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Exponential queueing networks with an attached inventory under (r, Q)- or (r, S)-policy

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Abstract

We integrate a server with attached inventory under (r, Q)- or (r, S)-policy into Jackson or Gordon-Newell networks. Replenishment lead times are non–zero and random and depend on the load of the system. While the inventory is depleted the server with attached inventory does not accept new customers (lost sales situation). We pursue three different approaches to handle routing with respect to this node during the time the inventory is empty.
We derive stationary distributions of joint queue lengths and inventory processes in explicit product form. The stationary distributions are then used to calculate performance measures of the respective systems.

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Key words: Jackson networks; Gordon-Newell networks; product form solution; stationary distributions; inventory systems; performance analysis; inventory policy; service level; inventory level.

1 Introduction

Recently there has been a considerable interest in inventory management for improving the quality of service (QoS) of today's service systems. An important question is how inventory management has to react to queueing of demands, resp. customers who present that demand, which is due to incorporated service facilities. An early contribution is [SSL92] where approximation procedures are used to find performance descriptions for models in which the interaction of queueing for service and inventory control is integrated. In a sequence of papers Berman and his coauthors ([BK99], [BK01], [BS00], [BS01], [BS02]) investigate the behavior of service systems with an attached inventory. A common assumption is the continuous review structure for the inventory. Furthermore, all those models assume that customers who arrive during the time the inventory is depleted join the queue. This is parallel to backordering shortages in the classical inventory management models. In the standard pure inventory system theory the alternative regime to handle shortages of the inventory is termed 'lost sales'. Here unsatisfied customer demand is lost without influencing the future behavior of the lost customer (see [Min00]). To model the lost sales situation when combining queueing for service and inventory management we stop the demand process during stock-outs.

The number of papers contributing to the problem of how the classical performance measures (e.g. queue length, waiting time, etc.) are influenced by the management of attached inventory and vice versa is still limited. Even more, following Mohebbi and Posner [MP98] the area of continuous review inventory systems with lost sales remains largely unexplored.

For the lost sales case some first results are obtained in [SSD+03]: Various M/M/1/∞- resp. M/M/1/N- servers with attached inventory under several replenishment policies, and with exponentially distributed replenishment lead times and lost sales were investigated. The inventory management policies include the well-known (r, Q)- and (r, S)-strategies and a policy with reorder point 0 and random size of replenishment orders. The stationary distributions of these systems were explicitly computed and show a product form. To the best of our knowledge these are the first closed form solutions to service facilities with attached inventories and stochastic lead times.

Integration of such a service station with an attached inventory into a complex queueing network is a challenging problem. The present paper is a first attempt to construct models where steady state distributions are explicitly available. The importance of having direct access to these distributions is obvious: almost all first order performance measures can be obtained from this. We therefore pursue the objective of obtaining easy-to-handle product form results for the steady states. Performance measures of the system can then easily be computed.

We assume that lost sales at the dedicated server with the inventory are not necessarily lost to the whole system, resp., the network. This is in line with the principles of flexible manufacturing, where stations can to a certain extend substitute duties and work of other stations. Moreover this enables customers after being served at other nodes to which they have been directed to try to reenter the dedicated node, i.e., we have to incorporate retrial of customers after a random time, which may depend on the routing control of the network. An essential feature of our models is therefore how to reregulate the customer flows when the inventory indicates a stockout. To resolve this we suggest several rerouting principles which are called STALLING, BLOCKING, and SKIPPING and investigate these in detail. Similar principles are used when dealing with certain overflow and blocking nodes in communication networks and were further developed in [SD03a] to regulate customer flows in networks with unreliable servers.

We will integrate one server with an attached inventory under (r, Q)- or (r, S)-policy into Jackson or Gordon-Newell networks. We investigate models with different replenishment lead time intensities while the inventory is depleted. The lead time intensities are state dependent this enables the system to react to the actual load observed in the network. For all models we derive explicit expressions for the corresponding stationary distributions which turn out to be of product form.

The rest of the paper is organized as follows: In Section 2 we quote as our starting point results on classical exponential networks with stochastic routing and state-dependent service rates. In Section 3 we introduce inventory management policies and present a state space, which allows for a Markovian description of the system. In Sections 4 and 5 we state and prove our main results on product form steady states using the three rerouting principles. In Section 6 we derive for the different systems performance measures related to inventory and service of customers. The paper is rounded out with some conclusions and an outlook in Section 7. Table 1 at the end of this paper gives an overview of the different models under investigation.

Throughout the paper we set empty sums to 0 and empty products to 1, $\subset$ means subset or equal. $\mathbb{N} := \{1, 2, 3, \dots\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{R}$ are the real numbers. We denote by

$$\delta_{ab} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases}$$

the Kronecker delta symbol. The indicator function for a set A is denoted by

$$1_A(x) = 1_{\{x\in A\}} = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

2 Standard separable networks

In this section we summarize definitions and theorems on classical exponential networks which will be needed in the sequel. The open network has been introduced by Jackson [Jac57]. Its closed analogue, the Gordon-Newell network, has already been introduced by Jackson some years later, see [Jac63], and has been rediscovered by Gordon and Newell [GN67].

Definition 2.1 (Jackson network) *A Jackson network is a network of service stations (nodes) numbered* $\{1, 2, \ldots, J\} =: \bar{J}$. *Station* j *is a single server with infinite waiting room under FCFS regime. Customers in the network are indistinguishable. At node* j *there is an external Poisson–*λ_j *arrival stream,* $\lambda_j \geq 0$. *Customers arriving at node* j *from the outside or from inside of the network request for an amount of service time which is exponentially distributed with mean* 1. *Service at node* j *is provided with intensity* $\mu_j(n_j) > 0$, *if there are* $n_j > 0$ *customers present at node* j, *and otherwise* $\mu_j(0) := 0$. *All service and inter-arrival times are assumed to be independent.*

Movements of the customers in the network are governed by a Markovian routing mechanism: a customer on leaving node i *selects with probability* $r(i,j) \geq 0$ *to visit node* j *next, and then enters node* j *immediately, commencing service if she finds the server idle, otherwise she joins the tail of the queue at node* j*; with probability* $r(i,0) \geq 0$ *this customer decides to leave the network immediately* ($\sum_{j=0}^{J} r(i,j) = 1$ *for all* $i \in \bar{J}$.) *Given the departure node* i *the customer's routing decision is made independently of the network's history.*

Let $\bar{J}_0 := \{0, 1, \ldots, J\}$. *With* $\lambda := \sum_{j=1}^{J} \lambda_j$, $r(0,j) := \frac{\lambda_j}{\lambda}$ *and* $r(0,0) := 0$ *we assume that the routing matrix* $\mathcal{R} := (r(i,j) : i, j \in \bar{J}_0)$ *is irreducible.*

Let $X_j(t)$ *denote the number of customers present at node* j *at time* $t \geq 0$, *either waiting or in service (local queue length at node* j*), then* $X(t) := (X_j(t) : j \in \bar{J})$ *is the joint queue length vector of the network at time* t *and* $X = (X(t) : t \geq 0)$ *is the joint queue length process of the Jackson network with state space* $E := \mathbb{N}_0^J$.

Remark 2.2 *Note that the class of exponential FCFS single servers with state dependent service rates encompass the classical exponential FCFS multi-servers with state-independent service rates.*

Theorem 2.3 ([Jac57]) *The joint queue length process* X *of the Jackson network is a Markov process with transition rates* $(q(x,y) : x, y \in E, x \neq y)$ *given by:*
for $i, j \in \bar{J}$ *and* $x = (n_1, \ldots, n_J) \in E$

$$\begin{aligned} q(n_1, \ldots, n_i, \ldots, n_J;\, n_1, \ldots, n_i + 1, \ldots, n_J) &= \lambda_i \\ \textit{and if} \quad n_i > 0: & \\ q(n_1, \ldots, n_i, \ldots, n_J;\, n_1, \ldots, n_i - 1, \ldots, n_J) &= \mu_i(n_i)\, r(i,0) \\ q(n_1, \ldots, n_i, \ldots, n_j, \ldots, n_J;\, n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_J) &= \mu_i(n_i)\, r(i,j). \end{aligned}$$

Furthermore

$$q(x,x) = -\sum_{y \in E \setminus \{x\}} q(x,y) \text{ and } q(x,y) = 0 \text{ otherwise.}$$

The traffic equation of the network

$$\eta_j = \lambda_j + \sum_{i=1}^{J} \eta_i r(i,j), \quad j \in \bar{J}, \tag{1}$$

has a unique solution which we denote by $\eta = (\eta_1, \dots, \eta_J)$. *For technical reasons we set* $\eta_0 := \lambda$. *We assume henceforth* X *to be ergodic implying that the unique stationary and limiting distribution* π *of* X *is given by*

$$\hat{\pi}(n_1, \dots, n_J) = K(J)^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right), \quad (n_1, \dots, n_J) \in \mathbb{N}_0^J$$

with finite normalization constant

$$K(J) \quad = \quad \sum_{(n_1,\dots,n_J) \in \mathbb{N}_0^J} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right). \tag{2}$$

Definition 2.4 (Gordon-Newell network) *A Gordon-Newell network consists of a set of single server nodes* $\bar{J} := \{1, 2, \dots, J\}$ *as described in Definition 2.1 of the Jackson network without external arrivals and departures, i.e. the probabilities* $r(j,0)$ *for all* $j \in \bar{J}$ *and the total network arrival rate* λ *are set to zero. There are* $D > 0$ *indistinguishable customers cycling in the network according to an irreducible Markov matrix* $\mathcal{R} = (r(i,j) : i,j \in \bar{J})$. *The service times are the same as in the open Jackson network case and the independence assumptions on service times and routing decisions are assumed to hold as well.*

Let $X_j(t)$ *denote the number of customers present at node* j *at time* $t \geq 0$, *either waiting or in service (local queue length at node* j*). Then* $X(t) := (X_j(t) : j \in \bar{J})$ *is the joint queue length vector of the network at time* t. *We denote by* $X = (X(t) : t \geq 0)$ *the joint queue length process of the Gordon-Newell network with state space* $E := S(J,D) := \{(n_1, n_2, \dots, n_J) \in \mathbb{N}_0^J : n_1 + n_2 + \dots + n_J = D\}$.

Theorem 2.5 ([Jac63], [GN67]) *The joint queue length process of the Gordon-Newell network* X *is a Markov process with transition rates* $(q(x,y) : x, y \in S(J,D), x \neq y)$ *given by:*
for $i, j \in \bar{J}$ *and* $x = (n_1, \dots, n_J) \in S(J,D)$:

$$q(n_1 \dots, n_i, \dots, n_j, \dots, n_J;\, n_1, \dots, n_i - 1, \dots, n_j + 1, \dots, n_J) = \mu_i(n_i) r(i,j) \ \textit{if } n_i > 0$$

and

$$q(x,x) = - \sum_{y \in S(J,D) \setminus \{x\}} q(x,y),$$
$$q(x,y) = 0, \ \textit{otherwise}.$$

The process X *is ergodic. Let* $\eta = (\eta_1, \dots, \eta_J)$ *denote the unique probability solution of the traffic equation*

$$\eta_j = \sum_{i=1}^{J} \eta_i r(i,j), \ j \in \bar{J}. \tag{3}$$

The unique stationary and limiting distribution $\pi = \pi(J,D)$ *of* X *on* $S(J,D)$ *is*

$$\pi(n_1, \dots, n_J) = G(J,D)^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) \ \textit{for } (n_1, \dots, n_J) \in S(J,D), \tag{4}$$

with $G(J,D)$ *as normalization constant.*

Remark 2.6 *In the following we will precisely distinguish in Jackson networks between the case of state-dependent service rates with normalization constant* $K^d(J)$ *given by*

$$K^d(J) := \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right),$$

and the case of state-independent service rates, associated with the former by:

$$\mu_j(n_j) := \begin{cases} \mu_j, & n_j > 0, \\ 0, & n_j = 0, \end{cases}$$

with normalization constant $K^i(J)$ *given by*

$$K^i(J) := \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} \prod_{j=1}^{J} \left(\frac{\eta_j}{\mu_j}\right)^{n_j} = \prod_{j=1}^{J} \frac{\mu_j}{\mu_j - \eta_j}.$$

Analogously we distinguish in Gordon-Newell networks between the case of state-dependent service rates with normalization constant $G^d(J, D)$ *and the case of state-independent service rates with normalization constant* $G^i(J, D)$.

3 Introducing inventory management with (r, Q)- or (r, S)-policy

In the networks described in Section 2 we now consider the server at the first node to have an attached inventory with (r, Q)- or (r, S)-policy. This means that this server needs for the service of each customer exactly one item from the attached inventory. The on-hand inventory, which is the stock that is physically on the shelf, decreases by one at the moment of service completion. If the inventory position reaches a predetermined value, the reorder point r, $r \geq 0$, a replenishment order is precautionarily triggered. In the (r, Q)-policy case the size of the replenishment order is fixed to Q units, with $Q > r$. In the (r, S)-policy case the inventory on hand is restocked to the order-up-to level S, with $S > r$ on each replenishment.

The service of customers is continued during the replenishment lead time as long as the inventory on hand is positive. But if the inventory is depleted the service of customers is stopped. Just before the inventory becomes empty the customer, who took the last piece from inventory, jumps to her next destination node $j \in \bar{J}_0$ according to the routing matrix $\mathcal{R}$. Note that in case of $r(1, 1) > 0$ the customer can rejoin the queue at node 1 for another service there.

As has been shown in [SSD+03] M/M/1/∞-systems with attached inventory under (r, Q)- or (r, S)-policy and exponential lead times with constant rate ν have a stationary distribution of product form if newly arriving customers are rejected during the time the inventory is depleted (lost sales). In case of a $\cdot$/M/1/∞-server with (r, Q)- or (r, S)-policy embedded as node 1 into a Jackson resp. Gordon-Newell network we will therefore assume that the server does not accept new customers while the inventory is empty. We will pursue different approaches to handle routing during this time span below. In Sections 4.1 and 5.1 we will force all customers' activities to grind to a halt, using the Stalling principle, whereas in Sections 4.2, 4.3 and 5.2 we will discuss two rerouting principles which originate from the theory of information blocking, namely Blocking and Skipping.

The assumption $r < Q$ resp. $r < S$, which is a common restriction of stochastic lost sales models [BL85], avoids that more than one order is outstanding at a time. In our case this restriction entails that in order to determine the moment a replenishment order has to be triggered we can simply work with the on-hand inventory instead of the common inventory position which means the on-hand stock plus outstanding orders minus backorders.

To describe the inventory process at the first node in addition to the queue lengths at all nodes from $\bar{J}$ in Jackson resp. Gordon-Newell networks we have to enlarge the state spaces $E = \mathbb{N}_0^J$ resp. $E = S(J, D)$ of the standard Jackson resp. Gordon-Newell network processes by adding a component which carries information about the on-hand inventory position.

This information is sufficient for a Markovian description of the system. An even more detailed system description is introduced in Section 5, where we quote in addition every time the inventory is depleted the node $j \in \bar{J}_0$ to which the last customer departing from node 1 has jumped just before the inventory became empty. This will lead to more detailed information on the stationary behavior of the system via more explicit balance equations. Even more important, we will be able to resolve in this way the lost sales problem in situations which are not accessible to the Markovian description in this section.

We summarize all positive inventory positions in set $\bar{I}$ and all possible inventory positions in set $\bar{I}_0 := \bar{I} \cup \{0\}$. Hence, in case the first server has an attached inventory under (r, Q)-policy $\bar{I} := \{1, 2, ..., Q + r\}$ and in case of an inventory under of (r, S)-policy $\bar{I} := \{1, 2, ..., S\}$.

Definition 3.1 (State space) *For describing the above introduced Jackson resp. Gordon-Newell networks with an inventory at the first node we introduce states of the form*

$$(n_1, ..., n_J; k) \in E \times \bar{I}_0, \tag{5}$$

summarized in state space $\widetilde{E} := E \times \bar{I}_0$. The meaning of such a prototype state is:
the numbers n_j indicate that there are n_j customers present at node $j \in \bar{J}$. The inventory on-hand at server 1 is at position $k \in \bar{I}_0$. Using this state space we will construct new Markovian network processes on $\widetilde{E}$ which are denoted by $\widetilde{X}$.

In the sequel we present two different approaches to model the system described above. The difference between the two models, denoted by M_{I} and M_{II}, primarily lies in the definition of the replenishment lead time intensity while the inventory is depleted. In model M_{I}, replenishment lead time rates are state-dependent for all inventory positions k, $0 \leq k \leq r$. In the second model M_{II} the replenishment lead time is exponentially distributed with constant parameter ν in case of depleted inventory. This is a desired assumption (see [BL85], [MP98]). In both models the stationary distribution is of product form.

The replenishment lead times for the two models are specified as follows.

Definition 3.2 (Replenishment rates) *The time until an ongoing lead time expires at the $\cdot/M/1/\infty$-server with attached inventory under (r, Q)- or (r, S)-policy in a Jackson resp. Gordon-Newell network is governed by state-dependent intensities.*

For all network states $x \in \widetilde{E}$ the state dependent intensity of the replenishment lead times are as follows:

for M_{I} :

$$\nu_{\mathrm{I}}(x) = \nu \left(r(1,0) + \sum_{l=1}^{J} r(1,l) \frac{\mu_l(n_l)}{\eta_l} \right) 1_{\{0 \leq k \leq r\}}, \qquad x = (n_1, ..., n_J; k) \in \widetilde{E},$$

for M_{II} :

$$\nu_{\mathrm{II}}(x)=\begin{cases}\nu\left(r(1,0)+\sum_{l=1}^{J} r(1,l)\frac{\mu_l(n_l)}{\eta_l}\right)1_{\{1\le k\le r\}}, & x=(n_1,...,n_J;k)\in\widetilde{E} \text{ with } k\in\bar{I},\\ \nu, & x=(n_1,...,n_J;0)\in\widetilde{E}.\end{cases}$$

Remark 3.3 *Note that in both models* M_{I} *and* M_{II} *the intensity of the replenishment lead time in states* $x=(n_1,...,n_J;k)\in\widetilde{E}$ *with* $1\le k\le r$ *is at any rate positive, if* $r(1,0)>0$ *independently of the number and positions of customers in the system. In case of* $r(1,0)=0$ *it may be* 0 *if there are no customers in the system or if customers are only present at nodes* $l\in\bar{J}$ *for which* $r(1,l)=0$.

