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An invariance property of conditional sojourn time distributions in cyclic networks of queues

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Abstract

We consider a closed cycle of M exponential single server nodes. The conditional joint sojourn time distribution, given the cycle time, is shown to be independent of the number of circulating customers. This holds especially in the presence of bottleneck stations. An explicit expression for the (degenerate) density of this distribution is given.

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1 Introduction

Closed queueing systems consisting of single-server stations with exponentially distributed and queue independent service rates, a finite population of indistinguishable customers (jobs) and random routing, i. e. queueing-systems of the Gordon-Newell type show distinct behaviour as the number of jobs increases: if there is one (or more than one) station for which the ratio of the product of the intensity of arriving customers and their mean service demand to the service capacity of the station is greater than for all other stations, then the system shows the so called bottleneck behaviour when the number of customers increases [GN67]. In the other case (same ratio for all stations) the customers share equally the stations independent of the population size (Koenigsberg called this the case of *balanced machines* [Koe58]). In order to determine sojourn times it is necessary to make the customers distinguishable. Clearly systems with distinguishable customers also show the bottleneck behaviour. A bottleneck-station which is visited by a customer on his itinerary, strongly influences his cycle time (the time for a complete run through the nodes $Q[1]$ to $Q[M]$). Nearly all of the cycle time is generated by the bottleneck-station. Boxma has investigated this in the case of a M-station tandem [Box88]. In the case of balanced machines the cycle time is generated by all stations as equal shares in contrast to the bottleneck behaviour [Kel84]. In the ladder case the conditional joint density of the successive sojourn times, conditioned on the cycle time, is constant and also independent of the number of customers. In this paper the conditional joint density of the successive sojourn times in the case of bottleneck behaviour is determined. Surprisingly this density shows not only a simple functional form, but is also independent of the number of customers as well.
The organisation of the paper is as follows: Section 2 gives the systemdescription, section 3 reviews the relevant statements concerning cycle time and sojourn time, section 4 contains the conditional joint density for the bottleneck case, finally section 5 contains the proof of the main result given in section 4.

2 Description of the System

Consider a closed cyclic queueing system with M nodes and a fixed number N of clockwise circulating customers (i. e. customers move through the nodes in ascending order of the node-numbers). The station $Q[i]$ represents a queue-independent single-server with service rate μ_i, infinite waiting room and first-come-first-serve (FCFS) queueing discipline. So the system operates in the following way: if the service of a customer at node $Q[i]$ terminates, he moves directly and instantaneousley to node $Q[i+1]$ (resp. to $Q[1]$ if a service ends at node $Q[M]$). The customers at station $Q[i]$ (if there are any) move one position forward and the service of the customer at the head of the queue begins immediately. If there are further customers present at queue $Q[i]$ at the arrival instant of the moving customer, he joins the tail of the queue; if there aren't any further customers his service begins immediately. All jumps and the reorganisation of the queues happen without time lag. The service demand of all customers at every node is exp(1)-distributed. This constitutes no restriction. If the service demand (measured in serviceunits per customer) of a customer at node $Q[i]$ is $\exp(\tilde{\mu}_i)$-distributed and if the service-device at this node has the capacity s_i (measured in serviceunits per timeunit) put $\mu_i := \tilde{\mu}_i s_i$. The essential point is, that the service demands at one and the same station are exponentially distributed with the same parameter for all customers. In this sense the circulating customers are *indistinguishable*. Finally the family of all service demands is assumed to be an independent family and independent of anything else.
With respect to sojourn time investigations, it is necessary to be able to distinguish the customers. For this purpose one of the customers is tagged (the so called *test-customer*). One possibility to obtain this distinguishability is the introduction of internal customer types. Formally this is a variant of the deterministic routing according to Kelly [Kel79], but the internal types now have the function to enable the distinction of the customers instead of determining their route through the network. There are other possibilities of marking a customer, but because of the specific topology of the cyclic network under consideration, which prohibits the overtaking of customers, every marking leads to the distinctiveness of all customers present. Formally let $\Theta_i := \{i\}$, $i \in \underline{N}$, the type-set corresponding to customer $C(i)$. The type i determines the route $w(i) = (1, \ldots, M)$ of customer $C(i)$ through the net. After this route is traversed entirely by the customer $C(i)$ with the service-completion of $C(i)$ at queue $Q[M]$, he moves to $Q[1]$ to begin his next itinerary.

