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Abstract

In this paper we introduce isotone differences stochastic ordering of Markov processes on partially ordered state spaces both in discrete and continuous time in order to compare standard Jackson networks and degradable Jackson networks in which the nodes are subject to random breakdowns and repairs.

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1 Introduction

Let $Y = (Y(t), t \geq 0)$, $\widetilde{Y} = (\widetilde{Y}(t), t \geq 0)$ be two stationary homogeneous Markov processes with state space E, possessing identical invariant (stationary) distribution π. We assume that E is a lattice ordered Polish space equipped with a partial ordering $\prec$. Especially, we shall consider $E = \mathbb{R}$ with standard $\leq$ ordering, $E = \mathbb{N}^J$ with coordinatewise ordering $\leq^J$ and $E = \mathcal{P}(\mathrm{J})$, the power set of $\mathrm{J} = \{1, \ldots, J\}$, with inclusion ordering $\subseteq$.

We shall use the following notation. We denote by $E^{(n)} = \times_{i=1}^n E_i$ a product of partially lattice ordered spaces $(E_i, \prec_i)$, $i = 1, \ldots, n$. The partial ordering $\prec^{(n)}$ on $E^{(n)}$ is the coordinatewise ordering, i.e. for $\mathbf{x} = (x_1, \ldots, x_n) \in E^{(n)}$, $\mathbf{y} = (y_1, \ldots, y_n) \in E^{(n)}$, $\mathbf{x} \prec^{(n)} \mathbf{y}$ if and only if $x_i \prec_i y_i$ for all $i = 1, \ldots, n$. If $(E_i, \prec_i) = (E, \prec)$ for $i = 1, \ldots, n$ then we write E^n instead of $E^{(n)}$ and $\prec^n$ instead of $\prec^{(n)}$. Clearly, if E_i are lattice ordered spaces then $(E^{(n)}, \prec^{(n)})$ is lattice ordered as well.

We shall consider integral stochastic orderings on the lattice ordered space $(E^{(n)}, \prec^{(n)})$. A function $f : E^{(n)} \to \mathbb{R}$ is *supermodular* if

$$f(x \wedge y) + f(x \vee y) \geq f(x) + f(y) . \tag{1}$$

Note that the definition of supermodular functions does not require that they are defined on a product space $E^{(n)}$. A function $f : E^{(2)} \to \mathbb{R}$ has *isotone differences* if for $x_1 \prec_1 x_1'$, $x_2 \prec_2 x_2'$ we have

$$f(x_1', x_2') - f(x_1, x_2') \geq f(x_1', x_2) - f(x_1, x_2). \tag{2}$$

A function $f : E^{(n)} \to \mathbb{R}$ has *isotone differences* if (2) is satisfied for any pair i, j of coordinates, whereas the remaining variables are fixed. If E_i, $i = 1, \ldots, n$ are totally ordered then both definitions are equivalent. However, for partially ordered spaces conditions (1) and (2) are not equivalent as mentioned in Heyman and Sobel (1984). We discuss both definitions in Section 2.1. We show that the class of functions with isotone differences on a product of partially ordered state space possesses the same closure properties as on products of totally ordered spaces, whereas for supermodular functions some of those properties may fail.

We say that two random elements X, Y of E are ordered in supermodular ordering ($X <_{\mathrm{sm}} Y$) if $\mathbb{E}[f(X)] \leq \mathbb{E}[f(Y)]$ for all supermodular functions f on E, for which the expectations exist. Supermodular ordering on a totally ordered space is a dependence ordering in the sense of Joe (1997). In particular, if two real valued stationary processes $Y = (Y(t), t \geq 0)$ and $\widetilde{Y} = (\widetilde{Y}(t), t \geq 0)$ are comparable w.r.t $<_{\mathrm{sm}}$ then for each $t \geq 0$, $\mathbf{Cov}[Y(0), Y(t)] \leq \mathbf{Cov}[\widetilde{Y}(0), \widetilde{Y}(t)]$. Moreover one dimensional marginal distributions for both processes coincide. In case of uniform Markov processes in continuous (discrete) time on countable subsets of $\mathbb{R}$ supermodular ordering is characterized in terms of generators (transition probabilities) by Hu and Pan (2000), see also Szekli *et al.* (1994), Bäuerle (1997), Bäuerle and Rolski (1998) for previous results in this area. In order to obtain comparison results it is needed to assume stochastic monotonicity of both Y and $\widetilde{Y}$. We are not aware of any results concerning supermodular ordering of Markov processes without this restriction, except of the recent papers by Kulik (2004) and Daduna and Szekli (2005a). We refer to Müller and Stoyan (2002) for a review of recent work and applications of the supermodular ordering in case of totally ordered state spaces. For partially ordered state spaces, some results extending Hu and Pan (2000)- concerning dependence orderings for Markov processes are given in the paper by Daduna and Szekli (2005a).

The motivating example for this paper is that of the Markov process defined by the joint queue lengths (in other words, network process) in a standard Jackson network as described in Jackson (1957). It is possible that two network processes have the same stationary distribution but their time behavior is completely different otherwise. Moreover, in equilibrium, for a fixed time instant the correlation in space (between nodes) vanishes. However, it is known that correlation in time does not vanish, see for instance Disney and Kiessler (1987). An investigation of dependence structure is treated systematically in Daduna and Szekli (1995), see also Kanter (1985). Comparison of dependence in different network processes with the same equilibrium (stationary) distribution is a natural question. Our results concerning comparison of Markov processes allow us to compare two Jackson networks with the same stationary distribution but with different routing matrices. Moreover, we derive comparison results for two degradable Jackson networks where the nodes are subject to different types of breakdowns and repairs as described and studied by Sauer and Daduna (2003). Related results for standard Jackson type networks are given by Daduna and Szekli (2005b). In Kulik and Sauer (2005) correlation inequalities concerning dependence in lag for the unreliable networks are derived.

Our paper is organized as follows. In Section 2.1 we study properties of supermodular-type functions on partially ordered spaces and collect definitions of the dependence orderings. The results concerning orderings of Markov processes both in discrete and continuous-time case are given in the rest of Section 2. In Section 3 we apply our results to the comparison of networks.

2 Isotone differences ordering for Markov processes on partially ordered state spaces

We shall start this section with definitions and preliminary properties of orderings.

2.1 Functions with isotone differences and related orderings

Assume that E_i, $i = 1, \dots, n$, are lattice ordered Polish spaces with Borel σ-algebra $\mathcal{E}_i$ and a closed partial ordering $\prec_i$, respectively. Then $(E^{(n)}, \prec^{(n)})$ is a lattice and hence for any $x, y \in E^{(n)}$ there exists a largest lower bound $x \wedge y \in E^{(n)}$ and a smallest upper bound $x \vee y \in E^{(n)}$, which are determined uniquely by

$$x \wedge y := \sup\{w : w \prec^{(n)} x,\ w \prec^{(n)} y\}\,, x \vee y := \inf\{w : x \prec^{(n)} w,\ y \prec^{(n)} w\}\,.$$

In this paper we shall use the following particular cases.

(i) Let $(E_i, \prec_i) = (E, \prec) = (\mathbb{R}, \leq)$ for $i = 1, \ldots, n$, with $\leq$ being the standard linear ordering in $\mathbb{R}$. Then for $x = (x_1, \ldots, x_n) \in E^n$, $y = (y_1, \ldots, y_n) \in E^n$, $x \wedge y$ is defined as $x \wedge y = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$ with $x_i \wedge y_i = \min\{x_i, y_i\}$, $i = 1, \ldots, n$ and $x \vee y$ is defined as $x \vee y = (x_1 \vee y_1, \ldots, x_n \vee y_n)$ with $x_i \vee y_i = \max\{x_i, y_i\}$, $i = 1, \ldots, n$.

(ii) Let $\mathrm{J} = \{1, \ldots, J\}$ and denote by $\mathcal{P}(\mathrm{J})$ its power set, i.e. the set of all subsets of J. Taking $(E_i, \prec_i) = (E, \prec) = (\mathcal{P}(\mathrm{J}), \subseteq)$, $i = 1, \ldots, n$, with $\subseteq$ being inclusion ordering, we have for $\mathrm{A}_i, \mathrm{B}_i \in E_i$ that $\mathrm{A}_i \wedge \mathrm{B}_i = \mathrm{A}_i \cap \mathrm{B}_i$, $\mathrm{A}_i \vee \mathrm{B}_i = \mathrm{A}_i \cup \mathrm{B}_i$, $i = 1, \ldots, n$. It is clear that in this case $(E, \subseteq)$ and $(E^n, \subseteq^n)$ are lattices.

We denote by $\mathcal{I}^*(E, \prec)$ $(\mathcal{D}^*(E, \prec))$ the set of all real valued $\prec$-increasing (decreasing) functions on $(E, \mathcal{E})$. Moreover, by $\mathcal{L}_{\mathrm{sm}}(E^{(n)}, \prec^{(n)})$ we denote the class of all real valued supermodular functions on $E^{(n)}$ as defined by (1). If additionally f is $\prec$-increasing then we write $f \in \mathcal{L}_{\mathrm{ism}}(E^{(n)}, \prec^{(n)})$. Related to this class is the class of functions with isotone differences, defined by (2). We denote this class by $\mathcal{L}_{\mathrm{idif}}(E^{(n)}, \prec^{(n)})$. Note that the definition of a function with isotone differences does not require that E_i are lattices. If, additionally, f is taken to be increasing we shall write $f \in \mathcal{L}_{\mathrm{i-idif}}(E^{(n)}, \prec^{(n)})$. If it is clear which space and which ordering we mean we shall omit $(E^{(n)}, \prec^{(n)})$ in these notations. The following lemma is due to Heyman and Sobel (1984).

