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On the structure of roundtrip time distributions in discrete time networks

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Abstract

In a closed cycle of Bernoulli single server nodes we investigate a customer's roundtrip behaviour, i.e. his cycle time and the successive sojourn times during one cycle. We derive a structural semi-invariance property for the conditional joint sojourn time distribution, given the cycle time. The main result is that the conditional joint sojourn time distribution is independent of the number of circulating customers if the feasibility region of the cycle time variable is taken into account. This holds in the balanced machines case as well as in the presence of bottleneck stations. An explicit expression for the counting density of this distribution is given, which is of independent interest.

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Keywords: sojourn times, cycle times, discrete time network, geometric convolutions, conditional distributions of geometric variables

1 Introduction

Queueing networks in discrete time have found much interest in recent years in the literature due to the occurence of network applications which are based on a generic internal lattice structured time scale. E.g. in the ATM protocol for high speed networks at least three relevant levels with different time scales are considered, call level, burst level, and cell level. The latter two levels can be modeled on a basis of generic discrete scales. For a discussion see the editorial introduction and several papers in the special volume [TGBT94]. A more recently evolved class of applications for such models is the analysis and control of Wireless Local Area Networks (WLANs), for more details see the relevant contributions in the Proceedings of the 18th International Teletraffic Congress (2003) [CLTG03], especially [SV03], [NSB$^+$03], [MKC03] und [KN03]. For surveys on discrete time network theory see [Woo94], [BK93], [CMP99], [Dad01].

A fundamental model in performance analysis of teletraffic network theory is a closed cycle of queues which is a model which suitably reflects many features of a transmission line under window flow control [Rei79]. In this note we investigate the structure of these cyclic networks in discrete time mainly under the assumption of heavy traffic.
We consider a closed cycle of a fixed number M of geometric servers when the population size increases. The problems arising in this situation of increasing load have been subject of many reseach effort in the classical

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situation of continuous time Gordon-Newell networks, starting with the first paper of Gordon and Newell on the subject [GN67]. It is well known that there are two essentially different pictures.
(i) All servers have the same load: Then the total population in system is shared equally by all nodes up to random fluctuations.
(ii) Differently loaded servers exist: Then bottlenecks occur, which in the simplest case with exactly one slowest server means that almost the whole population is queued up at this slowest server.

Similar pictures are observed in discrete time networks of queues. In the cyclic network case under consideration this means that
(i) all servers have the same service intensity (the case of balanced machines) , and
(ii) if there is exactly one server with slowest service which then this node evolves as a unique bottleneck.
In case of balanced machines we find as steady state *not* a uniform distribution on the state space [PR94]. In case of unequal service rates we observe occurrence of bottlenecks as in the continuous time networks. The relevant bottleneck behaviour is determined in the appendix.

The aim of this paper is to investigate further the structural behaviour of discrete time geometric cyclic networks in both cases. Our main interest is in the travel time behaviour of individual customers: The starting point is the steady state cycle time distribution of a customer and (observed during one cycle) the joint vector distribution of his successive sojourn times at the different nodes. These distributions are well known and given in the transform domain by their respective z-transforms (generating functions) [Dad01].

In case of a unique bottleneck from Theorem 5.1 it follows by computations parallel to those in [Box88] that the cycle time of a customer is almost completely determined by his sojourn time at the bottleneck node if the total number N of customers in the network is sufficiently high. Moreover it can be shown that the influence of the other servers decreases geometrically in N.
We investigate in more detail the structure of the joint sojourn time vector and derive especially the conditional joint sojourn time distribution at the different nodes for a customer during one cycle given his total cycle time.

In the continuous counterpart network this problem was solved for the case (i) of balanced machines by Kelly [Kel83]. The result is as one would have expected: The prescribed cycle time is uniformly distributed on the nodes, independently of the number of customers in the system. This is clearly in accordance with our intuition about the network's physical behaviour. The case (ii) of unbalanced machines was investigated in [MD03], the result was a surprising invariance property. Although with increasing population the cycle time is more and more dominated by the bottleneck server, any prescribed cycle time is shared by the successive nodes at a ratio which is independent of the population size. Therefore the conditional sojourn time distribution of the successive sojourn times can be computed from the case of a network with only one customer cycling, which means in a network without any competition for network resources.

From the physical structure of the discrete time network it follows that a completely parallel result cannot be expected, because any customer at any node has to stay there for at least one time unit. Additionally, a customer's cycle can expire only if all other customers in the network have departed from the last node of the cycle. We conclude that the cycle time consists of at least $\max(M, N)$ units of time and the conditional sojourn time distribution given a prescribed cycle time will always depend on M and N through $\max(M, N)$. Nevertheless we shall prove a modified semi-invariance principle which describes the conditional distributions in explicit form and which is rather surprising and simple as well.

For all *feasible* values of the cycle time in a fixed network with M nodes the conditional joint distribution of a test customer's sojourn times at the successively visited nodes given that cycle time is independent of the number N of circulating customers. But the *feasibility region* for the values of the cycle time depends on $\max(M, N)$, and so we cannot dispense our main statement from referencing to the population size of the network as it is

possible in the network's continuous time counterpart. This holds for the case of balanced machines as well as for the unbalanced cycle.
Nevertheless, the semi-inavariance property is again in contrast to intuition with respect to the bottleneck behaviour, which occurs in the discrete time case as well, when there are servers working with different speed. As a byproduct and on the way to our main result we compute the pmf (probability mass function) of the joint sojourn time distribution and the cycle time distribution as well. These explicit expressions show a surprising nice structure which is of independent interest and which cannot be seen from the z-transform directly. As another byproduct we obtain some interesting properties of geometric convolutions.

The outline of the paper: In Section 2 we describe the system under consideration in detail and collect the required necessities on stationary behaviour, the arrival theorem, and the customer stationary travel time behaviour in the z-transform domain from the literature. We then invert the z-transforms and discuss the special systems' structure in light of these explicit expressions for the respective probability mass functions. These explicit expressions are utilized in Section 3 to derive the semi-invariance principle for the conditional distributions of the joint sojourn time vector. In Section 4 we present some technical lemmata on geometrical distributions which shed another light on our results and are partly needed for the derivations. These give en passant a simple proof of a recently proved property of geometric convolutions [SB99]. In the appendix we compute the bottleneck behaviour of the joint queue length vector process of the network in case of having exactly one slowest server.