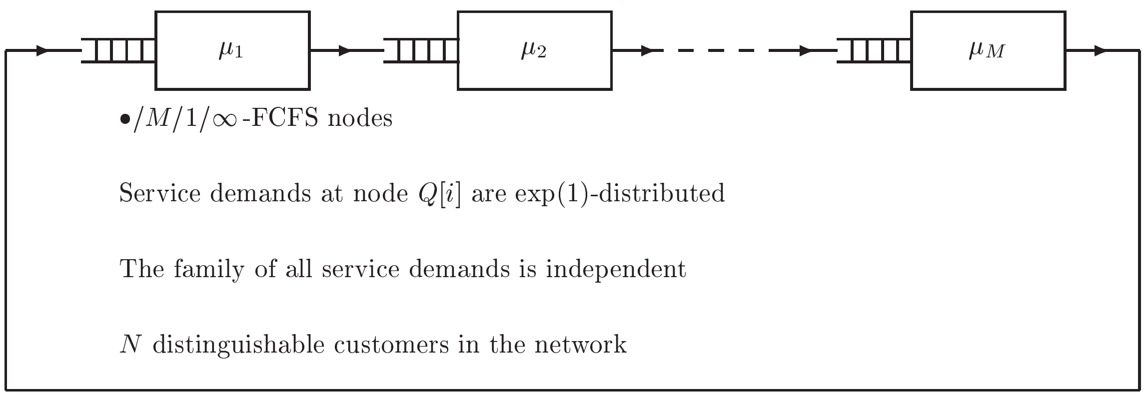

Figure 1: Cyclic network consisting of $\bullet/M/1/\infty$-FCFS single-server stations

The states of the characterising network-process are:

$$\begin{aligned} x = (x_1,\ldots,x_M) &= (((t_{11},1),\ldots,(t_{1n_1},1)),((t_{21},2),\ldots,(t_{2n_2},2)),\ldots,((t_{M1},M),\ldots,(t_{Mn_M},M))) \\ &\equiv ((t_{11},\ldots,t_{1n_1}),(t_{21},\ldots,t_{2n_2}),\ldots,(t_{M1},\ldots,t_{Mn_M})), \end{aligned}$$

with $(n_1,\ldots,n_M) \in Z(M,N) := \{(n_1,\ldots,n_M) \in \mathbb{N}_0^M \mid n_1 + \cdots + n_M = N\}$. A local state with $n_i = 0$ is interpreted as empty node. If further the customers are in the order $(1,2,\ldots,N)$ at some instance (what can be assumed without restriction), the N-Tupel $(t_{11},\ldots,t_{1n_1},t_{21},\ldots,t_{2n_2},\ldots,t_{M1},\ldots,t_{Mn_M})$ is a cyclic permutation of $(1,2,\ldots,N)$ due to the impossibility of overtaking. Hence the feasible states of the enlarged process are obtainable in the following manner: put $M-1$ tags between the customers in the cyclic permutations $(p(1),p(2),\ldots,p(N))$ of $(1,2,\ldots,N)$. Hence the number of feasible states is $N\binom{N+M-1}{N}$. The steady-state distribution of the enlarged queueing-process $(Z_t)_{t\in\mathbb{R}_+}$ is (remember that we deal with a special case of a Kelly-network with deterministic routing) given by

$$\hat{\pi}(x_1,\ldots,x_M) = \frac{1}{K(M,N)} \prod_{i=1}^{M} \mu_i^{-\mathbf{n}(x_i)}$$

with $K(M,N) = \sum_{\vec{x}\in E} \prod_{i=1}^{M} \mu_i^{-\mathbf{n}(x_i)}$ where E is the state-space of the process Z and $\mathbf{n}$ is the mapping who maps the (finite) string $x_i = (t_{i1},\ldots,t_{in_i})$ onto his lenghts. With other words, $\mathbf{n}(x_i) = n_i$ is the number of customers at node $Q[i]$. The steady-state distribution of the accompanying queue-length process (X_t) is given by

$$\pi(n_1,\ldots,n_M) = \sum_{\vec{x}\in E \mid \substack{\mathbf{n}(x_i)=n_i \\ \forall i\in \underline{M}}} \frac{1}{K(M,N)} \prod_{i=1}^{M} \mu_i^{-\mathbf{n}(x_i)} = \frac{N}{K(M,N)} \prod_{i=1}^{M} \mu_i^{-n_i} = \frac{1}{G(M,N)} \prod_{i=1}^{M} \mu_i^{-n_i} \tag{1}$$

with $G(M,N) = \sum_{\vec{n}\in Z(M,N)} \prod_{i=1}^{M} \mu_i^{-n_i}$. This is the steady-state distribution of a cyclic Gordon-Newell network with the same network topology and N (indistinguishable) customers [GN67]. Here and in the following the vector-notation $x = (x_1,\ldots,x_k) = \vec{x}$ for elements $x \in \mathbb{R}^k$ is used occasionally.

3 Sojourn time and cycle time distributions

Now tag one customer to follow him on his itineraries through the cycle. Let $S_i^{(n)}$, $i \in \underline{M}$, $n \in \mathbb{N}$ denote the sojourn time of the tagged (or test-) customer on his n-th cycle through his route $W = (1,\ldots,M)$ at node $Q[i]$, after the system is started. These sojourn times are the sum of two components, the waiting time at node $Q[i]$ and the subsequent service time (which is $\exp \mu_i$-distributed). The n-th cycle time of the test-customer is the sum $S^n = S_1^n + \cdots + S_M^n$ of all subsequent sojourn times during the n-th cycle. The asymptotic behaviour of the family $(S_1^n,\ldots,S_M^n)_{n\in\mathbb{N}}$ of the successive sojourn times is stated (in the transform domain) in the following theorem.