Lemma 2.1 .

(i) *Let $E_1, E_2, \ldots, E_n$ be lattices. If f is supermodular on $(E^{(n)}, \prec^{(n)})$ then it has also isotone differences.*

(ii) *Let $E_1, \ldots, E_n$ be totally ordered. If f has isotone differences on $(E^{(n)}, \prec^{(n)})$ then it is also supermodular.*

The above lemma implies that in case of totally ordered spaces both notions are equivalent. This is not the case when E_i, $i = 1 \ldots, n$, are only partially ordered. We have however the following preservation results which are in parallel to the totally ordered case (cf. Theorem 3.9.3 in Müller and Stoyan (2002)).

Lemma 2.2 *Let $f, g : (E^{(n)}, \prec^{(n)}) \to \mathbb{R}$ and assume that E_i, $i = 1, \ldots, n$ are lattices.*

(i) *If $f, g \in \mathcal{L}_{\mathrm{sm}}(E^{(n)}, \prec^{(n)})$ then $\alpha f + \beta g \in \mathcal{L}_{\mathrm{sm}}(E^{(n)}, \prec^{(n)})$.*

(ii) *If $f \in \mathcal{L}_{\mathrm{ism}}(E^{(n)}, \prec^{(n)})$ and $\varphi : \mathbb{R} \to \mathbb{R}$ is increasing and convex then $\varphi \circ f \in \mathcal{L}_{\mathrm{ism}}(E^{(n)}, \prec^{(n)})$.*

Lemma 2.3 *Let $f, g : (E^{(n)}, \prec^{(n)}) \to \mathbb{R}$ and assume that E_i, $i = 1, \ldots, n$ are partially ordered.*

(i) *If $f, g \in \mathcal{L}_{\mathrm{idif}}(E^{(n)}, \prec^{(n)})$ then $\alpha f + \beta g \in \mathcal{L}_{\mathrm{idif}}(E^{(n)}, \prec^{(n)})$.*

(ii) *If* $f \in \mathcal{L}_{\text{i-idif}}(E^{(n)}, \prec^{(n)})$ *and* $\varphi : \mathbb{R} \to \mathbb{R}$ *is increasing and convex then* $\varphi \circ f \in \mathcal{L}_{\text{i-idif}}(E^{(n)}, \prec^{(n)})$.

(iii) *If* $f \in \mathcal{L}_{\text{idif}}(E^{(n)}, \prec^{(n)})$ *and* $g_i : E_i \to E_i$ *are* $\prec_i$*-increasing then* $f(g_1(\cdot), \dots, g_n(\cdot)) \in \mathcal{L}_{\text{idif}}(E^{(n)}, \prec^{(n)})$.

(iv) *If* $f, g \in \mathcal{L}_{\text{i-idif}}(E^{(n)}, \prec^{(n)})$ *then* $fg \in \mathcal{L}_{\text{i-idif}}(E^{(n)}, \prec^{(n)})$. *In particular, if* $f \in \mathcal{I}^*(E_1, \prec_1)$, $g \in \mathcal{I}^*(E_2, \prec_2)$ *then for* fg *defined on* $E^{(2)} = E_1 \times E_2$ *by* $(fg)(x,y) = f(x)g(y)$ *for* $x \in E_1, y \in E_2$ *holds* $fg \in \mathcal{L}_{\text{i-idif}}(E^{(2)}, \prec^{(2)})$. *If* $f \in \mathcal{D}^*(E_1, \prec_1)$, $g \in \mathcal{D}^*(E_2, \prec_2)$ *then* $fg \in \mathcal{L}_{\text{idif}}(E^{(2)}, \prec^{(2)})$.

The following example shows however that increasing transformations do not preserve supermodularity and a product of increasing functions need not to be supermodular on partially ordered spaces.

Example 2.4 Let $E_1 = E_2 = \{\{0\}, \{0,1\}, \{0,2\}, \{0,1,2\}\}$ with standard substitution $\prec := \subseteq$. Define $g(\{0\}) = \{0,1\}$, $g(\{0,1\}) = g(\{0,2\}) = g(\{0,1,2\}) = \{0,1,2\}$ and $h(\mathrm{B}) = \mathrm{B}$ for each $\mathrm{B} \in E_1$. Both functions are increasing. The function $f(\mathrm{A},\mathrm{B}) = \mathbb{I}_{(\mathrm{A}\cap\mathrm{B}=\mathrm{C})}(\mathrm{A},\mathrm{B})$ for $\mathrm{C} = \{0,1,2\}$ is supermodular on $(E^{(2)}, \prec^{(2)})$. Define $\tilde{f}(\mathrm{A},\mathrm{B}) = f(g(\mathrm{A}), h(\mathrm{B}))$. The function $\tilde{f}$ is not supermodular. Indeed, for $\mathrm{A}_1 = \{0,1\}$, $\mathrm{A}_2 = \{0,2\}$ and $\mathrm{B}_1 = \mathrm{B}_2 = \{0,1,2\}$ we have

$$\begin{aligned}
&\tilde{f}(\mathrm{A}_1 \cap \mathrm{A}_2, \mathrm{B}_1 \cap \mathrm{B}_2) + \tilde{f}(\mathrm{A}_1 \cup \mathrm{A}_2, \mathrm{B}_1 \cup \mathrm{B}_2) - \tilde{f}(\mathrm{A}_1, \mathrm{B}_1) - \tilde{f}(\mathrm{A}_2, \mathrm{B}_2) \\
&\quad = f(g(\mathrm{A}_1 \cap \mathrm{A}_2), h(\mathrm{B}_1 \cap \mathrm{B}_2)) + f(g(\mathrm{A}_1 \cup \mathrm{A}_2), h(\mathrm{B}_1 \cup \mathrm{B}_2)) \\
&\qquad - f(g(\mathrm{A}_1), h(\mathrm{B}_1)) - f(g(\mathrm{A}_2), h(\mathrm{B}_2)) \\
&\quad = \mathbb{I}_{(g(\mathrm{A}_1\cap\mathrm{A}_2)\cap h(\mathrm{B}_1\cap\mathrm{B}_2)=\mathrm{C})} + \mathbb{I}_{(g(\mathrm{A}_1\cup\mathrm{A}_2)\cap h(\mathrm{B}_1\cup\mathrm{B}_2)=\mathrm{C})} \\
&\qquad - \mathbb{I}_{(g(\mathrm{A}_1)\cap h(\mathrm{B}_1)=\mathrm{C})} - \mathbb{I}_{(g(\mathrm{A}_2)\cap h(\mathrm{B}_2)=\mathrm{C})} \\
&\quad = \mathbb{I}_{(\{0,1\}\cap\{0,1,2\}=\{0,1,2\})} + \mathbb{I}_{(\{0,1,2\}\cap\{0,1,2\}=\{0,1,2\})} \\
&\qquad - \mathbb{I}_{(\{0,1,2\}\cap\{0,1,2\}=\{0,1,2\})} - \mathbb{I}_{(\{0,1,2\}\cap\{0,1,2\}=\{0,1,2\})} \\
&\quad = -1
\end{aligned}$$

In the above setting, a sufficient condition for $\tilde{f}$ to be supermodular is $k(A_1) \cap k(A_2) = k(A_1 \cap A_2)$ for both $k = h$ and $k = g$. Here, this is not the case. Moreover, since $\mathbb{I}_{(g(\mathrm{A})\cap\mathrm{B}=\mathrm{C})}(\mathrm{A},\mathrm{B}) = \mathbb{I}_{(g(\mathrm{A})=\mathrm{C})}(\mathrm{A})\mathbb{I}_{(\mathrm{B}=\mathrm{C})}(\mathrm{B})$, $\tilde{f}$ is a product of increasing functions not being supermodular. However $\tilde{f}$, using Lemma 2.3 (iv), has isotone differences. Therefore, we also showed that there exists a function which has isotone differences but is not supermodular.

From the above example it follows that if we define supermodular ordering on a partially ordered state space then in general it is not preserved under increasing transformations. Therefore, the class of supermodular functions may be restrictive in the study of dependence orderings of random processes on general partially ordered spaces. Hence, we limit ourselves to the class of functions with isotone differences and we call the corresponding ordering the *isotone differences ordering.*

Definition 2.5 *Let $Y = (Y_1, \ldots, Y_n)$, $\widetilde{Y} = (\widetilde{Y}_1, \ldots, \widetilde{Y}_n)$ be random vectors with values in $(E^{(n)}, \mathcal{E}^{(n)}, \prec^{(n)})$.*

(i) *Y is smaller than $\widetilde{Y}$ in the isotone differences ordering ($Y <_{\mathrm{idif}(\prec^{(n)})} \widetilde{Y}$) if*

$$\mathbb{E}\left[f(Y_1 \ldots, Y_n))\right] \leq \mathbb{E}\left[f(\widetilde{Y}_1, \ldots, \widetilde{Y}_n)\right]$$

for all $f \in \mathcal{L}_{\mathrm{idif}}(E^{(n)}, \prec^{(n)})$.

(ii) *Y is smaller than $\widetilde{Y}$ in the concordance ordering ($Y <_{\mathrm{cc}(\prec^{(n)})} \widetilde{Y}$) if*

$$\mathbb{E}\left[\prod_{i=1}^{n} f_i(Y_i)\right] \leq \mathbb{E}\left[\prod_{i=1}^{n} f_i(\widetilde{Y}_i)\right]$$

both for all functions f_i in $\mathcal{I}^*(E_i, \prec_i)$, $i = 1, \ldots, n$, and all f_i in $\mathcal{D}^*(E_i, \prec_i)$, $i = 1, \ldots, n$.

