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On convergence of throughput in large networks with state-dependent service rates

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1 Introduction

[1] The behavior of large networks under random influences has found much interest in the area of applied probability and especially queueing theory. Ever since Jackson [Jac63] and Gordon and Newell [GN67] computed explicitly the steady-state distributions of closed networks of exponential queueing systems, an important part of the research has been devoted to the behavior of the network when either the number of customers in the network increased without bound while the number of nodes was held fixed or the number of customers and the number of nodes both grew without bound. The main objective is to study asymptotic behavior of performance measures, especially queue-length distributions at the nodes. An interesting feature is the possibility of bottlenecks, namely nodes where, asymptotically, the queue lengths tend to infinity. So at these nodes, we observe traffic jams and condensation phenomena (see [MY96]). For a recent survey and further references see [BBK99].

When the number of nodes is held fixed and the number of customers tends to infinity, a common observation which dates back to the findings of Gordon and Newell [GN67] is that the bottlenecks are determined by the minimum of the maximal ratio of arrival rate over service rate that the nodes can provide. Moreover, the (asymptotic) throughput (mean departure rate) of the network is determined essentially by that minimum. For networks with a fixed number of nodes, another approach to asymptotic analysis is the calculation of fluid limits or diffusion limits by a suitable scaling of time and queue lengths. (See [CY01], [Yao94], and the references given there.)

For the case when the number of customers and the number of nodes both grow without bound, the research in the field of queueing theory is more recent. One approach, as used by Malyshev and Yakovlev in the interesting investigation [MY96], is to allow individual nodes and their behavior to be persistent during the growth process. Thus the microscopic behavior can be studied in the limit.

A contrasting approach is to re-scale time, queue lengths, and also the space where the queues are located. This common technique in the analysis of interacting particle systems over regular lattices (Liggett [Lig85]) leads to a fluid or hydrodynamic limit which describes the asymptotic macroscopic behavior. (See also [KL99] and [DP91].) Under this limit, the individual local behavior of a particle or node disappears from the observation space.

The setting in this paper is a generalization of that in [MY96], and it facilitates the analysis of the microscopic behavior of the nodes. Our interest is in general Gordon-Newell networks (closed

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Jackson networks), where at each node there is an exponential server with a service rate which is a non-decreasing (not necessarily bounded) function of the queue length. The customers' movements are governed by an irreducible stochastic routing matrix. For simplicity of presentation, we restrict ourselves to closed, cyclic networks. This is possible since we can incorporate the relative visit ratios into the service rates to obtain an equivalent closed cyclic network. For details, see Section 3.5.

Our primary objective in this investigation is to study the convergence of throughputs for a sequence of closed cyclic networks. Each network has a *customer density*, i.e., the ratio of the number of customers to the number of nodes, and a *service rate measure* induced by the proportion of nodes having each given service rate. The set of all service rates is equipped with the topology of pointwise convergence. We assume that the sequence of customer densities converges and that the sequence of service rate measures converges weakly.

In [MY96], Malyshev and Yakovlev studied asymptotic queue-length distributions for Gordon-Newell networks with a single server at each node and with the service rate independent of the queue length. They worked with parameters called relative utilizations which, for the case of cyclic networks, is equivalent to scaling all service rates so that the slowest service rate equals 1. Their Assumption W on the existence of a limiting relative utilization measure on $[0, 1]$ is, for state-independent servers, equivalent to our assumption of weak convergence of the service-rate measures. For our investigation of asymptotic throughput, the notion of relative utilization does not seem appropriate, especially since our service rates are state-dependent and possibly unbounded. Moreover, by not relativizing, we are able to assert the existence of limiting queue-length distributions for some sequences of networks with state-independent servers that do not satisfy the assumptions in [MY96]. (See Examples 3.13 and 3.22.)

Our main theorems, which appear in Section 3, are stated in terms of the *asymptotic profile* for a sequence of networks. An asymptotic profile is a triple $(\mathcal{L}, \alpha, \beta)$, where $\mathcal{L}$ is the set of persistent service rates in the sequence, α is the limiting customer density, and β is the limiting service rate measure. Let $\lambda^*_{\mathcal{L}}$ be the least upper bound of the lower envelope of $\mathcal{L}$. Then $\lambda^*_{\mathcal{L}}$ (which can be thought of as the *horizon* of $\mathcal{L}$) is the largest θ such that for $M/M(n)/1/\infty$ queues having arrival rate less than θ and service rates μ belonging to $\mathcal{L}$, the expected queue length is uniformly bounded in μ.

Our characterization of limiting throughput is conveniently described in terms of the correspondence g_β between arrival rates θ and the mean expected queue length for an $M/M(n)/1/\infty$ queue having random service rate μ with distribution β. This correspondence is one-to-one on its domain of finiteness.

We prove in Theorem 3.11 that for finite α, where $g_\beta^{-1}(\alpha)$ is no greater than the horizon $\lambda^*_{\mathcal{L}}$, the limiting throughput must exist, and it equals $g_\beta^{-1}(\alpha)$. For other finite α, provided that g_β is infinite beyond the horizon, the limiting throughput must exist and equal $\lambda^*_{\mathcal{L}}$.

We also establish, in Theorem 3.16, that another sufficient condition for the existence of limiting throughput is the presence of slow, persistent service rates along every subsequence. This condition is always satisfied in the framework of [MY96], since every network contains the slow rate 1.

Our method of proof is surprisingly elementary and hinges on relationships between throughputs and likelihood ratios of queue lengths that stem from the product form of the steady-state distributions. From these relationships, we are able to compute bounds for possible limit points of throughputs. With similar techniques, under the conditions of either Theorem 3.11 or Theorem 3.16, we are able to compute in Theorem 3.20 the limiting queue-length distribution for each persistent service rate. Thus this theorem extends the queue-length-distribution results of [MY96] to networks with state-dependent service rates. Malyshev and Yakovlev had commented ([MY96], p.94) that *we believe that generalizations for networks with state dependent service rates ... are possible.*

For a sequence of networks whose asymptotic profile satisfies neither the conditions of Theorem 3.11 nor those of Theorem 3.16, the situation is more intricate. Here, it is not possible to assert convergence of throughputs or convergence of queue-length distributions without information beyond the asymptotic profile. Indeed, we can always construct (Theorem 3.17) a sequence of networks, with the same asymptotic profile, for which the limiting throughput does not exist.

While the networks considered in the literature usually live on a continuous time scale there are some recent papers with discrete time networks where the asymptotic throughput is characterized under various conditions on the structure of the nodes; see [PR94] [PR99], [DPR03]. In these papers, the setting and the questions addressed are similar in spirit to our present work. In [DPR03], we studied in discrete time the special case of a sequence of cyclic networks with only a finite number of possible node types. Whether all of the results of this current paper have analogs in discrete time still remains open.

2 Description of the system

2.1 Single-node system

Suppose we have an isolated single server $M/M(n)/1/\infty$ FCFS queue having infinite waiting room and state-dependent service rates and satisfying the following assumptions. Indistinguishable customers arrive in a Poisson(θ) stream. Service requests and interarrival times are independent. Service to a customer in service is provided at rate $\mu(h) \geq 0$ if there are h customers present either waiting or in service, where $\mu = (\mu(h) : h \in \mathbb{N}) \in \mathbb{R}_+^{\mathbb{N}}$ is the *service rate* for the node, and $\mathbb{N} = \{0, 1, 2, \dots\}$.

Definition 2.1 *A* service rate *is a non-decreasing sequence $\mu = (\mu(h) : h \in \mathbb{N}) \in \mathbb{R}_+^{\mathbb{N}}$ with $\mu(0) = 0$. For a service rate μ we define*

$$\mu^* := \sup(\mu(h) : h \in \mathbb{N}),$$

the asymptotic maximal rate of service, and

$$ULI(\mu) := \inf\{h \in \mathbb{N} : \mu(h+1) > 0\},$$

the upper level of the server's inactivity. When μ is identically 0, we set $ULI(\mu) = \infty$. Whenever $ULI(\mu) = 0$, we call μ a positive service rate.

Remark 2.2 *The case when $ULI(\mu)$ is positive represents a standard queue under the so–called N-policy with $N = ULI(\mu)$, and this may reflect an optimal control of service systems where the costs of restarting service is not negligible. For more details, see [DS96].*

Proposition 2.3 *Consider the $M/M(n)/1/\infty$ queue having service rate μ and having arrival rate θ, where $0 < \theta < \mu^*$. Let $a = ULI(\mu)$, and let $\pi_{\mu,\theta}$ be the probability measure (on the set of non-negative integers) defined by*

$$\pi_{\mu,\theta}(x) = \left(\prod_{h=a+1}^{x} \frac{\theta}{\mu(h)}\right) \cdot C(\mu,\theta)^{-1} \qquad for \quad x \geq a. \tag{2.1}$$

and by

$$\pi_{\mu,\theta}(x) = 0 \qquad for \quad 0 \leq x < a,$$

where

$$C(\mu,\theta) = \sum_{y=a}^{\infty} \prod_{h=a+1}^{y} \frac{\theta}{\mu(h)} < \infty \tag{2.2}$$

is the normalizing constant. Then the queue length process $X = (X(t) : t \geq 0)$ on $\{x \in \mathbb{N} : x \geq a\}$ is ergodic with steady state distribution $\pi_{\mu,\theta}$.