2 Description of the System, Steady States, and Sojourn Times

Consider a closed cyclic queueing network with M nodes and a fixed number N of indistinguishable, clockwise circulating customers. The time scale for the system under consideration is $\mathbb{N}_0$. Here and in the following $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the set of nonnegative integers. The servers at the nodes $Q[i], i = 1. \dots, M$, are state independent Bernoulli servers, which are working independently, i. e. station $Q[i]$ is a single-server queue with service probability $p_i \in (0,1)$, infinite waiting room and first-come-first-serve (FCFS) queueuing discipline. If at time instant n at node $Q[i]$ a customer is in service, then his service ends at time instant $n+1$ with probability p_i. With probability $q_i := 1 - p_i$ this customer will stay at node $Q[i]$ requesting at least one more unit of service time. Customers being served at station $Q[i]$ jump instantaneousley to node $Q[i+1]$ (resp. to $Q[1]$ if a service ends at node $Q[M]$). If at some node at the same time an arrival and a departure occur we assume that the departure event takes place first (*late arrivals, departures first*, LA-DF). As a consequence, a customer arriving at the end of time slot $[n-1, n)$ at $Q[i]$ has to stay there for at least until the end of time slot $[n, n+1)$, that is, his sojourn time at any queue is one time unit at minimum. A sojourn time at $Q[i]$ is the sum of the waiting time at node $Q[i]$ and the subsequent service time there.
For station $Q[i]$ let $X_i(n)$ denote the queue length (= number of waiting customers + customer in service, if any) there at time $n \in \mathbb{N}_0$. Then the joint queue length process $X = ((X_1(n), \dots, X_M(n)) : t \geq 0)$ of the system is a discrete time Markov process with state space

$$Z(M,N) := \{(n_1, \dots, n_M) \in \mathbb{N}_0^M \mid n_1 + \dots + n_M = N\}.$$

Here $(X_1(t), \dots, X_M(t)) = (n_1, \dots, n_M)$ indicates that at time t there are n_i customers present at node $Q[i]$. X is ergodic with unique stationary and limiting distribution [PR94]

$$\pi^{M,N}(n_1, \dots, n_M) = G(M,N)^{-1} \prod_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)}, \quad (n_1, \dots, n_M) \in Z(M,N)$$

with the normalizing constant

$$G(M,N)^{-1} = \sum_{(n_1,\ldots,n_M)\in Z(M,N)} \prod_{i=1}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)},$$

with $\eta(n,m) := 1-\delta(n,m)$ defined as the complementary Kronecker delta. Now tag one customer to follow him on his itineraries through the cycle. Let $S_i^{(n)}$, $i \in \underline{M}$, $n \in \mathbb{N}$, denote the sojourn time at node $Q[i]$ of the tagged customer during his n-th cycle through his route $W = (1,\ldots,M)$ after the system is started (as usual we shall refer to the tagged customer as a test customer whose passage through the system will be observed). The n-th cycle time of the test customer is the sum $S^{(n)} = S_1^{(n)} + \cdots + S_M^{(n)}$ of all subsequent sojourn times during the n-th cycle. If necessary, we use the subscript (M,N) to indicate, that the underlying distribution $P = P_{(M,N)}$ refers to M nodes and a total population of N circulating customers (including the test customer).
To determine the joint sojourn time distribution, the knowledge of the arrival distribution $\pi_1^{(M,N)}$ at arrival instants of the tagged customer at node $Q[1]$ is necessary. Let $\pi_k^{(M,N)}$ be defined as the conditional distribution of the other customers seen by the test customer jumping from node $Q[k-1]$ to node $Q[k]$ at his arrival instant at node $Q[k]$, given that such a transition of the tagged customer takes place ($k=1$ means a jump from node $Q[M]$ to $Q[1]$).

Theorem 2.1. *[Dad96]*
*Let $X = (X_n)_{n\in\mathbb{Z}}$ be the stationary continuation of $(X_n)_{n\in\mathbb{N}_0}$ under $\pi^{M,N}$. Further let $A(k)$ be the event "*At time instant $n=0$ an arrival at node $Q[k]$ takes place*". Then*

$$\begin{aligned}
\pi_k^{M,N}(n_1,\ldots,n_m) &:= P\big(X_0 = (n_1,\ldots,n_{k-1},n_k+1,n_{k+1},\ldots,n_m) \mid A(k)\big) \\
&= G_k(M,N)^{-1} \cdot \left(\frac{q_k}{p_k}\right)^{n_k} \prod_{\substack{i=1\\ i\neq k}}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)}
\end{aligned} \tag{1}$$

for all $(n_1,\ldots,n_M) \in Z(M,N-1)$.

The arrival probability $\pi_k^{M,N}$ depends on k, the number of the node where the arrival appears: the expression in (1) is asymmetric with respect to k and the arrival probability keeps trace of the arriving customer. Note that the value of the normalizing constant

$$G_k(M,N) = \sum_{\vec{n}\in Z(M,N-1)} \left(\frac{q_k}{p_k}\right)^{n_k} \prod_{\substack{i=1\\ i\neq k}}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)}$$

is independent of the node number $k \in \underline{M}$, see [Dad01], lemma 7.3. The common value is denoted by $G_{\mathrm{A}}(M,N)$. The asymptotic behaviour of the family $(S_1^{(n)},\ldots,S_M^{(n)})_{n\in\mathbb{N}}$ of the successive sojourn times in the transform domain ($g^{(M,N)}\colon [0,1)^M \to \mathbb{R}$ is the corresponding generating function) is, see [Dad97],