(iii) *Let $Y = (Y(t), t \in T)$, $\widetilde{Y} = (\widetilde{Y}(t), t \in T)$, $T \subseteq \mathbb{R}$ be stochastic processes. Then we write $Y <_{\mathrm{a}(\prec^{(\infty)})} \widetilde{Y}$ if for all $n \geq 2$, all $t_1 < \cdots < t_n \in T$ we have $(Y(t_1), \ldots, Y(t_n)) <_{\mathrm{a}(\prec^{(n)})} (\widetilde{Y}(t_1), \ldots, \widetilde{Y}(t_n))$, where $<_{\mathrm{a}(\prec^{(n)})}$ is one of the above orderings.*

For simplicity, we shall skip $(\prec^{(n)})$ in the above notations if it will be clear which orderings $\prec_i$, $i = 1, \ldots, n$ are used. By Lemma 2.3 products of monotone functions have isotone differences, hence we have the following property.

Lemma 2.6 *If $Y = (Y_1, \ldots, Y_n)$, $\widetilde{Y} = (\widetilde{Y}_1, \ldots, \widetilde{Y}_n)$ are random vectors of $E^{(n)}$ then*

$$Y <_{\mathrm{idif}(\prec^{(n)})} \tilde{Y} \Longrightarrow Y <_{\mathrm{cc}(\prec^{(n)})} \tilde{Y}\,.$$

2.2 Discrete time Markov chains

The theorem stated below for $<_{\mathrm{idif}}$-ordering is in a sense parallel to that for $\prec_{cc}$ ordering treated in Daduna and Szekli (2005a), however the lemma stated below, needed here, is new. For $E = \mathbb{R}$ the theorem was proved in Hu and Pan (2000).

Let $Y = (Y_n, n \in \mathbb{N})$, $\widetilde{Y} = (\widetilde{Y}_n, n \in \mathbb{N})$ be two different stationary Markov processes on a partially ordered state space $(E, \prec)$, with a common invariant distribution π and one step transition kernels $K^Y, K^{\widetilde{Y}} : (E \times \mathcal{E}) \to [0,1]$, respectively. Let $K^Y_{\leftarrow}, K^{\widetilde{Y}}_{\leftarrow} : (E \times \mathcal{E}) \to [0,1]$ be transition kernels for the corresponding time reversed processes. Recall that a transition kernel K is $\prec$-monotone if a map $x \to \int_E K(x, dy) f(y)$ is $\prec$-increasing provided that f is $\prec$-increasing.

Lemma 2.7 *Assume that kernel K is $\prec$-monotone and f has isotone differences on $(E^{n+1}, \prec^{n+1})$. Then a function $h : E^n \to \mathbb{R}$ defined by*

$$h(x_0, \ldots, x_{n-1}) = \int_E K(x_{n-1}, dx_n) f(x_0, \ldots, x_n)$$

has isotone differences on $(E^n, \prec^n)$.

Proof. Isotone differences property of h w.r.t. $(x_0,\dots,x_{n-2})$ is obvious. So, we have to show this property only w.r.t. (x_i,x_{n-1}), $i=0,\dots,n-2$. So, it is sufficient to show that for all $x_0\prec x_0'$, $x_{n-1}\prec x_{n-1}'$ we have

$$h(x_0,x_{n-1})+h(x_0',x_{n-1}')-h(x_0',x_{n-1})-h(x_0,x_{n-1}')\geq 0$$

with

$$h(x_0,x_{n-1})=\int_E K(x_{n-1},dx_n)f(x_0,x_{n-1},x_n)\,,$$

$f\in\mathcal{L}_{\text{idif}}(E^3,\prec^3)$. We have

$$\begin{aligned}
&h(x_0,x_{n-1})+h(x_0',x_{n-1}')-h(x_0',x_{n-1})-h(x_0,x_{n-1}')=\\
&= \int K(x_{n-1},dx_n)f(x_0,x_{n-1},x_n)+\int K(x_{n-1}',dx_n)f(x_0',x_{n-1}',x_n)\\
&\quad -\int K(x_{n-1},dx_n)f(x_0',x_{n-1},x_n)-\int K(x_{n-1}',dx_n)f(x_0,x_{n-1}',x_n)\\
&= \int K(x_{n-1}',dx_n)[f(x_0',x_{n-1}',x_n)-f(x_0,x_{n-1}',x_n)]\\
&\quad -\int K(x_{n-1},dx_n)[f(x_0',x_{n-1},x_n)-f(x_0,x_{n-1},x_n)]\\
&\overset{(*)}{\geq} \int K(x_{n-1}',dx_n)[f(x_0',x_{n-1},x_n)-f(x_0,x_{n-1},x_n)]\\
&\quad -\int K(x_{n-1},dx_n)[f(x_0',x_{n-1},x_n)-f(x_0,x_{n-1},x_n)]\\
&= \int K(x_{n-1}',dx_n)[g_{x_0,x_0',x_{n-1}}(x_n)]-\int K(x_{n-1},dx_n)[g_{x_0,x_0',x_{n-1}}(x_n)]\\
&\overset{(**)}{\geq} \int K(x_{n-1},dx_n)[g_{x_0,x_0',x_{n-1}}(x_n)]-\int K(x_{n-1},dx_n)[g_{x_0,x_0',x_{n-1}}(x_n)]\\
&= 0\,.
\end{aligned}$$

Where (*) follows from isotone differences property of f, i.e.

$$f(x_0',x_{n-1}',x_n)-f(x_0,x_{n-1}',x_n)\geq f(x_0',x_{n-1},x_n)-f(x_0,x_{n-1},x_n)\,,$$

(**) follows from the fact that for each fixed x_0,x_0',x_{n-1} the function

$$g_{x_0,x_0',x_{n-1}}(x_n):=f(x_0',x_{n-1},x_n)-f(x_0,x_{n-1},x_n)$$

is increasing in x_n, which in turns follows from the isotone differences property of f. Finally, we use in (**) monotone kernel property.

□

Theorem 2.8 *Assume that either K^Y and $K_{\leftarrow}^{\widetilde{Y}}$ are $\prec-$ monotone or $K_{\leftarrow}^{Y}$ and $K^{\widetilde{Y}}$ are $\prec-$ monotone and*

$$(Y_0,Y_1)<_{\text{idif}}(\widetilde{Y}_0,\widetilde{Y}_1),$$

then $Y<_{\text{idif}}\widetilde{Y}$.

Proof. We shall show that for any $n \geq 1$ we have $(Y_0, \ldots, Y_n) <_{\text{idif}} (\widetilde{Y}_0, \ldots, \widetilde{Y}_n)$. We proceed by induction in the case when K^Y and $K^{\widetilde{Y}}_{\leftarrow}$ are monotone. The result is obviously true for $n = 1$. Now, denoting the distribution of $(Y_0, \ldots, Y_n)$ by $P^{(Y_0,\ldots,Y_n)}$ (and using similar notation for other distributions) we have for all functions f with isotone differences:

$$
\begin{aligned}
&\mathbb{E}[f(Y_0, \ldots, Y_n)] \;= \\
&\quad = \int_{E^n} P^{(Y_0,\ldots,Y_{n-1})} d(x_0, \ldots, x_{n-1}) \int_E P^{Y_n|(Y_0=x_0,\ldots,Y_{n-1}=x_{n-1})} f(x_0, \ldots, x_n) dx_n \\
&\quad = \int_{E^n} P^{(Y_0,\ldots,Y_{n-1})} d(x_0, \ldots, x_{n-1}) \int_E P^{Y_n|(Y_{n-1}=x_{n-1})} f(x_0, \ldots, x_n) dx_n \\
&\quad = \int_{E^n} P^{(Y_0,\ldots,Y_{n-1})} d(x_0, \ldots, x_{n-1}) \underbrace{\int_E f(x_0, \ldots, x_n) K^Y(x_{n-1}, dx_n)}_{=:h_1(x_0,\ldots,x_{n-1})} \\
&\quad = \mathbb{E}\left[h_1(Y_0, \ldots, Y_{n-1})\right] \\
&\quad \overset{(1)}{\leq} \mathbb{E}\left[h_1(\widetilde{Y}_0, \ldots, \widetilde{Y}_{n-1})\right] \\
&\quad = \int_{E^n} P^{(\widetilde{Y}_0,\ldots,\widetilde{Y}_{n-1})} d(x_0, \ldots, x_{n-1}) \int_E f(x_0, \ldots, x_n) K^Y(x_{n-1}, dx_n) \\
&\quad = \int_E \pi(dx_{n-1}) \int_E K^Y(x_{n-1}, dx_n) \int_{E^{n-1}} P^{(\widetilde{Y}_0,\ldots,\widetilde{Y}_{n-2})|\widetilde{Y}_{n-1}=x_{n-1}} f(x_0, \ldots, x_n) d(x_0, \ldots, x_{n-2}) \\
&\quad = \int_{E^2} P^{(Y_{n-1},Y_n)} d(x_{n-1}, x_n) \underbrace{\int_{E^{n-1}} P^{(\widetilde{Y}_0,\ldots,\widetilde{Y}_{n-2})|\widetilde{Y}_{n-1}=x_{n-1}} f(x_0, \ldots, x_n) d(x_0, \ldots, x_{n-2})}_{=:h_2(x_{n-1},x_n)} \\
&\quad = \mathbb{E}\left[h_2(Y_{n-1}, Y_n)\right] \\
&\quad \overset{(2)}{\leq} \mathbb{E}\left[h_2(\widetilde{Y}_{n-1}, \widetilde{Y}_n)\right] \\
&\quad = \int_{E^2} P^{(\widetilde{Y}_{n-1},\widetilde{Y}_n)} d(x_{n-1}, x_n) \int_{E^{n-1}} P^{(\widetilde{Y}_0,\ldots,\widetilde{Y}_{n-2})|\widetilde{Y}_{n-1}=x_{n-1}} f(x_0, \ldots, x_n) d(x_0, \ldots, x_{n-2}) \\
&\quad = \mathbb{E}[f(\widetilde{Y}_0, \ldots, \widetilde{Y}_n)]\,.
\end{aligned}
$$

The inequality (1) follows from isotone differences property of $h_1(x_0, \ldots, x_{n-1})$ (Lemma 2.7) and induction assumption. The inequality (2) follows from the facts that $K^{\widetilde{Y}}_{\leftarrow}$ is monotone and that h_2 has isotone differences, which can be observed using the same argument as in Lemma 2.7.