Theorem 3.1 *[BKK84] (theorem 1, p. 130)*

$$\begin{aligned} \phi^{(M,N)}(\theta_1,\ldots,\theta_M) &:= \lim_{n\to\infty} E\left(e^{-(\theta_1 S_1^{(n)}+\cdots+\theta_M S_M^{(n)})}\right) \\ &= \sum_{\vec{n}\in Z(M,N-1)} \pi^{(M,N-1)}(\vec{n}) \prod_{j=1}^{M} \left(\frac{\mu_j}{\mu_j+\theta_j}\right)^{n_j}, \quad 0 \leqq \theta_j < 1. \end{aligned} \tag{2}$$

Here $\pi^{(M,N-1)}$ is the steady-state distribution, that in arrival-instants of the tagged customer at node $Q[1]$ there are n_1 further customers present at node $Q[1]$ (without counting the tagged customer), and n_j customers at the nodes $Q[j]$ for $j \in \{2, \ldots, M\}$ respectively, that is

$$\pi^{(M,N-1)}(\vec{n}) = \frac{\prod_{j=1}^{M} \mu_j^{-n_j}}{\sum_{\vec{k} \in Z(M,N-1)} \prod_{j=1}^{M} \mu_j^{-k_j}} = \frac{\prod_{j=1}^{M} \mu_j^{-n_j}}{G(M,N-1)}. \tag{3}$$

Furthermore $G(M, N-1) := \sum_{\vec{k} \in Z(M,N-1)} \prod_{j=1}^{M} \mu_j^{-k_j}$ is the normalizing constant and $\pi^{(M,N-1)}$ is actually the steady-state distribution of a cyclic Gordon-Newell network with the same topology, but only $N-1$ customers circulating. A random variable $(S_1, \ldots, S_M)$ in $\mathbb{R}_+^M$, with Laplace-transform (2) is now interpreted as the family of successive sojourn times under steady-state conditions. This is justified because $\phi^{(M,N)}$ is actually the Laplace-transform under the corresponding Palm distribution. Note: the distribution of the cycle time $S := S_1 + \cdots + S_M$ is independent of the choice of the start node and the arrangement of the nodes with respect to their capacity, i. e. the values of μ_i. So it is no restriction to assume $\mu_1 \leqq \mu_2 \leqq \cdots \leqq \mu_M$ what shall be done.
This result is generalized in [SD87] to the case of so called quasi-overtake free routes traversed by a tagged customer in a mixed network (for the notion *overtake free* see [WV80] and [Mel82]; see also [BD90]). The tagged customer is allowed to change his type in between, to pass through other routes, determined by its actual type, and also exponential multiservers in a specific arrangement are allowed as stations on the route in question. This generalisation opens the way to carry over the results concerning the conditional sojourn time distribution given in the following section to quasi-overtake free routes in mixed networks.

Taking $\theta_1 = \cdots = \theta_M =: \theta$ in equation (2) leads to the Laplace-transform $\psi^{(M,N)}$ of the cycle time S under steady-state conditions, which was given in [SD83].

Corollary 3.1 *([SD83], theorem, p. 149)*

$$\psi^{(M,N)}(\theta) := \phi^{(M,N)}(\theta, \ldots, \theta) = \sum_{\vec{n} \in Z(M,N-1)} \pi^{(M,N-1)}(\vec{n}) \prod_{j=1}^{M} \left(\frac{\mu_j}{\mu_j + \theta} \right)^{n_j+1}. \tag{4}$$

4 The conditional sojourn time distribution

The observation that the Laplace-transforms $\phi^{(M,N)}$ of the joint distribution of the successive sojourn times (2) can be inverted (with a closed form expression) is the starting point for the further investigations.

Lemma 4.1 *The distribution of the successive sojourn times $(S_1, \ldots, S_M)$, experienced by the test-customer on his itinerary under steady-state conditions posesses the density*

$$f^{(M,N)}(x_1, \ldots, x_M) = G(M, N-1)^{-1} \frac{(x_1 + \cdots + x_M)^{N-1}}{(N-1)!} \prod_{j=1}^{M} \mu_j e^{-\mu_j x_j} \mathbf{1}_{[0,\infty)}(x_j), \quad \vec{x} \in \mathbb{R}_+^M. \tag{5}$$

Proof: The distribution of the vector $(S_1, \ldots, S_M)$ is given by

$$P^{(S_1,\ldots,S_M)} = \sum_{\vec{n} \in Z(M,N-1)} \pi^{(M,N-1)}(\vec{n}) \bigotimes_{j=1}^{M} \Gamma_{\mu_j, n_j+1}. \tag{6}$$

Hence the density of $(S_1, \ldots, S_M)$ is given by

$$\begin{aligned}
f^{(M,N)}(x_1, \ldots, x_M) &= \sum_{\vec{n} \in Z(M,N-1)} \pi^{(M,N-1)}(\vec{n}) \prod_{j=1}^{M} \mu_j^{n_j+1} \frac{x_j^{n_j}}{n_j!} e^{-\mu_j x_j} \mathbf{1}_{[0,\infty)}(x_j) \\
&= \sum_{\vec{n} \in Z(M,N-1)} G(M, N-1)^{-1} \prod_{j=1}^{M} \frac{x_j^{n_j}}{n_j!} \mu_j e^{-\mu_j x_j} \mathbf{1}_{[0,\infty)}(x_j) \\
&= G(M, N-1)^{-1} \left(\sum_{\vec{n} \in Z(M,N-1)} \prod_{j=1}^{M} \frac{x_j^{n_j}}{n_j!} \right) \prod_{j=1}^{M} \mu_j e^{-\mu_j x_j} \mathbf{1}_{[0,\infty)}(x_j) \\
&= G(M, N-1)^{-1} \frac{(x_1 + \cdots + x_M)^{N-1}}{(N-1)!} \prod_{j=1}^{M} \mu_j e^{-\mu_j x_j} \mathbf{1}_{[0,\infty)}(x_j). \qquad (7)
\end{aligned}$$