$$
\begin{aligned}
g^{(M,N)}(u_1,\dots,u_M) &= \lim_{n\to\infty} E\left(\prod_{i=1}^{M} u_i^{S_i^{(n)}}\right) = E_{\pi_1^{M,N}}\left(\prod_{i=1}^{M} u_i^{S_i^{(1)}}\right)\\
&= \sum_{(n_1,\dots,n_M)\in Z(M,N-1)} G_{\mathrm{A}}(M,N)^{-1}\left(\frac{q_1}{p_1}\right)^{n_1}\prod_{i=2}^{M}\left[\left(\frac{q_i}{p_i}\right)^{n_i}\left(\frac{1}{q_i}\right)^{\eta(0,n_i)}\right]\\
&\quad\cdot\left(\frac{p_1u_1}{1-q_1u_1}\right)^{n_1+1}\prod_{i=2}^{M}\left[\left(\frac{p_iu_i}{1-q_iu_i}\right)^{n_i+1}\left(\frac{1}{u_i}\right)^{\eta(0,n_i)}\right]\\
&= G_{\mathrm{A}}(M,N)^{-1}\sum_{(n_1,\dots,n_M)\in Z(M,N-1)}\left(\frac{q_1}{p_1}\right)^{n_1}\left(\frac{p_1u_1}{1-q_1u_1}\right)^{n_1+1} \qquad (2)\\
&\quad\cdot\prod_{i=2}^{M}\left(\frac{q_i}{p_i}\right)^{n_i} q_i^{\delta(0,n_i)-1}\left(\frac{p_iu_i}{1-q_iu_i}\right)^{n_i+1} u_i^{\delta(0,n_i)-1}.
\end{aligned}
$$

Taking $u_1 = \cdots = u_M =: u$ yields the generating function $h^{(M,N)}(u) = g^{(M,N)}(u,\dots,u)$ of the cycle time under steady-state conditions, which was given in [Dad96]. Again with lemma 7.3 in [Dad01] it can be seen, that the generating function $g^{(M,N)}$ is the same for all $k \in \underline{M}$, that is the sojourn time distribution doesn't depend neither on the number of the starting node nor on the arrangement of the nodes with respect to their service probabilities.
In our lemma 2.1 below we give the probability mass function (pmf) of the joint sojourn time distribution. In that expression the apparently asymmetry with respect to k vanishes.

In the following a random vector $(S_1,\dots,S_M)$ in $\mathbb{R}_+^M$, with generating function $g^{(M,N)}$ is interpreted as the family of successive sojourn times under steady-state conditions. Similarly a random variable S with generating function $h^{(M,N)}$ is interpreted as the steady-state cycle time.

The observation that the generating function $g^{(M,N)}$ of the joint distribution of the successive sojourn times can be inverted with a closed form expression for the corresponding pmf is the starting point for the further investigations. See section 4 for the utilized properties concerning geometric distributions.
Recall that $Z(M,N) = \{(x_1,\dots,x_M)\in\mathbb{N}_0^M \mid x_1+\cdots+x_M = N\}$. Let $Z^\star(M,N)$ be defined as

$$
Z^\star(M,N) := \{(x_1,\dots,x_M)\in\mathbb{N}^M \mid x_1+\cdots+x_M = N\}
$$

Lemma 2.1. *The distribution of the successive sojourn times $(S_1,\dots,S_M)$ experienced by the test-customer on his itinerary under steady-state conditions possesses the pmf $p^{(M,N)}$, given by*

$$
p^{(M,N)}(s_1,\dots,s_M) = G_{\mathrm{A}}(M,N)^{-1}\binom{s_1+\cdots+s_M-1}{N-1}\prod_{i=1}^{M}p_iq_i^{s_i-1}\mathbf{1}[(s_1,\dots,s_M)\in\mathbb{N}^M]. \qquad (3)
$$

The cycle time $S := S_1+\cdots+S_M$ has the pmf $q^{(M,N)}$, given by

$$
q^{(M,N)}(s) = G_{\mathrm{A}}(M,N)^{-1}\binom{s-1}{N-1}\sum_{(s_1,\dots,s_M)\in Z^\star(M,s)}\prod_{i=1}^{M}p_iq_i^{s_i-1}. \qquad (4)
$$

Note that $q^{(M,N)}(s) = 0$ for $s < \max(M,N)$: if $s < M$ holds, then $Z^\star(M,s) = \varnothing$, hence the empty sum is null by definition. If $s \geqq M$ but $s < N$ holds, then the binomial coefficient $\binom{s-1}{N-1}$ vanishes.
Proof . From (2) one gets the following joint sojourn time distribution:

$$
\begin{aligned}
P^{(S_1,\dots,S_M)} &= G_{\mathrm{A}}(M,N)^{-1} \sum_{(n_1,\dots,n_M)\in Z(M,N-1)} \left(\frac{q_1}{p_1}\right)^{n_1} (\mathrm{Nb}(n_1+1,p) * \epsilon_{n_1+1}) \\
&\quad \otimes \bigotimes_{i=2}^{M} \left(\frac{q_i}{p_i}\right)^{n_i} q_i^{\delta(0,n_i)-1} \left(\mathrm{Nb}(n_i+1,p_i) * \epsilon_{n_i+\delta(0,n_i)}\right),
\end{aligned}
$$

where $\otimes$ denotes the product and $*$ the convolution of measures. Furthermore ϵ_x is the Dirac Measure in x and $\mathrm{Nb}(n,p)$ denotes the negative binomial distribution with parameters n and p (see section 4).
Hence for $(s_1,\dots,s_M) \in \mathbb{Z}^M$ the pmf $p^{(M,N)}$ of $(S_1,\dots,S_M)$ is given by

$$
\begin{aligned}
& p^{(M,N)}(s_1,\dots,s_M) \\
&= G_{\mathrm{A}}(M,N)^{-1} \sum_{(n_1,\dots,n_M)\in Z(M,N-1)} p_1 q_1^{s_1-1} \binom{s_1-1}{s_1-1-n_1} \prod_{i=2}^{M} p_i q_i^{s_i-1} \binom{s_i-\delta(0,n_i)}{s_i-n_i-\delta(0,n_i)}.
\end{aligned} \tag{5}
$$