2.3 Continuous time

Let $Y = (Y(t), t \geq 0)$, $\widetilde{Y} = (\widetilde{Y}(t), t \geq 0)$ be stationary homogeneous continuous time Markov processes with a countable state space $(E, \mathcal{E}, \prec)$ and transition kernel families $\mathbf{P}^Y := (P_t^Y : E \times \mathcal{E} \to [0,1], t \geq 0)$ and $\mathbf{P}^{\widetilde{Y}} := (P_t^{\widetilde{Y}} : E \times \mathcal{E} \to [0,1], t \geq$

0), respectively. Denote by $\mathbf{P}^{Y}_{\leftarrow}$, $\mathbf{P}^{\widetilde{Y}}_{\leftarrow}$ transition families for the time reversed processes. Denote by $Q^Y = (q^Y(x,y),\ x,y \in E)$, $Q^{\widetilde{Y}} = (q^{\widetilde{Y}}(x,y),\ x,y \in E)$, $Q^Y_{\leftarrow} = (q^Y_{\leftarrow}(x,y),\ x,y \in E)$, $Q^{\widetilde{Y}}_{\leftarrow} = (q^{\widetilde{Y}}_{\leftarrow}(x,y),\ x,y \in E)$ the corresponding infinitesimal generators. Throughout the rest of the paper we assume that all processes are uniform (i.e. their generators are bounded and conservative). From now on we shall assume that either $\mathbf{P}^Y$ and $\mathbf{P}^{\widetilde{Y}}_{\leftarrow}$ or $\mathbf{P}^Y_{\leftarrow}$ and $\mathbf{P}^{\widetilde{Y}}$ are families of $\prec$ – monotone kernels (and shall refer to this property as to the monotonicity assumption).

The proof of the following theorem follows the same arguments as Theorem 3.6 in Daduna and Szekli (2005a) in the case of $\prec_{cc}$-ordering, therefore we shall skip it.

Theorem 2.9 *Under the above assumptions the following properties are equivalent:*

(i) $Y <_{\text{idif}} \widetilde{Y}$;

(ii) $(Y_0, Y_t) <_{\text{idif}} (\widetilde{Y}_0, \widetilde{Y}_t)$, *for all* $t \geq 0$;

(iii) *For any* $f \in \mathcal{L}_{\text{idif}}(E^2, \prec^2)$ *we have*

$$\sum_{x\in E} \pi(x) \sum_{y\in E} f(x,y) q^Y(x,y) \leq \sum_{x\in E} \pi(x) \sum_{y\in E} f(x,y) q^{\widetilde{Y}}(x,y)\,. \tag{3}$$

2.4 ε transformation

Consider now a stationary Markov process Y with intensities $q^Y(x,y)$ and stationary distribution π. For $x_1 \prec x_2$, $y_1 \prec y_2$ and fixed $\varepsilon > 0$ such that the quantities in (4) define a new transition matrix of a Markov process $\widetilde{Y^\varepsilon}$ (called ε transformation of Y) set

$$q^{\widetilde{Y^\varepsilon}}(x,y) = \begin{cases} q^Y(x,y) + \frac{\varepsilon}{\pi(x)}, & \text{if } (x = x_1, y = y_1) \text{ or } (x = x_2, y = y_2)\,, \\ q^Y(x,y) - \frac{\varepsilon}{\pi(x)}, & \text{if } (x = x_1, y = y_2) \text{ or } (x = x_2, y = y_1)\,, \\ q^Y(x,y), & \text{otherwise}\,. \end{cases} \tag{4}$$

Lemma 2.10 *The Markov processes* Y *and* $\widetilde{Y^\varepsilon}$ *with the transition intensities given by (4) have equal invariant distribution* π.

Proof . From assumptions we have for the intensity matrix of Y: $\pi Q^Y = \mathbf{0}$, where $\mathbf{0}$ is a null-vector. It means that for any $y \in E$ we have

$$\sum_{x\in E} \pi(x) q^Y(x,y) = 0\,.$$

Now, for $y \neq y_1, y_2$ we have

$$\sum_{x\in E} \pi(x) q^{\widetilde{Y^\varepsilon}}(x,y) = \sum_{x\in E} \pi(x) q^Y(x,y) = 0\,.$$

Moreover, for $y = y_1$

$$\begin{aligned}\sum_{x\in E} \pi(x) q^{\widetilde{Y^\varepsilon}}(x,y_1) &= \sum_{x\neq x_1,x_2} \pi(x) q^Y(x,y_1) \\ &\quad + \pi(x_1)\left(q^Y(x_1,y_1) + \frac{\varepsilon}{\pi(x_1)}\right) + \pi(x_2)\left(q^Y(x_2,y_1) - \frac{\varepsilon}{\pi(x_2)}\right) \\ &= \sum_{x\in E} \pi(x) q^Y(x,y_1) = 0\,.\end{aligned}$$

It follows analogously that $\sum_{x\in E} \pi(x) q^{\widetilde{Y^\varepsilon}}(x,y_2) = 0$.

□

Proposition 2.11 *Consider Markov processes Y and $\widetilde{Y^\varepsilon}$ (with intensities given by (4)). Under the monotonicity assumption, $Y <_{\text{idif}} \widetilde{Y^\varepsilon}$.*

Proof. From Theorem 2.9 it is sufficient to prove that

$$\sum_{x\in E} \pi(x) \sum_{y\in E} q^Y(x,y) f(x,y) \leq \sum_{x\in E} \pi(x) \sum_{y\in E} q^{\widetilde{Y^\varepsilon}}(x,y) f(x,y)$$

for all $f \in \mathcal{L}_{\text{idif}}(E^2, \prec^2)$.

The difference between left-hand-side and right-hand-side is

$$\varepsilon\Big(f(x_1,y_1) + f(x_2,y_2) - f(x_1,y_2) - f(x_2,y_1)\Big)$$

being nonnegative due to the assumptions on f.

□

It is clear that the ε transformation can be iterated to obtain a sequence of ordered processes.

3 Comparison of queueing networks

3.1 Standard Jackson networks

Consider a Jackson network which consists of J numbered nodes, denoted by $\mathrm{J} = \{1, \ldots, J\}$. Station $j \in \mathrm{J}$, is a single server queue with infinite waiting room under FCFS regime. Customers in the network are indistinguishable. There is an external Poisson arrival stream with intensity $\lambda > 0$ and arriving customers are sent to node j with probability r_{0j}, $\sum_{j=1}^J r_{0j} = r \leq 1$. The quantity $r_{00} := 1-r$ is then the rejection probability with that customers immediately leave the network again. Customers arriving at node j from the outside or from other nodes request a service which is exponentially distributed with mean 1. Service at node j is provided with intensity $\mu_j(n_j) > 0$ ($\mu_j(0) := 0$), where n_j is the number of customers at node j including the one being served. All service times and arrival processes are assumed to be independent.

A customer departing from node j immediately proceeds to node i with probability $r_{ji} \geq 0$ or departs from the network with probability r_{j0}. The routing is independent of the past of the system given the momentary node where the customer is. Let $\mathrm{J}_0 := \mathrm{J} \cup \{0\}$. We assume that the matrix $R := (r_{ij}, i, j \in \mathrm{J}_0)$ is aperiodic and irreducible.

Let $X_j(t)$ be the number of customers present at node j at time $t \geq 0$. Then $X(t) = (X_1(t), \ldots, X_J(t))$ is the joint queue length vector at time instant $t \geq 0$ and $X := (X(t), t \geq 0)$ is the joint queue length process with state space $(E, \prec) := (\mathbb{N}^J, \leq^J)$ (where $\leq^J$ denotes the standard coordinatewise ordering).

The following theorem is classical (Jackson (1957)).

Theorem 3.1 *Under the above assumptions the queueing process X is a Markov process with infinitesimal operator $Q^X = (q(x,y) : x, y \in E)$ given by*

$$q^X(n_1, \ldots, n_i, \ldots, n_J; n_1, \ldots, n_i + 1, \ldots, n_J) = \lambda r_{0i}$$

and for $n_i > 0$

$$q^X(n_1, \ldots, n_i, \ldots, n_J; n_1, \ldots, n_i - 1, \ldots, n_J) = \mu_i(n_i) r_{i0}\,,$$

$$q^X(n_1, \ldots, n_i, \ldots, n_j, \ldots, n_J; n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_J) = \mu_i(n_i) r_{ij}\,.$$

Furthermore

$$q^X(x,x) = -\sum_{y \in E \setminus \{x\}} q^X(x,y) \text{ and } q^X(x,y) = 0 \text{ otherwise.}$$

We assume henceforth X to be ergodic. The unique invariant and limiting distribution π of X is then

$$\pi_0(n_1, \ldots, n_J) = K(J)^{-1} \prod_{j=1}^{J} \prod_{k=1}^{n_j} \left(\frac{\eta_j}{\mu_j(k)} \right), \quad (n_1, \ldots, n_J) \in \mathbb{N}^J \tag{5}$$

with normalization constant $K(J)$ and with $\eta = (\eta_1, \ldots, \eta_J)$ as unique solution of the traffic equation of the network:

$$\eta_j = r_{0j}\lambda + \sum_{i=1}^{J} \eta_i r_{ij}\ ,\ j \in \mathrm{J}.$$

Consider now two network processes X, and its ε transformation $\widetilde{X^\varepsilon}$. It is well-known that for Jackson networks with state-independent service rates (i.e. $\mu_j(n_j) = \mu_j$, if $n_j > 0$, for $j \in \mathrm{J}$) both the network process and its time-reversal counterpart are stochastically monotone w.r.t. standard coordinatewise ordering $\leq^J$ (see Daduna and Szekli (1995)). Therefore from Lemma 2.10 and Proposition 2.11 we have the following result.