The relation (2.1) is standard (see, for example, Chen and Yao [CY01]) and follows because the balance equations for the Markov chain imply

$$\frac{\pi_{\mu,\theta}(x+1)}{\pi_{\mu,\theta}(x)} = \frac{\theta}{\mu(x+1)}, \quad \text{for} \quad x \geq a. \tag{2.3}$$

In Proposition 2.3 and in the sequel, we make the convention that empty products of the type "$\prod_{h=a+1}^{a}$" equal 1. Also, we take empty sums of the type "$\sum_{h=a+1}^{a}$" to be equal to 0.

Definition 2.4 *The mean steady-state queue length for a service rate μ and for an arrival rate θ satisfying $0 < \theta < \mu^*$ is*

$$m(\mu,\theta) = \sum_{x=0}^{\infty} x \cdot \pi_{\mu,\theta}(x);$$

meanwhile if $\theta > 0$ and $\theta \geq \mu^*$, we set $m(\mu,\theta) = \infty$.

Example 2.5 ($M/M/1/\infty$) *If H is a positive constant, let μ^H be the service rate which satisfies*

$$\mu^H(x) = H \quad \text{if} \quad x \geq 1.$$

If $0 < \theta < H$, then $\pi_{\mu^H,\theta}$ is the probability distribution of the number of successes prior to the first failure in independent Bernoulli trials, where the success probability is θ/H. Therefore

$$m(\mu^H,\theta) = \frac{\theta}{H-\theta} \qquad \text{for} \quad 0 < \theta < H.$$

Also, $m(\mu^H,\theta) = \infty$ for $\theta \geq H$.

Example 2.6 (*equivalent to $M/M/\infty$*) *Suppose μ is defined by $\mu(h) = h$ for $h \in \mathbb{N}$. Then for each positive real number θ, $\pi_{\mu,\theta}$ is the distribution of a Poisson random variable with parameter θ. Thus $m(\mu,\theta) = \theta$.*

Example 2.7 *Consider the $M/M(n)/1/u-1$ queue with limited waiting room of size $u-1$, where u is a positive integer, with service rate μ, arrival rate $\theta > 0$, and $ULI(\mu) = a < u$. Here, θ need not be less than μ^*. For this queue, calculations as in (2.3) from the balance equations will show that the equilibrium distribution $\pi^u_{\mu,\theta}$ satisfies*

$$\frac{\pi^u_{\mu,\theta}(x+1)}{\pi^u_{\mu,\theta}(x)} = \frac{\theta}{\mu(x+1)}, \quad a \leq x \leq u-1 \tag{2.4}$$

and

$$\pi^u_{\mu,\theta}(x) = 0 \quad \text{for} \quad 0 \leq x < a \quad \text{or} \quad x > u. \tag{2.5}$$

As in (2.1), $\pi^u_{\mu,\theta}$ *can be explicitly determined from* (2.4) *and* (2.5). *Let* $m^u(\mu,\theta)$ *denote the mean steady-state queue length defined by*

$$m^u(\mu,\theta) = \sum_{x=0}^{\infty} x \cdot \pi^u_{\mu,\theta}(x);$$

It follows from standard results on likelihood ratio ordering and stochastic ordering (see Chen and Yao [CY01] and Müller and Stoyan [MS02]) that the queue-length random variable is monotone increasing in u in the stochastic ordering and hence, as $u \to \infty$, *we have*

$$m^u(\mu,\theta) \nearrow m(\mu,\theta). \tag{2.6}$$

The relation (2.6) *can be readily obtained from the following well-known likelihood ratio lemma (see [MS02] and [SS94]). This lemma will also be useful for us in the sequel.*

Lemma 2.8 *Let* π_1 *and* π_2 *probabilities on the set of nonnegative integers with supports* $\{x : a_1 \le x < b_1\}$, *and* $\{x : a_2 \le x < b_2\}$, *respectively, where* $0 \le a_2 \le a_1 \le b_2 \le b_1 \le \infty$. *Assume that for all* x *satisfying* $a_1 \le x < b_2$, *we have*

$$\frac{\pi_1(x+1)}{\pi_1(x)} \ge \frac{\pi_2(x+1)}{\pi_2(x)}. \tag{2.7}$$

Then for each $x \ge 0$,

$$\sum_{y=x}^{\infty} \pi_1(y) \ge \sum_{y=x}^{\infty} \pi_2(y),$$

and hence

$$\sum_{x=0}^{\infty} x\pi_1(x) \ge \sum_{x=0}^{\infty} x\pi_2(x), \tag{2.8}$$

where strict inequality holds in (2.8) *unless* $\pi_1 = \pi_2$.

Proof. See [MS02] and [SS94].

Remark 2.9 *It is easy to see that for every service rate* μ, *if* $0 < \theta < \mu^*$, *then* $m(\mu,\theta) < \infty$. *It is an immediate consequence of Lemma 2.8 that if* μ *and* ν *are service rates such that* $\mu(h) \le \nu(h)$ *for each positive integer* h, *then* $m(\nu,\theta) \le m(\mu,\theta)$ *for all* θ. *Also, it follows from Lemma 2.8 that for fixed* μ, *the function* $m(\mu,\theta)$ *is a strictly-increasing, continuous function of* θ *on* $0 < \theta < \mu^*$. *Furthermore,*

$$\lim_{\theta \to \mu^*} m(\mu,\theta) = \infty. \tag{2.9}$$

Assertion (2.9) *can be seen in the case when* $\mu^* < \infty$ *as follows. We have*

$$\mu \le \mu^{\mu^*},$$

where μ^{μ^*} *is* μ^H *defined in Example 2.5 with* $H = \mu^*$. *Therefore, for* $\theta < \mu^*$,

$$m(\mu,\theta) \ge m(\mu^{\mu^*},\theta) = \frac{\theta}{\mu^* - \theta},$$

and the right side goes to infinity as $\theta \nearrow \mu^*$. To see assertion (2.9) in the case when $\mu^* = \infty$, use Lemma 2.8 to conclude that for each positive integer H and for $\theta > \mu(H)$,

$$m(\mu, \theta) \geq m^H(\mu^\theta, \theta) = \frac{H}{2}.$$

The last equality follows because the equilibrium distribution $\pi^H_{\mu^\theta,\theta}$ is uniform on $\{x : 0 \leq x \leq H\}$ as can be seen from (2.4) and (2.5).

Proposition 2.10 *For fixed $\theta > 0$, the function $m(\mu, \theta)$ is continuous in the following sense: If $\{\mu_j\}$ is a sequence of service rates such that for each positive integer h,*

$$\lim_{j\to\infty} \mu_j(h) = \mu(h), \tag{2.10}$$

where μ is a service rate, then

$$\lim_{j\to\infty} m(\mu_j, \theta) = m(\mu, \theta). \tag{2.11}$$

To prove the proposition, we first require a lemma establishing geometric bounds for certain products:

Lemma 2.11 *Suppose that $\mu_j \longrightarrow \mu$ pointwise, where $\{\mu_j\}$ are service rates, μ is a service rate, $ULI(\mu) = b$, and for all j, $ULI(\mu_j) \leq b$. Suppose further that $0 < \theta < \theta' < \mu^*$. Then for some positive real number L and some positive integer j_0 and for all $y > b$ and all $j \geq j_0$,*

$$\prod_{h=b+1}^{y} \frac{\theta}{\mu_j(h)} \leq L \cdot \left(\frac{\theta}{\theta'}\right)^{y-b}. \tag{2.12}$$

Proof. Observe first that since $\theta' < \mu^*$, we can use (2.16) and (2.10) to find positive integers k_0 and j_0 such that $k_0 > b$ and

$$\mu_j(k_0) \geq \theta' \qquad \text{for all } j \geq j_0. \tag{2.13}$$

Let $r = \min\{\mu_j(b+1) : j \in \mathbb{N}\}$. Since

$$\lim_{j\to\infty} \mu_j(b+1) = \mu(b+1) > 0$$

and since $ULI(\mu_j) \leq b$, we must have $r > 0$. Now suppose y is a positive integer satisfying $y > b$. If $y \geq k_0 > b$, then because $\mu_j(\cdot)$ is increasing and because of (2.13), we must have for each j such that $j \geq j_0$,

$$\prod_{h=b+1}^{y} \frac{\theta}{\mu_j(h)} \leq \left(\frac{\theta}{r}\right)^{k_0-b} \left(\frac{\theta}{\theta'}\right)^{y-k_0} = \left(\frac{\theta'}{r}\right)^{k_0-b} \left(\frac{\theta}{\theta'}\right)^{y-b}. \tag{2.14}$$

Otherwise, if $k_0 > y > b$, then for $j \geq j_0$,

$$\prod_{h=b+1}^{y} \frac{\theta}{\mu_j(h)} = \prod_{h=b+1}^{y} \frac{\theta'}{\mu_j(h)} \cdot \left(\frac{\theta}{\theta'}\right)^{y-b} \leq \left(\frac{\theta'}{r}\right)^{y-b} \left(\frac{\theta}{\theta'}\right)^{y-b}. \tag{2.15}$$

If $\theta' \leq r$, then we take $L = 1$, while if $\theta' > r$, we take

$$L = \left(\frac{\theta'}{r}\right)^{k_0-b}.$$

Thus in both cases (2.14) and (2.15), the relation (2.12) must be true for $j \geq j_0$.