If $\mu_l(\cdot)$ *is increasing in the number of customers at node* l*, then so is the intensity of the replenishment lead time. Hence, if* $\mu_l(\cdot)$ *is increasing for all* $l\in\bar{J}$*, then the more customers there are in the system the shorter the expected length of the replenishment lead time will be. This means: the duration of the lead times reacts to the load of exactly those nodes where an expected departure from node* 1 *will occur with positive probability.*

If in addition the routing matrix is reversible this behavior of the system has an appealing interpretation. Reversibility means

$$\eta_j\, r(j,i)=\eta_i\, r(i,j) \text{ for } i,j\in\bar{J}_0 \text{ with } \eta_j\in\bar{J} \text{ from (1) and } \eta_0=\lambda. \tag{6}$$

Thus, using $r(1,0)=\lambda\, r(0,1)/\eta_1=\lambda_1/\eta_1$ *the state-dependent intensity of the exponentially distributed lead time in a state* $x=(n_1,...,n_J;k)$ *with* $0\le k\le r$ *for model* M_{I} *and* $1\le k\le r$ *for model* M_{II} *is given by*

$$\nu(x)=\nu\left(r(1,0)+\sum_{l=1}^{J} r(1,l)\frac{\mu_l(n_l)}{\eta_l}\right)=\frac{\nu}{\eta_1}\left(\lambda_1+\sum_{l=1}^{J} r(l,1)\mu_l(n_l)\right).$$

$\lambda_1+\sum_{l=1}^{J}\mu_l(n_l)r(l,1)$ *is the actual arrival rate to server 1 depending on the customers in the system* $(n_1,...,n_J)\in E$ *and* η_1 *is the overall arrival rate to server 1 in steady state. Hence, the larger the actual intensity of customers' arrivals to server 1 the larger the arrival intensity for replenishment orders. (Reversibility of the routing matrix will be assumed in Section 4.2).*

Example 3.4 *Assume* $r(1,j)=0$ *for all* $j\in\bar{J}$*. Then, after being served at node 1 customers leave the network with probability 1. In this special case, models* M_{I} *and* M_{II} *coincide. The intensity of the replenishment lead time is equal to* ν *if and only if* $0\le k\le r$*. This situation can be looked upon as considering node* 1 *as a single exponential server with attached inventory as dealt with in [SSD+03] being fed in part by the departure process from the residual network, which is a Poisson process.*

Furthermore, we define a function $C:\bar{I}_0\to\mathbb{R}_+$ as follows. It always holds: $C(0):=\dfrac{\eta_1}{\nu}$.
In case of using (r,Q)-policy we set

$$\begin{aligned} C(k) &:= \left(\frac{\eta_1+\nu}{\eta_1}\right)^{k-1} && \text{for } k=1,...,r,\\ C(k) &:= \left(\frac{\eta_1+\nu}{\eta_1}\right)^{r} && \text{for } k=r+1,...,Q,\\ C(k) &:= \left(\frac{\eta_1+\nu}{\eta_1}\right)^{r}-\left(\frac{\eta_1+\nu}{\eta_1}\right)^{k-Q-1} && \text{for } k=Q+1,...,Q+r.\end{aligned}$$

Then the following fundamental equation for the (r,Q)-policy system derived in [SSD+03] equation (14) is satisfied for all $1 \le k \le Q+r$:

$$C(k+1)(1-\delta_{Q+r,k}) = C(k)\left(1+\frac{\nu}{\lambda}1_{\{1\le k\le r\}}\right) - 1_{\{k=Q\}} - \frac{\nu}{\lambda}C(k-Q)1_{\{k>Q\}}. \tag{7}$$

In case of using (r,S)-policy we set

$$\begin{aligned} C(k) &:= \left(\frac{\eta_1+\nu}{\eta_1}\right)^{k-1} && \text{for } k=1,...,r, \\ C(k) &:= \left(\frac{\eta_1+\nu}{\eta_1}\right)^{r} && \text{for } k=r+1,...,S, \end{aligned}$$

such that the fundamental equation for the (r,S)-policy system derived in [SSD+03] equation (31) is satisfied for all $1 \le k \le S$:

$$C(k)\left(\eta_1+\nu 1_{\{1\le k\le r\}}\right) = \eta_1 C(k+1)(1-\delta_{Sk}) + \eta_1 1_{\{k=S\}} + \nu\sum_{h=1}^{r} C(h)1_{\{k=S\}}. \tag{8}$$

Using these functions we now define probability measures on the enlarged state space $\widetilde{E}$. It will be shown in Section 4 that these probability measures are the unique stationary and limiting distributions of the describing supplemented Markov process $\widetilde{X}$ in models M_{I} and M_{II}, respectively, with various inventory management policies.

Definition 3.5 (Product form equilibrium states) *On state space $\widetilde{E}$ we define two probability measures as follows. For $(n_1,...,n_J;k) \in \widetilde{E}$ we set:*

$$\pi_{\mathrm{I}}(n_1,...,n_J;k) = K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)C(k), \tag{9}$$

$$\pi_{\mathrm{II}}(n_1,...,n_J;k) = \begin{cases} K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)C(k) & \text{for } k\in\bar{I}, \\ K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)C(0)\left(r(1,0)+\sum_{j\in\bar{J}}\frac{\mu_j(n_j)}{\eta_j}r(1,j)\right) & \text{for } k=0, \end{cases} \tag{10}$$

where $C(k)$, $k\in\bar{I}_0$, is the value of the function C defined above for the (r,Q)- resp. (r,S)-policy. $\eta=(\eta_1,...,\eta_J)$ is the solution of the traffic equation (1) resp. (3) of the standard Jackson resp. Gordon-Newell network. K is the normalization constant:

$$K = K^d(J)\sum_{k\in\bar{I}_0} C(k)$$

in case of open networks, where we assume $K^d(J)$ to be finite, and

$$K = G^d(J,D)\sum_{k\in\bar{I}_0} C(k)$$

in case of closed networks.

4 The steady state results for networks with attached inventory

In this section we define three rerouting regimes, namely Stalling, Blocking and Skipping which regulate customers' movements in the network in times the inventory at node 1 is depleted. We show that the describing Markov process $\widetilde{X}$ of model M_{I} with replenishment rate $\nu_{\mathrm{I}}(\cdot)$ according to Definition 3.2 has steady state probabilities π_{I} from Definition 3.5 (9) for all three rerouting regimes. The describing Markov process $\widetilde{X}$ of model M_{II} with replenishment rate $\nu_{\mathrm{II}}(\cdot)$ has stationary distribution π_{II} from Definition 3.5 (10) only if the Stalling principle is used.

4.1 Stalling in case of depleted inventory

We establish in this section that in case of depleted inventory the total movement of customers in the network is brought to a stop. Only the replenishment process is ongoing. In particular, all service intensities at the nodes are set to zero. In addition customers who want to enter the network from the outside in open networks are rerouted to an immediate departure or equivalently the overall arrival rate λ is set to zero. Service at all nodes is not resumed until the replenishment order has arrived at the first node.

The underlying principle, which we call Stalling, following [Sch84], is well known in the literature, see e.g. [Mik88], [Mit74], [Dij93][Subsection 3.6.2], and moreover applied in the "real world" of highly integrated production processes as well, where it is a common practice to handle failure situations. If, for example, some production machine in an integrated production network breaks down and must be repaired, due to the complexity of the interactions the whole production process is often interrupted until the defect is removed. The Stalling principle can be summarized as

Definition 4.1 (Stalling) *If the inventory at the first server is depleted and the customer who took the last item has performed her jump, then all arrival streams to the network and all service processes at the nodes are completely interrupted and resumed only when the replenishment order at server 1 has arrived.*

Definition 4.2 ($\mathbf{M_{I}^{st}}$, $\mathbf{M_{II}^{st}}$) *A Jackson network according to Definition 2.1 with attached inventory under (r,Q)- or (r,S)-policy at the first server, rerouting according to Stalling (Definition 4.1) and state-dependent replenishment rate $\nu_{\mathrm{I}}(\cdot)$ resp. $\nu_{\mathrm{II}}(\cdot)$ according to Definition 3.2 is denoted by M_{I}^{st} and M_{II}^{st}, respectively.*

Theorem 4.3 *The describing Markov process $\widetilde{X}$ of models M_{I}^{st} resp. M_{II}^{st} has a stationary distribution on state space $\widetilde{E}$ given by (9) and (10), respectively.*

Proof. We first carry out the proof for the Jackson network with attached inventory under (r,S)-policy at the first server using model M_{I}^{st}. Here the steady state equations for the network process $\widetilde{X}$ are given by:

$$
\begin{aligned}
&\pi_{\mathrm{I}}(n_1,...,n_J;k)\left(\sum_{j=1}^{J}\lambda_j+\sum_{j=1}^{J}(1-\delta_{0n_j})\mu_j(n_j)+\nu_{\mathrm{I}}(n_1,...,n_J;k)\right)\\
&= \sum_{j=1}^{J}(1-\delta_{0n_j})\pi_{\mathrm{I}}(n_1,...,n_j-1,...,n_J;k)\lambda_j\\
&\quad+\sum_{j=1}^{J}(1-\delta_{0n_j})\sum_{\substack{i=2\\ i\neq j}}^{J}\pi_{\mathrm{I}}(n_1,...,n_i+1,...,n_j-1,...,n_J;k)\mu_i(n_i+1)r(i,j)\\
&\quad+\sum_{j=2}^{J}(1-\delta_{0n_j})\pi_{\mathrm{I}}(n_1,...,n_J;k)\mu_j(n_j)r(j,j)\\
&\quad+\sum_{j=2}^{J}\pi_{\mathrm{I}}(n_1,...,n_j+1,...,n_J;k)\mu_j(n_j+1)r(j,0)\\
&\quad+\sum_{j=2}^{J}(1-\delta_{0n_j})\pi_{\mathrm{I}}(n_1+1,...,n_j-1,...,n_J;k+1)\mu_1(n_1+1)r(1,j)(1-\delta_{Sk})\\
&\quad+(1-\delta_{0n_1})\pi_{\mathrm{I}}(n_1,...,n_J;k+1)\mu_1(n_1)r(1,1)(1-\delta_{Sk})\\
&\quad+\pi(n_1+1,...,n_J;k+1)\mu_1(n_1+1)r(1,0)(1-\delta_{Sk})\\
&\quad+\sum_{h=0}^{r}\pi_{\mathrm{I}}(n_1,...,n_J;h)\,\nu_{\mathrm{I}}(n_1,...,n_J;h)1_{\{k=S\}} \qquad \text{for all } (n_1,...,n_J;k)\in\widetilde{E},\ k\in\bar{I}, \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
&\pi_{\mathrm{I}}(n_1,...,n_J;0)\nu_{\mathrm{I}}(n_1,...,n_J;0)\\
&= \sum_{j=2}^{J}(1-\delta_{0n_j})\pi_{\mathrm{I}}(n_1+1,...,n_j-1,...,n_J;1)\mu_1(n_1+1)r(1,j)\\
&\quad+(1-\delta_{0n_1})\pi_{\mathrm{I}}(n_1,...,n_J;1)\mu_1(n_1)r(1,1)\\
&\quad+\pi_{\mathrm{I}}(n_1+1,...,n_J;1)\mu_1(n_1+1)r(1,0) \qquad \text{for all } (n_1,...,n_J;0)\in\widetilde{E}. \qquad (12)
\end{aligned}
$$

We have to show that distribution (9) solves these equations (11) and (12). In the first equality, (11), canceling the term $K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)$ leads to

$$
\begin{aligned}
&C(k)\left(\sum_{j=1}^{J}\lambda_j+\sum_{j=1}^{J}(1-\delta_{0n_j})\mu_j(n_j)+\nu_{\mathrm{I}}(n_1,...,n_J;k)\right)\\
&= \sum_{j=1}^{J}(1-\delta_{0n_j})C(k)\frac{\mu_j(n_j)}{\eta_j}\lambda_j\\
&\quad+\sum_{j=1}^{J}(1-\delta_{0n_j})\sum_{\substack{i=2\\ i\neq j}}^{J}C(k)\frac{\mu_j(n_j)}{\eta_j}\eta_i r(i,j)+\sum_{j=2}^{J}(1-\delta_{0n_j})C(k)\mu_j(n_j)r(j,j)\\
&\quad+\sum_{j=2}^{J}C(k)\eta_j r(j,0)+\sum_{j=2}^{J}(1-\delta_{0n_j})C(k+1)\eta_1\frac{\mu_j(n_j)}{\eta_j}r(1,j)(1-\delta_{Sk})
\end{aligned}
$$

$$+(1-\delta_{0n_1})C(k+1)\mu_1(n_1)r(1,1)(1-\delta_{Sk})+C(k+1)\eta_1 r(1,0)(1-\delta_{Sk})$$
$$+\sum_{h=0}^{r} C(h)\,\nu_{\mathrm{I}}(n_1,...,n_J;h)1_{\{k=S\}}.$$

Note that summing up the traffic equation, (1), of the standard Jackson network over all nodes $j \in \bar{J}$ yields

$$\sum_{j=1}^{J}\eta_j=\sum_{j=1}^{J}\lambda_j+\sum_{j=1}^{J}\sum_{i=1}^{J}\eta_i\,r(i,j)=\sum_{j=1}^{J}\lambda_j+\sum_{i=1}^{J}\eta_i\,(1-r(i,0))$$

or equivalently

$$\sum_{j=1}^{J}\lambda_j=\sum_{j=1}^{J}\eta_j\,r(j,0). \tag{13}$$

Using (13) and factoring out $C(k)$ and $C(k+1)$ we obtain from the previous:

$$\begin{aligned}
&C(k)\left(\sum_{j=1}^{J}(1-\delta_{0n_j})\mu_j(n_j)+\nu_{\mathrm{I}}(n_1,...,n_J;k)\right)\\
&= C(k)\left(-\eta_1 r(1,0)+\sum_{j=1}^{J}(1-\delta_{0n_j})\mu_j(n_j)\,\frac{\lambda_j+\sum_{i=2}^{J}\eta_i r(i,j)}{\eta_j}\right)\\
&\quad +C(k+1)\eta_1\left(r(1,0)+\sum_{j=1}^{J}(1-\delta_{0n_j})\frac{\mu_j(n_j)}{\eta_j}r(1,j)\right)(1-\delta_{Sk})\\
&\quad +\sum_{h=0}^{r}C(h)\,\nu_{\mathrm{I}}(n_1,...,n_J;h)1_{\{k=S\}}.
\end{aligned}$$

Noting that $\frac{\lambda_j+\sum_{i=2}^{J}\eta_i r(i,j)}{\eta_j}=1-\frac{\eta_1 r(1,j)}{\eta_j}$ and canceling $C(k)\sum_{j=1}^{J}(1-\delta_{0n_j})\mu_j(n_j)$ yields:

$$\begin{aligned}
&C(k)\nu_{\mathrm{I}}(n_1,...,n_J;k)\\
&= \eta_1\Big(C(k+1)(1-\delta_{Sk})-C(k)\Big)\left(r(1,0)+\sum_{j=1}^{J}(1-\delta_{0n_j})\frac{\mu_j(n_j)}{\eta_j}\,r(1,j)\right)\\
&\quad +\sum_{h=0}^{r}C(h)\,\nu_{\mathrm{I}}(n_1,...,n_J;h)1_{\{k=S\}}.
\end{aligned}$$

After canceling the term $r(1,0)+\sum_{j=1}^{J}(1-\delta_{0n_j})\frac{\mu_j(n_j)}{\eta_j}\,r(1,j)$ on both sides of the above equation only the fundamental equation of the (r,S)-policy from (8) remains which is satisfied by the quantities $C(k)$, $k=1,...,S$.

The second equality (12) is satisfied by the proposed stationary distribution from (9), since:

$$\begin{aligned}
&\sum_{j=2}^{J}(1-\delta_{0n_j})\pi_{\mathrm{I}}(n_1+1,...,n_j-1,...,n_J;1)\mu_1(n_1+1)r(1,j)\\
&\quad+(1-\delta_{0n_1})\pi_{\mathrm{I}}(n_1,...,n_J;1)\mu_1(n_1)r(1,1)+\pi_{\mathrm{I}}(n_1+1,...,n_J;1)\mu_1(n_1+1)r(1,0)\\
&=\sum_{j=2}^{J}\pi_{\mathrm{I}}(n_1,...,n_j,...,n_J;1)\eta_1(1-\delta_{0n_j})\frac{\mu_j(n_j)}{\eta_j}r(1,j)\\
&\quad+\pi_{\mathrm{I}}(n_1,...,n_J;1)\eta_1(1-\delta_{0n_1})\frac{\mu_1(n_1)}{\eta_1}r(1,1)+\pi_{\mathrm{I}}(n_1,...,n_J;1)\eta_1 r(1,0)\\
&\overset{\star}{=}\pi_{\mathrm{I}}(n_1,...,n_J;0)\,\nu\left(r(1,0)+\sum_{j=1}^{J}(1-\delta_{0n_j})\frac{\mu_j(n_j)}{\eta_j}r(1,j)\right)\\
&=\pi_{\mathrm{I}}(n_1,...,n_J;0)\,\nu_{\mathrm{I}}(n_1,...,n_J;0),
\end{aligned}$$

where $\overset{\star}{=}$ follows from $C(1)=\frac{\nu}{\eta_1}C(0)$.