Theorem 3.2 *Suppose that a Jackson network X has state-independent service intensities. Then* $X <_{\mathrm{idif}((\leq^{\mathrm{J}})^{(\infty)})} \widetilde{X^{\varepsilon}}$.

In particular, if $f \in \mathcal{L}_{\mathrm{i-idif}}(E^k, (\leq^J)^k)$, $g \in \mathcal{L}_{\mathrm{i-idif}}(E^{n-k}, (\leq^J)^{n-k})$, then by Lemma 2.3, $fg \in \mathcal{L}_{\mathrm{i-idif}}(E^n, (\leq^J)^n)$, hence

$$\begin{aligned} &\mathbb{E}\left[f\left(X(t_1),\ldots,X(t_k)\right) g\left(X(t_{k+1}),\ldots,X(t_n)\right)\right] \leq \\ &\qquad \mathbb{E}\left[f\left(\widetilde{X^{\varepsilon}}(t_1),\ldots,\widetilde{X^{\varepsilon}}(t_k)\right) g\left(\widetilde{X^{\varepsilon}}(t_{k+1}),\ldots,\widetilde{X^{\varepsilon}}(t_n)\right)\right] \end{aligned} \tag{6}$$

which is related to Corollary 5.1 in Daduna and Szekli (1995).

Example 3.3 Let $\mathbf{e}_i$ be an n-dimensional vector with 1 on ith coordinate and 0 otherwise. Consider ε transformation specified in the following way: Fix $\mathbf{n} = (n_1, \ldots, n_J) \in \mathbb{N}^J$ with $n_i > 0$ and i, j such that $1 \leq i < j \leq J$. For simplicity assume that λ and service intensities are equal to 1. Take in (4) $x_1 = \mathbf{n} - \mathbf{e}_i, x_2 = \mathbf{n}, y_1 = \mathbf{n} - \mathbf{e}_i, y_2 = \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j$ to obtain

$$\begin{aligned} q^{\widetilde{X^{\varepsilon}}}(\mathbf{n}-\mathbf{e}_i, \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) &= q^X(\mathbf{n}-\mathbf{e}_i, \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) - \varepsilon/\pi(\mathbf{n}-\mathbf{e}_i) = r_{0j} - \varepsilon/\pi(\mathbf{n}-\mathbf{e}_i) \\ q^{\widetilde{X^{\varepsilon}}}(\mathbf{n}-\mathbf{e}_i, \mathbf{n}-\mathbf{e}_i) &= q^X(\mathbf{n}-\mathbf{e}_i, \mathbf{n}-\mathbf{e}_i) + \varepsilon/\pi(\mathbf{n}-\mathbf{e}_i) \\ &= (1-r) + \sum_{i=1}^{J}(1-\delta_{0n_i})(1-r_{ii}) \\ q^{\widetilde{X^{\varepsilon}}}(\mathbf{n}, \mathbf{n}-\mathbf{e}_i) &= q^X(\mathbf{n}, \mathbf{n}-\mathbf{e}_i) - \varepsilon/\pi(\mathbf{n}) = r_{i0} - \varepsilon/\pi(\mathbf{n}) \\ q^{\widetilde{X^{\varepsilon}}}(\mathbf{n}, \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) &= q^X(\mathbf{n}, \mathbf{n}-\mathbf{e}_i+\mathbf{e}_j) + \varepsilon/\pi(\mathbf{n}) = r_{ij} + \varepsilon/\pi(\mathbf{n}) \end{aligned}$$

We observe that the changes in transition intensities according to (4) correspond here to changes in routing probabilities in a similar way with the following meaning. We increase the rejection probability $r_{00} = 1 - r$ and the transition probability r_{ij} and at the same time we decrease the arrival probability r_{0j} and the departure probability r_{i0}. From Theorem 3.2 it follows that then the stationary joint queue length distribution increases in isotone differences ordering.

3.2 Queueing networks with breakdowns

We consider the servers at the nodes in the Jackson network to be unreliable, i.e., the nodes may break down. The breakdown event may occur in different ways. Nodes may break down as an isolated event or in groups simultaneously, and the repair of the nodes may end for each node individually or in groups as well. It is not required that those nodes which stopped service simultaneously return to service at the same time instant. To describe the system's evolution we have to enlarge the state space for the network process as will be described below. For a detailed description we refer to Sauer and Daduna (2003).

Control of breakdowns and repairs is as follows:

Let $\mathrm{I} \subset \mathrm{J}$ be the set of nodes in down status and $\mathrm{H} \subset \mathrm{J}\backslash\mathrm{I}, \mathrm{H} \neq \emptyset$, be some subset of nodes in up status. Then the nodes of H break down concurrently with intensity $\alpha(\mathrm{I}, \mathrm{I} \cup \mathrm{H})$.

Nodes in down status neither accept new customers nor continue serving the old customers which will wait for the server's return. (At nodes i under repair the service intensities $\mu_i(n_i)$ are set to 0). Therefore, the routing matrix has to be changed so that customers attending to join a node in down status are rerouted to nodes in up status or to the outside. We will give three possible rerouting schemes below. These originate from mechanismus from the theory of information blocking.

Assume the nodes in I are under repair, $\mathrm{I} \subset \mathrm{J}, \mathrm{I} \neq \emptyset$. Then if $\mathrm{H} \subset \mathrm{I}, \mathrm{H} \neq \emptyset$, the nodes of H return from repair as a batch group with intensity $\beta(\mathrm{I}, \mathrm{I} \setminus \mathrm{H})$ and immediately resume their services. Routing then has to be updated again as will be described below.

The intensities for occurrence of breakdowns and repairs have to be set under constraints. A rather general versatile class is defined as follows.

Definition 3.4 *Let* I *be the set of nodes in down status. The intensities for breakdowns, resp. repairs for* $\mathrm{H} \neq \emptyset$ *are defined by*

$$\alpha(\mathrm{I}, \mathrm{I} \cup \mathrm{H}) := \frac{a(\mathrm{I} \cup \mathrm{H})}{a(\mathrm{I})}, \quad \textit{resp.} \quad \beta(\mathrm{I}, \mathrm{I} \setminus \mathrm{H}) := \frac{b(\mathrm{I})}{b(\mathrm{I}\backslash\mathrm{H})}, \tag{7}$$

where a *and* b *are any functions,* $a, b : \mathcal{P}(\mathrm{J}) \to \mathbb{R}_+ = [0, \infty)$ *whereas* $\frac{0}{0} := 0$. *The above intensities are assumed henceforth to be finite.*

The rerouting matrices of interest are as follows.

Definition 3.5 (BLOCKING) *Assume that the routing matrix of the original process is reversible. Assume the nodes in* I *are the nodes of the Jackson network presently under repair. Then the routing probabilities are redefined on* $\mathrm{J}_0\backslash\mathrm{I}$ *according to*

$$\tilde{r}^{\mathrm{I}}_{ij} = \begin{cases} r_{ij}, & i, j \in \mathrm{J}_0\backslash\mathrm{I},\ i \neq j, \\ r_{ii} + \sum_{k \in \mathrm{I}} r_{ik}, & i \in \mathrm{J}_0\backslash\mathrm{I},\ i = j. \end{cases} \tag{8}$$

Note that even in case of $r_{00} = 0$, *external arrivals may be now rejected with positive probability to an immediate departure, because arrivals to nodes under repair are rerouted:*

$$\tilde{r}^{\mathrm{I}}_{00} = r_{00} + \sum_{k \in \mathrm{I}} r_{0k} \ \geq 0.$$

Definition 3.6 (STALLING) *If there is any breakdown of either a single node or a group of nodes, then all arrival streams to the network and all service processes at the nodes in up status are completely interrupted and resumed only when all nodes are repaired again.*

Definition 3.7 (SKIPPING) *Assume that the nodes in* I *are the nodes presently under repair. Then the routing matrix is redefined on* $\mathrm{J}_0 \setminus \mathrm{I}$ *according to:*

$$\begin{array}{ll} \hat{r}^{\mathrm{I}}_{jk} = r_{jk} + \sum_{i\in \mathrm{I}} r_{ji}\hat{r}^{\mathrm{I}}_{ik}, & k,j \in \mathrm{J}_0\backslash\mathrm{I}, \\ \hat{r}^{\mathrm{I}}_{ik} = r_{ik} + \sum_{l\in \mathrm{I}} r_{il}\hat{r}^{\mathrm{I}}_{lk}, & i \in \mathrm{I}, k \in \mathrm{J}_0\backslash\mathrm{I}. \end{array}$$

For describing the breakdown of nodes in Jackson networks we have to attach to the state spaces $E = \mathbb{N}^J$ of the corresponding network processes X an additional component which carries information of the reliability behavior of the system described by a process Y. We introduce states of the form

$$(\mathrm{I};\ n_1, n_2, \ldots, n_J) \in \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J\,.$$

The meaning of such a prototype state is:

I is the set of nodes under repair. For $j \in \mathrm{J}\backslash\mathrm{I}$, the numbers $n_j \in \mathbb{N}$ indicate that at nodes j which work in a normal up status, there are n_j customers present; for $i \in \mathrm{I}$ the numbers $n_i \in \mathbb{N}$ indicate that at each node i which is in down status there are n_i customers that wait at node i for the return of the repaired server. Collecting these states we define for the networks new Markov processes $Z = (Y, X)$ on

$$\widetilde{E} = \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J\,. \tag{9}$$

For such general models with breakdowns and repairs and with the above rerouting principles it was shown in Sauer and Daduna (2003) that on the state space $\widetilde{E}$ the steady state distribution for Z is of product form. Note that the breakdown/repair process Y is Markovian on the state space $\mathcal{P}(\mathrm{J})$ of all subsets of J, but that the network process component X is in this setting not a Markov process.