Proof of Proposition 2.10. We consider two cases: (i) $0 < \theta < \mu^*$, and (ii) $\theta \geq \mu^*$. For case (i), let $ULI(\mu) = b < \infty$, and for each j, let $a_j = ULI(\mu_j)$. Because of (2.10), we must have $\limsup_{j\to\infty} a_j \leq b$. Also,

$$\liminf_{j\to\infty} \mu_j^* \geq \sup_h \liminf_{j\to\infty} \mu_j(h) = \sup_h \mu(h) = \mu^*. \tag{2.16}$$

So without loss of generality, we can assume that for each j, $a_j \leq b$, and that $\theta < \mu_j^*$. We have

$$\pi_{\mu_j,\theta}(x) = \prod_{h=a_j+1}^{x} \frac{\theta}{\mu_j(h)} \cdot C(\mu_j,\theta)^{-1} \qquad \text{for } x \geq a_j, \tag{2.17}$$

where $C(\mu_j,\theta)$ is described in (2.2). For $x \geq b$, we can write

$$\pi_{\mu_j,\theta}(x) = \left(\prod_{h=b+1}^{x} \frac{\theta}{\mu_j(h)}\right)\left(\prod_{h=a_j+1}^{b} \frac{\theta}{\mu_j(h)}\right)\left[\sum_{y=a_j}^{b-1}\prod_{h=a_j+1}^{y}\frac{\theta}{\mu_j(h)} + \sum_{y=b}^{\infty}\prod_{h=a_j+1}^{y}\frac{\theta}{\mu_j(h)}\right]^{-1}.$$

If we let

$$A_j = \sum_{y=a_j}^{b-1}\prod_{h=y+1}^{b}\frac{\mu_j(h)}{\theta} \quad \text{and} \quad B_j = \sum_{y=b}^{\infty}\prod_{h=b+1}^{y}\frac{\theta}{\mu_j(h)},$$

then

$$\pi_{\mu_j,\theta}(x) = \prod_{h=b+1}^{x} \frac{\theta}{\mu_j(h)} \cdot (A_j + B_j)^{-1} \qquad \text{for } x \geq b. \tag{2.18}$$

Note that, in view of (2.17),

$$\pi_{\mu_j,\theta}(x) = \prod_{h=x+1}^{b} \frac{\mu_j(h)}{\theta} \cdot \pi_{\mu_j,\theta}(b) \quad \text{for} \quad a_j \leq x < b.$$

¿From (2.18) we have

$$\pi_{\mu_j,\theta}(x) = (A_j + B_j)^{-1} \qquad \text{for } x = b$$

and hence

$$\pi_{\mu_j,\theta}(x) = \prod_{h=x+1}^{b} \frac{\mu_j(h)}{\theta} \cdot (A_j + B_j)^{-1} \quad \text{for} \quad a_j \leq x < b. \tag{2.19}$$

The right side of the equality in (2.19) reduces to 0 when $x < a_j$. Therefore we can write

$$\pi_{\mu_j,\theta}(x) = \prod_{h=x+1}^{b} \frac{\mu_j(h)}{\theta} \cdot (A_j + B_j)^{-1} \qquad \text{when } 0 \leq x < b. \tag{2.20}$$

Now suppose θ' satisfies $\theta < \theta' < \mu^*$. It follows from Lemma 2.11 and bounded convergence that $B_j \to C(\mu,\theta)$. Because of (2.10) and the fact that $\mu(h) = 0$ for $h \leq b$, we have $A_j \to 0$. Thus we can use (2.18) to conclude that for $x \geq b$,

$$\lim_{j\to\infty} \pi_{\mu_j,\theta}(x) = \pi_{\mu,\theta}(x). \tag{2.21}$$

Meanwhile, if $0 \le x < b$, then $\mu_j(x) \to 0$ and, with the help of (2.20), we get $\lim_{j\to\infty} \pi_{\mu_j,\theta}(x) = 0 = \pi_{\mu,\theta}(x)$. Thus we have shown that (2.21) holds for all x and all $\theta < \mu^*$. We can then use (2.12) and (2.21) to conclude that

$$\lim_{M\to\infty} \sup_{j\ge j_0} \sum_{y=M}^{\infty} y \cdot \pi_{\mu_j,\theta}(y) = 0.$$

Hence (2.11) will also be true, by uniform integrability. This finishes the proof in case (i).

In case (ii), assume that $\mu^* \le \theta$. Then $m(\mu,\theta) = \infty$. We must show that

$$\lim_{j\to\infty} m(\mu_j,\theta) = \infty. \tag{2.22}$$

When $\mu^* > 0$, we have by Remark 2.9 and by case (i),

$$\liminf_{j\to\infty} m(\mu_j,\theta) \ge \sup_{0<\theta'<\mu^*} \liminf_{j\to\infty} m(\mu_j,\theta') = \sup_{0<\theta'<\mu^*} m(\mu,\theta') = \infty$$

On the other hand when $\mu^* = 0$, given any positive integer H, we can find a positive integer j_0 such that

$$\mu_j(h) \le \theta \quad \text{for all } j \ge j_0 \text{ and } 1 \le h \le H.$$

Therefore, by Lemma 2.8, for each $j \ge j_0$, the quantity $m(\mu_j,\theta)$ is at least as large as the expected queue length of the $M/M(n)/1/H-1$ queue with constant service rate θ for $1 \le h \le H$ and arrival rate θ. Since the equilibrium distribution for this $M/M(n)/1/H-1$ queue is uniform on $\{h : 0 \le h \le H\}$, its expected queue length equals $H/2$. So we have

$$m(\mu_j,\theta) \ge \frac{H}{2} \quad \text{for} \quad j \ge j_0.$$

Since H was an arbitrary positive integer, the proof is complete. □

2.2 Cyclic networks of queues

We consider closed cycles of single server queues with queue-length-dependent service rates. In a cyclic network $\mathcal{C}$, the nodes are numbered $1, 2, \ldots, n$ and there are k indistinguishable customers cycling in the network. The set of all service times is an independent set of variables. Node j has a single service channel, an infinite waiting room which is organized according to the first–come–first–served (FCFS) regime, and customers finding at their arrivals no other customers there immediately start service. With each node j is associated a service rate μ_j according to Definition 2.1: If there are x_j customers present at node j then service is provided to the customer in service at rate $\mu_j(x_j) \ge 0$. A customer departing from node j immediately proceeds to node $j+1$ (we set $n+1 = 1$). Let $X = (X(t) : t \ge 0)$ denote the vector process recording the joint queue lengths in the cycle for time variable t. $X(t) = (X_1(t), \ldots, X_n(t)) \in \mathbb{N}^n$ reads: at time t there are $X_j(t)$ customers present at node j, either in service or waiting. The assumptions put on the system imply that X is a strong Markov process with state space

$$S(k,n) := \{(x_1, \ldots, x_n) : x_j \in \mathbb{N}, j = 1, \ldots, n, x_1 + \cdots + x_n = k\}. \tag{2.23}$$

If $k \le \sum a_j$, where $a_j = ULI(\mu_j)$, then any state $(x_1, \ldots, x_n) \in S(k,n)$ with $x_j \le a_j$ for each j $(1 \le j \le n)$ is an absorbing state for X. If $k > \sum a_j$, then X can be shown to be ergodic, and, in the limit, there are at least a_j customers at node j $(1 \le j \le n)$. The limiting behavior of the process X is equivalently obtained from a cyclic network with $k - \sum_{j=1}^{n} a_j$ customers and in which the service rate at node j $(1 \leq j \leq n)$ is ν_j, where

$$\nu_j(h) = \mu_j(a_j + h)$$

for $h \geq 0$. As a result, for ease of presentation, we shall place the following assumption in force:

Assumption 2.12 *For each cyclic network $\mathcal{C}$, the service rate at each node is a positive service rate.*

The results to follow in this subsection are well-known and are essentially contained in chapters 2 and 3 of Chen and Yao [CY01]:

Theorem 2.13 *X is ergodic on $S(k,n)$, and its unique steady–state and limiting distribution $\pi = (\pi(x_1,\ldots,x_n) : (x_1,\ldots,x_n) \in S(k,n))$ is given by*

$$\pi(x_1,\ldots,x_n) = \left(\prod_{j=1}^{n} \prod_{h=1}^{x_j} \frac{\theta}{\mu_j(h)} \right) \cdot G_\theta(k,n)^{-1}, \tag{2.24}$$

where $\theta > 0$ is any constant and

$$G_\theta(k,n) = \sum_{(x_1,\ldots,x_n)\in S(k,n)} \prod_{j=1}^{n} \prod_{h=1}^{x_j} \frac{\theta}{\mu_j(h)} \tag{2.25}$$

is the normalizing constant. (Note that π does not depend on θ.)