We now prove that the describing Markov process $\widetilde{X}$ of model M_{II}^{st} (Jackson network with attached inventory at the first server, rerouting according to Stalling and replenishment rate $\nu_{\mathrm{II}}(\cdot)$) under (r,S)-policy has stationary distribution (10) on state space $\widetilde{E}$. The steady state equations for the network process here arise from (11) and (12) by replacing $\pi_{\mathrm{I}}(\cdot)$ and $\nu_{\mathrm{I}}(\cdot)$ by $\pi_{\mathrm{II}}(\cdot)$ and $\nu_{\mathrm{II}}(\cdot)$, respectively. We have to show that distribution (10) solves these slightly modified equations. In case of the first equality we can refer to our proof above since $\pi_{\mathrm{II}}(\cdot)$ and $\nu_{\mathrm{II}}(\cdot)$ coincide with $\pi_{\mathrm{I}}(\cdot)$ and $\nu_{\mathrm{I}}(\cdot)$ for all $(n_1,...,n_J;k)\in\widetilde{E}$ with $k\in\bar{I}$ and since

$$\begin{aligned}
&\pi_{\mathrm{II}}(n_1,...,n_J;0)\,\nu_{\mathrm{II}}(n_1,...,n_J;0)\\
&\quad=K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)\eta_1\left(r(1,0)+\sum_{j\in\bar{J}}\frac{\mu_j(n_j)}{\eta_j}r(1,j)\right)\\
&\quad=\pi_{\mathrm{I}}(n_1,...,n_J;0)\,\nu_{\mathrm{I}}(n_1,...,n_J;0).
\end{aligned}$$

For the second equality we note that

$$\begin{aligned}
&\sum_{j=2}^{J}(1-\delta_{0n_j})\pi_{\mathrm{II}}(n_1+1,...,n_j-1,...,n_J;1)\mu_1(n_1+1)r(1,j)\\
&\quad+(1-\delta_{0n_1})\pi_{\mathrm{II}}(n_1,...,n_J;1)\mu_1(n_1)r(1,1)+\pi_{\mathrm{II}}(n_1+1,...,n_J;1)\mu_1(n_1+1)r(1,0)\\
&=\sum_{j=2}^{J}\pi_{\mathrm{II}}(n_1,...,n_j,...,n_J;1)\eta_1(1-\delta_{0n_j})\frac{\mu_j(n_j)}{\eta_j}r(1,j)\\
&\quad+\pi_{\mathrm{II}}(n_1,...,n_J;1)\eta_1(1-\delta_{0n_1})\frac{\mu_1(n_1)}{\eta_1}r(1,1)+\pi_{\mathrm{II}}(n_1,...,n_J;1)\eta_1 r(1,0)\\
&\overset{\star}{=}\pi_{\mathrm{II}}(n_1,...,n_J;0)\,\nu\\
&=\pi_{\mathrm{II}}(n_1,...,n_J;0)\,\nu_{\mathrm{II}}(n_1,...,n_J;0),
\end{aligned}$$

where $\overset{\star}{=}$ follows from $C(1)=1$.

With arguments analogous to those above the theorem is proved for an attached inventory under (r,Q)-policy for models M_{I}^{st} and M_{II}^{st}. In the first global balance equation, (11), we have to replace $(1-\delta_{Sk})$ by $(1-\delta_{Q+r,k})$ and in the last term the sum is deleted since h must be replaced by $k-Q$ and $1_{\{S=k\}}$ is replaced by $1_{\{k>Q\}}$. The calculation now follows along the same lines as for the (r,S)-policy case, with the only difference that we now have to involve the fundamental equation for the (r,Q)-policy system given in (7) in the last step. There is no change in the second balance equation, (12), for M_{I}^{st} and M_{II}^{st} under (r,Q)-policy, respectively.

By standard summation the normalization constant is for both models given by the same expression since in case of model M_{II} it follows:

$$
\begin{aligned}
K &= \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J}\left[\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)\left(\sum_{k\in\bar{I}}C(k)+C(0)\left(r(1,0)+\sum_{j=1}^{J}r(1,j)\frac{\mu_j(n_j)}{\eta_j}\right)\right)\right]\\
&= K^d(J)\sum_{k\in\bar{I}}C(k)+K^d(J)C(0)\,r(1,0)+C(0)\sum_{j=1}^{J}r(1,j)\sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J}\left[\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)\frac{\mu_j(n_j)}{\eta_j}\right]\\
&= K^d(J)\sum_{k\in\bar{I}}C(k)+K^d(J)C(0)\,r(1,0)+C(0)\sum_{j=1}^{J}r(1,j)K^d(J)\\
&= K^d(J)\sum_{k\in\bar{I}_0}C(k). \qquad \square
\end{aligned}
$$

Remark 4.4 *The normalization constant K factorizes into two parts, one depending only on the queue lengths and the other on the inventory process. This results in the following remarkable product structure for model M_{I}*

$$
\pi_{\mathrm{I}}(n_1,\dots,n_J;k)=\hat{\pi}(n_1,\dots,n_J)\,C(k)\left(\sum_{k\in\bar{I}_0}C(k)\right)^{-1}\qquad \text{for } (n_1,\dots,n_J;k)\in\widetilde{E}
$$

with $\hat{\pi}(n_1,\dots,n_J)$ as steady state distribution in the standard Jackson network. Hence, the limiting (marginal) distributions of the queue length processes coincide with those of the standard Jackson network.

The result of Theorem 4.3 carries directly over to the case of closed networks:

Theorem 4.5 *Consider a Gordon-Newell network according to Definition 2.4 with attached inventory under (r,Q)- or (r,S)-policy at the first server, rerouting according to Stalling, and state-dependent replenishment rates $\nu_{\mathrm{I}}(\cdot)$ or $\nu_{\mathrm{II}}(\cdot)$ according to Definition 3.2. Then the describing process in this Gordon-Newell network has a stationary distribution on state space $\widetilde{E}$ given by (9) or (10), respectively, with adjusted normalization constant.*

Proof. The proof follows immediately from the proof of Theorem 4.3 by setting $\lambda=0$ and $r(j,0)=0$ for all $j\in\bar{J}$. $\square$

4.2 Rerouting in case of depleted inventory: Blocking

In this section we do not stop all customers' activities in the network in case of depleted inventory. We only block the customers on the way to node 1 when the inventory is empty. To be more specific: we use the blocking principle Repetitive Service – Random Destination (rs–rd) to obtain a new routing regime for our networks in case that the inventory at the first node is depleted and so this server can neither continue his service nor accept new customers at present. We forbid customers to enter node 1 and redirect them to other nodes in the network so that we have to deal with a situation like blocking. This principle was originally used to resolve blocking situations in networks with resource constraints (see [Onv90], [BDNPO01]) and is a well established approximation for some common communication protocols. It was adapted to the context of unreliable servers in the frame of exponential networks in [SD03a] and in the frame of BCMP networks in [SD03b].

If customers are not allowed to join the queue of node 1 the RS–RD–BLOCKING rule for their jumps is: a customer after being served at node $i \in \bar{J}_0 \setminus \{1\}$ chooses her next destination node $j \in \bar{J}_0$ according to her routing instruction. If she wants to enter node $j = 1$ in times the inventory is depleted, she remains at node i to obtain another service there. When this additional service expires she selects her destination node anew according to her routing rule. Formalizing the RS–RD concept to our setting of Jackson resp. Gordon-Newell networks yields the following behavior:

Definition 4.6 (Blocking with reversible routing in exponential networks) *Assume that the inventory at node* 1 *of the Jackson (resp. Gordon-Newell) network is presently depleted. Then the routing probabilities are redefined on* $\bar{J}_0 \setminus \{1\}$ *according to*

$$\tilde{r}(i,j) = \begin{cases} r(i,j), & i \neq j,\ i,j \in \bar{J}_0 \setminus \{1\} \quad (\textit{resp. } \bar{J} \setminus \{1\}), \\ r(i,i) + r(i,1), & i = j, \quad i \in \bar{J}_0 \setminus \{1\} \quad (\textit{resp. } \bar{J} \setminus \{1\}), \\ 0, & i = 1 \textit{ or } j = 1. \end{cases}$$

In case of a Jackson network it follows for the external arrival rates

$$\tilde{\lambda}_j = \lambda\, \tilde{r}(0,j) = \lambda\, r(0,j) = \lambda_j \qquad \textit{for nodes } j \in \bar{J} \setminus \{1\},$$
$$\tilde{\lambda}_1 = \lambda\, \tilde{r}(0,1) = 0 \qquad \textit{for node } 1.$$

Note that external arrivals may be rejected with positive probability to an immediate departure, because arrivals to node 1 are rerouted while the inventory is depleted:

$$\tilde{r}(0,0) = r(0,0) + r(0,1) = \frac{\lambda_1}{\lambda} \geq 0.$$

We assume that the routing matrix of the original process, see Definition 2.1 (resp. Definition 2.4), is reversible:

$$\eta_j r(j,i) = \eta_i r(i,j) \textit{ for } i,j \in \bar{J}_0 \quad (\textit{resp. } \bar{J}). \tag{14}$$

with η_j *from (1) (resp. (3)).*

Definition 4.7 ($\mathbf{M_I^{bl}}$) *A Jackson network according to Definition 2.1 with reversible routing (14), attached inventory under* (r,Q)*- or* (r,S)*-policy at the first server, rerouting according to Blocking (Definition 4.6) and state-dependent replenishment rate* $\nu_{\mathrm{I}}(\cdot)$ *according to Definition 3.2 is denoted by* M_{I}^{bl}*.*

Theorem 4.8 *The describing network process* $\widetilde{X}$ *of model* M_{I}^{bl} *has a stationary distribution of product form on state space* $\widetilde{E}$ *given by (9).*

Proof. We first carry out the proof for an attached inventory under (r,S)-policy at the first server. The steady state equations for the network process of model M_{I}^{bl} are (11) and

$$\begin{aligned}
&\pi_{\mathrm{I}}(n_1,...,n_J;0)\left(\sum_{j=2}^J\tilde{\lambda}_j+\sum_{j=2}^J\mu_j(n_j)(1-\delta_{0n_j})+\nu_{\mathrm{I}}(n_1,...,n_J;0)\right)\\
=\;&\sum_{j=2}^J\pi_{\mathrm{I}}(n_1,...,n_j-1,...,n_J;0)\tilde{\lambda}_j(1-\delta_{0n_j})\\
&+\sum_{j=2}^J(1-\delta_{0n_j})\sum_{l=2,l\neq j}^J\pi_{\mathrm{I}}(n_1,...,n_l+1,...,n_j-1,...,n_J;0)\mu_l(n_l+1)\,\tilde{r}(l,j)\\
&+\sum_{j=2}^J(1-\delta_{0n_j})\pi(n_1,...,n_J;0)\mu_j(n_j)\,\tilde{r}(j,j)\\
&+\sum_{j=2}^J\pi_{\mathrm{I}}(n_1,...,n_j+1,...,n_J;0)\mu_j(n_j+1)\,\tilde{r}(j,0)\\
&+\sum_{j=2}^J(1-\delta_{0n_j})\pi_{\mathrm{I}}(n_1+1,...,n_j-1,...,n_J;1)\mu_1(n_1+1)r(1,j)\\
&+(1-\delta_{0n_1})\pi_{\mathrm{I}}(n_1,...,n_J;1)\mu_1(n_1)r(1,1)\\
&+\pi_{\mathrm{I}}(n_1+1,...,n_J;1)\mu_1(n_1+1)r(1,0)\qquad\text{for all }(n_1,...,n_J;0)\in\widetilde{E}.
\end{aligned}\tag{15}$$

(11) is solved by the distribution given in (9) as has already been shown in the proof of Theorem 4.3. It remains to show that (9) solves equation (15). In (15) the proposed stationary distribution from (9) satisfies three so called local balance equations. The first local balance equation is (12), which was shown to be satisfied by (9) in the proof of Theorem 4.3. Furthermore, the term at the left hand side

$$\pi_{\mathrm{I}}(n_1,...,n_J;0)\sum_{j=2}^J\tilde{\lambda}_j$$

equals the following term at the right hand side

$$\sum_{j=2}^J\pi_{\mathrm{I}}(n_1,...,n_j+1,...,n_J;0)\mu_j(n_j+1)\,\tilde{r}(j,0)=\pi_{\mathrm{I}}(n_1,...,n_J;0)\sum_{j=2}^J\eta_j\tilde{r}(j,0).$$

This can be seen from the following computations: the solutions of the traffic equations of the standard Jackson network of Definition 2.1 and of a Jackson network with node set $\bar{J}\backslash\{1\}$, unchanged service rates $\mu_j(\cdot)$, external arrival rates $\tilde{\lambda}_j$ and rerouting probabilities $\tilde{r}(i,j)$ for $i,j\in\bar{J}\backslash\{1\}$ according to Definition 4.6, are up to a constant equal, i.e.,

$$\eta_j=\tilde{\lambda}_j+\sum_{i\in\bar{J}\backslash\{1\}}\eta_i\,\tilde{r}(i,j)\text{ for all }j\in\bar{J}\backslash\{1\},\qquad\text{since}\tag{16}$$

$$\begin{aligned}
\tilde{\lambda}_j+\sum_{i\in\bar{J}\backslash\{1\}}\eta_i\,\tilde{r}(i,j)&=\lambda_j+\sum_{\substack{i\in\bar{J}\backslash\{1\}\\ i\neq j}}\eta_i\,r(i,j)+\eta_j\Big(r(j,j)+r(j,1)\Big)\\
&=\lambda_j+\sum_{i\in\bar{J}\backslash\{1\}}\eta_i\,r(i,j)+\underbrace{\eta_j\,r(j,1)}_{=\eta_1\,r(1,j)}=\eta_j\text{ for }j\in\bar{J}\backslash\{1\},
\end{aligned}$$

because of the premise of reversibility, (14). Summing equation (16) up over all $j \in \bar{J}\backslash\{1\}$ yields:

$$\sum_{j=2}^{J} \eta_j = \sum_{j=2}^{J} \tilde{\lambda}_j + \sum_{i=2}^{J} \eta_i \sum_{j=2}^{J} \tilde{r}(i,j) = \sum_{j=2}^{J} \tilde{\lambda}_j + \sum_{i=2}^{J} \eta_i \left(1 - \tilde{r}(i,0)\right)$$

or equivalently

$$\sum_{j=2}^{J} \tilde{\lambda}_j \quad = \quad \sum_{j=2}^{J} \eta_j \, \tilde{r}(j,0). \tag{17}$$

With (17) the quoted second local balance equation is satisfied.

Thirdly, it holds for each $j \in \bar{J} \setminus \{1\}$ with $n_j > 0$:

$$\begin{aligned}
&\pi_{\mathrm{I}}(n_1,...,n_j-1,...,n_J;0)\tilde{\lambda}_j + \sum_{l=2,l\neq j}^{J} \pi_{\mathrm{I}}(n_1,...,n_l+1,...,n_j-1,...,n_J;0)\mu_l(n_l+1)\, \tilde{r}(l,j) \\
&\quad + \pi_{\mathrm{I}}(n_1,...,n_J;0)\mu_j(n_j)\, \tilde{r}(j,j) \\
&= \pi_{\mathrm{I}}(n_1,...,n_J;0)\frac{\mu_j(n_j)}{\eta_j}\left(\tilde{\lambda}_j + \sum_{l=2}^{J} \eta_l \tilde{r}(l,j)\right) \\
&= \pi_{\mathrm{I}}(n_1,...,n_J;0)\mu_j(n_j),
\end{aligned}$$

where the last equality follows from (16).

The proof for the parallel system with an attached inventory under (r,Q)-policy can be carried out in the same way as in the proof of Theorem 4.3 since the first balance equation, (11), for models M_{I}^{st} and M_{I}^{bl} with (r,Q)-policy coincide. The second balance equation for model M_{I}^{bl} with (r,Q)-policy is the same as for M_{I}^{bl} with (r,S)-policy, given by (15). It is satisfied by the proposed stationary distribution (9) as shown above. □

The result of Theorem 4.8 carries directly over to the case of closed networks:

Theorem 4.9 *Consider a Gordon-Newell network according to Definition 2.4 with reversible routing, attached inventory under (r,Q)- or (r,S)-policy at the first server, rerouting according to Blocking (Definition 4.6) and state-dependent replenishment rate $\nu_{\mathrm{I}}(\cdot)$ according to Definition 3.2. Then the describing network process in this Gordon-Newell network has a stationary distribution of product form on state space $\widetilde{E}$ given by (9) with adjusted normalization constant.*

Proof. The proof follows immediately from the proof of Theorem 4.8 by setting $\lambda = 0$ and $r(j,0) = 0$ for all $j \in \bar{J}$. □

Remark 4.10 *Note that $\pi_{\mathrm{II}}(\cdot)$ in conjunction with $\nu_{\mathrm{II}}(\cdot)$ cannot serve as stationary distribution for a Jackson network where rerouting is done according to Blocking. This is due to the fact that the second balance equation, (15), is not satisfied if $\pi_{\mathrm{I}}(\cdot)$ is replaced by $\pi_{\mathrm{II}}(\cdot)$ and $\nu_{\mathrm{I}}(\cdot)$ is replaced by $\nu_{\mathrm{II}}(\cdot)$. Even assuming state-independent service rates μ_j for all nodes $j \in \bar{J}$ is no remedy for the proof. In Section 5.2. the state space is extended such that a replenishment rate similar to $\nu_{\mathrm{II}}(\cdot)$ but with extended information content leads to product from results when using the Blocking principle.*

We cannot expect to remove the requirement to have reversible routing for the original network of this subsection. Even in the simpler case of blocking control this is a standard assumption necessary to obtain explicit expressions for the steady state distribution. It is an advantage of the routing control mechanism of the next section that this assumption is not needed to obtain simple to use equilibria.