Theorem 3.8 *A Jackson network with breakdown and repair intensities given by (7) and rerouting according to either* BLOCKING *or* STALLING, *or* SKIPPING *has a stationary distribution of product form given by:*

$$\pi(\mathrm{I};\ n_1, n_2, \ldots, n_J) = \pi^Y(\mathrm{I})\ \pi_0(n_1, n_2, \ldots, n_J)$$

with

$$\pi^Y(\mathrm{I}) = \left(1 + \sum_{\substack{\mathrm{K}\subset \mathrm{J}\\ \mathrm{K}\neq\emptyset}} \frac{a(\mathrm{K})}{b(\mathrm{K})}\right)^{-1} \frac{a(\mathrm{I})}{b(\mathrm{I})} \qquad \textit{for } \mathrm{I} \subset \mathrm{J}$$

and $\pi_0(n_1, n_2, \ldots, n_J)$ *which coincides with the equilibrium distribution in the standard Jackson network (5).*

Nevertheless, the behavior of the queueing process X is different in all cases (standard Jackson, BLOCKING, STALLING, SKIPPING). Moreover, it is possible to change the breakdown/repair intensities in such a way that the stationary distribution for the joint process Z remains the same. However, the behavior in time may be completely different and presents itself for comparison in dependence orderings.

3.3 Comparison of queueing networks with breakdowns

For a given set K denote by $\{\mathrm{K}\}_{+}^{\uparrow}$, $\{\mathrm{K}\}_{-}^{\downarrow}$, $\{\mathrm{K}\}^{\prec}$ the sets of its ancestors, descendants and relatives, respectively, i.e.

$$\begin{aligned}\{\mathrm{K}\}_{+}^{\uparrow} &:= \{\mathrm{I} \subseteq \mathrm{J} : \mathrm{K} \subset \mathrm{I}, \mathrm{K} \neq \mathrm{I}\}\,,\\ \{\mathrm{K}\}_{-}^{\downarrow} &:= \{\mathrm{I} \subseteq \mathrm{J} : \mathrm{I} \subset \mathrm{K}, \mathrm{K} \neq \mathrm{I}\}\,,\\ \{\mathrm{K}\}^{\prec} &:= \{\mathrm{K}\}_{-}^{\downarrow} \cup \{\mathrm{K}\}_{+}^{\uparrow}\,.\end{aligned}$$

Recall that $Y = (Y(t), t \geq 0)$ is a cadlag Markov process on the state space $\mathcal{P}(\mathrm{J})$ which describes availability of the network's components over time, i.e. $Y(t) = \mathrm{K}$, $\mathrm{K} \in \mathcal{P}(\mathrm{J})$, means that at time t the set K consists of the nodes which are under repair. We have

$$q^{Y}(\mathrm{K},\mathrm{H}) = \begin{cases} \alpha(\mathrm{K},\mathrm{H}) = \frac{a(\mathrm{H})}{a(\mathrm{K})}, & \text{if } \mathrm{H} \in \{\mathrm{K}\}_{+}^{\uparrow}\,,\\ \beta(\mathrm{K},\mathrm{H}) = \frac{b(\mathrm{K})}{b(\mathrm{H})}, & \text{if } \mathrm{H} \in \{\mathrm{K}\}_{-}^{\downarrow}\,,\\ -\sum_{\mathrm{I}\in\{\mathrm{K}\}_{+}^{\uparrow}} \frac{a(\mathrm{I})}{a(\mathrm{K})} - \sum_{\mathrm{I}\in\{\mathrm{K}\}_{-}^{\downarrow}} \frac{b(\mathrm{K})}{b(\mathrm{I})}, & \text{if } \mathrm{H} = \mathrm{K},\\ 0, & \text{otherwise.}\end{cases} \tag{10}$$

Note that such a defined Y has up-and-down property. According to (4) we define for fixed $\mathrm{I}_1 \subset \mathrm{I}_2$, $\mathrm{J}_1 \subset \mathrm{J}_2$ new intensities by

$$q^{\widetilde{Y}^{\varepsilon}}(\mathrm{K},\mathrm{H}) = \begin{cases} q^{Y}(\mathrm{K},\mathrm{H}) + \frac{\varepsilon}{\pi(\mathrm{K})}, & \text{if } (\mathrm{K} = \mathrm{I}_1, \mathrm{H} = \mathrm{J}_1) \text{ or } (\mathrm{K} = \mathrm{I}_2, \mathrm{H} = \mathrm{J}_2)\,,\\ q^{Y}(\mathrm{K},\mathrm{H}) - \frac{\varepsilon}{\pi(\mathrm{K})}, & \text{if } (\mathrm{K} = \mathrm{I}_1, \mathrm{H} = \mathrm{J}_2) \text{ or } (\mathrm{K} = \mathrm{I}_2, \mathrm{H} = \mathrm{J}_1)\,,\\ q^{Y}(\mathrm{K},\mathrm{H}), & \text{otherwise}\,.\end{cases} \tag{11}$$

Consider the processes Y, $\widetilde{Y}^{\varepsilon}$ on state space $(\mathcal{P}(\mathrm{J}), \subseteq)$. Applying Proposition 2.11 we have the following comparison result.

Proposition 3.9 *Assume that the breakdown/repair intensity matrix $Q^{Y} = (q^{Y}(\mathrm{K},\mathrm{H}),$ $\mathrm{K},\mathrm{H} \subseteq \mathrm{J})$ and the matrix $Q_{\leftarrow}^{\widetilde{Y}^{\varepsilon}}$ of the time-reversal counterpart of $\widetilde{Y}^{\varepsilon}$ are stochastically monotone. Then $Y <_{\mathrm{idif}((\subseteq)^{(\infty)})} \widetilde{Y}^{\varepsilon}$.*

Example 3.10 1. Let $f(\mathrm{K},\mathrm{H}) = \mathbb{I}_{(\mathrm{K}\in\mathcal{B},\mathrm{H}\in\mathcal{C})}(\mathrm{K},\mathrm{H})$, where $\mathcal{B}, \mathcal{C} \subseteq \mathcal{P}(\mathrm{J})$ are $\subseteq$-increasing. Then f has isotone differences on $((\mathcal{P}(\mathrm{J}))^2, \subseteq^2)$. By Proposition 3.9 we obtain

$$\mathbb{P}(Y(t_1) \in \mathcal{B}, Y(t_2) \in \mathcal{C}) \leq \mathbb{P}(\widetilde{Y}^{\varepsilon}(t_1) \in \mathcal{B}, \widetilde{Y}^{\varepsilon}(t_2) \in \mathcal{C})\,.$$

2. Let $g(\mathrm{K},\mathrm{H}) = f(|\mathrm{K}|,|\mathrm{H}|)$ with $f \in \mathcal{L}_{\mathrm{idif}}(\mathrm{J}^2, \leq^2)$. Then $g \in \mathcal{L}_{\mathrm{idif}}((\mathcal{P}(\mathrm{J}))^2,$ $\subseteq^2)$. Especially, by Proposition 3.9 we obtain by taking $f(|\mathrm{K}|,|\mathrm{H}|) = |\mathrm{K}||\mathrm{H}|$

$$\mathbf{Cov}\,[|Y(t_1)|, |Y(t_2)|] \leq \mathbf{Cov}\left[|\widetilde{Y}^{\varepsilon}(t_1)|, |\widetilde{Y}^{\varepsilon}(t_2)|\right]\,.$$

3. Recall that $\{\mathrm{C}\}^{\uparrow} = \{\mathrm{A} : \mathrm{A} \supseteq \mathrm{C}\}$, $\{\mathrm{C}\}^{\downarrow} = \{\mathrm{A} : \mathrm{A} \subseteq \mathrm{C}\}$. Let $f(\mathrm{A}_1, \ldots, \mathrm{A}_k) = \mathbb{I}_{\{\mathrm{A}_1 \cap \cdots \cap \mathrm{A}_k \in \{\mathrm{C}\}^{\uparrow}\}}$ for some $\mathrm{C} \in \mathcal{P}(\mathrm{J})$. Then $f \in \mathcal{L}_{\mathrm{idif}}((\mathcal{P}(\mathrm{J}))^k, \subseteq^k)$ and hence

$$\mathbb{P}(Y(t_1) \cap \cdots \cap Y(t_k) \in \{\mathrm{C}\}^{\uparrow}) \leq \mathbb{P}(\widetilde{Y}^{\varepsilon}(t_1) \cap \cdots \cap \widetilde{Y}^{\varepsilon}(t_k) \in \{\mathrm{C}\}^{\uparrow}),$$

i.e. the set of nodes in the first network which are under repair at time instants $t_1, \ldots, t_k$ is smaller (in the stochastic sense) than the corresponding set for the second network. On the other hand, because $f(\mathrm{A}_1, \ldots, \mathrm{A}_k) = \mathbb{I}_{\{\mathrm{A}_1 \cup \cdots \cup \mathrm{A}_k \in \{\mathrm{C}\}^{\downarrow}\}}$ has isotone differences as well, we obtain

$$\mathbb{P}(Y(t_1) \cup \cdots \cup Y(t_k) \in \{\mathrm{C}\}^{\downarrow}) \leq \mathbb{P}(\widetilde{Y}^{\varepsilon}(t_1) \cup \cdots \cup \widetilde{Y}^{\varepsilon}(t_k) \in \{\mathrm{C}\}^{\downarrow}),$$

i.e. the set of nodes in the first network which are under repair at least in one of the time instants $t_1, \ldots, t_k$ is larger (in the stochastic sense) than the corresponding set for the second network.