Note that for the last equation we have used the multinomial theorem. The density of the distribution of the successive sojourn times is therefore the product of two functions, namely the density of a family of independent exponentially distributed random variables with parameters $\mu_1, \ldots, \mu_M$ respectively, and a function which depends on the one hand in a relative simple way on the number of the customers present and on the other hand (only) on the sum of the x_i:

$$f^{(M,N)}(x_1, \ldots, x_M) = \tilde{f}^{(M,N)}(x_1 + \ldots + x_M) \prod_{j=1}^{M} \mu_j e^{-\mu_j x_j} \mathbf{1}_{[0,\infty)}(x_j), \qquad (8)$$

with

$$\tilde{f}^{(M,N)}(x) = G(M, N-1)^{-1} \frac{x^{N-1}}{(N-1)!}. \qquad (9)$$

Having this expression for the joint density at hand we can write the density of the cycle time in the following form:

$$\begin{aligned}
g^{(M,N)}(t) &= \int_0^t \int_0^{t-x_1} \cdots \int_0^{t-x_1-\cdots-x_{M-2}} f^{(M,N)}\left(\gamma_{M-1}^t(x_1, \ldots, x_{M-1})\right) dx_{M-1} \cdots dx_2\, dx_1 \;, \\
&= \int_{\Delta_{M-1}^t} f^{(M,N)}\left(\gamma_{M-1}^t(x_1, \ldots, x_{M-1})\right) d(x_1, \cdots, x_{M-1}) \\
&= \int_{\Delta_{M-1}^t} G(M, N-1)^{-1} \frac{t^{N-1}}{(N-1)!} \mu_M e^{-\mu_M t} \prod_{j=1}^{M-1} \mu_j e^{-x_j(\mu_j - \mu_M)} d(x_1, \cdots, x_{M-1}) \qquad (10)
\end{aligned}$$

with $\gamma_{M-1}^t : \mathbb{R}^{M-1} \to \mathbb{R}^M$ defined by

$$\gamma_{M-1}^t(x_1, \ldots, x_{M-1}) := (x_1, \ldots, x_{M-1}, t - x_1 - \cdots - x_{M-1}) \qquad (11)$$

and Δ_k^t defined by

$$\Delta_k^t := \{(x_1, \ldots, x_k) \in \mathbb{R}_+^k \mid x_1 + \cdots + x_k \leqq t\}, \quad k \in \mathbb{N}. \qquad (12)$$

The functional relationship stated in equations (8) and (10) suggests to consider the conditional joint sojourn time distribution, given the cycle time, i. e. $P^{(S_1,\dots,S_M)|S_1+\dots+S_M=t}$. This leads to the following theorem.

Theorem 4.1 *Let $f\colon \mathbb{R}_+^M \to \mathbb{R}$ be measurable and bounded. The conditional expectation*

$$E\left(f\circ(S_1,\dots,S_M)\mid S_1+\dots+S_M=t\right),\quad t\in\mathbb{R}_+,\ M\in\mathbb{N},\ M\geqq 2$$

of the joint sojourn time distribution, given the cycle time, is independent of the number N of circulating customers. A version of the conditional expectation is given by

$$\begin{aligned}
&E\left(f\circ(S_1,\dots,S_M)\mid S_1+\dots+S_M=t\right) &(13)\\
&= \int\limits_{\Delta_{M-1}^t} f\left(\gamma_{M-1}^t(x_1,\dots,x_{M-1})\right) f^{(M,N)}\left(\gamma_{M-1}^t(x_1,\dots,x_{M-1})\right)\left(g^{(M,N)}(t)\right)^{-1} d(x_1,\dots,x_{M-1})\\
&= \frac{\int\limits_{\Delta_{M-1}^t} f\left(\gamma_{M-1}^t(x_1,\dots,x_{M-1})\right)\mu_M e^{-\mu_M t}\prod\limits_{j=1}^{M-1}\mu_j e^{-x_j(\mu_j-\mu_M)}\,d(x_1,\dots,x_{M-1})}{\int\limits_{\Delta_{M-1}^t}\mu_M e^{-\mu_M t}\prod\limits_{j=1}^{M-1}\mu_j e^{-x_j(\mu_j-\mu_M)}\,d(x_1,\dots,x_{M-1})}. &(14)
\end{aligned}$$

The proof of the theorem is postponed to section 5. The independence of the conditional joint sojourn time distribution given the cycle time of the number N of circulating customers is remarkable, because it means, if the cycle time is given, the distribution of the successive sojourn times the tagged customer experiences at the nodes $Q[1],\dots,Q[M]$ of his cycle is the same, irrespective of many other or nor other customers present. But if the tagged customer is the only customer present, his sojourn times coincides with his service times and their joint distribution is a distribution of an independent family $(X_1,\dots,X_M)$ the components of which are exponentially distributed with parameters μ_i respectively. So, if the tagged customer's cycle time once is given, he moves apparently through the nodes, as if he was not affected by the other customers.