4.3 Rerouting in case of depleted inventory: Skipping

In case of depleted inventory at the first server we here determine the customers' rerouting process by the Skipping principle, which relies on methods from the theory of taboo probabilities for Markov chains. These principles seem to have been introduced into the realm of queueing network theory independently several times and were used to resolve blocking, see e.g. [EF98] and [Ser99][Chapter 3.6], (where it is called blocking and rerouting) and the references therein. As a general methodology Skipping was already introduced by Schassberger [Sch84] and later on, based on Schassberger's result, it was used in [DS96] to construct general abstract network processes.

For a graphical understanding of the Skipping principle consider the routing process of the Jackson network from Definition 2.1 as an irreducible, homogeneous Markov chain $V = (V_n : n \in \mathbb{N})$ on state space $\bar{J}_0 := \{0, 1, \ldots, J\}$ with transition matrix $\mathcal{R} = (r(i,j) : i, j \in \bar{J}_0)$. Assume that traveling customers in the network are no longer allowed to enter node 1. Graphically spoken, the Skipping rule for the customers' jumps under this restriction is:
if the customer present at node $i \in \bar{J}_0 \backslash \{1\}$ selects (with probability $r(i,j)$) for her jump's destination node $j \in \bar{J}_0 \backslash \{1\}$, this jump is allowed and immediately performed and the customer joins node j for service. If (with probability $r(i,1)$) she selects for her jump's destination node 1, she only performs an imaginary jump to that node, spending no time there, but jumping on immediately according to the routing matrix $\mathcal{R}$, i.e. with probability $r(1,l)$ she selects another possible successor node l; if $l \in \bar{J}_0 \backslash \{1\}$ then the jump is performed and the customer joins node l for service, but if $l = 1$ again the customer has to perform another random choice as if she would depart from 1; and so on. We formalize this principle for our case in the next definition:

Definition 4.11 (Skipping) *Assume that the inventory at node* 1 *of the Jackson (resp. Gordon-Newell) network is presently depleted. Then the Skipping principle yields new Markov routing matrices*

$$\hat{\mathcal{R}} = \left(\hat{r}(i,k) = r(i,k) + r(i,1) \frac{r(1,k)}{1 - r(1,1)} : i, k \in \bar{J}_0 \setminus \{1\} \ (\textit{resp. } \bar{J} \setminus \{1\}) \right). \tag{18}$$

It follows for the external arrival rates in Jackson networks

$$\hat{\lambda}_i = \lambda \hat{r}(0,i) = \lambda_i + \lambda_1 \frac{r(1,i)}{1 - r(1,1)} \ \textit{for } i \in \bar{J}_0 \setminus \{1\} \quad \textit{and} \quad \tilde{\lambda}_1 = 0. \tag{19}$$

Definition 4.12 ($\mathbf{M_I^{sk}}$) *A Jackson network according to Definition 2.1 with attached inventory under (r, Q)- or (r, S)-policy at the first server, rerouting according to Skipping (Definition 4.11) and state-dependent replenishment rate $\nu_I(\cdot)$ according to Definition 3.2 is denoted by M_I^{sk}.*

Theorem 4.13 *The describing network process $\widetilde{X}$ in model M_I^{sk} has a stationary distribution of product form on state space $\widetilde{E}$ given by (9).*

Proof. The steady state equations for the network processes of model M_I^{sk} with (r, S)-policy are (11) and (15). With arguments analogous to those in the proof of Theorem 4.8 it here suffices to show that the solutions of the traffic equations of a standard Jackson network and of a Jackson network with node set $\bar{J} \setminus \{1\}$, service rates $\mu_j(\cdot)$, external arrival rates $\hat{\lambda}_j$ from (19) and rerouting probabilities $\hat{r}(i,j)$ from (18) for $i, j \in \bar{J} \backslash \{1\}$ are up to a constant equal, i.e. that

$$\eta_j = \hat{\lambda}_j + \sum_{i \in \bar{J} \backslash \{1\}} \eta_i \hat{r}(i,j) \text{ for } j \in \bar{J} \setminus \{1\}.$$

This is done in [SD03a][Theorem 6.4]. The proof for the model M_{I}^{sk} with (r,Q)-policy is carried out similarly. $\square$

The result of Theorem 4.13 carries over to the case of closed networks by setting $\lambda = 0$ and $r(j,0) = 0$ for all $j \in \bar{J}$.

Theorem 4.14 *Consider a Gordon-Newell network according to Definition 2.4 with attached inventory under (r,Q)- or (r,S)-policy at the first server, rerouting according to Skipping (Definition 4.11) and state-dependent replenishment rate $\nu_{\mathrm{I}}(\cdot)$ according to Definition 3.2. Then the describing network process in this Gordon-Newell network has a stationary distribution of product form on state space $\widetilde{E}$ given by (9) with adjusted normalization constant.*

5 Extension of the state space and lumpability

As has already been noted in Section 3 it is possible to extend state space $\widetilde{E}$ and to obtain a more detailed Markovian description of Jackson resp. Gordon-Newell networks with attached inventory under (r,Q)- or (r,S)-policy at the first node. The states of the extended state space additionally provide the information of the successive destination node for the customer who empties the inventory. We define the extended state space as follows:

Definition 5.1 (Extended state space) *For describing Jackson (resp. Gordon-Newell) networks with attached inventory under (r,Q)- or (r,S)-policy at the first node we introduce extended states of the form*

$$(n_1, ..., n_J; k) \in E \times \bar{I}, \tag{20}$$

$$(n_1, ..., n_J; 0; j) \in E \times \{0\} \times \bar{J}_0 \,(resp.\ \bar{J}), \tag{21}$$

summarized in state space

$$\widetilde{E}_D := E_{\bar{I}} \cup E_0, \ \ where\ E_{\bar{I}} := E \times \bar{I},\ E_0 := E \times \{0\} \times \bar{J}_0 \,(resp.\ \bar{J}).$$

The meaning of such a prototype state is :
in states (20) and (21) the numbers n_j indicate that there are n_j customers present at node $j \in \bar{J}$. In state (20) the inventory on-hand at server 1 is at position $k \in \bar{I}$. In state (21) the inventory is depleted and therefore the first server is actually out of service. The last customer who has been served at node 1 has left for server $j \in \bar{J}_0$ (resp. $j \in \bar{J}$).
Collecting these states we define for the networks new Markov processes $\widetilde{X}_D$ on $\widetilde{E}_D$.

Definition 5.2 *The time until an ongoing lead time expires at the $\cdot/M/1/\infty$-server with attached inventory under (r,Q)- or (r,S)-policy in a Jackson resp. Gordon-Newell network with extended state space $\widetilde{E}_D$ is governed by state-dependent intensities. For technical reasons we set $n_0 := \infty$ and define replenishment rates $\nu_D(x)$ for all network states $x \in \widetilde{E}_D$ as follows:*

$$\nu_D(x) = \begin{cases} \nu \left(r(1,0) + \sum_{l=1}^{J} r(1,l) \frac{\mu_l(n_l)}{\eta_l} \right) 1_{\{1 \le k \le r\}}, & x = (n_1, ..., n_J; k) \in E_{\bar{I}}, \\ \nu \, (1 - \delta_{0 n_j}), & x = (n_1, ..., n_J; 0; j) \in E_0. \end{cases}$$

Remark 5.3 *Note that if $x = (n_1, ..., n_J; 0; j) \in E_0$ with $n_j = 0$ no replenishment order is on its way. More clearly spoken, if the customer whose service has taken the last item from the inventory and who has jumped on to node $j \in \bar{J} \setminus \{1\}$, departs from this node j leaving no other customer behind her there, the replenishment order is stopped and resumed only when another customer enters node j.*

5.1 Extension of the state space for rerouting according to Stalling

Definition 5.4 ($\mathbf{D^{st}}$) *A Jackson network according to Definition 2.1 with attached inventory under (r,Q)- or (r,S)-policy at the first server, extended state space $\widetilde{E}_D$ according to Definition 5.1, replenishment lead times as in Definition 5.2 and rerouting according to Stalling (Definition 4.1) is denoted by D^{st}.*

Remark 5.5 *Note that if rerouting is done according to Stalling then for all states of the form $(n_1,...,n_J;0;j) \in E_0$, $j \in \bar{J}$, we necessarily have $n_j > 0$, because the customer who took the last item from inventory joins the queue of server j just before the system is stalled. Consequently, the Jackson resp. Gordon-Newell network now has state space $\widetilde{E}_D^{st} := E_{\bar{I}} \cup E_0^{st}$ where*

$$E_0^{st} := E_0 \setminus \{(n_1,...,n_J;0;j) \in E_0 \,:\, n_j = 0\}.$$

However, in the interest of notational simplicity, we shall use state space $\widetilde{E}_D$, because our formulas will correctly assign a steady-state probability equal to zero to states in $E_0 \setminus E_0^{st}$.

We explicitly present the states with their steady state probabilities in the theorem below, which is an explicit product form result for the described network:

Theorem 5.6 *In model D^{st} the describing network process $\widetilde{X}_D$ has a stationary distribution of product form given by*

$$\pi_D(n_1,...,n_J;k) = K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right) C(k) \text{ with } (n_1,...,n_J;k) \in E_{\bar{I}}, \tag{22}$$

$$\pi_D(n_1,...,n_J;0;j) = K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right) C(0)\,\frac{\mu_j(n_j)}{\eta_j}\,r(1,j) \quad \text{with } (n_1,...,n_J;0;j) \in E_0, j \in \bar{J}, \tag{23}$$

$$\pi_D(n_1,...,n_J;0;0) = K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right) C(0)\,r(1,0) \quad \text{with } (n_1,...,n_J;0;0) \in E_0, \tag{24}$$

where $C(k)$, $k \in \bar{I}_0$, is the value of the function C defined in Section 3 for the (r,Q)- resp. (r,S)-policy. $\eta = (\eta_1,...,\eta_J)$ is the solution of the traffic equation (1) of the standard Jackson network. K is the normalization constant

$$K = K^d(J)\sum_{k\in\bar{I}_0} C(k),$$

which we assume to be finite.

At first sight it is surprising that the appearing normalization constant K is the same as in Definition 3.5.

By setting $n_0 := \infty$ and $\mu_0(n_0) := \eta_0$ formulas (23) and (24) can be summarized in one expression.

Proof. We first prove the theorem in case of (r,S)-policy. The global balance equations for the network process are given by

$$\begin{aligned}
&\pi_D(n_1,...,n_J;k)\left(\sum_{j=1}^{J}\lambda_j+\sum_{j=1}^{J}(1-\delta_{0n_j})\mu_j(n_j)+\nu_D(n_1,...,n_J;k)\right)\\
= & \sum_{j=1}^{J}(1-\delta_{0n_j})\pi_D(n_1,...,n_j-1,...,n_J;k)\lambda_j\\
& +\sum_{j=1}^{J}(1-\delta_{0n_j})\sum_{\substack{i=2\\ i\neq j}}^{J}\pi_D(n_1,...,n_i+1,...,n_j-1,..,n_J;k)\mu_i(n_i+1)r(i,j)\\
& +\sum_{j=2}^{J}(1-\delta_{0n_j})\pi_D(n_1,...,n_J;k)\mu_j(n_j)r(j,j)\\
& +\sum_{j=2}^{J}\pi_D(n_1,...,n_j+1,...,n_J;k)\mu_j(n_j+1)r(j,0)\\
& +\sum_{j=2}^{J}(1-\delta_{0n_j})\pi_D(n_1+1,...,n_j-1,...,n_J;k+1)\mu_1(n_1+1)r(1,j)(1-\delta_{Sk})\\
& +(1-\delta_{0n_1})\pi_D(n_1,...,n_J;k+1)\mu_1(n_1)r(1,1)(1-\delta_{Sk})\\
& +\pi_D(n_1+1,...,n_J;k+1)\mu_1(n_1+1)r(1,0)(1-\delta_{Sk})\\
& +\sum_{j=1}^{J}(1-\delta_{0n_j})\pi_D(n_1,...,n_J;0;j)\nu 1_{\{k=S\}}+\pi_D(n_1,...,n_J;0;0)\nu 1_{\{k=S\}}\\
& +\sum_{h=1}^{r}\pi_D(n_1,...,n_J;h)1_{\{k=S\}}\nu_D(n_1,...,n_J;h) \qquad \text{for all } (n_1,...,n_J;k)\in E_{\bar{I}}, \qquad (25)
\end{aligned}$$

$$\begin{aligned}
&\pi_D(n_1,...,n_J;0;j)\nu(1-\delta_{0n_j})\\
= & \ \pi_D(n_1+1,...,n_j-1,...,n_J;1)\mu_1(n_1+1)r(1,j)(1-\delta_{0n_j})\\
& \qquad \text{for all } (n_1,...,n_J;0;j)\in E_0 \text{ with } j\neq 0, \qquad (26)
\end{aligned}$$

$$\begin{aligned}
&\pi_D(n_1,...,n_J;0;0)\nu\\
= & \ \pi_D(n_1+1,...,n_J;1)\mu_1(n_1+1)r(1,0) \qquad \text{for all } (n_1,...,n_J;0;0)\in E_0. \qquad (27)
\end{aligned}$$

Note that for the sake of clarity we have added the term $(1-\delta_{0n_j})$ even if it is superfluous because of the factor $\mu_j(n_j)$ which is 0 itself whenever $n_j=0$. We have to show that the stationary distribution given in (22) – (24) satisfies (25) – (27). We begin with (26):
for all $(n_1,...,n_J;0;j)\in E_0$ with $j\neq 0$ it holds:

$$\begin{aligned}
&\pi_D(n_1,...,n_J;0;j)\ \nu\ (1-\delta_{0n_j})\\
&= K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)C(0)\,\frac{\mu_j(n_j)}{\eta_j}\,r(1,j)\nu\ (1-\delta_{0n_j})\\
&= K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)\frac{\eta_1}{\mu_1(n_1+1)}\,\frac{\mu_j(n_j)}{\eta_j}\,\mu_1(n_1+1)\ r(1,j)\ (1-\delta_{0n_j})\\
&= \pi_D(n_1+1,...,n_j-1,...,n_J;1)\ \mu_1(n_1+1)\ r(1,j)\ (1-\delta_{0n_j}),
\end{aligned}$$

since $C(1) = 1$. Analogously it follows in case of equation (27) for all $(n_1, ..., n_J; 0; 0) \in E_0$:

$$\begin{aligned}
&\pi_D(n_1, ..., n_J; 0; 0)\nu \\
&= K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(0)\, r(1,0)\, \nu \\
&= K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) \frac{\eta_1}{\mu_1(n_1+1)}\, \mu_1(n_1+1)\, r(1,0) \\
&= \pi_D(n_1+1, ..., n_J; 1)\mu_1(n_1+1)\, r(1,0).
\end{aligned}$$

In order to prove that (25) is satisfied we split up $\nu_D(n_1, ..., n_J; k)$ into its components and show that the proposed stationary distribution satisfies the following three types of local balance equations, which constitute (25):

$$\begin{aligned}
&\pi_D(n_1, ..., n_J;\ k) \left(\sum_{j=1}^{J} \lambda_j + \nu 1_{\{1 \le k \le r\}} r(1,0) \right) \\
&\overset{\star}{=} K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(k) \left(\sum_{j=2}^{J} \eta_j\, r(j,0) \ + (\eta_1 + \nu 1_{\{1 \le k \le r\}}) r(1,0) \right) \\
&\overset{\bullet}{=} K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(k) \sum_{j=2}^{J} \eta_j\, r(j,0) \\
&\quad + K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) \eta_1 \left(C(k+1)(1-\delta_{Sk}) + 1_{\{k=S\}} + \frac{\nu}{\eta_1} \sum_{h=1}^{r} C(h) 1_{\{k=S\}} \right) r(1,0) \\
&= K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(k) \sum_{j=2}^{J} \frac{\eta_j}{\mu_j(n_j+1)}\, \mu_j(n_j+1)\, r(j,0) \\
&\quad + K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) \frac{\eta_1}{\mu_1(n_1+1)}\, C(k+1)\mu_1(n_1+1)(1-\delta_{Sk}) r(1,0) \\
&\quad + K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) \frac{\eta_1}{\nu}\, r(1,0)\nu 1_{\{k=S\}} + K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) \sum_{h=1}^{r} C(h) \nu 1_{\{k=S\}} r(1,0)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=2}^{J} \pi_D(n_1, ..., n_j+1, ..., n_J; k)\mu_j(n_j+1) r(j,0) \\
&\quad + \pi_D(n_1+1, ..., n_J; k+1)\mu_1(n_1+1) r(1,0)(1-\delta_{Sk}) + \pi_D(n_1, ..., n_J; 0; 0)\nu 1_{\{k=S\}} \\
&\quad + \sum_{h=1}^{r} \pi_D(n_1, ..., n_J; h)\nu 1_{\{k=S\}} r(1,0),
\end{aligned}$$

where $\overset{\star}{=}$ follows from (13) and $\overset{\bullet}{=}$ follows from the fundamental equation (8) of the (r,S)-policy.