Take now the transition intensities in the special form:

$$q^Y(\mathrm{K}, \mathrm{H}) = \begin{cases} \lambda(\mathrm{K})\delta(\mathrm{K}, \mathrm{H}), & \text{if } \mathrm{H} \in \{\mathrm{K}\}^{\prec}, \\ 0, & \text{otherwise}. \end{cases}$$

Here, given the set K being under repair, $\lambda(\mathrm{K})$ is interpreted as the intensity of having a jump in Y and $\delta(\mathrm{K}, \mathrm{H})$ is the conditional probability that after the jump the set H is under repair. Especially, we put

$$\delta(\mathrm{K}, \mathrm{H}) = \frac{\mathbb{I}_{(\mathrm{H} \in \{\mathrm{K}\}^{\prec})}}{|\{\mathrm{K}\}^{\prec}|}$$

and $\lambda(\mathrm{K}) = |\{\mathrm{K}\}^{\prec}|$. If $\{T_n\}$ is a sequence of jump points of Y, then it means that $Y(T_n)$ is uniformly distributed on the set $\{\mathrm{K}\}^{\prec}$ given that $Y(T_n-) = \mathrm{K}$. Then $q^Y(\mathrm{K}, \mathrm{H}) = 1$ if and only if $\mathrm{H} \in \{\mathrm{K}\}^{\prec}$. Note that in this case π^Y is uniformly distributed on $\mathcal{P}(\mathrm{J})$. Note that we can also represent our intensities as

$$q^Y(\mathrm{K}, \mathrm{H}) = \frac{\mathbb{I}_{(\mathrm{H} \in \{\mathrm{K}\}_{-}^{\downarrow})}}{|\{\mathrm{K}\}_{-}^{\downarrow}|} \cdot |\{\mathrm{K}\}_{-}^{\downarrow}| + \frac{\mathbb{I}_{(\mathrm{H} \in \{\mathrm{K}\}_{+}^{\uparrow})}}{|\{\mathrm{K}\}_{+}^{\uparrow}|} \cdot |\{\mathrm{K}\}_{+}^{\uparrow}|.$$

Here, $|\{\mathrm{K}\}_{-}^{\downarrow}|$, $|\{\mathrm{K}\}_{+}^{\uparrow}|$ play the role of the repair and breakdown intensity, respectively. Then, for instance $\frac{\mathbf{I}_{(\mathrm{H} \in \{\mathrm{K}\}_{-}^{\downarrow})}}{|\{\mathrm{K}\}_{-}^{\downarrow}|}$ serves as repair probability for the set of nodes $\mathrm{K} \backslash \mathrm{H}$ given that we have repair. Now, we take another Markov process $\widetilde{Y}^{\varepsilon}$ with the same jump intensity $\lambda(\mathrm{K})$ and modified transition probabilities. Let $\mathrm{I}_1 \subset \mathrm{I}_2$, $\mathrm{J}_1 \subset \mathrm{J}_2$. Then we put

$$\widetilde{\delta}^{\varepsilon}(\mathrm{K}, \mathrm{H}) = \begin{cases} \frac{\mathbf{I}_{(\mathrm{H} \in \{\mathrm{K}\}^{\prec})} + \varepsilon}{|\{\mathrm{K}\}^{\prec}|}, & \text{if } (\mathrm{K} = \mathrm{I}_1, \mathrm{H} = \mathrm{J}_1) \text{ or } (\mathrm{K} = \mathrm{I}_2, \mathrm{H} = \mathrm{J}_2), \\ \frac{\mathbf{I}_{(\mathrm{H} \in \{\mathrm{K}\}^{\prec})} - \varepsilon}{|\{\mathrm{K}\}^{\prec}|}, & \text{if } (\mathrm{K} = \mathrm{I}_1, \mathrm{H} = \mathrm{J}_2) \text{ or } (\mathrm{K} = \mathrm{I}_2, \mathrm{H} = \mathrm{J}_1), \\ \delta(\mathrm{K}, \mathrm{H}), & \text{otherwise}. \end{cases}$$

From Proposition 3.9 these both models are $<_{\mathrm{idif}}$-comparable under the monotonicity assumption.

Now, we turn our attention to the joint network process $Z = (Y, X)$. Recall that its state space is $\widetilde{E} = \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J$. Consider unreliable networks with the same routing matrices but different breakdown/repair processes Y and $\widetilde{Y}^\varepsilon$ as in Proposition 3.9.

Lemma 3.11 *Assume that (3) holds for Y and $\widetilde{Y}^\varepsilon$ with $(E, \prec) = (\mathcal{P}(\mathrm{J}), \subseteq)$. Then (3) holds for Z and $\widetilde{Z}^\varepsilon$ with $(E, \prec) = (\widetilde{E}, \subseteq \times \leq^J)$.*

Proof. Let $\mathrm{I}, \mathrm{I}_1, \mathrm{I}_2 \in \mathcal{P}(\mathrm{J})$, $\mathbf{n}, \mathbf{n}_1, \mathbf{n}_2 \in \mathbb{N}^J$. Recall that Z and $\widetilde{Z}^\varepsilon$ are Markov processes with intensities

$$q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) = \begin{cases} q^Y(\mathrm{I}_1, \mathrm{I}_2), & \text{if } \mathrm{I}_1 \neq \mathrm{I}_2 \text{ and } \mathbf{n}_1 = \mathbf{n}_2, \\ q_{\mathrm{I}_1}(\mathbf{n}_1, \mathbf{n}_2), & \text{if } \mathrm{I}_1 = \mathrm{I}_2 \text{ and } \mathbf{n}_1 \neq \mathbf{n}_2, \\ -q^Y(\mathrm{I}_1, \mathrm{I}_1) - q_{\mathrm{I}_1}(\mathbf{n}_1, \mathbf{n}_1), & \text{if } \mathrm{I}_1 = \mathrm{I}_2 \text{ and } \mathbf{n}_1 = \mathbf{n}_2, \\ 0, & \text{otherwise} \end{cases}$$

and

$$q^{\widetilde{Z}^\varepsilon}(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) = \begin{cases} q^{\widetilde{Y}^\varepsilon}(\mathrm{I}_1, \mathrm{I}_2), & \text{if } \mathrm{I}_1 \neq \mathrm{I}_2 \text{ and } \mathbf{n}_1 = \mathbf{n}_2, \\ q_{\mathrm{I}_1}(\mathbf{n}_1, \mathbf{n}_2), & \text{if } \mathrm{I}_1 = \mathrm{I}_2 \text{ and } \mathbf{n}_1 \neq \mathbf{n}_2, \\ -q^{\widetilde{Y}^\varepsilon}(\mathrm{I}_1, \mathrm{I}_1) - q_{\mathrm{I}_1}(\mathbf{n}_1, \mathbf{n}_1), & \text{if } \mathrm{I}_1 = \mathrm{I}_2 \text{ and } \mathbf{n}_1 = \mathbf{n}_2, \\ 0, & \text{otherwise} \end{cases}$$

respectively. Here $q_{\mathrm{I}}(\mathbf{n}_1, \mathbf{n}_2)$ are intensities of a standard Jackson network with valid node set $\mathrm{J} \setminus \mathrm{I}$, in which the routing matrix depends on the current state I of Y $(\widetilde{Y}^\varepsilon)$ and is restricted to the set $\mathrm{J}_0 \setminus \mathrm{I}$ consistently according to BLOCKING, STALLING or SKIPPING. Therefore components belonging to the node set I are kind of frozen and must coincide in $\mathbf{n}_1$ and $\mathbf{n}_2$ in order to obtain a feasible transition. Due to our assumptions, the intensities $q_{\mathrm{I}}(\mathbf{n}_1, \mathbf{n}_2)$ are the same for the networks Z and $\widetilde{Z}^\varepsilon$ for each fixed I. Furthermore we have from Theorem 3.8 and Lemma 2.10:

$$\pi^Z(\mathrm{I}, \mathbf{n}) = \pi^Y(\mathrm{I})\, \pi_0(\mathbf{n}) = \pi^{\widetilde{Y}^\varepsilon}(\mathrm{I})\, \pi_0(\mathbf{n}) = \pi^{\widetilde{Z}^\varepsilon}(\mathrm{I}, \mathbf{n})\,.$$