Because of the binomial coefficients the RHS of (5) vanishes if $s_i < 1$ for some $i \in \underline{M}$. This reflects a consequence of the LA-DF regime: the test customer has to stay at least one quantum of time at every station.
For $(s_1,\dots,s_M) \in \mathbb{N}^M$ one has

$$
\begin{aligned}
& p^{(M,N)}(s_1,\dots,s_M) \\
&= G_{\mathrm{A}}(M,N)^{-1} \sum_{(n_1,\dots,n_M)\in Z(M,N-1)} p_1 q_1^{s_1-1} \binom{s_1-1}{n_1} \prod_{i=2}^{M} p_i q_i^{s_i-1} \binom{s_i-\delta(0,n_i)}{n_i} \\
&= G_{\mathrm{A}}(M,N)^{-1} \left(\prod_{i=1}^{M} p_i q_i^{s_i-1}\right) \sum_{(n_1,\dots,n_M)\in Z(M,N-1)} \binom{s_1-1}{n_1} \prod_{i=2}^{M} \binom{s_i}{n_i} \\
&= G_{\mathrm{A}}(M,N)^{-1} \binom{s_1+\dots+s_M-1}{N-1} \prod_{i=1}^{M} p_i q_i^{s_i-1} \mathbf{1}[(s_1,\dots,s_M)\in\mathbb{N}^M].
\end{aligned}
$$

For the third equation we have used *Cauchy's polynomial formula* [Jor65]

$$
\sum_{(n_1,\dots,n_M)\in Z(M,N)} \prod_{j=1}^{M} \binom{s_j}{n_j} = \binom{s_1+\dots+s_M}{N}, \quad (s_1\dots,s_M)\in\mathbb{N}_0^M,\ N\in\mathbb{N}_0, \tag{6}
$$

which in case $M = 2$ is known as *identity of Vandermonde*, see [Hen74].
To compute the pmf $q^{(M,N)}$ of the cycle time S note that $q^{(M,N)}(s) = 0$ if $s < M$, due to the indicator $\mathbf{1}[(s_1,\dots,s_M)\in\mathbb{N}^M]$ in (6). Hence

$$
\begin{aligned}
q^{(M,N)}(s) &= \sum_{(s_1,\dots,s_M)\in Z^\star(M,s)} p^{(M,N)}(s_1,\dots,s_M) \\
&= G_{\mathrm{A}}(M,N)^{-1} \sum_{(s_1,\dots,s_M)\in Z^\star(M,s)} \binom{s_1+\ldots+s_M-1}{N-1} \prod_{i=1}^{M} p_i q_i^{s_i-1} \\
&= G_{\mathrm{A}}(M,N)^{-1} \binom{s-1}{N-1} \sum_{(s_1,\dots,s_M)\in Z^\star(M,s)} \prod_{i=1}^{M} p_i q_i^{s_i-1}, \qquad s\in\mathbb{Z}.
\end{aligned}
$$

$\square$

That $p^{(M,N)}(s_1,\dots,s_M)=0$ for $(s_1,\dots,s_M)\in\mathbb{N}^M$ with $s_1+\dots+s_M<\max(M,N)$ follows from the physical behaviour of the network: the test customer C has finished his round trip at that time instant at which his service at $Q[M]$ terminates. Necessarily all other customers ($N-1$ in number) must have been served at $Q[M]$ up to the time instant C entered service at $Q[M]$, hence it lasted at least N units of time up to the moment C can leave $Q[M]$, and it lasted for at least $M-1$ units of time to arrive at $Q[M]$. Hence the test customer's roundtrip time lasted for at least $\max(M,N)$ units of time.

Special cases:
(1) balanced machines, that is $p_1=\dots=p_M=:p$. Then the normalizing constant $G_A(M,N)$ takes on the following form:

$$\begin{aligned} G_A(M,N) &= \sum_{\vec{n}\in Z(M,N-1)} \left(\frac{q}{p}\right)^{n_M} \prod_{i=1}^{M-1} \left(\frac{q}{p}\right)^{n_i} \left(\frac{1}{q}\right)^{\eta(0,n_i)} \\ &= \left(\frac{q}{p}\right)^{N-1} \sum_{\vec{n}\in Z(M,N-1)} \left(\frac{1}{q}\right)^{\eta(0,n_1)+\dots+\eta(0,n_{M-1})} \\ &= \left(\frac{1}{p}\right)^{N-1} q^{N-M} \sum_{\vec{n}\in Z(M,N-1)} q^{\delta(0,n_1)+\dots+\delta(0,n_{M-1})} \end{aligned}$$

Now from (3) or (4) the cycle time pmf in the case of balanced machines can easily be computed:

$$q^{(M,N)}(s)=\frac{p^{M+N-1}q^{s-N}\binom{s-1}{N-1}\binom{s-1}{M-1}}{\sum\limits_{\vec{n}\in Z(M,N-1)} q^{\delta(0,n_1)+\dots+\delta(0,n_{M-1})}}, \tag{7}$$

We have used the relation

$$\#\big(Z^*(M,s)\big)=\#\big(Z(M,s-M)\big)=\binom{s-1}{M-1},$$

which holds for $s,M\in\mathbb{N}$, because of $(s_1,\dots,s_M)\in Z^*(M,s)\Leftrightarrow(s_1-1,\dots,s_M-1)\in Z(M,s-M)$, where $\#(A)$ denotes the number of elements of the set A.
To compute the denominator in (7), note that $\delta(0,n_1)+\dots+\delta(0,n_{M-1})$ is the number of empty nodes among the first $M-1$ stations. So, for $M\geqq 2$, $N\geqq 1$

$$\begin{aligned} \sum_{\vec{n}\in Z(M,N-1)} q^{\delta(0,n_1)+\dots+\delta(0,n_{M-1})} &= q^{M-1}+\sum_{k=0}^{N-2}\sum_{l=0}^{M-2}\binom{M-1}{l}\binom{N-2-k}{M-2-l}q^l \\ &= \sum_{l=0}^{M-1}\binom{M-1}{l}\binom{N-1}{M-1-l}q^l \end{aligned} \tag{8}$$

The expression in (8) has the following combinatorial interpretation: the binomial coefficient $\binom{M-1}{l}$ is the number of ways to choose the l nodenumbers of the empty nodes among the first $M-1$ nodes. If there are exactly l of the first $M-1$ nodes empty, we need $M-1-l$ customers to ensure that the remaining $M-l-1$ nodes among the first $M-1$ are nonempty and $\binom{N-1}{M-1-l}$ is the number of elements of $Z(M-l,N-M+l)$, that is the number of arrangements of the remaining $N-M+l$ customers at the available positions at $M-l-1+1$ stations.