It holds for each $j \in \bar{J} \setminus \{1\}$ with $n_j > 0$:

$$\begin{aligned}
&\pi_D(n_1, ..., n_j - 1, ..., n_J; k)\lambda_j \\
&+ \sum_{s=2, s\neq j}^{J} \pi_D(n_1, ..., n_s + 1, ..., n_j - 1, ..., n_J; k)\mu_s(n_s + 1)r(s, j) \\
&+ \pi_D(n_1, ..., n_J; k)\mu_j(n_j)r(j, j) \\
&+ \pi_D(n_1 + 1, ..., n_j - 1, ..., n_J; k + 1)\mu_1(n_1 + 1)r(1, j)(1 - \delta_{Sk}) \\
&+ \pi_D(n_1, ..., n_J; 0; j)\nu 1_{\{k=S\}} + \sum_{h=1}^{r} \pi_D(n_1, ..., n_J, h)\nu 1_{\{k=S\}} r(1, j)\frac{\mu_j(n_j)}{\eta_j} 1_{\{1\le h\le r\}} \\
&= K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(k)\mu_j(n_j) \underbrace{\left(\frac{\lambda_j + \sum_{\substack{s=2\\ s\neq j}}^{J} \eta_s r(s, j) + \eta_j r(j, j)}{\eta_j}\right)}_{=1-\frac{\eta_1 r(1,j)}{\eta_j}} \\
&+ K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(k + 1)\frac{\eta_1}{\mu_1(n_1 + 1)}\frac{\mu_j(n_j)}{\eta_j}\mu_1(n_1 + 1)r(1, j)(1 - \delta_{Sk}) \\
&+ K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) \frac{\mu_j(n_j)}{\eta_j}\frac{\eta_1}{\nu} r(1, j)\nu 1_{\{k=S\}} + \sum_{h=1}^{r} K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(h)\nu 1_{\{k=S\}} r(1, j)\frac{\mu_j(n_j)}{\eta_j} \\
&= K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(k)\mu_j(n_j) \\
&+ K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) \frac{\mu_j(n_j)}{\eta_j} r(1, j) \left(-\eta_1 C(k) + \eta_1 C(k + 1)(1 - \delta_{Sk}) + \eta_1 1_{\{k=S\}} + \nu \sum_{h=1}^{r} C(h)1_{\{k=S\}}\right) \\
&\stackrel{\bullet}{=} K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(k) \left(\mu_j(n_j) + \nu 1_{\{1\le k\le r\}} r(1, j)\frac{\mu_j(n_j)}{\eta_j}\right) \\
&= \pi_D(n_1, ..., n_J;\ k) \left(\mu_j(n_j) + \nu 1_{\{1\le k\le r\}} r(1, j)\frac{\mu_j(n_j)}{\eta_j}\right)
\end{aligned}$$

where $\stackrel{\bullet}{=}$ follows from (8).

Analogously, we compute for node 1 with $n_1 > 0$:

$$\begin{aligned}
&\pi_D(n_1 - 1, ..., n_J; k)\lambda_1 \\
&+ \sum_{s=2}^{J} \pi_D(n_1 - 1, ..., n_s + 1, ..., n_J; k)\mu_s(n_s + 1)r(s, 1) \\
&+ \pi_D(n_1, ..., n_J; k + 1)\mu_1(n_1)r(1, 1)(1 - \delta_{Sk}) \\
&+ \pi_D(n_1, ..., n_J; 0; 1)\nu 1_{\{k=S\}} + \sum_{h=1}^{r} \pi_D(n_1, ..., n_J, h)\nu 1_{\{k=S\}} r(1, 1)\frac{\mu_1(n_1)}{\eta_1} 1_{\{1\le h\le r\}}
\end{aligned}$$

$$= K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(k)\mu_1(n_1) \underbrace{\left(\frac{\lambda_1 + \sum_{s=2}^{J} \eta_s r(s,1)}{\eta_1}\right)}_{=1-\frac{\eta_1 r(1,1)}{\eta_1}}$$

$$+ K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(k+1)\mu_1(n_1) r(1,1)(1-\delta_{Sk})$$

$$+ K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) \frac{\mu_1(n_1)}{\eta_1}\frac{\eta_1}{\nu} r(1,1)\nu 1_{\{k=S\}} + \sum_{h=1}^{r} K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(h)\nu 1_{\{k=S\}} r(1,1)\frac{\mu_1(n_1)}{\eta_1}$$

$$= K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(k)\mu_1(n_1)$$

$$+ K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) \frac{\mu_1(n_1)}{\eta_1} r(1,1) \left(-\eta_1 C(k) + \eta_1 C(k+1)(1-\delta_{Sk}) + \eta_1 1_{\{k=S\}} + \nu \sum_{h=1}^{r} C(h) 1_{\{k=S\}}\right)$$

$$\stackrel{\bullet}{=} K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)}\right) C(k) \left(\mu_1(n_1) + \frac{\mu_1(n_1)}{\eta_1} r(1,1)\nu 1_{\{1\le k\le r\}}\right)$$

$$= \pi_D(n_1,...,n_J;\ k) \left(\mu_1(n_1) + \nu 1_{\{1\le k\le r\}} r(1,1)\frac{\mu_1(n_1)}{\eta_1}\right)$$

where $\stackrel{\bullet}{=}$ follows from (8).

With arguments analogous to those above the theorem can be proved for the (r,Q)-policy case of model D^{st} as well. In the first global balance equation (25) we have to replace $(1-\delta_{Sk})$ by $(1-\delta_{Q+r,k})$, $1_{\{k=S\}}$ twice by $1_{\{k=Q\}}$ and in the last term the sum is deleted since h has to be replaced by $k-Q$ and here $1_{\{k=S\}}$ by $1_{\{k>Q\}}$. The calculation now follows the same lines as for the (r,S)-policy case, with the only difference that we now have to involve the fundamental equation for the (r,Q)-policy system given in (7). The second and third balance equations (26) and (27) stay the same and are satisfied form our results above.

The normalization constant K follows from standard summations. □

Remark 5.7 *The normalization constant K factorizes into two parts. This results in the following product structure*

$$\pi_D(n_1,...,n_J;k) = \hat{\pi}(n_1,...,n_J)C(k)\left(\sum_{k\in\bar{I}_0} C(k)\right)^{-1} \quad \text{for } (n_1,...,n_J)\in E, k\in\bar{I},$$

$$\pi_D(n_1,...,n_J;0;j) = \hat{\pi}(n_1,...,n_j-1,...,n_J)C(0)\,r(1,j)\left(\sum_{k\in\bar{I}_0} C(k)\right)^{-1}$$
$$\text{for } (n_1,...,n_J;0;j)\in E_0, j\in\bar{J} \text{ with } n_j>0,$$

$$\pi_D(n_1,...,n_J;0;0) = \hat{\pi}(n_1,...,n_J)C(0)\,r(1,0)\left(\sum_{k\in\bar{I}_0} C(k)\right)^{-1} \quad \text{for } (n_1,...,n_J)\in E,$$

with $\hat{\pi}(n_1,...,n_J)$ as steady state distribution in the standard Jackson network.

The result of Theorem 5.6 carries over to the case of closed networks without any problems:

Theorem 5.8 *Consider a Gordon-Newell network according to Definition 2.4 with attached inventory under (r, Q)- or (r, S)-policy at the first server, extended state space $\widetilde{E}_D$ according to Definition 5.1, replenishment lead times as in Definition 5.2 and rerouting according to Stalling (Definition 4.1). Then the enlarged network process of this Gordon-Newell network has a stationary distribution of product form given by* (22) *and* (23) *with adjusted normalization constant.*

Proof. The proof follows immediately from the proof of Theorem 5.6 by setting $\lambda = 0$ and $r(j, 0) = 0$ for all $j \in \bar{J}$. □

We will now investigate whether the state space $\widetilde{E}$ of our original processes $\widetilde{X}$ from Sections 3 and 4 can be obtained from $\widetilde{E}_D$ using the notion of *ordinary lumpability.* Lumpability allows to reduce a Markov process with a large number of states to a Markov process with a smaller number of states (see [KS83] for the case of Markov chains with finite state space and [BY93] for the case of Markov processes with denumerable state space). This aggregation does not affect the performance measures of the original and the lumped process. We cite the definition of ordinary lumpability from [HHK02] and [RS94].

Definition 5.9 (Ordinary lumpability) *Denote by $\mathcal{F} = \{\omega_1, \omega_2, \omega_3, ... | \omega_i \cap \omega_j = \emptyset, i \neq j \in \mathbb{N}, \cup_{i \in \mathbb{N}} \omega_i = F, \omega_i \subset F\}$ a partition of state space F of a Markov process. The Markov process is ordinary lumpable with respect to $\mathcal{F}$ if for any subset $\omega_1 \in \mathcal{F}$ and for any two states $\xi_1, \xi_2 \in \omega_1$ the following equation holds*

$$\sum_{\zeta \in \omega_2} q(\xi_1, \zeta) \quad = \quad \sum_{\zeta \in \omega_2} q(\xi_2, \zeta) \textit{ for all } \omega_2 \in \mathcal{F}. \tag{28}$$

Here $q(\xi_1, \zeta)$ denotes the transition rate from state ξ_1 to state ζ. The definition says that for any two states from the same subset ω_1 the cumulative rate of moving to another subset, ω_2, has to be equal. The lumped Markov process then has state space $\mathcal{F}$ with steady state probability for being in state $\omega \in \mathcal{F}$ given by $\sum_{\xi \in \omega} \pi(\xi)$ for all $\omega \in \mathcal{F}$ (see [RS94]).

We will now apply the concept of lumpability to the detailed Markov process $\widetilde{X}_D$ with state space $\widetilde{E}_D$. I.e. we investigate whether lumpability of the network processes of model D^{st} with respect to a certain partition can lead to a Markov process which is defined on state space $\widetilde{E}$.

Definition 5.10 *Denote the partition of state space $F = \widetilde{E}_D = E_{\bar{I}} \cup E_0$ by $\mathcal{F}$. Each state $x = (n_1, ..., n_J; k) \in E_{\bar{I}}$ is represented by a singleton $\omega_{(n_1,...,n_J;k)} := \{(n_1, ..., n_J; k)\}$ in partition $\mathcal{F}$. States from E_0 are aggregated into sets: $\omega_{(n_1,...,n_J;0)} := \{(n_1, .., n_J; 0; j) : j \in \bar{J}_0\}$ for fixed $(n_1, ..., n_J) \in E$.*
The specific partition for the Stalling principle, denoted by $\mathcal{F}^{st}$, has the restriction that a set $\omega_{(n_1,...,n_J;0)} \in \mathcal{F}^{st}$ only includes elements from E_0^{st}, i.e.,

$$\omega_{(n_1,...,n_J;0)} = \{(n_1, .., n_J; 0; j) : (n_1, ..., n_J; 0; j) \in E_0^{st}, j \in \bar{J}_0\}.$$

Remark 5.11 *There exists a bijection between state space $\widetilde{E}$ and the partition $\mathcal{F}$.*

Theorem 5.12 *The describing Markov process in model D^{st} is ordinary lumpable with respect to partition $\mathcal{F}^{st}$.*

Proof. We carry out the proof for the Jackson network with (r, S)-policy. We have to show that for any subset $\omega_1 = \omega_{(n_1,...,n_J;k)} \in \mathcal{F}^{st}$ and for any two states $\xi_1, \xi_2 \in \omega_1$ equation (28) holds. We divide the proof into three parts depending on ω_1 and ω_2 with $k \in \bar{I}$ or $k = 0$:

(a) $\omega_1 = \omega_{(n_1,...,n_J;k)}$, $k \in \bar{I}$, $\omega_2 = \omega_{(m_1,...,m_J;h)}$, $h \in \bar{I}_0$. ω_1 is a singleton. This entails that both sides of (28) are the same for partition $\mathcal{F}^{st}$.

(b) $\omega_1 = \omega_{(n_1,...,n_J;0)}$, $\omega_2 = \omega_{(m_1,...,m_J;h)}$, $h \in \bar{I}$. Let $\xi_1, \xi_2 \in \omega_1$ with $\xi_1 = (n_1, ..., n_J; 0; i)$ and $\xi_2 = (n_1, ..., n_J; 0; j)$, $i, j \in \bar{J}_0$. ω_2 is a singleton. We have with $\zeta = (m_1, ..., m_J; h)$

$$q(\xi_1, \zeta) = \nu(1 - \delta_{0n_i})1_{\{n_1=m_1\}} \cdot ... \cdot 1_{\{n_J=m_J\}} \cdot 1_{\{h=S\}}$$

$$\text{and} \qquad q(\xi_2, \zeta) = \nu(1 - \delta_{0n_j})1_{\{n_1=m_1\}} \cdot ... \cdot 1_{\{n_J=m_J\}} \cdot 1_{\{h=S\}}.$$

Since $\omega_1 \in \mathcal{F}^{st}$, $(1 - \delta_{0n_k}) = 1$ for all $\xi = (n_1, ..., n_J; 0; k) \in E_0^{st}, k \in \bar{J}_0$. Thus, equation (28) is satisfied.

(c) $\omega_1 = \omega_{(n_1,...,n_J;0)} \in \mathcal{F}^{st}$, $\omega_2 = \omega_{(m_1,...,m_J;0)} \in \mathcal{F}^{st}$. Hence, if $\omega_1 \neq \omega_2$ then $\sum_{\zeta \in \omega_2} q(\xi, \zeta) = 0$ for all $\xi \in \omega_1$. If $\omega_1 = \omega_2$ then $\sum_{\zeta \in \omega_2} q(\xi, \zeta) = q(\xi, \xi)$ for all $\xi \in \omega_1$.

The proof for the Jackson network with (r, Q)-policy can be carried out on the same lines. $\square$

Corollary 5.13 *Consider the lumped Markov process of model D^{st} with respect to partition $\mathcal{F}^{st}$. We identify the singleton $\omega_{(n_1,...,n_J;k)}, k \in \bar{I}$ with the new state $(n_1, ..., n_J; k)$ and the set $\omega_{(n_1,...,n_J;0)}$ with the state $(n_1, ..., n_J; 0)$ for all $(n_1, ..., n_J) \in E$. Then, this process has replenishment rate $\nu_{\mathrm{II}}(\cdot)$ according to Definition 3.2 and stationary distribution (10).*

5.2 Extension of the state space for rerouting according to Blocking or Skipping

In order to successfully derive computations below we have to assume in this subsection that service-rates are state-independent.

Definition 5.14 ($\mathbf{D^{bl,i}}$) *A Jackson network according to Definition 2.1 with state-independent service rates, reversible routing, attached inventory under (r, Q)- or (r, S)-policy at the first server, extended state space $\widetilde{E}_D$ according to Definition 5.1, replenishment rates from Definition 5.2 and rerouting according to Blocking (Definition 4.6) is denoted by $D^{bl,i}$ ("bl" for Blocking, "i" for state-independent).*

Remark 5.15 *If rerouting is done according to Blocking or Skipping then states $(n_1, ..., n_J; 0; 1)$ with $n_1 = 0$ cannot be attained. Since, if the customer who took the last item from inventory joins the queue of server 1 once again then she cannot leave this node before the replenishment order has arrived. Consequently, the Jackson resp. Gordon-Newell network with inventory has describing state space $\widetilde{E}_D^{bl} := E_{\bar{I}} \cup E_0^{bl}$ where*

$$E_0^{bl} := E_0 \setminus \{(0, n_2, ..., n_J; 0; 1) \in E_0\}.$$

As before we will, in the interest of notational simplicity, operate on the whole state space $\widetilde{E}_D$, because our formulas will correctly assign a steady-state probability equal to zero to states in $E_0 \setminus E_0^{bl}$.

Theorem 5.16 *The supplemented network process in model $D^{bl,i}$ has a stationary distribution of product form given by*

$$\pi_d(n_1,...,n_J;k) \quad = \quad \overline{K}^{-1}\prod_{l=1}^{J}\left(\frac{\eta_l}{\mu_l}\right)^{n_l} C(k) \text{ with } (n_1,...,n_J;k)\in E_{\bar{I}}, \tag{29}$$

$$\pi_d(n_1,...,n_J;0;j) \quad = \quad \overline{K}^{-1}\prod_{l=1}^{J}\left(\frac{\eta_l}{\mu_l}\right)^{n_l} C(0)\,\frac{\mu_j}{\eta_j}\,r(1,j)$$
$$\text{with } (n_1,...,n_J;0;j)\in E_0, j\in \bar{J_0}\setminus\{1\}, \tag{30}$$

$$\pi_d(n_1,...,n_J;0;1) \quad = \quad \overline{K}^{-1}\prod_{l=1}^{J}\left(\frac{\eta_l}{\mu_l}\right)^{n_l} C(0)\,\frac{\mu_1}{\eta_1}\,(1-\delta_{0n_1})r(1,1)$$
$$\text{with } (n_1,...,n_J;0;1)\in E_0. \tag{31}$$

Here we formally define $\mu_0 := \eta_0$. $C(k)$, $k\in\bar{I}_0$, is the value of the function C defined in Section 3 for the (r,Q)- resp. (r,S)-policy. $\eta=(\eta_1,...,\eta_J)$ is the solution of the traffic equation (1) of the standard Jackson network. $\overline{K}$ is the normalization constant:

$$\overline{K} = K^i(J)\left(\sum_{k\in\bar{I}_0} C(k) + C(0)\sum_{j=2}^{J}\mathsf{P}^{i(J)}(X_j=0)\frac{\mu_j}{\eta_j}\,r(1,j)\right),$$

where $\mathsf{P}^{i(J)}(X_j=0)$ is the steady state probability for no customers at node $j\in\bar{J}$ in the standard Jackson network with state-independent service rates. We assume $\overline{K}$ to be finite.