Remark 2.14 *¿From* (2.24) *we conclude that the equilibrium distribution of the system is independent of the relative position of the nodes.*

In the following unless otherwise specified we always assume that the systems under consideration are in equilibrium.

Definition 2.15 *The* throughput $T_{k,n}(i)$ *at any node i is defined by*

$$T_{k,n}(i) = \sum_{(x_1,\ldots,x_n)\in S(k,n)} \pi(x_1,\ldots,x_n) \cdot \mu_i(x_i). \tag{2.26}$$

Thus the throughput at node i is the expected rate of progress at node i in equilibrium under π at any given time instant.

Lemma 2.16 *For $1 \leq i \leq n$ and for each $\theta > 0$, we have*

$$T_{k,n}(i) = \frac{\sum_{(x_1,\ldots,x_n)\in S(k,n)} \prod_{j=1}^{n} \prod_{h=1}^{x_j} \left(\frac{\theta}{\mu_j(h)}\right) \cdot \mu_i(x_i)}{\sum_{(x_1,\ldots,x_n)\in S(k,n)} \prod_{j=1}^{n} \prod_{h=1}^{x_j} \frac{\theta}{\mu_j(h)}} = \frac{G_1(k-1,n)}{G_1(k,n)}, \tag{2.27}$$

which is independent of i.

Definition 2.17 *The cycle's (average)* throughput $T_{k,n}$ *is the throughput $T_{k,n}(i)$ at any node i.*

Lemma 2.18 [vdW89] *For a fixed set of nodes $1, 2, \ldots, n$ the throughput $T_{k,n}$ is increasing in k.*

Lemma 2.19 *For a given set of nodes $1, 2, \ldots, n$ and a fixed number $k \geq 1$ of cycling customers, the throughput $T_{k,n}$ of the cycle is increased by deleting* any *node of the cycle and skipping across the gap.*

Proof: Similar to [DPR03], Prop. 3.5.

2.3 Sequences of cyclic networks

We now describe the setting for our study of the asymptotic behavior of sequences $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ of closed cyclic networks. Each $\mathcal{C}(N)$ is as defined in Section 2.2 ($\mathbb{N}^+$ is the set of positive integers). The set of distinct service rates that occur in network $\mathcal{C}(N)$ is denoted by $\mathcal{L}(N)$. Any network $\mathcal{C}(N)$ determines a probability measure $\beta(N)$, defined on $\mathcal{L}(N)$, where for each μ in $\mathcal{L}(N)$,

$$\beta(N)(\{\mu\})$$

is the proportion of all nodes in $\mathcal{C}(N)$ having service rate μ. We regard $\beta(N)$ as the *service-rate measure* of the network $\mathcal{C}(N)$. We let $k(N)$ and $n(N)$ denote the total number of customers, and the total number of nodes, respectively, in that network.

The asymptotics of the sequence $(\mathcal{C}(N), N \in \mathbb{N}^+)$ will depend on the *persistent* rates and will be determined under the following assumptions on the existence of *limiting customer density* and on the *limiting service-rate measure.* Towards defining these notions, we shall equip the set of all service rates with the topology of pointwise convergence. This topology is the same as that inherited from the usual topology on $\mathbb{R}_+^{\mathbb{N}}$. We let

$$\mathcal{L} = \bigcap_{M=1}^{\infty} \overline{\bigcup_{N=M}^{\infty} \mathcal{L}(N)}, \tag{2.28}$$

where for any set A of service rates, $\overline{A}$ denotes the closure of A with respect to this topology. The elements of $\mathcal{L}$ may be regarded as the "persistent service rates" of the sequence. (We note that, although Assumption 2.12 is in force, and each service rate in each $\mathcal{L}(N)$ is positive, the closed set $\mathcal{L}$ could contain service rates μ where $ULI(\mu) > 0$.)

We shall find it convenient to identify a probability measure defined on a closed subset of the set of all service rates with its unique probability extension to a larger closed set.

Assumption 2.20 *The networks grow without bound:*

$$n(N) \to \infty, \quad \textit{for} \quad N \to \infty.$$

The "limiting customer density" exists:

$$\frac{k(N)}{n(N)} \longrightarrow \alpha, \quad \textit{for} \quad N \to \infty, \tag{2.29}$$

where $\alpha = 0$ or $\alpha = \infty$ is allowed.
The "limiting service rate measure" exists: For some probability measure β on $\mathcal{L}$ we have

$$\beta(N) \xrightarrow{w} \beta, \quad \textit{for} \quad N \to \infty, \tag{2.30}$$

where $\xrightarrow{w}$ denotes weak convergence.

Relation (2.30) in Assumption 2.20 is the Assumption **(W)** of [MY96][p. 94], for the case of state independent service rates which can be parametrized by values in $[0, 1]$. Indeed, Malyshev and Yakovlev consider relative utilizations, but this is in their setting equivalent to considering constant service rates.

Definition 2.21 *Let $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ be any sequence of cyclic networks satisfying Assumption 2.20. We shall refer to the sequence $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ as "having asymptotic profile $(\mathcal{L}, \alpha, \beta)$."*

2.4 The summary parameters

We introduce some quantities which will be useful in characterizing the asymptotic throughput of sequences of networks. Given an arbitrary non-empty set $\mathcal{L}$ of service rates, let $\lambda_{\mathcal{L}}$ be the service rate such that for all h,

$$\lambda_{\mathcal{L}}(h) = \inf\{\mu(h) : \mu \in \mathcal{L}\},$$

and let

$$\lambda^*_{\mathcal{L}} = \sup_h \lambda_{\mathcal{L}}(h).$$

Remark 2.22 *The service rate $\lambda_{\mathcal{L}}$, which is the lower envelope of $\mathcal{L}$, need not belong to $\mathcal{L}$, even if $\mathcal{L}$ is closed. The quantity $\lambda^*_{\mathcal{L}}$ may be regarded as the "horizon" of $\mathcal{L}$, and it can take any non-negative extended-real value.*

Remark 2.23 *For any set $\mathcal{L}$ of service rates, if θ satisfies $0 < \theta < \lambda^*_{\mathcal{L}}$, then by Remark 2.9, for all μ in $\mathcal{L}$,*

$$m(\mu,\theta) \leq m(\lambda_{\mathcal{L}},\theta) < \infty,$$

and hence $m(\lambda_{\mathcal{L}},\theta)$ is an upper bound for the set $\{m(\mu,\theta) : \mu \in \mathcal{L}\}$.

Lemma 2.24 *Let $\mathcal{L}$ and $\hat{\mathcal{L}}$ be closed sets of service rates with $\mathcal{L} \subseteq \hat{\mathcal{L}}$, and suppose that for each h,*

$$\lambda_{\mathcal{L}}(h) = \lambda_{\hat{\mathcal{L}}}(h).$$

Then

$$\inf\{\mu^* : \mu \in \mathcal{L}\} = \inf\{\mu^* : \mu \in \hat{\mathcal{L}}\}. \tag{2.31}$$

Proof. Since for each non-negative integer h,

$$\lambda_{\mathcal{L}}(h) = \lambda_{\hat{\mathcal{L}}}(h) \leq \inf\{\mu^* : \mu \in \hat{\mathcal{L}}\}, \tag{2.32}$$

it follows that when $\lambda^*_{\mathcal{L}} = \infty$, both expressions in (2.31) are infinite. Now assume that $\lambda^*_{\mathcal{L}} < \infty$. For each positive integer M, use the hypothesis to find $\mu_M \in \mathcal{L}$ such that

$$\mu_M(M) < \lambda_{\mathcal{L}}(M) + \frac{1}{M}.$$

So we have a sequence (μ_M) such that if $1 \leq h \leq M$ and $M \geq 1$, then

$$\mu_M(h) \leq \mu_M(M) < \lambda^*_{\mathcal{L}} + \frac{1}{M}. \tag{2.33}$$

Thus for each h, the sequence $(\mu_M(h) : M \in \mathbb{N}^+)$ is bounded, and we can use a standard diagonal argument (Billingsley [Bil95], Appendix A14) to find an increasing sequence $n_1, n_2, \ldots$ of positive integers such that $(\mu_{n_j} : j \in \mathbb{N}^+)$ converges pointwise to some service rate ν. Since $\mathcal{L}$ is closed we must have $\nu \in \mathcal{L}$. Furthermore, from (2.32) and (2.33), we have

$$\nu^* \leq \lambda^*_{\mathcal{L}} \leq \inf\{\mu^* : \mu \in \hat{\mathcal{L}}\}.$$

Thus (2.31) holds. □

Lemma 2.25 *If $\mathcal{L}$ is a closed set of service rates, then*

$$\lambda^*_{\mathcal{L}} = \inf\{\mu^* : \mu \in \mathcal{L}\}.$$

Proof. In Lemma 2.24, set $\hat{\mathcal{L}} = \mathcal{L} \cup \{\lambda_{\mathcal{L}}\}$. Then by the definition of $\lambda_{\mathcal{L}}$ and the conclusion of Lemma 2.24,

$$\inf\{\mu^* : \mu \in \mathcal{L}\} = \inf\{\mu^* : \mu \in \hat{\mathcal{L}}\} = \lambda^*_{\mathcal{L}}.$$