Inserting (8) the cycle time pmf becomes

$$q^{(M,N)}(s) = \frac{p^{M+N-1}q^{s-N}\binom{s-1}{N-1}\binom{s-1}{M-1}}{\sum\limits_{l=0}^{M-1}\binom{M-1}{l}\binom{N-1}{M-1-l}q^l}.$$

(2) Pairwise distinct service probabilities: now $p_i \neq p_j$ for $i \neq j$ holds and one can use (12) to obtain

$$\begin{aligned} q^{(M,N)}(s) &= G_{\mathrm{A}}(M,N)^{-1}\binom{s-1}{N-1}\left(\prod_{i=1}^{M} p_i\right)\sum_{i=1}^{M}\frac{q_i^{s-1}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{M}(q_i-q_j)}\mathbf{1}[s \geqq M] \\ &= G_{\mathrm{A}}(M,N)^{-1}\binom{s-1}{N-1}\sum_{i=1}^{M} p_i q_i^{s-1}\prod_{\substack{j=1\\ j\neq i}}^{M}\frac{p_j}{p_j-p_i}\mathbf{1}[s \geqq M]. \end{aligned}$$

If there are some servers having the same service probabilities, but not all of them are equal, one needs a generalization of formula (12). In that case the resulting expression is considerably more involved, see (15).

3 The conditional sojourn time distribution

Having the functional relationships expressed in equations (3) and (4) at hand we are now prepared to compute directly the conditional distribution of the tagged customer's successive sojourn times, given his cycle time. This leads to the following *semi-invariance theorem* which resembles the results for continuous time networks in [Kel83] and [MD03].

Theorem 3.1. *Let $(S_1,\ldots,S_M)$ be the successive sojourn times at the nodes $Q[1],\ldots,Q[M]$ of a typical customer on his itinerary under steady-state conditions through a closed cyclic queueing system with a number of M FCFS state independent Bernoulli single-server nodes and a fixed number N of indistinguishable customers with time scale $\mathbb{N}_0$ and LA-DF regime.*
The conditional joint distribution $P_{(M,N)}^{(S_1,\ldots,S_M)|S}$ of the tagged customer's successive sojourn times, given his cycle time $S = S_1 + \cdots + S_M$, is distributed as an independent familiy $(X_1,\ldots,X_M)$, the components of which are geometrically distributed on $\mathbb{N}$ (that is $X_i \sim \mathrm{Geo}_{(1)}(p_i)$, see equation (10)) given their sum $X_1 + \cdots + X_M$. That is, for $s \geqq \max(M,N)$, the pmf $p_{(M,N;s)}$ of $P_{(M,N)}^{(S_1,\ldots,S_M)|S=s}$ is given by

$$\begin{aligned} p_{(M,N;s)}(s_1,\ldots,s_M) &:= P_{(M,N)}(S_1 = s_1,\ldots,S_M = s_M \mid S = s) \\ &= \frac{\mathbf{1}[(s_1,\ldots,s_M) \in Z^\star(M,s)]\prod\limits_{i=1}^{M} p_i q_i^{s_i-1}}{\sum\limits_{(t_1,\ldots,t_M)\in Z^\star(M,s)}\prod\limits_{i=1}^{M} p_i q_i^{t_i-1}}. \end{aligned} \tag{9}$$

Remarks: This result is striking. It means that looking at $p_{(M,N;s)}$ as a function of the total network population N, for feasible values s of the cycle time the conditional joint distribution is independent of the numbers of customers circulating: let $M \leqq s$. Then

$$p_{(M,N;s)} = p_{(M,1;s)}, \qquad N \leqq s.$$

If a feasible cycle time is given, the distribution of the successive sojourn times which the tagged customer experiences at the nodes $Q[1], \ldots, Q[M]$ of his cycle is the same, irrespective of whether there are many, few, or no other customers present, provided $N \leqq s$.
Saying it the other way round: if the tagged customer is the only customer cycling, his sojourn times coincide with his service times. If we compute in this trivial case the conditional distribution of the tagged customer's successive sojourn times, given his cycle time, then the theorem says that the result of the computation applies without any changes to *feasible* vectors and cycle time values for the case of further customers present in the system as well. So we can compute the (due to customer interactions) complicated conditional vector distribution for networks with large population sizes in a network where only one customer lives who has not to compete for network resources. This is unexpected and seems to be contrary to the intuition guided by the bottleneck behaviour of queueing networks. This can be described as follows: assume there are servers working with different service probabilities p_i and assume that the server at $Q[1]$ is the only one with smallest service probability p_1. Then the slowest server (that is the server at $Q[1]$) becomes the bottleneck of the system (see the appendix). The number of customers at the slowest queue grows unboundedly if the total network population N tends to infinity, while the queue lengths of the other nodes jointly remain finite with probability one. The limiting distribution of the other nodes is the steady state distribution of an open $M-1$ stage Bernoulli server tandem with Bernoulli input stream of intensity p_1 and service probabilities $p_2, \ldots, p_M$ at the stages $2, \ldots, M$ respectively. From this observation we can conclude similar statements on the dominating behaviour of the bottleneck node for the cycle times as in the case of continuous time systems [Box88].
Although our result is not as smooth as in the continuous time setting it reveals to some extent the behaviour of the corresponding continuous time cyclic network, see [MD03]. In the continuous time case the conditional joint distribution of the tagged customer's successive sojourn times, given his cycle time is distributed as an independent familiy $(X_1, \ldots, X_M)$, the components of which are exponentially distributed with parameter μ_i respectively, where μ_i is the service rate at station $Q[i]$.