In another wording theorem 4.1 takes on the following form.

Corollary 4.1 *The conditional joint distribution $P^{(S_1,\dots,S_M)|S_1+\dots+S_M=t}$ of the tagged customer's successive sojourn times, given his cycle time has, with respect to the $M-1$-dimensional Lebesgue-measure, the density*

$$d_t^M(x_1,\dots,x_{M-1}) \quad := \quad \frac{\left(f^{(M,N)}\circ\gamma_{(M-1)}^t\right)(x_1,\dots,x_{M-1})}{g^{(M,N)}(t)}\mathbf{1}_{\Delta_{M-1}^t}(x_1,\dots,x_{M-1}). \qquad (15)$$

4.1 Special cases

Let's consider the following special cases: (1) balanced machines and (2) pairwise distinct service rates.

(1) In case of balanced machines, that is $\mu_1=\dots=\mu_M=:\mu$, it follows from (5), that the density of the joint sojourn time distribution, with

$$G(M,N-1)=\sum_{\vec{n}\in Z(M,N-1)}\prod_{i=1}^M\mu^{-n_i}=\sum_{\vec{n}\in Z(M,N-1)}\mu^{-(N-1)}=\binom{M+N-2}{M-1}\mu^{-(N-1)}$$

is given by

$$f^{(M,N)}(x_1,\ldots,x_M) = \frac{(M-1)!}{(M+N-2)!}\mu^{M+N-1}(x_1+\cdots+x_M)^{N-1}e^{-\mu(x_1+\cdots+x_M)}\mathbf{1}_{\mathbb{R}_+^M}(\vec{x}).$$

The density $g^{(M,N)}$ of the cycle time is obtained by direct inversion of the Laplace-transform $\psi^{(M,N)}$ given in (4):

$$\psi^{(M,N)}(\theta) = \sum_{\vec{n}\in Z(M,N-1)} \binom{M+N-2}{M-1}^{-1} \prod_{j=1}^{M} \left(\frac{\mu}{\mu+\theta}\right)^{n_j+1} = \left(\frac{\mu}{\mu+\theta}\right)^{M+N-1},$$

that is, in the case of identical service-rates the the cycle time is $\Gamma_{\mu,M+N-1}$-distributed with density

$$g^{(M,N)}(t) = \mu^{M+N-1}\frac{t^{M+N-2}}{(M+N-2)!}e^{-\mu t}\mathbf{1}_{[0,\infty)}(t).$$

So the density of the conditional distribution is given by

$$d_t^M(x_1,\ldots,x_{M-1}) = \frac{(M-1)!}{t^{M-1}}\mathbf{1}_{\Delta_{M-1}^t}(x_1,\ldots,x_{M-1}). \tag{16}$$

After transforming this density by the mapping $(x_1,\ldots,x_{M-1}) \mapsto (\frac{x_1}{t},\ldots,\frac{x_{M-1}}{t})$, the transformed density

$$\tilde{d}^M(x_1,\ldots,x_{M-1}) = (M-1)!\mathbf{1}_{\Delta_{M-1}}(x_1,\ldots,x_{M-1}), \tag{17}$$

is independent of t. That means, in the case of balanced machines the joint sojourn time distribution is distributed as the the family $(WU_1,\ldots,WU_M)$ with W and $(U_1,\ldots,U_M)$ independent, W with $\Gamma_{\mu,M+N-1}$-distribution and $(U_1,\ldots,U_M)$ uniformly distributed on Δ_{M-1}^1. This case (with $\mu=1$) was treated in [Kel84].

(2) When all service rates are pairwise distinct one can use the formula

$$\sum_{(n_1,\ldots,n_M)\in Z(M,N)} \prod_{i=1}^{M} x_i^{n_i} = \sum_{i=1}^{M} \frac{x_i^{N+M-1}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{M}(x_i-x_j)}, \tag{18}$$

which holds for pairwise distinct real numbers $x_1,\ldots,x_M$, to invert the Laplace-transform of the cycle time. This formula is apparently due to [Koe58], who gave it without proof. A proof can be found in [Har85].
The inversion of the Laplace-transform $\psi^{(M,N)}$ of the cycle time leads now to the following density $g^{(M,N)}$.