□

Lemma 2.26 *If $(\mathcal{L}_M)$ is a non-increasing sequence of closed sets of service rates, and if $\mathcal{L}_\infty$ denotes $\bigcap_{M=1}^{\infty} \mathcal{L}_M$, then*

$$\lim_{M\to\infty} \lambda^*_{\mathcal{L}_M} = \lambda^*_{\mathcal{L}_\infty}.$$

Proof. Since $\lambda^*_{\mathcal{L}_\infty}$ is an upper bound of the non-decreasing sequence $(\lambda^*_{\mathcal{L}_M})$, we need consider only the case when $\lim_{M\to\infty} \lambda^*_{\mathcal{L}_M}$ is finite, and it is enough to show

$$\lambda^*_{\mathcal{L}_\infty} \leq \lim_{M\to\infty} \lambda^*_{\mathcal{L}_M}.$$

For each M, use Lemma 2.25 to find $\mu_M \in \mathcal{L}_M$ such that

$$\mu^*_M < \lambda^*_{\mathcal{L}_M} + \frac{1}{M}. \tag{2.34}$$

Since $(\mu^*_M : M \in \mathbb{N}^+)$ is bounded, we can get a subsequence (μ_{M_j}) which converges pointwise to a service rate ν. Since for every M, the elements of the subsequence are eventually in $\mathcal{L}_M$, it follows that $\nu \in \mathcal{L}_\infty$. And by Lemma 2.25, (2.16), and (2.34),

$$\lambda^*_{\mathcal{L}_\infty} \leq \nu^* \leq \limsup_{j\to\infty} \mu^*_{M_j} \leq \lim_{M\to\infty} \lambda^*_{\mathcal{L}_M}.$$

□

Let $\mathcal{L}$ be a closed set of service rates and let β be a probability measure on $\mathcal{L}$.

Definition 2.27 *For $\theta \in (0, \infty)$ define*

$$g_\beta(\theta) = \int_{\mathcal{L}} m(\mu, \theta) d\beta(\mu), \tag{2.35}$$

where $m(\mu, \theta)$ is the mean queue length of an $M/M(n)/1/\infty$ queue according to Definition 2.4. (We use the convention that $0 \cdot \infty = 0$ and note that $g_\beta(\theta) = \infty$ is a possibility.) Further define

$$g_\beta(0) = \inf\{g_\beta(\theta) : \theta > 0\}$$

and $g_\beta(\infty) = \infty$ to extend the domain of g_β to $[0, \infty]$.

For any sequence of networks having $\mathcal{L}$ as its set of persistent rates and β as its limiting service rate measure, the function $g_\beta(\cdot)$ is the steady-state mean expected asymptotic queue length. Here the expectation is taken with respect to the local (asymptotic) steady state queue length distribution at the node, while the mean is taken with respect to the limiting service rate measure. The function $g_\beta(\cdot)$ will help us determine the local asymptotics in terms of suitably-selected, state-dependent $M/M(n)/1/\infty$ queues as described in Subsection 2.1.

For the case of state-independent service rates and with the notation of [MY96], the function $g_\beta(\cdot)$ reduces to

$$g_\beta(\theta) = \theta \cdot h(\theta) = \int_0^1 \frac{\theta r}{1-\theta r} dI(r)$$

in (2.1) of [MY96][p. 96], where the integrand can be identified as the mean steady-state queue length corresponding to the constant service rate $1/r$ (see Example 2.5) and $I(\cdot)$ is the limiting relative utilization measure that corresponds to our $\beta(\cdot)$.

Notice that if β gives positive mass to the set of μ having $ULI(\mu) > 0$, then $g_\beta(0)$ will be strictly positive. Let

$$\Phi_\beta = \inf\{\theta : 0 \le \theta \le \infty \text{ and } g_\beta(\theta) = \infty\}.$$

We will see below that the domain of finiteness of the mean expected asymptotic queue length function $g_\beta(\cdot)$ is either the closed interval $[0, \Phi_\beta]$ or the half-open interval $[0, \Phi_\beta)$, depending on whether $g_\beta(\Phi_\beta)$ is finite or not. The following example shows that $g_\beta(\Phi_\beta) < \infty$ is possible, even when all service rates are constant.

Example 2.28 *Note first that if μ_n is the service rate satisfying*

$$\mu_n(h) = \frac{n+1}{n} \quad \textit{if} \quad h \ge 1,$$

then as in Example 2.5,

$$m(\mu_n, \theta) = \begin{cases} \dfrac{n\theta}{(n+1) - n\theta} & \textit{for} \quad 0 < \theta < \dfrac{n+1}{n} \\[2ex] \infty & \textit{for} \quad \theta \ge \dfrac{n+1}{n}. \end{cases} \tag{2.36}$$

Let $\mathcal{L} = \{\mu_1, \mu_2, \dots\}$ and let

$$\beta(\{\mu_n\}) = \frac{1}{2^n}, \qquad n \ge 1.$$

Then

$$g_\beta(\theta) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot m(\mu_n, \theta).$$

Note that $g_\beta(\theta) = \infty$ if $\theta > 1$, since for such θ, $m(\mu_n, \theta) = \infty$ for all sufficiently large n. Also,

$$g_\beta(1) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot n = 2.$$

Hence $\Phi_\beta = 1$ and $g_\beta(\Phi_\beta) = 2$.

Proposition 2.29 *The function g_β defined above with domain $[0, \infty]$ is non-decreasing and left-continuous everywhere, and it is continuous except possibly at Φ_β. Also, g_β is continuous at Φ_β if and only if $g_\beta(\Phi_\beta) = \infty$. Furthermore, g_β is strictly-increasing on $[0, \Phi_\beta]$.*

Proof. We first show that g_β is non-decreasing and left-continuous on $[0, \Phi_\beta]$. If $\Phi_\beta < \infty$, let $\theta \in [0, \Phi_\beta]$, and suppose $\theta_n \nearrow \theta$. It is enough to show that $g_\beta(\theta_n) \nearrow g_\beta(\theta)$. In view of the monotone convergence theorem, it suffices to show that for each μ in $\mathcal{L}$,

$$m(\mu, \theta_n) \nearrow m(\mu, \theta). \tag{2.37}$$

If $\theta \leq \mu^*$, then (2.37) is true by Remark 2.9. Meanwhile, if $\theta > \mu^*$, then from Definition 2.4 $m(\mu, \theta) = \infty$ and also, for sufficiently large n, $\theta_n > \mu^*$. Hence for such n, we have $m(\mu, \theta_n) = \infty$, and again (2.37) must hold.

If $\Phi_\beta = \infty$, the above argument shows that g_β is non-decreasing and left-continuous on $[0, \Phi_\beta)$. Moreover, if $\theta_n \nearrow \infty = \Phi_\beta$, then by Remark 2.9, we must have $m(\mu, \theta_n) \nearrow \infty$ for all μ. Hence $g_\beta(\theta_n) \nearrow g_\beta(\Phi_\beta) = \infty$.

Next we show that g_β is strictly-increasing on $[0, \Phi_\beta)$. If $\theta < \Phi_\beta$, then $g_\beta(\theta) < \infty$ and therefore $\beta\{\mu \in \overline{\mathcal{L}} : m(\mu, \theta) = \infty\} = 0$. So for θ_1 and θ_2 satisfying $0 < \theta_1 < \theta_2 < \Phi_\beta$ and for $\beta-$almost all μ, we have $m(\mu, \theta_1) < m(\mu, \theta_2)$ and hence $g_\beta(\theta_1) < g_\beta(\theta_2)$. On $[0, \Phi_\beta)$, g_β is right-continuous as well, again by monotone convergence.

Since $g_\beta(w) = \infty$ for all $w \in (\Phi_\beta, \infty]$, the rest of the assertions readily follow. □

Lemma 2.30 *(i) The parameters $\lambda^*_{\mathcal{L}}$ and Φ_β satisfy*

$$\lambda^*_{\mathcal{L}} \leq \Phi_\beta.$$

(ii) The value $g_\beta^{-1}(w)$ is well-defined and finite for each w in $[g_\beta(0), g_\beta(\Phi_\beta))$, and it is also well-defined and finite for $w = g_\beta(\Phi_\beta)$ if $g_\beta(\Phi_\beta) < \infty$.

Proof. (i) For any θ satisfying $0 < \theta < \lambda^*_{\mathcal{L}}$, by Remark 2.23,

$$g_\beta(\theta) = \int_{\mathcal{L}} m(\mu, \theta) d\beta(\mu) \leq m(\lambda_{\mathcal{L}}, \theta) < \infty.$$

Hence by definition of Φ_β, we have $\lambda^*_{\mathcal{L}} \leq \Phi_\beta$.
Assertion (ii) follows because g_β is strictly-increasing and continuous on $\{\theta : g_\beta(\theta) < \infty\}$. □

Remark 2.31 *Even when $\mathcal{L}$ is finite, $\lambda^*_{\mathcal{L}}$ could be less than Φ_β. To see this, let $\mathcal{L} = \{\mu_1, \mu_2\}$ with $\mu_1^* < \mu_2^*$ and let β be the point mass at $\{\mu_2\}$. Then $\lambda^*_{\mathcal{L}} = \mu_1^*$ and $\Phi_\beta = \mu_2^*$.*

3 The limiting throughput

Let $\mathcal{L}$ be a closed set of service rates, α be a non-negative extended real number, and β be a probability measure on $\mathcal{L}$. Our aim is to find conditions under which, for every sequence $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ of cyclic networks having asymptotic profile $(\mathcal{L}, \alpha, \beta)$, the sequence $(T(N) = T_{k(N),n(N)} : N \in \mathbb{N}^+)$ of throughputs (as defined in (2.26)) has a limit. In the presence of these conditions, we plan to characterize the limiting throughput, and in the absence of these conditions, we plan to construct sequences of networks for which the limiting throughput does not exist.