Proof. Let $s \geqq \max(M, N)$, hence $q^{(M,N)}(s) > 0$. Then

$$\begin{aligned}
&p_{(M,N;s)}(s_1, \ldots, s_M) = P_{(M,N)}(S_1 = s_1, \ldots, S_M = s_M \mid S_1 + \cdots + S_M = s) \\
&= \frac{P_{(M,N)}(S_1 = s_1, \ldots, S_M = s_M)\mathbf{1}[s_1 + \cdots + s_M = s]}{q^{(M,N)}(s)} \\
&= \frac{\mathbf{1}[s_1 + \cdots + s_M = s]\mathbf{1}[(s_1, \ldots, s_M) \in \mathbb{N}^M] G_A(M,N)^{-1} \binom{s_1 + \cdots + s_M - 1}{N-1} \prod_{i=1}^{M} p_i q_i^{s_i - 1}}{G_A(M,N)^{-1} \binom{s-1}{N-1} \sum_{(t_1,\ldots,t_M)\in Z^\star(M,s)} \prod_{i=1}^{M} p_i q_i^{t_i-1}} \\
&= \frac{\mathbf{1}[(s_1, \ldots, s_M) \in Z^\star(M,s)] \prod_{i=1}^{M} p_i q_i^{s_i-1}}{\sum_{(t_1,\ldots,t_M)\in Z^\star(M,s)} \prod_{i=1}^{M} p_i q_i^{t_i-1}}.
\end{aligned}$$

The denominator in (9) can be written as follows

$$\sum_{(t_1,\ldots,t_M)\in Z^\star(M,s)} \prod_{i=1}^{M} p_i q_i^{t_i-1} = \sum_{(t_1,\ldots,t_M)\in Z(M,s-M)} \prod_{i=1}^{M} p_i q_i^{t_i},$$

and this is the pmf at s of the sum $X_1 + \cdots + X_M$ of an independent family $(X_1, \ldots, X_M)$ of random variables which are $\mathrm{Geo}_{(1)}(p_i)$ distributed respectively (see (14) in subsection 4.1). $\square$

4 Geometric Distributions

Extending the definition of geometric distributions formally to the set $\mathbb{Z}$ of integers simplifies many of the computations in our proofs and circumvents considering many special cases. The geometric distribution $\mathrm{Geo}_{(0)}(p)$ with parameter $p \in (0,1)$ is defined by its corresponding pmf (probability mass function) $\mathrm{geo}_{(0)}(p)$ on the set $\mathbb{Z}$ of integers by

$$\mathrm{geo}_{(0)}(p)\,(n) := pq^n \mathbf{1}[n \geqq 0] = \begin{cases} 0 & \text{if } n < 0 \\ pq^n & \text{if } n \geqq 0 \end{cases}, \qquad p \in (0,1),$$

with $q := 1-p$. The generating function $\mathrm{GF}\big(\mathrm{geo}_{(0)}(p)\big) =: \gamma_{(0)}(p)$ of $\mathrm{Geo}_{(0)}(p)$ is then given by

$$\gamma_{(0)}(p)(z) = \sum_{n\in\mathbb{Z}} \mathrm{geo}_{(0)}(p)\,(n)\,z^n = \sum_{n=0}^{\infty} pq^n z^n = \frac{p}{1-qz}, \qquad |z| < 1.$$

Note that $\mathrm{Geo}_{(0)}(p)$ is the special case of the Negative Binomial Distribution $\mathrm{Nb}(1,p)$ with parameters 1 and p (the notation in the literature is not unique, see for example [JKK92], Chapter 5).
Define now the family $\mathrm{Geo}_{(r)}(p)$, $r \in \mathbb{N}_0$, of discrete distributions on $\mathbb{Z}$ by

$$\mathrm{Geo}_{(r)}(p) := \mathrm{Geo}_{(0)}(p) * \epsilon_r, \qquad r \in \mathbb{N}_0\,. \tag{10}$$

Here ϵ_r is the Dirac-measure in r and $*$ denotes the convolution of measures. The pmf $\mathrm{geo}_{(r)}(p)$ of $\mathrm{Geo}_{(r)}(p)$ is then given by

$$\mathrm{geo}_{(r)}(p)\,(n) = \mathrm{geo}_{(0)}(p)(n-r)\,\mathbf{1}[n \geqq r] = \begin{cases} 0 & \text{if } n < r \\ pq^{n-r} & \text{if } n \geqq r \end{cases}, \quad r \in \mathbb{N}_0,\ p \in (0,1), \qquad n \in \mathbb{Z}.$$

Binomial coefficient: define the binomial coefficient B as a function on $\mathbb{Z}^2$ to $\mathbb{Z}$ in the following way: $B\colon\ \mathbb{Z}^2 \to \mathbb{Z},\quad (n,k) \mapsto B(n,k) := \binom{n}{k}$ with

$$\begin{aligned} \binom{n}{k} &= \prod_{1\leqq j\leqq k} \frac{n+1-j}{j}, \qquad \text{if } k \geqq 0; \\ \binom{n}{k} &= 0, \qquad \text{if } k < 0. \end{aligned}$$

As usual an empty product is defined to be one. For particular cases we have

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \forall n \in \mathbb{Z}, \text{ and } \binom{n}{k} = 0, \quad \text{if } 0 \leqq n < k. \tag{11}$$

4.1 Convolution of geometrics

Recently the explicit computation of convolutions of geometric distributions with different parameters has found renewed interest fom the point of computational purposes as well as from applications in reliability theory [SB99]. As we have seen in the previous Sections these problems reappear in our framework again. We therefore sketch some details here and simplify the proofs of [SB99].
(a) same parameter: $\underbrace{\mathrm{Geo}_{(0)}(p) * \cdots * \mathrm{Geo}_{(0)}(p)}_{M-\text{times}} = \big(\mathrm{Geo}_{(0)}(p)\big)^{*M}$ has the pmf

$$\Big(\mathrm{geo}_{(0)}(p)\Big)^{*M}(n) = \binom{M+n-1}{n} p^M (1-p)^n, \qquad n \in \mathbb{Z}.$$

Proof. Clearly $\left(\mathrm{geo}_{(0)}(p)\right)^{*M}(n) = 0$ for negative values of n from the definition of a convolution. For $n \geqq 0$ we have

$$\left(\mathrm{geo}_{(0)}(p)\right)^{*M}(n) = \sum_{(n_1,\dots,n_M)\in Z(M,n)} \prod_{i=1}^{M} p(1-p)^{n_i} = \binom{M+n-1}{n} p^M (1-p)^n.$$

□

That is, $\mathrm{Nb}(1,p)^{*M} = \mathrm{Nb}(M,p)$, the negative binomial distribution with parameters M and p.