Proof. The first global balance equation is the same as in the Jackson network where rerouting is done according to Stalling for both inventory management policies. This is due to the fact that the definition of the state-dependent replenishment lead times $\nu_D(\cdot)$ are zero for states from $E_0\setminus E_0^{st}$. From Theorem 5.6 this equation is solved by the proposed distribution given in (29) – (31). The second global balance equation is for both inventory management policies and for all $(n_1,...,n_J;0;i)\in E_0$ given by

$$\begin{aligned}
&\pi_d(n_1,...,n_J;0;i)\left(\sum_{j=2}^{J}\tilde{\lambda}_j + \sum_{j=2}^{J}\mu_j(1-\delta_{0n_j}) + \nu(1-\delta_{0n_i})\right)\\
&= \sum_{j=2}^{J}\pi_d(n_1,...,n_j-1,...,n_J;0;i)\tilde{\lambda}_j(1-\delta_{0n_j})\\
&\quad+\sum_{j=2}^{J}(1-\delta_{0n_j})\sum_{l=2,l\neq j}^{J}\pi_d(n_1,...,n_l+1,...,n_j-1,..,n_J;0;i)\mu_l\,\tilde{r}(l,j)\\
&\quad+\sum_{j=2}^{J}(1-\delta_{0n_j})\pi_d(n_1,...,n_J;0;i)\mu_j\,\tilde{r}(j,j)\\
&\quad+\sum_{j=2}^{J}\pi_d(n_1,...,n_j+1,...,n_J;0;i)\mu_j\,\tilde{r}(j,0)\\
&\quad+\pi_d(n_1+1,...,n_i-1,...,n_J;1)\mu_1 r(1,i)(1-\delta_{0n_i}).
\end{aligned}\tag{32}$$

We have to show that the distribution from (29) – (31) satisfies (32). First note that for $i=1$ and $n_1=0$ both sides of (32) vanish. The proposed stationary distribution satisfies the following three types of local balance equations, which constitute (32):

$$\pi_d(n_1,...,n_J;0;i)\nu(1-\delta_{0n_i}) = \pi_d(n_1+1,...,n_i-1,...,n_J;1)\mu_1 r(1,i)(1-\delta_{0n_i}),$$

since $C(1)=1$ for the (r,Q)- and the (r,S)-policy.

$$\pi_d(n_1,...,n_J;0;i)\sum_{j=2}^{J}\tilde\lambda_j \overset{\star}{=} \pi_d(n_1,...,n_J;0;i)\sum_{j=2}^{J}\eta_j\tilde r(j,0) = \sum_{j=2}^{J}\pi_d(n_1,...,n_j+1,...,n_J;0;i)\mu_j\tilde r(j,0)$$

where $\overset{\star}{=}$ follows from (17).

Furthermore, involving (16) it follows for each $j\in \bar J\backslash\{1\}$ with $n_j>0$:

$$\begin{aligned}
&\pi_d(n_1,...,n_j-1,...,n_J;0,i)\tilde\lambda_j + \sum_{\substack{l=2\\ l\neq j}}^{J}\pi_d(n_1,...,n_l+1,...,n_j-1,...,n_J;0,i)\mu_l\,\tilde r(l,j)\\
&\quad+\pi_d(n_1,...,n_J;0,i)\mu_j\,\tilde r(j,j)\\
&= \pi_d(n_1,...,n_J;0,i)\left(\frac{\mu_j}{\eta_j}\tilde\lambda_j + \sum_{\substack{l=2\\ l\neq j}}^{J}\frac{\mu_j}{\eta_j}\frac{\eta_l}{\mu_l}\mu_l\tilde r(l,j) + \mu_j\tilde r(j,j)\right)\\
&= \pi_d(n_1,...,n_J;0,i)\,\mu_j\,\frac{\tilde\lambda_j+\sum_{l=2}^{J}\eta_l\tilde r(l,j)}{\eta_j}\\
&= \pi_d(n_1,...,n_J;0,i)\mu_j.
\end{aligned}$$

The normalization constant is given by

$$\begin{aligned}
\overline{K} &= \sum_{(n_1,...,n_J)\in\mathbb{N}_0^J}\left[\prod_{l=1}^{J}\left(\frac{\eta_l}{\mu_l}\right)^{n_l}\left(\sum_{k\in\bar I}C(k)+C(0)\left(r(1,0)+r(1,1)\frac{\mu_1}{\eta_1}(1-\delta_{0n_1})+\sum_{j=2}^{J}\frac{\mu_j}{\eta_j}r(1,j)\right)\right)\right]\\
&= K^i(J)\left(\sum_{k\in\bar I}C(k)+C(0)\,r(1,0)\right)+C(0)\sum_{n_1=1}^{\infty}\sum_{(n_2,..,n_J)\in\mathbb{N}_0^{J-1}}\prod_{l=1}^{J}\left(\frac{\eta_l}{\mu_l}\right)^{n_l}\frac{\mu_1}{\eta_1}r(1,1)\\
&\quad+C(0)\sum_{j=2}^{J}\sum_{n_j=1}^{\infty}\sum_{(n_1,..,n_{j-1},n_{j+1},..,n_J)\in\mathbb{N}_0^{J-1}}\prod_{l=1}^{J}\left(\frac{\eta_l}{\mu_l}\right)^{n_l}\frac{\mu_j}{\eta_j}r(1,j)\\
&\quad+C(0)\sum_{j=2}^{J}\sum_{(n_1,..,n_{j-1},n_{j+1},..,n_J)\in\mathbb{N}_0^{J-1}}\prod_{\substack{l=1\\ l\neq j}}^{J}\left(\frac{\eta_l}{\mu_l}\right)^{n_l}\frac{\mu_j}{\eta_j}r(1,j)\\
&= K^i(J)\left(\sum_{k\in\bar I}C(k)+C(0)\sum_{j\in\bar J_0}r(1,j)\right)+C(0)\sum_{j=2}^{J}\mathsf{P}^{i(J)}(X_j=0)K^i(J)\frac{\mu_j}{\eta_j}r(1,j)\\
&= K^i(J)\left(\sum_{k\in\bar I_0}C(k)+C(0)\sum_{j=2}^{J}\mathsf{P}^{i(J)}(X_j=0)\frac{\mu_j}{\eta_j}\,r(1,j)\right). \qquad\square
\end{aligned}$$

The occurrence of the term $\mathsf{P}^{i(J)}(X_j=0)$ indicates an inter-dependence of inventory and queue length processes which is non vanishing even in the long-range behavior of the system. This is in contrast to the mutual dependence of the queue lengths which seem to asymptotically vanish although they naturally strongly interact by the moving customers (see [Dad01][Remark 2.7]).

The result of Theorem 5.16 for open exponential networks can easily be applied to closed networks as well by modifying the proof of Theorem 5.16 by just setting $\lambda = 0$ and $r(j,0) = 0$ for all $j \in \bar{J}$:

Theorem 5.17 *The describing network process in a Gordon-Newell network according to Definition 2.4 with state-independent service rates, reversible routing, attached inventory under (r,Q)- or (r,S)-policy at the first server, extended state space $\widetilde{E}_D$ according to Definition 5.1, replenishment rates from Definition 5.2 and rerouting according to Blocking (Definition 4.6) has a stationary distribution of product form given by (29) – (31) with adjusted normalization constant.*

We quote the results for rerouting according to Skipping:

Definition 5.18 ($\mathbf{D^{sk,i}}$) *A Jackson network according to Definition 2.1 with state-independent service rates, attached inventory under (r,Q)- or (r,S)-policy at the first server, extended state space $\widetilde{E}_D$ according to Definition 5.1, replenishment rates from Definition 5.2 and rerouting according to Skipping (Definition 4.11) is denoted by $D^{sk,i}$.*

Theorem 5.19 *The describing network process in model $D^{sk,i}$ has a stationary distribution of product form given by (29) – (31).*

Proof. The proof is analogous to the proof of Theorem 4.13 with reference to Theorem 5.6 instead of Theorem 4.3. □

Theorem 5.20 *The describing network process in a Gordon-Newell network according to Definition 2.4 with state-independent service rates, attached inventory under (r,Q)- or (r,S)-policy at the first server, extended state space $\widetilde{E}_D$ according to Definition 5.1, replenishment rates from Definition 5.2 and rerouting according to Skipping (Definition 4.11) has a stationary distribution of product form given by (29) – (31) with adjusted normalization constant.*

Proof. The proof can be done by setting $\lambda = 0$ and $r(j,0) = 0$ for all $j \in \bar{J}$. □

We will now investigate whether lumpability of the network processes of the models $D^{bl,i}$ and $D^{sk,i}$ with partition $\mathcal{F}$ (according to Definition 5.9) can lead to a Markov process which is defined on state space $\widetilde{E}$.

Definition 5.21 *We formally specify the partition for the Blocking and Skipping principles, denoted by $\mathcal{F}^{bl}$. A set $\omega_{(n_1,\dots,n_J;0)} \in \mathcal{F}^{bl}$ only includes elements from E_0^{bl}, i.e.*

$$\omega_{(n_1,\dots,n_J;0)} := \{(n_1,..,n_J;0;j) \,:\, (n_1,...,n_J;0;j) \in E_0^{bl}, j \in \bar{J}_0\}.$$

Theorem 5.22 *The describing Markov processes in models $D^{bl,i}$ and $D^{sk,i}$ are not ordinary lumpable with respect to partition $\mathcal{F}^{bl}$.*

Proof. We carry out the proof for the Jackson network with (r,S)-policy and treat the Blocking or Skipping principles at the same time. We show that equation (28) does not hold for $\omega_1 = \omega_{(n_1,\dots,n_J;0)} \in \mathcal{F}^{bl}$. Define $\omega_2 = \omega_{(m_1,\dots,m_J;h)} \in \mathcal{F}^{bl}$ with $h \geq 1$ and let $\xi_1, \xi_2 \in \omega_1$ with $\xi_1 = (n_1,...,n_J;0;i)$ and $\xi_2 = (n_1,...,n_J;0;j)$, $i,j \in \bar{J}_0$. Then ω_2 is a singleton. We have as before with $\zeta \in \omega_2$

$$q(\xi_1,\zeta) = \nu(1-\delta_{0n_i})1_{\{n_1=m_1\}} \cdot \ldots \cdot 1_{\{n_J=m_J\}} \cdot 1_{\{h=S\}}$$

and $$q(\xi_2,\zeta) = \nu(1-\delta_{0n_j})1_{\{n_1=m_1\}} \cdot \ldots \cdot 1_{\{n_J=m_J\}} \cdot 1_{\{h=S\}}.$$

If rerouting is done according to Blocking or Skipping and $\omega_1 \in \mathcal{F}^{bl}$, then it is possible that one side of (28) is positive whereas the other side is 0 since here $(n_1,...,n_J;0;k) \in \omega_1 = \omega_{(n_1,\dots,n_J;0)}$ even if $n_k = 0$.
The proof for the Jackson network with (r,Q)-policy can be carried out on the same lines. □

6 Computation of performance measures

The network processes $\widetilde{X}$ with their respective steady states include as observables the information on the inventory position and queue lengths. Explicit computation of performance measures in classical queueing and inventory theory is done using the stationary system processes. Our aim is to develop analogous formulas in the setting developed in the preceding sections: unified treatment of service with queueing and inventory management with lead time delays. Having determined the stationary distributions, we can construct several measures of operating characteristics for the different models. We introduce and compute the following measures of system performance for stationary systems: the expected reorder rate λ_R, the departure and arrival rates of customers from/to node 1, denoted by λ_D and λ_A, the safety stock s, α- and β-service levels, the average on-hand inventory $\overline{Inv}$, the expected lost sales per unit time $\overline{LS}$ and per cycle $\overline{LS}_c$ and the average queue length.

We confine ourselves to calculating performance measures for Jackson networks with attached inventory. We say that the system goes through one cycle in the time between the placing of two successive orders or the receipt of two successive procurements with respect to the mean cycle time λ_R^{-1}. Let us denote by $X_1, X_2, ..., X_J$ and Y random variables distributed like the stationary queue length distribution at nodes $1, 2, ..., J$ and the stationary inventory distribution, respectively.
We assume that an underlying probability space $(\Omega, \mathcal{A}, \mathsf{P})$ is given where all random variables are defined on. In order to distinguish between the different models we will add indices to the random variables where needed. For example Y_{M_I} denotes the random variable distributed like the stationary inventory distribution of model M_I.

Definition 6.1 (Safety stock) *The safety stock is the Palm stationary mean value*

$$s \quad = \quad \mathsf{E}_0(\textit{net inventory position just before the arrival of a replenishment order}).$$

Under the lost sales regime the net inventory position (often quoted by the process Y_{net}) is defined as the inventory on hand. Therefore $Y_{net} := Y$.

In the literature the notion of net inventory is commonly used with respect to safety stocks. Therefore, we follow this convention here as well.

Definition 6.2 (Service levels)

$$\begin{aligned}
\alpha_1 &= \mathsf{P}(\textit{net inventory position at the end of a cycle } > 0), \\
\alpha_2 &= \mathsf{P}(\textit{net inventory position at the end of a time unit } > 0), \\
\beta &= \frac{\mathsf{E}(\textit{demand satisfied per cycle})}{\mathsf{E}(\textit{total demand per cycle})} \stackrel{\star}{=} \frac{\mathsf{E}(\textit{demand satisfied per unit of time})}{\mathsf{E}(\textit{total demand per unit of time})}.
\end{aligned}$$

The definition of the α-service levels is an adapted combination of the definitions given by [Sch81] and [SP85]. The definition of the β-service level is standard and can be found in [Sch81] and [SP85]. Note that the underlying probability measures may be different casewise . Here $\stackrel{\star}{=}$ is justified by ergodicity of $\widetilde{X}$ and therefore is a property of regenerative processes.

We define $K_Y := K_{Y_I} + K_{Y_0}$, where

$$K_{Y_I} := \sum_{k \in \bar{I}} C(k) \quad \text{and} \quad K_{Y_0} := C(0).$$

The normalization constant K from Definition 3.5 and Theorem 5.6 is then given by $K = K^d(J)K_Y$. For models $M_{\mathrm{I}}^{st}, M_{\mathrm{II}}^{st}, M_{\mathrm{I}}^{bl}, M_{\mathrm{I}}^{sk}$ and the detailed system D^{st} we derive from standard summations:

$$\mathsf{P}(Y=0) = C(0)\left(\sum_{k\in\bar{I}_0} C(k)\right)^{-1}.$$

Lemma 6.3 (Performance measures for $\mathbf{M_I}$) *Model M_{I} has the following inventory related measures of system performance:*

$$\begin{aligned}
\lambda_R &= \left(\frac{\eta_1+\nu}{\eta_1}\right)^r \frac{\eta_1}{K_Y}, & \lambda_D &= \lambda_A = \eta_1 \mathsf{P}(Y>0),\\
s &= r - \frac{\eta_1}{\nu}\left(1-\left(\frac{\eta_1}{\eta_1+\nu}\right)^r\right), & &\\
\alpha_1 &= 1-\left(\frac{\eta_1}{\eta_1+\nu}\right)^r, & \alpha_2 &= \beta = \mathsf{P}(Y>0).
\end{aligned}$$

For the (r,Q)-policy

$$\overline{Inv} = K_Y^{-1}\left(\frac{\eta_1+\nu}{\eta_1}\right)^r Q\left(\frac{Q+1}{2}+s\right).$$

For the (r,S)-policy

$$\overline{Inv} = K_Y^{-1}\left(\frac{\eta_1+\nu}{\eta_1}\right)^r\left(\frac{\eta_1}{\nu}s + \frac{(S+1)S-(r+1)r}{2}\right).$$

Here $K_Y = Q\left(\frac{\eta_1+\nu}{\eta_1}\right)^r + \frac{\eta_1}{\nu}$ for the (r,Q)-policy case and $K_Y = \left(S-r+\frac{\eta_1}{\nu}\right)\left(\frac{\eta_1+\nu}{\eta_1}\right)^r$ for the (r,S)-policy case.

Proof. The proof of Lemma 6.3 is presented in the Appendix. □

Corollary 6.4 (Performance measures for $\mathbf{M_{II}^{st}}$) *The inventory related performance measures of model M_{II}^{st} are the same as for model M_{I} given in Lemma 6.3.*

Proof. Model M_{II}^{st} has the same state space $\widetilde{E}$ and the same sets D, A, R_q and R_l defined in the proof of Lemma 6.3 as model M_{I}. The intensities of arrivals to and departures from node 1 in both models are the same as well. Furthermore, the normalization constant K and the steady state probabilities of model M_{II}^{st} for states $(n_1,...,n_J;k)\in\widetilde{E}$ with $k\in\bar{I}$ coincide with those of model M_{I}. Hence, the computation of the intensities λ_D, λ_A and the average on-hand inventory position $\overline{Inv}$ also apply to model M_{II}^{st}. Since, $\pi_{\mathrm{II}}(x)\nu_{\mathrm{II}}(x) = \pi_{\mathrm{I}}(x)\nu_{\mathrm{I}}(x)$ for all $x\in\widetilde{E}$ the expected reorder rate λ_R, the safety stock s and the α_1-service level coincide in models M_{II}^{st} and M_{I}. The results for α_2 and β are now immediate. □

Corollary 6.5 (Performance measures for $\mathbf{D^{st}}$) *The inventory related performance measures of model D^{st} are the same as for model M_{I} given in Lemma 6.3.*

Proof. This result follows Corollary 5.13 and Corollary 6.4. The short proof of Corollary 6.5 is given in the Appendix for later reference. □

The results from Lemma 6.3 carry over easily to models $D^{bl,i}$ and $D^{sk,i}$ from Section 5.2. Here, we define $\overline{K}_Y := K_{Y_I} + K_{Y_0}(1+a)$, where

$$a := \sum_{j=2}^{J} \mathsf{P}^{i(J)}(X_j = 0)\frac{\mu_j}{\eta_j}\, r(1,j). \tag{33}$$

The normalization constant $\overline{K}$ is then given by $\overline{K} = K^i(J)\overline{K}_Y$. For models $D^{bl,i}$ and $D^{sk,i}$ it follows in analogy to the computations of the normalization constant $\overline{K}$ in the proof of Theorem 5.16 that

$$\mathsf{P}(Y=0) = C(0)\,(1+a)\left(\sum_{k\in\bar{I}} C(k) + C(0)\,(1+a)\right)^{-1} \quad \text{with } a \text{ from (33).}$$

Corollary 6.6 (Performance measures for $\mathbf{D^{bl,i}}$ and $\mathbf{D^{sk,i}}$) *The inventory related performance measures of models $D^{bl,i}$ and $D^{sk,i}$ are the same as for model M_{I} given in Lemma 6.3 with K_Y replaced by $\overline{K}_Y$.*

Proof. The calculation of the intensities λ_R, λ_D and λ_A can be carried out on the same lines as in the Proof Corollary 6.5 with π_D replaced by π_d and $K^d(J)$ replaced by $K^i(J)$. The calculation of the safety stock s, the α_1-service level and the average on-hand inventory $\overline{Inv}$ follows on the lines of the proof of Lemma 6.3. Results for α_1 and β are now obvious. □

Remark 6.7 (Invariance properties) *The safety stock s and service level α_1 only depend on η_1, ν and reorder point r. Furthermore, these measures are the same for all the different models defined above.*

All performance measures stated in Lemma 6.3 depend on the characteristics of the original network (without attached inventory) only through the parameter η_1.