3.1 Bounds for limit points of throughputs

In what follows, we let

$$\overline{\theta} = \limsup_{N\to\infty} T(N) \quad \text{and} \quad \underline{\theta} = \liminf_{N\to\infty} T(N).$$

We now obtain upper and lower bounds for $\overline{\theta}$ and $\underline{\theta}$, respectively.

Lemma 3.1 *For each sequence* $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ *of cyclic networks,*

$$\overline{\theta} \leq \limsup_{N\to\infty} \lambda^*_{\mathcal{L}(N)}.$$

Proof. It readily follows from Definitions 2.15 and 2.17 that

$$T(N) \leq \inf\{\mu^* : \mu \in \mathcal{L}(N)\} = \lambda^*_{\mathcal{L}(N)}$$

for each $N \in \mathbb{N}^+$. So the assertion holds. $\square$

Lemma 3.2 *For each sequence* $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ *of cyclic networks having asymptotic profile* $(\mathcal{L}, \alpha, \beta)$*, either*

$$\overline{\theta} = 0 \quad \text{or} \quad g_\beta(\overline{\theta}) \leq \alpha.$$

Proof. If $\alpha = \infty$ there is nothing to prove. Now assume that $\alpha < \infty$ and that θ satisfies $0 < \theta < \overline{\theta}$. We shall prove that $g_\beta(\theta) \leq \alpha$. The result will then follow from the left continuity of g_β (Proposition 2.29).

Let u be a positive integer. Select a subsequence

$$N_1 < N_2 < \cdots$$

such that

$$T(N_i) > \theta \qquad \text{for all } i \geq 1. \tag{3.1}$$

We may without loss of generality take $n(N_i) \geq 2$ for all i. We consider for each fixed N_i a network $\mathcal{C}^+(N_i)$ which is obtained from $\mathcal{C}(N_i)$ by inserting into the same sequence of nodes additionally u further customers, so the population size of $\mathcal{C}^+(N_i)$ is $k(N_i) + u$.

For making notation simple, consider node 1 in $\mathcal{C}^+(N_i)$. Then the likelihood ratio for the queue length X_1^+ at node 1 under the equilibrium distribution P^+ satisfies (for all $y = 0, 1, \ldots, u-1$)

$$\begin{aligned}
\frac{P^+(X_1^+ = y+1)}{P^+(X_1^+ = y)} &= \frac{1}{\mu_1(y+1)} \frac{\left\{\sum_{x_2+\cdots+x_{n(N_i)}=k(N_i)+u-y-1} \prod_{\ell=2}^{n(N_i)} \left(\prod_{h=1}^{x_\ell} \frac{1}{\mu_\ell(h)}\right)\right\}}{\left\{\sum_{x_2+\cdots+x_{n(N_i)}=k(N_i)+u-y} \prod_{\ell=2}^{n(N_i)} \left(\prod_{h=1}^{x_\ell} \frac{1}{\mu_\ell(h)}\right)\right\}} \qquad (3.2)\\
&\overset{(1)}{\geq} \frac{1}{\mu_1(y+1)} \frac{\left\{\sum_{x_2+\cdots+x_{n(N_i)}=k(N_i)-1} \prod_{\ell=2}^{n(N_i)} \left(\prod_{h=1}^{x_\ell} \frac{1}{\mu_\ell(h)}\right)\right\}}{\left\{\sum_{x_2+\cdots+x_{n(N_i)}=k(N_i)} \prod_{\ell=2}^{n(N_i)} \left(\prod_{h=1}^{x_\ell} \frac{1}{\mu_\ell(h)}\right)\right\}}\\
&\overset{(2)}{=} \frac{1}{\mu_1(y+1)} \cdot T(N_i)\\
&\overset{(3)}{\geq} \frac{1}{\mu_1(y+1)} \cdot \theta. \qquad (3.3)
\end{aligned}$$

Here $\overset{(1)}{\geq}$ follows from interpreting according to Lemma 2.16 and (2.26) the second ratio in (3.2) as a throughput $T^+(k(N_i)+u-y, n(N_i)-1)$ in a network $\mathcal{C}^+(N_i)$ which consists of nodes $2, 3, \ldots, n(N_i)$ with $k(N_i) + u - y$ customers cycling and then applying Lemma 2.18. The equality $\overset{(2)}{=}$ is obtained from Lemma 2.19 and (2.27), and $\overset{(3)}{\geq}$ follows from (3.1). As is seen from (2.4), the last expression in (3.3) is the likelihood ratio

$$\frac{\pi^u_{\mu_1,\theta}(y+1)}{\pi^u_{\mu_1,\theta}(y)}, \qquad y = 0, 1, \ldots, u-1$$

where π^u is the equilibrium distribution with limited waiting room of size $u-1$. The same arguments hold for all other nodes j ($1 \le j \le n(N_i)$). It then follows from Lemma 2.8 that for all j,

$$E^+[X_j^+] \ge m^u(\mu_j, \theta). \tag{3.4}$$

Since the number of customers in $\mathcal{C}^+(N_i)$ equals $k(N_i)+u$, we obtain, by summing the inequalities in (3.4) and by dividing by $n(N_i)$,

$$\frac{k(N_i)+u}{n(N_i)} \ge \int_{\mathcal{L}(N_i)} m^u(\mu, \theta) d\beta^+(N_i)(\mu). \tag{3.5}$$

For fixed u, as $i \to \infty$, the left side converges to α. For the convergence of the right side, note that the expectation $m^u(\mu, \theta)$ is a finite sum depending only on the values $\mu(h)$ ($h = 1, 2, \ldots, u$) and hence it is a continuous function of μ. Moreover, for all μ, $m^u(\mu, \theta)$ is bounded by u. Since $\beta^+(N_i) = \beta(N_i)$, the right side of (3.5) converges as well because $\beta(N_i) \xrightarrow{w} \beta$. Therefore we have

$$\alpha \ge \int_{\mathcal{L}} m^u(\mu, \theta) d\beta(\mu). \tag{3.6}$$

By (2.6) and monotone convergence, we conclude

$$\alpha \ge \int_{\mathcal{L}} m(\mu, \theta) d\beta(\mu) = g_\beta(\theta).$$

□

Corollary 3.3 *If $\alpha < \infty$ then*

$$\overline{\theta} \le \Phi_\beta. \tag{3.7}$$

Proof. The assertion is trivially true if $\overline{\theta} = 0$. Otherwise, by Lemma 3.2, $g_\beta(\overline{\theta}) \le \alpha$; hence by definition of Φ_β, we have $\overline{\theta} \le \Phi_\beta$.

□

Corollary 3.4 *If $\alpha < \infty$ and $\alpha \le g_\beta(0)$, then $\overline{\theta} = 0$.*

Proof. If $\overline{\theta}$ were positive, then since $\overline{\theta} \le \Phi_\beta$ (Corollary 3.3) and since g_β is strictly-increasing on $[0, \Phi_\beta]$ (Proposition 2.29), we must have

$$g_\beta(\overline{\theta}) > g_\beta(0) \ge \alpha.$$

This contradicts Lemma 3.2. □

Corollary 3.5 *If $\alpha < \infty$ and $g_\beta(0) \le \alpha$, then $g_\beta(\overline{\theta}) \le \alpha$.* □

Proof. When $\overline{\theta} = 0$, the assertion is part of the hypothesis; and when $\overline{\theta} > 0$, Lemma 3.2 gives the result.

Lemma 3.6 *For each sequence $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ of cyclic networks having asymptotic profile $(\mathcal{L}, \alpha, \beta)$, if $\underline{\theta} < \lambda^*_{\mathcal{L}}$, then*

$$\alpha \le g_\beta(\underline{\theta}).$$

Proof. We take θ satisfying $\underline{\theta} < \theta < \lambda^*_{\mathcal{L}}$ and will show that $\alpha \leq g_\beta(\theta)$. Obtain a subsequence of networks $(\mathcal{C}(N_i) : i \in \mathbb{N}^+)$ such that for all i we have $T(N_i) < \theta$.

