overtaken him at node 2 during his service there or choosing an overtaking path when departing from node 1.

Now, the number of customers found at node 3 before them determines the number of exponential waiting phases for the test customer there, which in network II is at least as great as in network I. It follows

$$S_3' = S_3 + D_3, \tag{10}$$

where D_3 has $\Gamma_{m_3\mu_3,d_3}$ distribution for some random $d_3 \in \mathbb{N}$.

If $d_3 = 0$, (no extra customers are waiting in network II before the test customer at node 3) then $S_3 = S_3'$ and at time $S_2 + S_3 = S_2' + S_3'$ in both systems the test customer departs from node 3, arriving immediately at node 4. Repeating in this case the arguments used to classify the situation at departure instant from node 2, we see that both test customers find the same number of regular customers at node 4, accompanied in network II by some extra customers. It follows

$$S_4' = S_4 + D_4, \tag{11}$$

where D_4 has $\Gamma_{m_4\mu_4,d_4}$ distribution for some random $d_4 \in I\!N$. It follows

$$S_3' + S_4' \geq S_3 + S_4, \tag{12}$$

and we have finished the proof in this case.
Note that the relations (10), (11), (12), are valid pathwise due to our assumption that the service times of the test customers are not sampled from the service time streames used for the other customers. If this would not be the case, all relations can only be read as distributional relations. The same remark applies to the following relations (13) through (16).

We turn to the case $d_3 > 0$, which at first guarantees that

$$S_3' \geq S_3, \tag{13}$$

which is *the correct inequality* for our desired result.

But if $S_3' = S_3 + s_3$ for some $s_3 > 0$, two events may occur, which possibly influence the duration of S_4' in *the wrong direction*; the first possibly dangerous event is:

(A) During $[S_3, S_3 + s_3)$ in network II some further regular customers may be served at node 4 which could diminish the delay of the test customer at node 4 – compared with the delay of the test customer experienced at node 4 in network I. To elucidate the problem we discuss separately the cases of single server and multiserver nodes 4:
(α) For node 4 being a single server, i.e., $m_4 = 1$, we conclude that the reduction of delay, say $z_4 \geq 0$, at node 4 in system II is maximally s_3. Therefore we still have

$$S_3' + S_4' \geq S_3 + s_3 + S_4 - z_4 \geq S_3 + S_4, \tag{14}$$

which is *the correct inequality* for our desired result again.
(β) For node 4 being a multiserver, i.e., $m_4 > 1$, the above argument does not hold because in a time intervall of length s_3 a reduction of work up to $s_3 \times m_4$ can be obtained. We therefore conclude as follows:
(β i) Assume the test customer in network I finds on his arrival at node 4 less or equal $m_4 - 1$ other customers there (which are all regular). He then enters service immediately and we have

$$S_4' \geq S_4, \tag{15}$$

which is sufficient.
(β ii) Assume the test customer in network I finds at least m_4 other customers at node 4 (which are all regular) on his arrival there. We then observe the networks at time $S_2' + S_3' = S_2 + S_3 + s_3$, when the test customer in network II departs from node 3. We have to distinguish two further cases.

(β ii a) At time $S_2' + S_3' = S_2 + S_3 + s_3$ the test customer's service in network I commenced up to that instant (possibly being finished already). Then from the construction of the test customers' service times via an extra vector we immediately obtain

$$S_3' + S_4' \geq S_3 + S_4, \tag{16}$$

(β ii b) At time $S_2' + S_3' = S_2 + S_3 + s_3$ the test customer's service in network I did not commence up to that time instant. This means he is still waiting because *regular* customers delayed his progress up to that moment. From the coupling construction and the absolute preemptive priority of regular customers over the extra customers in network II we conclude that those customers' *regular* counterparts in network II occupy the same positions and therefore enforce the test customer in network II to wait for service at least as long as the residual delay is of the test customer in network I. This guarantees that (16) holds in this case as well.

The second possibly dangerous event is:
(B) During $[S_3, S_3 + s_3)$ in network II some regular customers may be delayed by the test customer being still in service, while their counterpart regular customers in system I may pass node 3 without that additional delay because there the test customer already has left node 3. But due to the server's FCFS regime these early regular customers cannot delay the test customer in node 4 of network I.

Summarizing, this coupling argument shows that $h(\cdot)$ is increasing. $\odot$

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