From a system's point of view the possibly most important performance measure is the steady state throughput. For the standard Jackson network it is simply λ, see Definition 2.1. By simple flow balancing arguments we can derive throughput results for the general case of Jackson networks with an attached inventory in parallel to the computations for Jackson networks with unreliable servers in [SD03a].

Theorem 6.8 *The throughput TH in a Jackson network according to Definition 2.1 with (r,Q)- or (r,S)-policy at node 1 is the effective external arrival rate to the network. This is*

(a) for networks with rerouting according to Stalling

$$TH(st) = \lambda \mathsf{P}(Y>0),$$

(b) for networks with rerouting according to Blocking

$$TH(bl) = \lambda \mathsf{P}(Y>0) + (\lambda - \lambda_1)\mathsf{P}(Y=0) = \lambda - \lambda_1 \mathsf{P}(Y=0),$$

(c) for networks with rerouting according to Skipping

$$TH(sk) = \lambda \mathsf{P}(Y>0) + (\lambda - \lambda_1 \hat{r}(1,0))\mathsf{P}(Y=0) = \lambda - \lambda_1 \hat{r}(1,0)\mathsf{P}(Y=0),$$

where P is the underlying probability measure which governs the network process of the corresponding Jackson network in steady state. Note that the underlying probability measures may be different casewise as will be stated below.

Remark 6.9 *The expected number of lost sales per unit of time, which is the expected number of arrivals to the network which have to be rejected due to a depleted inventory is given by*

$$LS(rr) = \lambda - TH(rr), \qquad \text{with } rr \in \{st, bl, sk\}.$$

The expected number of lost sales per cycle is then given by $\overline{LS}_c(rr) = \overline{LS}(rr)/\lambda_R$.

In order to further investigate the throughput of the different systems, which have been introduced in this paper we once again note that for models $M_{\mathrm{I}}^{st}, M_{\mathrm{II}}^{st}, M_{\mathrm{I}}^{bl}, M_{\mathrm{I}}^{sk}$ and the detailed system D^{st} we have

$$\mathsf{P}(Y=0) = C(0)\left(\sum_{k\in\bar{I}_0} C(k)\right)^{-1}.$$

For models $D^{bl,i}$ and $D^{sk,i}$

$$\mathsf{P}(Y=0) = C(0)\,(1+a)\left(\sum_{k\in\bar{I}} C(k) + C(0)\,(1+a)\right)^{-1} \quad \text{with } a \text{ from (33).}$$

Hence, we have

$$TH(D^{st}) = TH(M_{\mathrm{I}}^{st}) = TH(M_{\mathrm{II}}^{st}) \le TH(M_{\mathrm{I}}^{bl}) \le TH(M_{\mathrm{I}}^{sk}) \le \lambda$$

and

$$TH(D^{bl,i}) \le TH(D^{sk,i}) \le \lambda.$$

Furthermore, since $a > 0$ we have $\mathsf{P}(Y_{M_{\mathrm{I}}} = 0) < \mathsf{P}(Y_{D^{bl,i}} = 0)$. This implies

$$TH(D^{bl,i}) \le TH(M_{\mathrm{I}}^{bl}) \quad \text{and} \quad TH(D^{sk,i}) \le TH(M_{\mathrm{I}}^{sk}).$$

So far there is no difference in the performance measures of models $M_{\mathrm{I}}^{st}, M_{\mathrm{II}}^{st}$ and the detailed system D^{st}. The next Theorem will reveal the fundamental difference between the various ways to model the Jackson network with attached inventory.

Theorem 6.10 *Let* $\mathsf{E}^{d(J)}(X_j)$ *denote the average queue length at node* $j \in \bar{J}$ *in the stationary standard Jackson network with state-dependent service rates. Then the average queue length at node* $j \in \bar{J}$ *is given by*

$$\mathsf{E}(X_j) = \begin{cases} \mathsf{E}^{d(J)}(X_j) & \text{for models } M_{\mathrm{I}}^{st}, M_{\mathrm{I}}^{bl}, M_{\mathrm{I}}^{sk}, \\ \mathsf{E}^{d(J)}(X_j) + \mathsf{P}(Y=0)r(1,j) & \text{for models } D^{st}, M_{\mathrm{II}}^{st}. \end{cases}$$

Proof. The calculation of the average queue length for models $M_{\mathrm{I}}^{st}, M_{\mathrm{I}}^{bl}$ and M_{I}^{sk} is straight forward. Therefore we only carry out the proof for model D^{st}:

$$\begin{aligned}
\mathsf{P}(X_j = n_j) &= \sum_{(n_1,..,n_{j-1},n_{j+1},..,n_J)\in\mathbb{N}_0^{J-1}} \left(\sum_{k\in\bar{I}} \pi(n_1,...,n_J;k) + \sum_{m\in\bar{J}_0} \pi(n_1,...,n_J;0;m)\right) \\
&= \sum_{(n_1,..,n_{j-1},n_{j+1},..,n_J)\in\mathbb{N}_0^{J-1}} K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)\left(\sum_{k\in\bar{I}} C(k) + C(0)\,r(1,0) + C(0)\sum_{m\in\bar{J}}\frac{\mu_m(n_m)}{\eta_m}r(1,m)\right) \\
&= \frac{\mathsf{P}^{d(J)}(X_j = n_j)}{\sum_{k\in\bar{I}_0} C(k)}\left(\sum_{k\in\bar{I}} C(k) + C(0)\,r(1,0) + C(0)\sum_{m\in\bar{J},m\ne j} r(1,m)\right) \\
&\quad + 1_{\{n_j>0\}}\sum_{(n_1,..,n_{j-1},n_{j+1},..,n_J)\in\mathbb{N}_0^{J-1}} K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)C(0)\,\frac{\mu_j(n_j)}{\eta_j}r(1,j)
\end{aligned}$$

$$= \mathsf{P}^{d(J)}(X_j = n_j) - \frac{\mathsf{P}^{d(J)}(X_j = n_j)}{\sum_{k\in\bar{I}_0} C(k)} C(0)\, r(1,j) + 1_{\{n_j>0\}} \frac{\mathsf{P}^{d(J)}(X_j = n_j - 1)}{\sum_{k\in\bar{I}_0} C(k)} C(0)\, r(1,j)$$
$$= \mathsf{P}^{d(J)}(X_j = n_j) + \mathsf{P}(Y=0) r(1,j) \left(1_{\{n_j>0\}} \mathsf{P}^{d(J)}(X_j = n_j - 1) - \mathsf{P}^{d(J)}(X_j = n_j)\right).$$

Hence,

$$\begin{aligned}
\mathsf{E}(X_j) &= \sum_{n_j=1}^{\infty} n_j \mathsf{P}(X_j = n_j) \\
&= \mathsf{E}^{d(J)}(X_j) + \sum_{n_j=1}^{\infty} n_j \mathsf{P}(Y=0) r(1,j) \left(\mathsf{P}^{d(J)}(X_j = n_j - 1) - \mathsf{P}^{d(J)}(X_j = n_j)\right) \\
&= \mathsf{E}^{d(J)}(X_j) + \mathsf{P}(Y=0) r(1,j) \sum_{n_j=1}^{\infty} \mathsf{P}^{d(J)}(X_j = n_j - 1) \\
&= \mathsf{E}^{d(J)}(X_j) + \mathsf{P}(Y=0) r(1,j).
\end{aligned}$$

The above calculations show that the probability for a certain queue length as well as the average queue length at the servers of the Jackson network are exactly the same for model M_{II}^{st}. □

Remark 6.11 $\mathsf{P}(Y=0)r(1,j)$ *is the probability that the system is stalled and that the customer who took the last item from inventory joins server* j.

Theorem 6.12 *Let* $\mathsf{E}^{i(J)}(X_j)$ *denote the average queue length at node* $j \in \bar{J}$ *in the stationary standard Jackson network with state-independent service rates. Then the average queue length at node* $j \in \bar{J}$ *for models* $D^{bl,i}$ *and* $D^{sk,i}$ *is given by*

$$\mathsf{E}(X_j) = \begin{cases} \mathsf{E}^{i(J)}(X_1) + \dfrac{\mathsf{P}(Y=0)}{1+a} r(1,1) & \textit{for } j = 1, \\ \mathsf{E}^{i(J)}(X_j) & \textit{for } j \in \bar{J} \setminus \{1\}. \end{cases}$$

Proof.

$$\mathsf{P}(X_1 = n_1) = \sum_{(n_2,..,n_J)\in\mathbb{N}_0^{J-1}} \left(\sum_{k\in\bar{I}} \pi(n_1,...,n_J;k) + \sum_{m\in\bar{J}_0} \pi(n_1,...,n_J;0;m) \right)$$
$$= \sum_{(n_2,..,n_J)\in\mathbb{N}_0^{J-1}} \overline{K}^{-1} \prod_{l=1}^{J} \left(\frac{\eta_l}{\mu_l}\right)^{n_l} \left(\sum_{k\in\bar{I}} C(k) + C(0)\, r(1,0) + C(0)\, \frac{\mu_1}{\eta_1} (1-\delta_{0n_1}) r(1,1) \right.$$
$$\left. + C(0) \sum_{m=2}^{J} \frac{\mu_m}{\eta_m}\, r(1,m) \right)$$
$$= \frac{\mathsf{P}^{i(J)}(X_1 = n_1)}{K_{Y_I} + K_{Y_0}(1+a)} \left(\sum_{k\in\bar{I}} C(k) + C(0)\, r(1,0) + C(0) \sum_{m=2}^{J} r(1,m) \right)$$
$$+ C(0) \sum_{m=2}^{J} \frac{\mathsf{P}^{i(J)}(X_1 = n_1) \mathsf{P}^{i(J)}(X_m = 0)}{K_{Y_I} + K_{Y_0}(1+a)} \frac{\mu_m}{\eta_m}\, r(1,m) + 1_{\{n_1>0\}} \frac{\mathsf{P}^{i(J)}(X_1 = n_1)}{K_{Y_I} + K_{Y_0}(1+a)} C(0)\, \frac{\mu_1}{\eta_1}\, r(1,1)$$

$$
\begin{aligned}
&= \mathsf{P}^{i(J)}(X_1 = n_1) - \frac{\mathsf{P}^{i(J)}(X_1 = n_1)}{K_{Y_I} + K_{Y_0}(1+a)} C(0)\, r(1,1) + 1_{\{n_1>0\}} \frac{\mathsf{P}^{i(J)}(X_1 = n_1)}{K_{Y_I} + K_{Y_0}(1+a)} C(0)\, \frac{\mu_1}{\eta_1}\, r(1,1) \\
&= \mathsf{P}^{i(J)}(X_1 = n_1) + \frac{K_{Y_0} r(1,1)}{K_{Y_I} + K_{Y_0}(1+a)} \left(1_{\{n_1>0\}} \frac{\mu_1}{\eta_1} - 1 \right) \mathsf{P}^{i(J)}(X_1 = n_1) \\
&= \mathsf{P}^{i(J)}(X_1 = n_1) + \frac{\mathsf{P}(Y=0)}{1+a} r(1,1) \left(1_{\{n_1>0\}} \frac{\mu_1}{\eta_1} - 1 \right) \mathsf{P}^{i(J)}(X_1 = n_1).
\end{aligned}
$$

Note that $\mathsf{P}(X_1 = 0) < \mathsf{P}^{i(J)}(X_1 = 0)$ and $\mathsf{P}(X_1 = n_1) > \mathsf{P}^{i(J)}(X_1 = n_1)$ for all $n_1 > 0$. Furthermore, $\left(1_{\{n_1>0\}} \frac{\mu_1}{\eta_1} - 1\right) = \frac{\mu_1 - \eta_1}{\eta_1} > 0$ for $n_1 > 0$, which is the reciprocal of the mean queue length at server 1 in a stationary standard Jackson network. The average queue length at server 1 is here given by:

$$
\begin{aligned}
\mathsf{E}(X_1) &= \sum_{n_1=1}^{\infty} n_1 \mathsf{P}(X_1 = n_1) = \mathsf{E}^{i(J)}(X_1) + \frac{\mathsf{P}(Y=0)}{1+a} r(1,1) \sum_{n_1=1}^{\infty} n_1 \frac{\mu_1 - \eta_1}{\eta_1} \mathsf{P}^{i(J)}(X_1 = n_1) \\
&= \mathsf{E}^{i(J)}(X_j) + \frac{\mathsf{P}(Y=0)}{1+a} r(1,1).
\end{aligned}
$$

We now calculate $\mathsf{P}(X_j = n_j)$ for $j = 2, ..., J$:

$$
\begin{aligned}
\mathsf{P}(X_j = n_j) &= \sum_{(n_1,..,n_{j-1},n_{j+1},..,n_J) \in \mathbb{N}_0^{J-1}} \left(\sum_{k \in \bar{I}} \pi(n_1, ..., n_J; k) + \sum_{m \in \bar{J}_0} \pi(n_1, ..., n_J; 0; m) \right) \\
&= \sum_{(n_1,..,n_{j-1},n_{j+1},..,n_J) \in \mathbb{N}_0^{J-1}} \overline{K}^{-1} \prod_{l=1}^{J} \left(\frac{\eta_l}{\mu_l} \right)^{n_l} \left(\sum_{k \in \bar{I}} C(k) + C(0)\, r(1,0) + \frac{\mu_1}{\nu} (1 - \delta_{0 n_1}) r(1,1) \right. \\
&\qquad \left. + C(0) \sum_{m=2}^{J} \frac{\mu_m}{\eta_m}\, r(1,m) \right) \\
&= \frac{\mathsf{P}^{i(J)}(X_j = n_j)}{K_{Y_I} + K_{Y_0}(1+a)} \left(\sum_{k \in \bar{I}} C(k) + C(0)\, r(1,0) + C(0)\, r(1,1) + C(0) \sum_{m=2, m \neq j}^{J} r(1,m) \right) \\
&\quad + C(0) \sum_{m=2, m \neq j}^{J} \frac{\mathsf{P}^{i(J)}(X_j = n_j) \mathsf{P}^{i(J)}(X_m = 0)}{K_{Y_I} + K_{Y_0}(1+a)} \frac{\mu_m}{\eta_m}\, r(1,m) + \frac{\mathsf{P}^{i(J)}(X_j = n_j)}{K_{Y_I} + K_{Y_0}(1+a)} C(0)\, \frac{\mu_j}{\eta_j}\, r(1,j) \\
&= \frac{\mathsf{P}^{i(J)}(X_j = n_j)}{K_{Y_I} + K_{Y_0}(1+a)} \left(\sum_{k \in \bar{I}_0} C(k) \right) - \frac{\mathsf{P}^{i(J)}(X_j = n_j)}{K_{Y_I} + K_{Y_0}(1+a)} C(0)\, r(1,j) \\
&\quad + C(0)\, \frac{\mathsf{P}^{i(J)}(X_j = n_j)}{K_{Y_I} + K_{Y_0}(1+a)} \sum_{m=2}^{J} \mathsf{P}^{i(J)}(X_m = 0) \frac{\mu_m}{\eta_m}\, r(1,m) \\
&\quad - C(0)\, \frac{\mathsf{P}^{i(J)}(X_j = n_j)}{K_{Y_I} + K_{Y_0}(1+a)} \mathsf{P}^{i(J)}(X_j = 0) \frac{\mu_j}{\eta_j}\, r(1,j) + \frac{\mathsf{P}^{i(J)}(X_j = n_j)}{K_{Y_I} + K_{Y_0}(1+a)} C(0)\, \frac{\mu_j}{\eta_j}\, r(1,j) \\
&= \mathsf{P}^{i(J)}(X_j = n_j) + \frac{K_{Y_0} r(1,j)}{K_{Y_I} + K_{Y_0}(1+a)} \mathsf{P}^{i(J)}(X_j = n_j) \left(-1 - \mathsf{P}^{i(J)}(X_j = 0) \frac{\mu_j}{\eta_j} + \frac{\mu_j}{\eta_j} \right)
\end{aligned}
$$

$$= \mathsf{P}^{i(J)}(X_j = n_j) + \frac{K_{Y_0} r(1,j)}{K_{Y_I} + K_{Y_0}(1+a)} \mathsf{P}^{i(J)}(X_j = n_j) \left(-1 + \mathsf{P}^{i(J)}(X_j > 0) \frac{\mu_j}{\eta_j} \right)$$
$$= \mathsf{P}^{i(J)}(X_j = n_j) + \frac{K_{Y_0} r(1,j)}{K_{Y_I} + K_{Y_0}(1+a)} \mathsf{P}^{i(J)}(X_j = n_j) \left(-1 + \mathsf{P}^{i(J)}(X_j \geq 0) \right)$$
$$= \mathsf{P}^{i(J)}(X_j = n_j).$$

Hence, $\mathsf{E}(X_j) = \mathsf{E}^{i(J)}(X_j)$. □

7 Conclusions

In this paper we have studied exponential Jackson and Gordon-Newell networks with an attached inventory under (r, Q)- or (r, S)-policy at one of the nodes.