We denote by $\mathcal{C}^+(N_i)$ a network obtained from $\mathcal{C}(N_i)$ by the following modification: The number of cycling customers, $k(N_i)$, is the same, and one node numbered $n(N_i)+1$, with service rate $\mu_{n(N_i)+1} = \lambda_{\mathcal{L}(N_i)}$ (see Remark 2.22), is added between nodes $n(N_i)$ and 1. Denote by $P^+ = P^+(N_i)$ the equilibrium distribution of the network $\mathcal{C}^+(N_i)$ and by X_j^+ the queue length at node j in $\mathcal{C}^+(N_i)$. To simplify notation we specify $j = 1$ in the next calculation. Then the likelihood ratio for X_1^+ satisfies (for $0 \leq y \leq k(N_i) - 1$)

$$\frac{P^+(X_1^+ = y+1)}{P^+(X_1^+ = y)} = \frac{1}{\mu_1(y+1)} \frac{\left\{\sum_{x_2+\cdots+x_{n(N_i)+1}=k(N_i)-y-1} \prod_{\ell=2}^{n(N_i)+1}\left(\prod_{h=1}^{x_\ell} \frac{1}{\mu_\ell(h)}\right)\right\}}{\left\{\sum_{x_2+\cdots+x_{n(N_i)+1}=k(N_i)-y} \prod_{\ell=2}^{n(N_i)+1}\left(\prod_{h=1}^{x_\ell} \frac{1}{\mu_\ell(h)}\right)\right\}} \tag{3.8}$$

$$\overset{(1)}{\leq} \frac{1}{\mu_1(y+1)} \frac{\left\{\sum_{x_2+\cdots+x_{n(N_i)+1}=k(N_i)-1} \prod_{\ell=2}^{n(N_i)+1}\left(\prod_{h=1}^{x_\ell} \frac{1}{\mu_\ell(h)}\right)\right\}}{\left\{\sum_{x_2+\cdots+x_{n(N_i)+1}=k(N_i)} \prod_{\ell=2}^{n(N_i)+1}\left(\prod_{h=1}^{x_\ell} \frac{1}{\mu_\ell(h)}\right)\right\}} \tag{3.9}$$

$$\overset{(2)}{\leq} \frac{1}{\mu_1(y+1)} \cdot T(N_i)$$

$$\overset{(3)}{\leq} \frac{1}{\mu_1(y+1)} \cdot \theta. \tag{3.10}$$

Here $\overset{(1)}{\leq}$ follows from interpreting the second ratio in (3.8) according to Lemma 2.16 and (2.26) as a throughput $T^\pm(k(N_i)-y, n(N_i))$ in a network $C^\pm(N_i)$ which consists of nodes $2, 3, \ldots, n(N_i), n(N_i)+1$, with $k(N_i) - y$ customers cycling and then applying Lemma 2.18.

Also, $\overset{(2)}{\leq}$ is obtained as follows: The second ratio in (3.9) is the throughput $T^\pm(k(N_i), n(N_i))$ in a network $C^\pm(N_i)$ which consists of nodes $2, 3, \ldots, n(N_i), n(N_i)+1$ with $k(N_i)$ customers cycling. We show

$$T^\pm(k(N_i), n(N_i)) \leq T(N_i) :$$

The network $C^\pm(N_i)$ with $T^\pm(k(N_i), n(N_i))$ is obtained from $\mathcal{C}(N_i)$ by substituting node 1 with service rate μ_1 by a node with service rate $\lambda_{\mathcal{L}(N_i)}$, which is the lower envelope of $\mathcal{L}(N_i)$. This is equivalent to decreasing the service rate at node 1 by an amount $\mu_1(h) - \lambda_{\mathcal{L}(N_i)}(h)$ for each $h \in \mathbb{N}$. From Theorem 14.B.13 in [SS94][p. 442/443] it follows that the number of customers present at nodes other than 1 decreases in the sense of likelihood ratio ordering, when transforming $\mathcal{C}(N_i)$ to $C^\pm(N_i)$, and consequently the throughput of the network decreases.

Further, $\overset{(3)}{\leq}$ follows from the assumption on θ.

Therefore, from Proposition 2.3, we have for all y $(0 \leq y < k(N_i))$,

$$\frac{P^+(X_1^+ = y+1)}{P^+(X_1^+ = y)} \leq \frac{\pi_{\mu_1,\theta}(y+1)}{\pi_{\mu_1,\theta}(y)}.$$

The same arguments hold for all other nodes j $(2 \leq j \leq n(N_i))$. We conclude from Lemma 2.8 that for $j = 1, \ldots, n(N_i)$,

$$E^+[X_j^+] \leq m(\mu_j, \theta). \tag{3.11}$$

Let $\beta^+(N_i)$ be the service-rate measure for $\mathcal{C}^+(N_i)$. Summing the inequalities in (3.11) over all j and dividing by $n(N_i)+1$, we obtain

$$\frac{k(N_i)}{n(N_i)+1} \leq \int_{\mathcal{L}(N_i)\cup\{\lambda_{\mathcal{L}(N_i)}\}} m(\mu, \theta) d\beta^+(N_i)(\mu). \tag{3.12}$$

The left side of (3.12) converges as $i \to \infty$ to α, and we have to show that the right side converges as well. By direct verification we see that as $i \to \infty$,

$$\beta^{+}(N_i) \xrightarrow{w} \beta. \tag{3.13}$$

For each positive integer M, let

$$\mathcal{L}^M = \overline{\bigcup_{N=M}^{\infty} \mathcal{L}(N)}.$$

Thus $(\mathcal{L}^M)$ is a decreasing sequence of closed sets whose intersection is $\mathcal{L}$. Because of Lemma 2.26, we have $\lim_{M\to\infty} \lambda^*_{\mathcal{L}^M} = \lambda^*_{\mathcal{L}}$. Since θ was chosen to satisfy $\theta < \lambda^*_{\mathcal{L}}$, we can select a positive integer M_0 such that $\theta < \lambda^*_{\mathcal{L}^{M_0}}$. Let

$$\hat{\mathcal{L}}^{M_0} = \overline{\bigcup_{N=M_0}^{\infty} \hat{\mathcal{L}}(N)},$$

where

$$\hat{\mathcal{L}}(N) = \begin{cases} \mathcal{L}(N) & \text{if} \quad (\forall i)\, N \neq N_i \\ \mathcal{L}(N_i) \cup \{\lambda_{\mathcal{L}(N_i)}\} & \text{if} \quad N = N_i. \end{cases}$$

Now for each h,

$$\inf\{\mu(h) : \mu \in \bigcup_{N=M_0}^{\infty} \mathcal{L}(N)\} = \inf\{\mu(h) : \mu \in \bigcup_{N=M_0}^{\infty} \hat{\mathcal{L}}(N)\}$$

and hence for each h,

$$\lambda_{\mathcal{L}^{M_0}}(h) = \lambda_{\hat{\mathcal{L}}^{M_0}}(h).$$

So by Lemma 2.24 and Lemma 2.25, we have

$$\lambda^*_{\mathcal{L}^{M_0}} = \lambda^*_{\hat{\mathcal{L}}^{M_0}}.$$

Now Lemma 2.30(i) guarantees that since θ satisfies $\underline{\theta} < \theta < \lambda_{\hat{\mathcal{L}}^{M_0}}$, the set $\{m(\mu,\theta) : \mu \in \hat{\mathcal{L}}^{M_0}\}$ is bounded. Thus weak convergence (3.13) gives

$$\lim_{i\to\infty} \int_{\hat{\mathcal{L}}^{M_0}} m(\mu,\theta)d\beta^{+}(N_i)(\mu) = \int_{\hat{\mathcal{L}}^{M_0}} m(\mu,\theta)d\beta(\mu) = g_\beta(\theta) < \infty. \tag{3.14}$$

When we put (3.14) together with (3.12), we obtain $\alpha \leq g_\beta(\theta)$. Finally, the right continuity of g_β at $\underline{\theta}$ (Proposition 2.29) yields $\alpha \leq g_\beta(\underline{\theta})$. $\square$

Corollary 3.7 *If $\alpha < \infty$ and $g_\beta(0) \leq \alpha \leq g_\beta(\lambda^*_{\mathcal{L}})$, then $\underline{\theta} \geq g_\beta^{-1}(\alpha)$.*

Proof. In the case when $\underline{\theta} \geq \lambda^*_{\mathcal{L}}$, the hypothesis implies by Proposition 2.29 and Lemma 2.30 that $\alpha \leq g_\beta(\lambda^*_{\mathcal{L}}) \leq g_\beta(\underline{\theta})$, and hence $\underline{\theta} \geq g_\beta^{-1}(\alpha)$. In the case when $\underline{\theta} < \lambda^*_{\mathcal{L}}$, Lemma 3.6 implies that $\alpha \leq g_\beta(\underline{\theta})$, and once again $\underline{\theta} \geq g_\beta^{-1}(\alpha)$. $\square$

Corollary 3.8 *If $\alpha \geq g_\beta(\lambda^*_{\mathcal{L}})$, then $\underline{\theta} \geq \lambda^*_{\mathcal{L}}$.*