Lemma 4.2 *[Har84], [Har85]*
In case of pairwise distinct service rates $\mu_1,\ldots,\mu_M$ the distribution of the cycle time S, experienced by the test-customer on his itinerary under steady-state conditions posesses the density

$$g^{(M,N)}(x) = G(M,N-1)^{-1}\frac{x^{N-1}}{(N-1)!}\sum_{i=1}^{M}\left(\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{\mu_j}{\mu_j-\mu_i}\right)\mu_i e^{-\mu_i x}\mathbf{1}_{[0,\infty)}(x). \tag{19}$$

Proof:

$$\begin{aligned}
\psi^{(M,N)}(\theta) &= \sum_{\vec{n}\in Z(M,N-1)} \pi^{(M,N-1)}(\vec{n}) \prod_{j=1}^{M} \left(\frac{\mu_j}{\mu_j+\theta}\right)^{n_j+1} \\
&= G(M,N-1)^{-1} \prod_{j=1}^{M} \frac{\mu_j}{\mu_j+\theta} \sum_{\vec{n}\in Z(M,N-1)} \prod_{j=1}^{M} \left(\frac{1}{\mu_j+\theta}\right)^{n_j} \\
&\overset{(*)}{=} G(M,N-1)^{-1} \prod_{j=1}^{M} \frac{\mu_j}{\mu_j+\theta} \sum_{i=1}^{M} \left(\frac{1}{\mu_i+\theta}\right)^{M+N-2} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{(\mu_i+\theta)(\mu_j+\theta)}{\mu_j-\mu_i} \\
&= G(M,N-1)^{-1} \sum_{i=1}^{M} \mu_i \left(\frac{1}{\mu_i+\theta}\right)^{N} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{\mu_j}{\mu_j-\mu_i}. \qquad (20)
\end{aligned}$$

In $(*)$ we have used formula (18). The last expression in equation (20) is invertable by inspection and leads to

$$P^S = G(M,N-1)^{-1} \sum_{i=1}^{M} \left(\frac{1}{\mu_i}\right)^{N-1} \left(\prod_{\substack{j=1\\ j\neq i}}^{M} \frac{\mu_j}{\mu_j-\mu_i}\right) \Gamma_{\mu_i,N}. \qquad (21)$$

So in this case the density of the conditional joint distribution has, with respect to the $M-1$-dimensional Lebesgue-measure, the density

$$d_t^M(x_1,\ldots,x_{M-1}) = \frac{\mu_M e^{-\mu_M t} \prod_{j=1}^{M-1} \mu_j e^{-x_j(\mu_j-\mu_M)}}{\sum_{i=1}^{M} \mu_i e^{-\mu_i t} \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{\mu_j}{\mu_j-\mu_i}} \mathbf{1}_{\Delta_{M-1}^t}(x_1,\ldots,x_{M-1}). \qquad (22)$$

Note that in the case of balanced machines the system do not show any bottleneck behaviour. In the other case, when there are servers working with different rates, the slowest server becomes the bottleneck of the system: if the number N of customers increases, the number of customers at the slowest queue grow to infinity, while the marginal distribution of the other nodes remains not only finite (see [GN67]), but also the number of customers remains comparatively small. With respect to the bottleneck behaviour and especially in view of the results of [Box88] concerning the influence of the slowest server on the sojourn times of the test customer (see next subsection), the independence of the joint conditional sojourn time distribution of the number N of circulating customers is astonishing and remarkable.

4.2 The Influence of the slowest server

In case of pairwise distinct service rates $\mu_1,\ldots,\mu_M$ one easily gets the mean cycle time from the Laplace-transform (4) of the cycle time $S = S(N)$ (N customers circulating).

$$E\big(S(N)\big) = N\frac{1}{\mu_1}\sum_{i=1}^{M}\left(\frac{\mu_1}{\mu_i}\right)^{N+1}\prod_{\substack{j=1\\ j\neq i}}^{M}\left(1-\frac{\mu_i}{\mu_j}\right)^{-1} \tag{23}$$

$$= N\frac{\sum_{i=1}^{M}\left(\frac{1}{\mu_i}\right)^{N+M-1}\prod_{\substack{j=1\\ j\neq i}}^{M}\left(\frac{1}{\mu_i}-\frac{1}{\mu_j}\right)^{-1}}{\sum_{i=1}^{M}\left(\frac{1}{\mu_i}\right)^{N+M-2}\prod_{\substack{j=1\\ j\neq i}}^{M}\left(\frac{1}{\mu_i}-\frac{1}{\mu_j}\right)^{-1}}. \tag{24}$$

In particular it follows that ([Box88])

$$E\big(S(N)\big) = N\mu_1^{-1}\big\{1+O\big([\frac{\mu_1}{\mu_2}]^N\big)\big\}, \quad N\to\infty. \tag{25}$$

To see this take (for example) formula (24) and use Buzen's algorithm [Buz73].
Similarly one gets for the second moment and the variance of $S(N)$ (see [Box88], p. 15)

$$E\big(S(N)^2\big) = N(N+1)\mu_1^2\big\{1+O\big([\frac{\mu_2}{\mu_1}]^N\big)\big\}, \quad N\to\infty, \tag{26}$$

$$\mathrm{Var}\big(S(N)\big) = N\mu_1^2\big\{1+O\big([\frac{\mu_2}{\mu_1}]^N\big)\big\}, \quad N\to\infty. \tag{27}$$

From (25) obviously in heavy traffic the slowest queue generates the main fraction of the cycle time of the test-customer. This clearly reflects the bottleneck behaviour with respect to the number of customers. So it is reasonable to approximate the distribution of the cycle time for great values of N by the sum of N consecutive service times at the slowest queue. Comparing this with the discussion in the last paragraph of subsection 4.1 we see that there remains a gap in the intuition for this approximation and the invariance property for the conditional sojourn time distribution stated in theorem 4.1.