(b) pairwise disjoint parameters:
In case of a family $\left(\mathrm{Geo}_{(0)}(p_1), \dots, \mathrm{Geo}_{(0)}(p_M)\right)$ of geometric distributions with pairwise distinct parameters $p_1, \dots, p_m$, $p_i \neq p_j$ if $i \neq j$, one can use the the formula

$$\sum_{(n_1,\dots,n_M)\in Z(M,N)} \prod_{i=1}^{M} x_i^{n_i} = \sum_{i=1}^{M} \frac{x_i^{N+M-1}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{M} (x_i - x_j)}, \tag{12}$$

which holds for pairwise distinct real numbers $x_1, \dots, x_M$, to compute the pmf of the desired convolution. Formula (12) was given without proof in [Koe58]. A proof can be found in [Har85].
It follows, that the pmf of $\mathrm{Geo}_{(0)}(p_1) * \dots * \mathrm{Geo}_{(0)}(p_M)$ is given by

$$\mathrm{geo}_{(0)}(p_1) * \dots * \mathrm{geo}_{(0)}(p_M)(n) = \sum_{i=1}^{M} p_i q_i^n \cdot \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{q_i p_j}{p_j - p_i} \mathbf{1}[n \geqq 0].$$

Proof. Clearly $\mathrm{geo}_{(0)}(p_1) * \dots * \mathrm{geo}_{(0)}(p_M)(n) = 0$ if $n < 0$. For $n \geqq 0$

$$\begin{aligned}&\mathrm{geo}_{(0)}(p_1) * \dots * \mathrm{geo}_{(0)}(p_M)(n)\\ &= \sum_{(n_1,\dots,n_M)\in Z(M,n)} \prod_{i=1}^{M} p_i q_i^{n_i} = p_1 \cdots p_M \sum_{i=1}^{M} \frac{q_i^{n+M-1}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{M} (q_i - q_j)} = \sum_{i=1}^{M} p_i q_i^n \cdot \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{q_i p_j}{p_j - p_i}.\end{aligned}$$

□

For our purposes we need convolutions of the form $\mathrm{Geo}_{(r_1)}(p_1) * \dots * \mathrm{Geo}_{(r_M)}(p_M)$. Because of the commutativity of the convolution

$$\mathrm{Geo}_{(r_1)}(p_1) * \dots * \mathrm{Geo}_{(r_M)}(p_M) = \mathrm{Geo}_{(0)}(p_1) * \dots * \mathrm{Geo}_{(0)}(p_M) * \epsilon_{r_1+\dots+r_M}$$

holds, and therefore the corresponding pmf in the case $p_1 = \dots = p_M = p$ is given by

$$\mathrm{geo}_{(r_1)}(p) * \dots * \mathrm{geo}_{(r_M)}(p)(n) = \mathrm{Nb}(M,p)\,(n-r) = \binom{n-r+M-1}{n-r} p^M q^{n-r}, \tag{13}$$

with $r := r_1 + \dots + r_M$. In the same way the pmf of $\mathrm{Geo}_{(r_1)}(p_1) * \dots * \mathrm{Geo}_{(r_M)}(p_M)$ in the case of pairwise distinct p_i's can be written down:

$$
\begin{aligned}
\mathrm{geo}_{(r_1)}(p_1) * \cdots * \mathrm{geo}_{(r_M)}(p_M)(n) &= \mathrm{geo}_{(0)}(p_1) * \cdots * \mathrm{geo}_{(0)}(p_M)(n-r) \\
&= \sum_{(t_1,\ldots,t_M)\in Z(M,n-r)} \prod_{i=1}^{M} p_i q_i^{t_i} \\
&= \sum_{i=1}^{M} p_i q_i^{n+M-1-r} \cdot \prod_{\substack{j=1\\ j\neq i}}^{M} \frac{p_j}{p_j - p_i} \mathbf{1}[n \geqq r].
\end{aligned}
\tag{14}
$$

This simplifies the proof, given in [SB99] for the computation of that pmf.
In case of service rates with any given values one could use the following generalisation of (12). Let $x_i \in \mathbb{R}$, $i \in \underline{m}$. Let $\bar{m}$ denote the number of the different values $\bar{x}_1, \ldots, \bar{x}_{\bar{m}}$ and let ν_l denote the number of $i \in \underline{m}$ with $x_i = \bar{x}_l$. Then

$$
\sum_{n\in Z(m,n)} \prod_{i=1}^{m} x_i^{n_i} = \sum_{l=1}^{\bar{m}} \sum_{\vec{a}\in Z(\bar{m},\nu_l-1)} (-1)^{a_l-\nu_l+1} \left(\bar{x}_l^{n+m-\nu_l}\right) \binom{n+a_l}{n} \left(\prod_{\substack{u=1\\ u\neq l}}^{\bar{m}} \bar{x}_u^{a_u} \binom{\nu_u + a_u - 1}{\nu_u - 1} (\bar{x}_l - \bar{x}_u)^{-\nu_u - a_u} \right) \tag{15}
$$

holds. For a proof see [Ser99] but note that in the expression given there typos occur (Proposition 1.32).

5 Appendix: Bottleneck behaviour

For completeness of the presentation we compute in this appendix the asymptotic bottleneck behaviour of the joint queue length vector process of the network in case of having exactly one slowest server. The restriction to having only one bottleneck is for simplicity of presentation only.

Theorem 5.1. *Let $(X_1, \ldots, X_M)$ be joint the queue-length process of a closed cyclic queueing system with M nodes and N indistinguishable customers in discrete time $\mathbb{N}_0$. The nodes are single-server state independent Bernoulli stations under FCFS with LA-DF regime. Let the service probability at queue $Q[i]$ be p_i, $i \in \underline{M}$ and assume $p_1 < \min(p_2, \ldots, p_M)$. Then the sequence $(\hat{\pi}^{(M,N)})_{N\in\mathbb{N}}$ of the $(M-1)$-dimensional marginal distributions of the queue-length at the stations $Q[2], \ldots, Q[M]$ under steady state conditions converges weakly to the equilibrium distribution of an open $(M-1)$ stage tandem with Bernoulli arrival process and service probabilities $p_2, \ldots, p_M$ respectively, that is*

$$
\lim_{N\to\infty} \hat{\pi}^{(M,N)}(n_2, \ldots, n_M) = \prod_{i=2}^{M} \left(1 - \frac{p_1}{p_i}\right) \left(\frac{p_1 q_i}{q_1 p_i}\right)^{n_i} \left(\frac{1}{q_i}\right)^{\eta(0,n_i)}, \quad \forall (n_2, \ldots, n_M) \in \mathbb{N}_0^{M-1}. \tag{16}
$$

The arrival probability is p_1, the service probability of the slowest queue $Q[1]$ in the closed network. That is in case of unboundedly increasing population size, the closed M-node tandem of Bernoulli servers becomes an open $(M-1)$ node tandem. The arrival probability equals the service probability of the slowest queue.