We considered three stochastic regimes for state dependent replenishment procedures, which lead to different stochastic lead time distributions governed by state-dependent intensities. The differences in the procedures mainly occur during the time the inventory is depleted.

We further assume that locally lost sales occur due to stockout of the inventory, but we did not require that the lost sales at the node with inventory are lost to the total network. We therefore constructed different rerouting regimes for customers rejected at the node with inventory, called Stalling, Blocking and Skipping. These rules enable customers even to retry for service later on after being rejected.

We proved that for a variety of models under the selected rerouting regimes stationary distributions of product form for the system processes exist. In some cases it could be shown that the limiting distributions of the queue length processes coincide in addition with that of the standard Jackson respectively Gordon-Newell network.

For the so obtained equilibrium systems which describe in a unified treatment service with queueing and inventory management with lead time delays various performance measures related to inventory and service of customers have been derived and compared for the different models.

Appendix

Proof of Lemma 6.3.
Let H be a subset of $\widetilde{E} \times \widetilde{E} -$ diag $(\widetilde{E} \times \widetilde{E})$ and let N_H be the point process counting the H-transitions of $\widetilde{X}$, i.e.

$$N_H(C) = \int_C 1_H(\widetilde{X}_{s-}, \widetilde{X}_s)\, N(ds),\ C \in \mathbb{B},$$

where $\mathbb{B}$ denotes the Borel σ-field of $\mathbb{R}$. Let us consider special sets that describe events of the queueing system M_{I} under investigation:

departure of customers from node 1:

$$D \quad = \{\big((n_1, ..., n_J; k), (n_1 - 1, ..., n_j + 1, ..., n_J; k-1)\big) : n_1 \geq 1, k \geq 1, j \in \bar{J}_0\},$$

arrival of customers to node 1, who are admitted to the node:

$$A \quad = \{\big((n_1, ..., n_J; k), (n_1 + 1, ..., n_j - 1, ..., n_J; k)\big) : k \geq 1, n_j \geq 1, j \in \bar{J}_0 \setminus \{1\}\}$$
$$\cup \{\big((n_1, ..., n_J; k), (n_1, ..., n_J; k-1)\big) : n_1 \geq 1, k \geq 1\},$$

arrival of replenishment orders for the order quantity (r, Q)-policy:

$$R_q \quad = \{\big((n_1, ..., n_J; k), (n_1, ..., n_J; k + Q)\big) : 0 \leq k \leq r\},$$

arrival of replenishment orders for the order level (r, S)-policy:

$$R_l \quad = \{\big((n_1, ..., n_J; k), (n_1, ..., n_J; S)\big) : 0 \leq k \leq r\},$$

Let R denote the set that describes the arrival of replenishment orders for any of the two systems. From formula (1.4.15) on page 38 of [BB03] the intensity of a marked point process N_H is given by

$$\lambda_H = E[N_H((0,1])] = \sum_{(e_1,e_2)\in H} \pi(e_1)q\big(e_1,e_2\big). \tag{34}$$

Hence,

$$\begin{aligned}
\lambda_R &= \sum_{(e_1,e_2)\in R} \pi_{\mathrm{I}}(e_1)q(e_1,e_2) = \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} \sum_{k=0}^{r} \pi_{\mathrm{I}}(n_1,\dots,n_J;k)\,\nu_{\mathrm{I}}(n_1,\dots,n_J;k) \\
&= \nu \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} \sum_{k=0}^{r} K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right) C(k) \left(r(1,0) + \sum_{j=1}^{J} r(1,j)\,\frac{\mu_j(n_j)}{\eta_j}\right) \\
&= \nu \left(\sum_{k=0}^{r} C(k)\right) \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right) \left(r(1,0) + \sum_{j=1}^{J} r(1,j)\,\frac{\mu_j(n_j)}{\eta_j}\right) \\
&= \nu \left(\sum_{k=0}^{r} C(k)\right) K^{-1}K^d(J) \sum_{j\in\bar{J}_0} r(1,j) = \frac{\eta_1}{K_Y}\left(\frac{\eta_1+\nu}{\eta_1}\right)^r,
\end{aligned}$$

since $\sum_{k=1}^{r} C(k) = \frac{\eta_1}{\nu}\left(\frac{\eta_1+\nu}{\eta_1}\right)^r - \frac{\eta_1}{\nu}$ for the (r,Q)- and the (r,S)-policy.

$$\begin{aligned}
\lambda_D &= \sum_{(e_1,e_2)\in D} \pi_{\mathrm{I}}(e_1)q(e_1,e_2) = \sum_{n_1=1}^{\infty} \sum_{(n_2,\dots,n_J)\in\mathbb{N}_0^{J-1}} \sum_{k\in\bar{I}} \pi_{\mathrm{I}}(n_1,\dots,n_J;k)\,\mu_1(n_1) \\
&= \sum_{n_1=1}^{\infty} \sum_{(n_2,\dots,n_J)\in\mathbb{N}_0^{J-1}} \sum_{k\in\bar{I}} K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right) C(k)\,\mu_1(n_1) \\
&= \eta_1 \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right) \sum_{k\in\bar{I}} C(k) \\
&= \eta_1 K^d(J)K^{-1} \sum_{k\in\bar{I}} C(k) = \eta_1 \mathsf{P}(Y>0).
\end{aligned}$$

Using (1) we further obtain

$$\begin{aligned}
\lambda_A &= \sum_{(e_1,e_2)\in A} \pi_{\mathrm{I}}(e_1)q(e_1,e_2) = \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} \sum_{k\in\bar{I}} \pi_{\mathrm{I}}(n_1,\dots,n_J;k) \left(\lambda_1 + \sum_{j\in\bar{J}} \mu_j(n_j)\,r(j,1)\right) \\
&= \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} K^{-1} \prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right) \left(\lambda_1 + \sum_{j\in\bar{J}} \mu_j(n_j)\,r(j,1)\right) \sum_{k\in\bar{I}} C(k) \\
&= K^{-1}K^d(J) \left(\lambda_1 + \sum_{j\in\bar{J}} \eta_j r(j,1)\right) \sum_{k\in\bar{I}} C(k) = K^{-1}K^d(J)\eta_1 \sum_{k\in\bar{I}} C(k) = \lambda_D.
\end{aligned}$$

From Definition 6.1 the safety stock is given by

$$
\begin{aligned}
s &= \sum_{k\geq 1} k\, \mathsf{P}^0_{N_R}(Y_{net,0-}=k) = \sum_{k=1}^{r} k \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} \mathsf{P}^0_{N_R}(\widetilde{X}_{0-}=(n_1,\dots,n_J;k))\\
&\stackrel{\star}{=} \frac{1}{\lambda_R}\sum_{k=1}^{r} k \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} \pi_{\mathrm{I}}(n_1,\dots,n_J;k)\,\nu_{\mathrm{I}}(n_1,\dots,n_J;k)\\
&= \frac{K_Y}{\eta_1}\left(\frac{\eta_1}{\eta_1+\nu}\right)^r \sum_{k=1}^{r} kC(k) \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)\nu\left(r(1,0)+\sum_{j=1}^{J} r(1,j)\,\frac{\mu_j(n_j)}{\eta_j}\right)\\
&= \frac{\nu}{\eta_1}\left(\frac{\eta_1}{\eta_1+\nu}\right)^r \sum_{k=1}^{r} kC(k)\\
&= \frac{\nu}{\eta_1}\left(\frac{\eta_1}{\eta_1+\nu}\right)^r \frac{1-(r+1)\left(\frac{\eta_1+\nu}{\eta_1}\right)^r + r\left(\frac{\eta_1+\nu}{\eta_1}\right)^{r+1}}{\left(1-\frac{\eta_1+\nu}{\eta_1}\right)^2}\\
&= \frac{\nu}{(\eta_1+\nu)^r}\,\frac{\eta_1^{r+1}-(r+1)\eta_1(\eta_1+\nu)^r + r(\eta_1+\nu)^{r+1}}{\nu^2}\\
&= r - \frac{\eta_1}{\nu}\left(1-\left(\frac{\eta_1}{\eta_1+\nu}\right)^r\right),
\end{aligned}
$$

since $C(k), 1\leq k\leq r$ are the same for the (r,Q)- and (r,S)-policy. $\stackrel{\star}{=}$ follows from formula (1.4.18) on page 39 of [BB03].

$$
\begin{aligned}
\alpha_1 &= \mathsf{P}^0_{N_R}(Y_{0-}>0) = 1 - \sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} \mathsf{P}^0_{N_R}(\widetilde{X}_{0-}=(n_1,\dots,n_J;0))\\
&= 1-\frac{1}{\lambda_R}\sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} \pi_{\mathrm{I}}(n_1,\dots,n_J;0)\nu_{\mathrm{I}}(n_1,\dots,n_J;0)\\
&= 1-\frac{1}{\lambda_R}\sum_{(n_1,\dots,n_J)\in\mathbb{N}_0^J} K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)\eta_1\left(r(1,0)+\sum_{j=1}^{J} r(1,j)\,\frac{\mu_j(n_j)}{\eta_j}\right)\\
&= 1-\left(\frac{\eta_1}{\eta_1+\nu}\right)^r,
\end{aligned}
$$

$$\alpha_2 = \mathsf{P}(Y>0),$$

$$\beta = \frac{\lambda_A}{\eta_1} = \mathsf{P}(Y>0).$$

Finally

$$\overline{Inv} = K^{-1}K^d(J)\sum_{k\in\overline{I}} kC(k) = K_Y^{-1}\sum_{k\in\overline{I}} kC(k).$$

The results for the (r,Q)- and (r,S)-policy follow from Theorem 3.5 and 4.4 of [SSD+03]. $\square$

Proof of Corollary 6.5.
Let us consider special sets that describe events of the queueing system under investigation:

departure of customers from node 1:

$$\begin{aligned} D &= \{\big((n_1,...,n_J;k),(n_1-1,...,n_j+1,...,n_J;k-1)\big) : n_1 \geq 1, k > 1, j \in \bar{J}_0\} \\ &\quad \cup \{\big((n_1,...,n_J;1),(n_1-1,...,n_j+1,...,n_J;0;j)\big) : n_1 \geq 1, j \in \bar{J}_0\}, \end{aligned}$$

arrival of customers to node 1, who are admitted to the node:

$$\begin{aligned} A &= \{\big((n_1,...,n_J;k),(n_1+1,...,n_j-1,...,n_J;k)\big) : k \geq 1, n_j \geq 1, j \in \bar{J}_0 \setminus \{1\}\} \\ &\quad \cup \{\big((n_1,...,n_J;k),(n_1,...,n_J;k-1)\big) : n_1 \geq 1, k > 1\} \\ &\quad \cup \{\big((n_1,...,n_J;1),(n_1,...,n_J;0;1)\big) : n_1 \geq 1\}, \end{aligned}$$

arrival of replenishment orders for the order quantity (r,Q)-policy:

$$\begin{aligned} R_q &= \{\big((n_1,...,n_J;k),(n_1,...,n_J;k+Q)\big) : 1 \leq k \leq r\} \\ &\qquad \cup \{\big((n_1,...,n_J;0;j),(n_1,...,n_J;Q)\big) : n_j \geq 1, j \in \bar{J}_0\}, \end{aligned}$$

arrival of replenishment orders for the order level (r,S)-policy:

$$\begin{aligned} R_l &= \{\big((n_1,...,n_J;k),(n_1,...,n_J;S)\big) : 1 \leq k \leq r\} \\ &\qquad \cup \{\big((n_1,...,n_J;0;j),(n_1,...,n_J;S)\big) : n_j \geq 1, j \in \bar{J}_0\}. \end{aligned}$$

Computations similar to those on the proof of Lemma 6.3 show that λ_D, λ_A and λ_R coincide in models M_{I} and D^{st}. Furthermore, $\pi_D(n_1,...,n_J;k) = \pi_{\mathrm{I}}(n_1,...,n_J;k)$ and $\nu_D(n_1,...,n_J;k) = \nu_{\mathrm{I}}(n_1,...,n_J;k)$ for all $(n_1,...,n_J;k) \in E_{\bar{J}}$ and the normalization constants K of both models are the same. Hence, from the formulas in the proof of Lemma 6.3 the safety stock s and the average on-hand inventory $\overline{Inv}$ are equal to the quantities given in Lemma 6.3. Since

$$\begin{aligned} &\pi_{\mathrm{I}}(n_1,...,n_J;0)\nu_{\mathrm{I}}(n_1,...,n_J;0) \\ &= K^{-1}\prod_{l=1}^{J}\prod_{i=1}^{n_l}\left(\frac{\eta_l}{\mu_l(i)}\right)\eta_1\left(r(1,0)+\sum_{j\in\bar{J}}\frac{\mu_j(n_j)}{\eta_j}r(1,j)\right) \\ &= \sum_{j\in\bar{J}_0}\pi(n_1,...,n_J;0;j)\,\nu_D(n_1,...,n_J;0;j). \end{aligned}$$

the α_1-service level is the same as in Lemma 6.3. The result for α_2 and β is immediate. □

Table 1: Overview of the different models

$M_{\mathrm{I}}^{st}, M_{\mathrm{I}}^{bl}, M_{\mathrm{I}}^{sk}$	
Definition 3.1	$\widetilde{E} = E \times \bar{I}_0$
Definition 3.2	$\nu_{\mathrm{I}}(n_1,...,n_J;k) = \nu \left(r(1,0) + \sum_{l=1}^{J} r(1,l) \frac{\mu_l(n_l)}{\eta_l} \right) 1_{\{0 \le k \le r\}}$
Definition 3.5	$\pi_{\mathrm{I}}(n_1,...,n_J;k) = K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(k)$
	$K = K^d(J) \sum_{k \in \bar{I}_0} C(k)$
M_{II}^{st}	
Definition 3.1	$\widetilde{E} = E \times \bar{I}_0$
Definition 3.2	$\nu_{\mathrm{II}}(n_1,...,n_J;k) = \begin{cases} \nu \left(r(1,0) + \sum\limits_{l=1}^{J} r(1,l) \dfrac{\mu_l(n_l)}{\eta_l} \right) 1_{\{1 \le k \le r\}}, & k \in \bar{I} \\ \nu, & k = 0 \end{cases}$
Definition 3.5	$\pi_{\mathrm{II}}(n_1,...,n_J;k) = \begin{cases} K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(k), & k \in \bar{I} \\ K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(0) \cdot & \\ \quad \cdot \left(r(1,0) + \sum_{j \in \bar{J}} \frac{\mu_j(n_j)}{\eta_j} r(1,j) \right), & k = 0 \end{cases}$
	$K = K^d(J) \sum_{k \in \bar{I}_0} C(k)$
D^{st}	
Definition 5.1	$\widetilde{E}_D := E_{\bar{I}} \cup E_0 := E \times \bar{I} \cup E \times \{0\} \times \bar{J}_0$
Definition 5.2	$\nu_D(x) = \begin{cases} \nu \left(r(1,0) + \sum\limits_{l=1}^{J} r(1,l) \dfrac{\mu_l(n_l)}{\eta_l} \right) 1_{\{1 \le k \le r\}}, & x = (n_1,...,n_J;k) \in E_{\bar{I}} \\ \nu\,(1 - \delta_{0 n_j}), & x = (n_1,...,n_J;0;j) \in E_0 \end{cases}$
Theorem 5.6	$\pi_D(x) = \begin{cases} K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(k), & x = (n_1,...,n_J;k) \in E_{\bar{I}} \\ K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(0)\, \frac{\mu_j(n_j)}{\eta_j} r(1,j), & x = (n_1,...,n_J;0;j) \in E_0, \\ & \quad j \in \bar{J} \\ K^{-1} \prod_{l=1}^{J} \prod_{i=1}^{n_l} \left(\frac{\eta_l}{\mu_l(i)} \right) C(0)\, r(1,0), & x = (n_1,...,n_J;0;0) \in E_0 \end{cases}$
	$K = K^d(J) \sum_{k \in \bar{I}_0} C(k)$
$D^{bl,i}, D^{sk,i}$	
Definition 5.1	$\widetilde{E}_D := E_{\bar{I}} \cup E_0 := E \times \bar{I} \cup E \times \{0\} \times \bar{J}_0$
Definition 5.2	$\nu_D(x) = \begin{cases} \nu \left(r(1,0) + \sum\limits_{l=1}^{J} r(1,l) \dfrac{\mu_l}{\eta_l} \right) 1_{\{1 \le k \le r\}}, & x = (n_1,...,n_J;k) \in E_{\bar{I}} \\ \nu\,(1 - \delta_{0 n_j}), & x = (n_1,...,n_J;0;j) \in E_0 \end{cases}$
Theorem 5.16	$\pi_d(x) = \begin{cases} \overline{K}^{-1} \prod_{l=1}^{J} \left(\frac{\eta_l}{\mu_l} \right)^{n_l} C(k), & x = (n_1,...,n_J;k) \in E_{\bar{I}} \\ \overline{K}^{-1} \prod_{l=1}^{J} \left(\frac{\eta_l}{\mu_l} \right)^{n_l} C(0)\, \frac{\mu_j}{\eta_j} r(1,j), & x = (n_1,...,n_J;0;j) \in E_0, \\ & \quad j \in \bar{J}_0 \setminus \{1\} \\ \overline{K}^{-1} \prod_{l=1}^{J} \left(\frac{\eta_l}{\mu_l} \right)^{n_l} C(0)\, \frac{\mu_1}{\eta_1} (1 - \delta_{0 n_1}) r(1,1), & x = (n_1,...,n_J;0;1) \in E_0 \end{cases}$
	$\overline{K} = K^i(J) \left(\sum_{k \in \bar{I}_0} C(k) + C(0) \sum_{j=2}^{J} \mathsf{P}^{i(J)}(X_j = 0) \frac{\mu_j}{\eta_j} r(1,j) \right)$

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