Another point of view is to consider the output process from a specific node. In the case of exponential single-server nodes it follows from corollary 3.1 that the cycle time distribution is independent of the choice of the start node. This is in contrast to the case, where not all service demands are exponentially distributed. In that case the cycle time distribution depends on the start node of the cycle, see [Box83].
So let the cycle time S now be defined as the time between N successive departures from node $Q[1]$, i. e.

$$S = i_1 + s_1 + i_2 + s_2 + \cdots + i_N + s_N, \tag{28}$$

with i_n the n-th idle time of server $Q[1]$ after service completion of the (arbitrarily choosen) test-customer and s_n the n-th service time, i. e. the service time at queue $Q[1]$ after the n-th idle time i_n.
This approach is discussed in detail in [Zaz97] using point process methods. Applying successively formulas for Palm and customer stationary distributions expressions for the cycle time distribution were derived by an iterative procedure with increasing population size.

5 Proof of theorem 4.1

Proof of theorem 4.1. First consider

$$
\begin{aligned}
& E\left(f \circ (S_1, \ldots, S_M) \mid S = t\right) && (29)\\
&= \int\limits_{\Delta^t_{M-1}} f\left(\gamma^t_{M-1}\left(\vec{x}_{M-1}\right)\right) f^{(M,N)}\left(\gamma^t_{M-1}\left(\vec{x}_{M-1}\right)\right)\left(g^{(M,N)}(t)\right)^{-1} d(x_1, \cdots, x_{M-1})\\
&= \int\limits_{\Delta^1_{M-1}} f\left(\gamma^t_{M-1}\left(t\vec{x}_{M-1}\right)\right) f^{(M,N)}\left(\gamma^t_{M-1}\left(t\vec{x}_{M-1}\right)\right) t^{M-1}\left(g^{(M,N)}(t)\right)^{-1} d(x_1, \cdots, x_{M-1})\\
&= \int\limits_{(0,1)^{M-1}} f\left(\gamma^t_{M-1}\left(tJ^{M-1}_1(\vec{x}), \ldots, tJ^{M-1}_{M-1}(\vec{x})\right)\right) f^{(M,N)}\left(\gamma^t_{M-1}\left(tJ^{M-1}_1(\vec{x}), \ldots, tJ^{M-1}_{M-1}(\vec{x})\right)\right)\\
&\qquad \cdot t^{M-1}\left(g^{(M,N)}(t)\right)^{-1} x_1^{M-2}(1-x_3)(1-x_4)^2\cdots(1-x_{M-1})^{M-3}\, d(x_1, \cdots, x_{M-1}).
\end{aligned}
$$

Here $J^k\colon \mathbb{R}^k \to \mathbb{R}^k$ is the Jacobi-mapping of dimension k, which is defined for $k \in \mathbb{N}$, $k \geqq 2$ by

$$
\begin{aligned}
J^2(x_1, x_2) &= \left(x_1\left(1-x_2\right), x_1 x_2\right), && (30)\\
J^M(x_1, \ldots, x_M) &= \left(J^{M-1}\left(x_1, \ldots, x_{M-1}\right)\left(1-x_M\right), x_1 x_M\right), \quad M \geqq 3, && (31)
\end{aligned}
$$

oder explicitly for $M \geqq 2$ by (the j-th coordinate $\mathrm{pr}_j \circ J^M$ is denoted by J^M_j)

$$
\begin{aligned}
J^M_1(x_1, \ldots, x_M) &= x_1 \prod_{k=2}^{M}(1-x_k), && (32)\\
J^M_j(x_1, \ldots, x_M) &= x_1 x_j \prod_{k=j+1}^{M}(1-x_k), \quad j = 2, \ldots, M. && (33)
\end{aligned}
$$

The Jacobi-mapping J^M maps $\mathbb{R}^*_+ \times (0,1)^{M-1}$ diffeomorph onto ${\mathbb{R}^*_+}^M$ and $(0,1)^M$ diffeomorph onto $\left(\Delta^1_M\right)^\circ$, the interior of Δ^1_M ($\mathbb{R}^*_+$ is defined as $\mathbb{R}^*_+ := \{x \in \mathbb{R} \mid x > 0\}$). The jacobian of the Jacobi-mapping is given by $\det\left(DJ^M\left(x_1, \ldots, x_M\right)\right) = x_1^{M-1} \prod\limits_{j=3}^{M}(1-x_j)^{j-2}$. So the second and the third equality are applications of the transformation-theorem.
After the Integration over Δ^t_{M-1} in (13) is led back to an Integration over $(0,1)^{M-1}$ one could proceed as follows:

$$\int\limits_{\mathbb{R}_+^*} \mathbf{1}_{[0,a]}(t) E\left(f \circ (S_1, \ldots, S_M) \mid S_1 + \cdots + S_M = t\right) dP^S(t) \tag{34}$$