Proof. Let $p_1 < p_2 \leqq \cdots \leqq p_M$. This is equivalent to $q_1/p_1 > q_2/p_2 \geqq \cdots \geqq q_M/p_M$, $(q_i = 1 - p_i)$. The marginal distribution $\hat{\pi}^{(M,N)}$ of the queue-length at the stations $Q[2], \ldots, Q[M]$ under steady state conditions is given by

$$
\begin{aligned}
\hat{\pi}^{(M,N)}(n_2,\dots,n_M) &= \pi^{(M,N)}\big((N-n_2-\dots-n_M),n_2,\dots,n_M\big) \\
&= \frac{\left(\frac{q_1}{p_1}\right)^{N-n_2-\dots-n_M}\left(\frac{1}{q_1}\right)^{\eta(0,N-n_2-\dots-n_M)}\prod_{i=2}^{M}\left(\frac{q_i}{p_i}\right)^{n_i}\left(\frac{1}{q_i}\right)^{\eta(0,n_i)}}{\sum_{\vec{n}\in Z(M,N)}\prod_{i=1}^{M}\left(\frac{q_i}{p_i}\right)^{n_i}\left(\frac{1}{q_i}\right)^{\eta(0,n_i)}} \\
&= \frac{\left(\frac{1}{q_1}\right)^{\eta(0,N-n_2-\dots-n_M)}\prod_{i=2}^{M}\left(\frac{p_1q_i}{q_1p_i}\right)^{n_i}\left(\frac{1}{q_i}\right)^{\eta(0,n_i)}}{\sum_{\vec{n}\in Z(M,N)}\left(\frac{1}{q_1}\right)^{\eta(0,n_1)}\prod_{i=2}^{M}\left(\frac{p_1q_i}{q_1p_i}\right)^{n_i}\left(\frac{1}{q_i}\right)^{\eta(0,n_i)}}. \qquad (17)
\end{aligned}
$$

To determine the limiting behaviour of the denominator in (17) consider the expression

$$
f(M,N) := \sum_{\vec{n}\in Z(M,N)} a_1^{\eta(0,n_1)}\prod_{i=2}^{M} a_i^{\eta(0,n_i)}x_i^{n_i},
$$

with $0 < a_1,\dots,a_M$ and $1 > \max(x_2,\dots,x_M) > 0$. Then $\lim_{N\to\infty} f(M,N) \in \mathbb{R}$ holds. This can be seen by induction on M: for $M=2$

$$
f(2,N) = \sum_{\vec{n}\in Z(2,N)} a_1^{\eta(0,n_1)}a_2^{\eta(0,n_2)}x_2^{n_2} = a_1 + \sum_{n=1}^{N-1} a_1a_2x_2^n + a_2x_2^N \underset{N\to\infty}{\longrightarrow} a_1 + a_1a_2\frac{x_2}{1-x_2}
$$

Suppose the assertion is valid for $M-1$ and assume additionally $x_2 > x_3 \geqq \dots \geqq x_M$. Then write

$$
\begin{aligned}
&\sum_{\vec{n}\in Z(M,N)} a_1^{\eta(0,n_1)}\prod_{i=2}^{M} a_i^{\eta(0,n_i)}x_i^{n_i} \\
&= \sum_{(n_2,\dots,n_M)\in Z(M-1,N)}\prod_{i=2}^{M} x_i^{n_i}a_i^{\eta(0,n_i)} + a_1\sum_{(n_1>0,n_2,\dots,n_M)\in Z(M,N)}\prod_{i=2}^{M} x_i^{n_i}a_i^{\eta(0,n_i)} \\
&= x_2^N\underbrace{\sum_{(n_2,\dots,n_M)\in Z(M-1,N)} a_2^{\eta(0,n_2)}\prod_{i=3}^{M}\left(\frac{x_i}{x_2}\right)^{n_i}a_i^{\eta(0,n_i)}}_{\text{convergent by induction hyphothesis}} + a_1\sum_{(n_1,n_2,\dots,n_M)\in Z(M,N-1)}\prod_{i=2}^{M} x_i^{n_i}a_i^{\eta(0,n_i)} \\
&\underset{N\to\infty}{\longrightarrow} a_1\prod_{i=2}^{M}\sum_{n_i=0}^{\infty} a_i^{\eta(0,n_i)}x_i^{n_i} = a_1\prod_{i=2}^{M}\left(1 + a_i\frac{x_i}{1-x_i}\right). \qquad (18)
\end{aligned}
$$

Note that x_2 and x_i/x_2 are smaller than one for $i \in \{3,\dots,M\}$. If $x_2 = x_3 \geqq \dots \geqq x_M$ holds, choose $\bar{x}_2$ with $1 > \bar{x}_2 > x_2$ and note that

$$
\sum_{(n_2,\dots,n_M)\in Z(M-1,N)}\prod_{i=2}^{M} x_i^{n_i}a_i^{\eta(0,n_i)} \leqq \sum_{(n_2,\dots,n_M)\in Z(M-1,N)} \bar{x}_2^{n_2}a_2^{\eta(0,n_2)}\prod_{i=3}^{M} x_i^{n_i}a_i^{\eta(0,n_i)}.
$$

Evaluating (18) with $a_i = 1/q_i$ and $x_i = p_1 q_i/(q_1 p_i)$, $i = 2, \ldots, M$ one gets as limit of the denominator in (17)

$$\frac{1}{q_1} \prod_{i=2}^{M} \left(1 - \frac{p_1}{p_i}\right)^{-1}.$$

□

The consequences of the theorem for cycle times of a customer under heavy traffic conditions, i.e. with a large number N of customers in the network, are obvious. On his arrival at the bottleneck node 1 the customer will find almost all other customers before him there. Therefore the cycle time is dominated by a term which is approximately the convolution of N geometric distributions $\text{geo}_{(1)}(p_1)$. This is in accordance with what Boxma [Box88] proved in the continuous time setting.

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