Assume now that $f : (\mathcal{P}(\mathrm{J}) \times \mathbb{N}^J)^2 \to \mathbb{R}$ has isotone differences. Then we have

$$\begin{aligned} \sum_{(\mathrm{I}_1, \mathbf{n}_1) \in \widetilde{E}} \pi(\mathrm{I}_1, \mathbf{n}_1) \sum_{(\mathrm{I}_2, \mathbf{n}_2) \in \widetilde{E}} q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) f(\mathrm{I}_1, \mathbf{n}_1, \mathrm{I}_2, \mathbf{n}_2) &= \\ = \sum_{(\mathrm{I}_1, \mathbf{n}_1)} \pi(\mathrm{I}_1, \mathbf{n}_1) \sum_{(\mathrm{I}_2 \neq \mathrm{I}_1, \mathbf{n}_2 = \mathbf{n}_1)} q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) f(\mathrm{I}_1, \mathbf{n}_1, \mathrm{I}_2, \mathbf{n}_2) & \\ + \sum_{(\mathrm{I}_1, \mathbf{n}_1)} \pi(\mathrm{I}_1, \mathbf{n}_1) \sum_{(\mathrm{I}_2 = \mathrm{I}_1, \mathbf{n}_2 \neq \mathbf{n}_1)} q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_2, \mathbf{n}_2) f(\mathrm{I}_1, \mathbf{n}_1, \mathrm{I}_2, \mathbf{n}_2) & \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{(\mathrm{I}_2=\mathrm{I}_1,\mathbf{n}_2=\mathbf{n}_1)} q^Z(\mathrm{I}_1,\mathbf{n}_1;\mathrm{I}_2,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_2)\\
&= \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathrm{I}_2\neq\mathrm{I}_1} q^Y(\mathrm{I}_1,\mathrm{I}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathbf{n}_2\neq\mathbf{n}_1} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_2)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) q^Z(\mathrm{I}_1,\mathbf{n}_1;\mathrm{I}_1,\mathbf{n}_1) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_1)\\
&= \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathrm{I}_2\neq\mathrm{I}_1} q^Y(\mathrm{I}_1,\mathrm{I}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathbf{n}_2\neq\mathbf{n}_1} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_2)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \left(- \sum_{(\mathrm{I}_2,\mathbf{n}_2)\neq(\mathrm{I}_1,\mathbf{n}_1)} q^Z(\mathrm{I}_1,\mathbf{n}_1;\mathrm{I}_2,\mathbf{n}_2) \right) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_1)\\
&= \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathrm{I}_2\neq\mathrm{I}_1} q^Y(\mathrm{I}_1,\mathrm{I}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathbf{n}_2\neq\mathbf{n}_1} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_2)\\
&\quad - \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \left(\sum_{\mathrm{I}_2\neq\mathrm{I}_1} q^Y(\mathrm{I}_1,\mathrm{I}_2) \right) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_1)\\
&\quad - \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \left(\sum_{\mathbf{n}_2\neq\mathbf{n}_1} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) \right) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_1)\\
&= \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathrm{I}_2} q^Y(\mathrm{I}_1,\mathrm{I}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1)\\
&\quad + \sum_{(\mathrm{I}_1,\mathbf{n}_1)} \pi(\mathrm{I}_1,\mathbf{n}_1) \sum_{\mathbf{n}_2} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_2)\\
&= \sum_{\mathbf{n}_1} \pi_0(\mathbf{n}_1) \left(\sum_{\mathrm{I}_1} \pi^Y(\mathrm{I}_1) \sum_{\mathrm{I}_2} q^Y(\mathrm{I}_1,\mathrm{I}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1) \right)\\
&\quad + \sum_{\mathrm{I}_1} \pi^Y(\mathrm{I}_1) \left(\sum_{\mathbf{n}_1} \pi_0(\mathbf{n}_1) \sum_{\mathbf{n}_2} q_{\mathrm{I}_1}(\mathbf{n}_1,\mathbf{n}_2) f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_1,\mathbf{n}_2) \right) .
\end{aligned}
$$

Now, for given $\mathbf{n}_1$ the function $(\mathrm{I}_1,\mathrm{I}_2) \to f(\mathrm{I}_1,\mathbf{n}_1,\mathrm{I}_2,\mathbf{n}_1)$ has isotone differences. According to our assumption that (3) holds for Y and $\widetilde{Y}^\varepsilon$ the first term in the latter expression is not bigger than the corresponding one for $\widetilde{Y}^\varepsilon$. Moreover, the second term is the same for both Z and $\widetilde{Z}^\varepsilon$. Therefore, the sufficient condition

(3) for ordering between Z and $\widetilde{Z}^\varepsilon$ is fulfilled.

$\square$

In order to use Theorem 2.9 for comparison of processes Z and $\widetilde{Z}^\varepsilon$ we need to have monotonicity of the corresponding transition kernels. However, in general this is not the case as the following example shows.

Example 3.12 Consider $\widetilde{E} = \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J$, inclusion ordering on $\mathcal{P}(\mathrm{J})$, coordinatewise ordering on $\mathbb{N}^J$, and $\prec$ - coordinatewise ordering on $\widetilde{E}$. In order to check $\prec$-monotonicity we have to check (among others) that for $\mathrm{I}_1 \subseteq \mathrm{I}_2$, $\mathbf{n}_1 \leq^J \mathbf{n}_2$ the infinitesimal generator of the network fulfills

$$q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathcal{A}_1 \times \mathcal{A}_2) \leq q^Z(\mathrm{I}_2, \mathbf{n}_2; \mathcal{A}_1 \times \mathcal{A}_2)$$

for any increasing sets $\mathcal{A}_1 \in \mathcal{P}(\mathrm{J})$, $\mathcal{A}_2 \in \mathbb{N}^J$, whereas

$$q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathcal{A}_1 \times \mathcal{A}_2) := \sum_{(\mathrm{I}_3, \mathbf{n}_3) \in \mathcal{A}_1 \times \mathcal{A}_2} q^Z(\mathrm{I}_1, \mathbf{n}_1; \mathrm{I}_3, \mathbf{n}_3).$$

Assume that $\mathrm{I}_1 \in \mathcal{A}_1$, $\mathbf{n}_1 \in \mathcal{A}_2$. We have as in Lemma 3.11

$$q^Z(\mathrm{I}, \mathbf{n}; \mathcal{A}_1 \times \mathcal{A}_2) = \sum_{\mathrm{K} \in \mathcal{A}_1} q^Y(\mathrm{I}, \mathrm{K}) + \sum_{\mathbf{m} \in \mathcal{A}_2} q_{\mathrm{I}}(\mathbf{n}, \mathbf{m})\,. \tag{12}$$

This expression (12) is clearly monotone w.r.t $\mathbf{n}$ since Q_{I} is a generator for a standard Jackson network without breakdowns with nodes $\mathrm{J} \setminus \mathrm{I}$. Take $\mathcal{A}_1 = \mathcal{P}(\mathrm{J})$. Then the first part in the above expression is 0. By considering STALLING principle Q_{I} is a zero matrix unless $\mathrm{I} = \emptyset$. For $\mathrm{I} = \emptyset$ the latter part in the above expression can be either positive or negative according to the choice of $\mathcal{A}_2$. Therefore expression (12) is not monotone w.r.t. I and the network process Z is not $\prec$-monotone. By considering BLOCKING or SKIPPING as underlying rerouting rule the situation gets more involved but does not lead to monotonicity either.

However, consider the following ordering: $(\mathrm{I}_1, \mathbf{n}_1) \prec (\mathrm{I}_2, \mathbf{n}_2)$ if and only if $\mathrm{I}_1 \subseteq_e \mathrm{I}_2$ and $\mathbf{n}_1 \leq^J \mathbf{n}_2$. Here, $\mathrm{I}_1 \subseteq_e \mathrm{I}_2$ if and only if $\mathrm{I}_1 = \mathrm{I}_2$. Then Z is clearly monotone w.r.t. to this ordering. The same holds for its time-reversal. Now, a function f which has isotone differences on $(\widetilde{E}^n, (\subseteq \times \leq^J)^n)$ has clearly isotone differences on $(\widetilde{E}^n, (\subseteq_e \times \leq^J)^n)$. Therefore, from Lemma 3.11 the inequality (3) holds for all functions with isotone differences on $(\widetilde{E}^n, (\subseteq_e \times \leq^J)^n)$. Because of the above mentioned monotonicity we have the following result.

Theorem 3.13 *Assume that two Jackson networks have the same arrival intensities, the same rerouting matrices according to either* BLOCKING *or* STALLING *or* SKIPPING *and breakdown/repair intensity matrices are given by (10) and (11). Assume also that breakdown/repair intensity matrices and its time-reversal counterparts are stochastically monotone. Then*

$$Z <_{\mathrm{idif}((\subseteq_{\mathrm{e}} \times \leq^{\mathrm{J}})^{(\infty)})} \widetilde{Z}^\varepsilon\,,$$

i.e. for all $n \geq 2$ and $t_1 \leq \cdots \leq t_n$,

$$\mathbb{E}\left[f\left(Z_{t_1}, \ldots, Z_{t_n}\right)\right] \leq \mathbb{E}\left[f\left(\widetilde{Z}^{\varepsilon}_{t_1}, \ldots, \widetilde{Z}^{\varepsilon}_{t_n}\right)\right],$$

for all functions f with isotone differences on $(\widetilde{E}^n, (\subseteq_e \times \leq^J)^n)$.

In particular, because a function g with isotone differences on $(\widetilde{E}^n, (\subseteq_e \times \leq^J)^n)$ has clearly isotone differences on $((\mathbb{N}^J)^n, (\leq^J)^n)$ we obtain the following corollary for the queue length process in a network with unreliable nodes which is not a Markov process.

Corollary 3.14 *Under conditions of Theorem 3.13 we have*

$$X <_{\mathrm{idif}((\leq^{\mathrm{J}})^{(\infty)})} \widetilde{X}^{\varepsilon}.$$

Example 3.15 As in standard Jackson network case it is possible to write the inequality (6) for the queue length process in a network which is subject to breakdowns and repairs.

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