$$\begin{aligned}
&= \int\limits_{\mathbb{R}_+^*} \mathbf{1}_{[0,a]}(t) \int\limits_{(0,1)^{M-1}} f\left(\gamma_{M-1}^t\left(tJ_1^{M-1}(\vec{x}), \ldots, tJ_{M-1}^{M-1}(\vec{x})\right)\right) f^{(M,N)}\left(\gamma_{M-1}^t\left(tJ_1^{M-1}(\vec{x}), \ldots, tJ_{M-1}^{M-1}(\vec{x})\right)\right) \\
&\qquad \cdot t^{M-1}\left(g^{(M,N)}(t)\right)^{-1} x_1^{M-2}(1-x_3)(1-x_4)^2 \cdots (1-x_{M-1})^{M-3}\, d(x_1, \cdots, x_{M-1})\, dP^S(t) \\
&= \int\limits_{\mathbb{R}_+^* \times (0,1)^{M-1}} \mathbf{1}_{[0,a]}(t) f\left(\gamma_{M-1}^t\left(tJ_1^{M-1}(\vec{x}), \ldots, tJ_{M-1}^{M-1}(\vec{x})\right)\right) f^{(M,N)}\left(\gamma_{M-1}^t\left(tJ_1^{M-1}(\vec{x}), \ldots, tJ_{M-1}^{M-1}(\vec{x})\right)\right) \\
&\qquad \cdot t^{M-1} x_1^{M-2}(1-x_3)(1-x_4)^2 \cdots (1-x_{M-1})^{M-3}\, d(t, x_1, \cdots, x_{M-1}) \\
&\overset{(*)}{=} \int\limits_{\mathbb{R}_+^* \times (0,1)^{M-1}} \mathbf{1}_{[0,a]}(t) f\left(J^M(t, x_2, x_3, \ldots, x_{M-1}, 1-x_1)\right) f^{(M,N)}\left(J^M(t, x_2, x_3, \ldots, x_{M-1}, 1-x_1)\right) \\
&\qquad \cdot t^{M-1} x_1^{M-2}(1-x_3)(1-x_4)^2 \cdots (1-x_{M-1})^{M-3}\, d(t, x_1, \cdots, x_{M-1}) \\
&= \int\limits_{\mathbb{R}_+^* \times (0,1)^{M-1}} \mathbf{1}_{[0,a]}(x_1) f\left(J^M(x_1, x_2, x_3, \ldots, x_{M-1}, x_M)\right) f^{(M,N)}\left(J^M(x_1, x_2, x_3, \ldots, x_{M-1}, x_M)\right) \\
&\qquad \cdot x_1^{M-1}(1-x_3)(1-x_4)^2 \cdots (1-x_{M-1})^{M-3}(1-x_M)^{M-2}\, d(x_1, x_2, \cdots, x_M)
\end{aligned}$$

$$\overset{(**)}{=} \int\limits_{\mathbb{R}_+^{*\,M}} \mathbf{1}_{[0,a]}(x_1 + \cdots + x_M) f(x_1, \ldots, x_M) \prod_{j=1}^M \mu_j e^{-\mu_j x_j} \cdot \frac{(x_1 + \cdots + x_M)^{N-1}}{G(M, N-1)(N-1)!}\, d(x_1, \cdots, x_M) \tag{35}$$

In $(*)$ we have utilized the relation

$$\gamma_{M-1}^t\left(tJ_1^{M-1}(x_1, x_2, \ldots, x_{M-1}), \ldots, tJ_{M-1}^{M-1}(x_1, x_2, \ldots, x_{M-1})\right) = J^M(t, x_2, \ldots, x_{M-1}, 1-x_1)$$

between γ_{M-1}^t and the mappings J^{M-1} and J^M. To see this relation consider

$$\begin{aligned}
&\gamma_{M-1}^t\left(tJ_1^{M-1}(x_1, x_2, \ldots, x_{M-1}), \ldots, tJ_{M-1}^{M-1}(x_1, x_2, \ldots, x_{M-1})\right) \\
&= \left(tJ_1^{M-1}(x_1, x_2, \ldots, x_{M-1}), \ldots, tJ_{M-1}^{M-1}(x_1, x_2, \ldots, x_{M-1}), t - t\sum_{j=1}^{M-1} J_j^{M-1}(x_1, \ldots, x_{M-1})\right) \\
&= \left(tJ^{M-1}(x_1, x_2, \ldots, x_{M-1}), t(1-x_1)\right) \\
&\overset{(\ddagger)}{=} \left(x_1 J^{M-1}(t, x_2, \ldots, x_{M-1}), t(1-x_1)\right) \\
&= J^M(t, x_2, \ldots, x_{M-1}, 1-x_1).
\end{aligned}$$

In $(\ddagger)$ we make use of $tJ^{M-1}(x_1, \ldots, x_{M-1}) = x_1 J^{M-1}(t, \ldots, x_{M-1})$ which follows directely from the explicit definition of the Jacobi-mapping (32) and (33).
Finally $(**)$ is again the application of the transformation-theorem. The expression in equation (35) coincides with

$$\int\limits_{S^{-1}[0,a]} f \circ (S_1, \ldots, S_M)\, dP,$$

which ends the proof.

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