Proof. If it were true that $\underline{\theta} < \lambda^*_{\mathcal{L}}$, then by Proposition 2.29,

$$g_\beta(\underline{\theta}) < g_\beta(\lambda^*_{\mathcal{L}}) \leq \alpha,$$

which contradicts Lemma 3.6. □

Proposition 3.9 *Let ℓ and υ be functions on $[0,\infty]$ defined by*

$$\ell(\alpha) = \begin{cases} 0 & \text{for} \quad 0 \leq \alpha < g_\beta(0) \\ g_\beta^{-1}(\alpha) & \text{for} \quad g_\beta(0) \leq \alpha < g_\beta(\lambda^*_{\mathcal{L}}) \\ \lambda^*_{\mathcal{L}} & \text{for} \quad g_\beta(\lambda^*_{\mathcal{L}}) \leq \alpha \leq \infty. \end{cases}$$

and

$$\upsilon(\alpha) = \begin{cases} 0 & \text{for} \quad 0 \leq \alpha < g_\beta(0) \\ g_\beta^{-1}(\alpha) & \text{for} \quad g_\beta(0) \leq \alpha < g_\beta(\Phi_\beta) \\ \Phi_\beta & \text{for} \quad g_\beta(\Phi_\beta) \leq \alpha < \infty \\ \infty & \text{for} \quad \alpha = \infty. \end{cases}$$

Then for all α,

$$\ell(\alpha) \leq \underline{\theta} \leq \overline{\theta} \leq \upsilon(\alpha). \tag{3.15}$$

Proof. Follows immediately from corollaries 3.3, 3.4, 3.5, 3.7, and 3.8. □

Remark 3.10 *Thus $\ell(\alpha)$ and $\upsilon(\alpha)$ provide lower and upper bounds, respectively, for the limit points of $(T(N) : N \in \mathbb{N}^+)$ for any sequence of cyclic networks with asymptotic profile $(\mathcal{L}, \alpha, \beta)$. It is worthwhile noting that $\ell(\alpha) \leq \lambda^*_{\mathcal{L}}$ and that if $\ell(\alpha) < \upsilon(\alpha)$, then $\alpha \geq g_\beta(\lambda^*_{\mathcal{L}})$ and $\ell(\alpha) = \lambda^*_{\mathcal{L}}$.*

3.2 Existence of the Limit

We are now prepared to formulate sufficient conditions on the asymptotic profile for the limiting throughput to exist. Hypothesis (ii) in the following theorem imposes the condition that the sequence is "eventually slow" in a probabilistic sense and hence that not all of the network's nodes can carry adequately the customer flow (measured by α) which is offered to the network.

Theorem 3.11 *Let $\mathcal{L}$ be a closed set of service rates, let β be a probability measure on $\mathcal{L}$, and suppose $\alpha \in [0,\infty]$ satisfies either*

(i) $0 \leq \alpha \leq g_\beta(\lambda^*_{\mathcal{L}})$ *and* $\alpha < \infty$
or

(ii) $g_\beta(\lambda^*_{\mathcal{L}}) < \alpha < \infty$ *and* $\lambda^*_{\mathcal{L}} = \Phi_\beta$
or

(iii) $\alpha = \infty$ *and* $\lambda^*_{\mathcal{L}} = \infty$.

Then for every sequence $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ of cyclic networks having asymptotic profile $(\mathcal{L}, \alpha, \beta)$, the limiting throughput exists, and

$$\lim_{N\to\infty} T(N) = \begin{cases} 0 & \text{when} \quad 0 \le \alpha < g_\beta(0) \\ g_\beta^{-1}(\alpha) & \text{when} \quad g_\beta(0) \le \alpha \le g_\beta(\lambda^*_{\mathcal{L}}) \text{ and } \alpha < \infty \\ \lambda^*_{\mathcal{L}} & \text{when} \quad g_\beta(\lambda^*_{\mathcal{L}}) \le \alpha \le \infty. \end{cases}$$

Remark 3.12 *The conclusion of Theorem 3.11 suggests that $g_\beta(\lambda^*_{\mathcal{L}})$ is the "critical customer density". Indeed, if $\alpha < g_\beta(\lambda^*_{\mathcal{L}})$, then α determines the throughput of the network as $g_\beta^{-1}(\alpha)$. On the other hand, increasing the customer density beyond $g_\beta(\lambda^*_{\mathcal{L}})$ will not further increase the throughput. (Thus in the special setting considered by Malyshev and Yakovlev [MY96], their critical density λ_{cr} is the same as the quantity $g_\beta(\lambda^*_{\mathcal{L}})$.)*

The following example demonstrates that even when service rates are constant, but the sequence of slowest service rates does not converge, the limiting throughput may exist.

Example 3.13 *If N is odd, suppose $C(N)$ has $N-1$ nodes of type μ^1 and one node of type $\mu^{1/2}$. If N is even, suppose $C(N)$ has $N-1$ nodes of type μ^1 and one node of type μ^2. (We use the notation of Example 2.5.) Then we have $\lambda^*_{\mathcal{L}} = 1/2$, and the limiting service rate measure β gives mass 1 to μ^1. Therefore if $0 < \theta < 1$, then*

$$g_\beta(\theta) = m(\mu^1, \theta) = \frac{\theta}{1-\theta}$$

and $\Phi_\beta = 1$. In case $\alpha < g_\beta(1/2)$, we can use Theorem 3.11 to conclude that the limiting throughput exists and equals $g_\beta^{-1}(\alpha) = \alpha/(1+\alpha)$.

Remark 3.14 *It is straightforward to check, from the definitions of ℓ and υ, that the hypotheses of Theorem 3.11 are equivalent to*

$$\ell(\alpha) = \upsilon(\alpha).$$

Proof of Theorem 3.11 The assertion follows immediately from inequality (3.15) and the remark above. □

Remark 3.15 *If $\mathcal{L}$ is finite and if $\beta(\{\mu\}) > 0$ for all μ in $\mathcal{L}$, then $g_\beta(\lambda^*_{\mathcal{L}}) = \infty$. Further, we have $g_\beta(0) = 0$. Hence, by Theorem 3.11, we must have*

$$\lim_{N\to\infty} T(N) = \begin{cases} g_\beta^{-1}(\alpha) & \text{if} \quad \alpha < \infty \\ \lambda^*_{\mathcal{L}} & \text{if} \quad \alpha = \infty. \end{cases}$$

The next theorem establishes that as long as every subsequence of networks has slow, persistent service rates, the limiting throughput must exist. We prove it only for the case not covered by Theorem 3.11.

Theorem 3.16 *Let $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ be any sequence of cyclic networks having asymptotic profile $(\mathcal{L}, \alpha, \beta)$ and satisfying*

$$\alpha \geq g_\beta(\lambda^*_{\mathcal{L}})$$

and

$$\lim_{N\to\infty} \lambda^*_{\mathcal{L}(N)} = \lambda^*_{\mathcal{L}}. \tag{3.16}$$

Then the limiting throughput exists, and

$$\lim_{N\to\infty} T(N) = \lambda^*_{\mathcal{L}}.$$

Proof. By Lemma 3.1 and the hypothesis, we have $\overline{\theta} \leq \lambda^*_{\mathcal{L}}$. By Corollary 3.8, we have $\underline{\theta} \geq \lambda^*_{\mathcal{L}}$. This completes the proof. □

3.3 Non-existence of the limit

In the light of Section 3.5 to follow, we see that applying Theorem 3.11 and Theorem 3.16 to networks with state-independent service rates in the setting of Malyshev and Yakovlev [MY96] always yields convergence of throughput. Indeed, we can easily verify that condition (3.16) is satisfied. Malyshev and Yakovlev had conjectured the possibility that their results on queue-length distributions could be extended to systems with state-dependent service rates as well. However, for systems that satisfy neither the hypotheses of Theorem 3.11 nor those of Theorem 3.16, the situation seems more delicate, even if all service rates are constant.

It follows from Theorem 3.11 and Remark 3.14 that $\ell(\alpha) = \upsilon(\alpha)$ is enough to guarantee that, for every sequence of cyclic networks having asymptotic profile $(\mathcal{L}, \alpha, \beta)$, the limiting throughput exists. When $\ell(\alpha) \neq \upsilon(\alpha)$, because of Remark 3.10, we have $\alpha \geq g_\beta(\lambda^*_{\mathcal{L}})$ and $\ell(\alpha) = \lambda^*_{\mathcal{L}} < \upsilon(\alpha)$. Under these conditions, we intend to construct a sequence of networks with asymptotic profile $(\mathcal{L}, \alpha, \beta)$ for which limiting throughput does not exist.

Theorem 3.17 *Let $\mathcal{L}$ be a closed set of service rates and β be a probability measure on $\mathcal{L}$. Suppose $\alpha \in (0, \infty]$ satisfies $\alpha \geq g_\beta(\lambda^*_{\mathcal{L}})$ and $\lambda^*_{\mathcal{L}} < \upsilon(\alpha)$, where υ is defined in Proposition 3.9. We can then find a sequence of cyclic networks $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ having asymptotic profile $(\mathcal{L}, \alpha, \beta)$ and satisfying*

$$\underline{\theta} = \lambda^*_{\mathcal{L}} \quad \text{and} \quad \overline{\theta} = \upsilon(\alpha).$$

Remark 3.18 *In view of* (3.15)*, it is not possible to construct an example with a larger $\overline{\theta}$ or smaller $\underline{\theta}$.*

The proof of Theorem 3.17 relies on the following lemma. Given a closed set $\mathcal{L}$ and a probability measure β on $\mathcal{L}$, fix a countable set $\mathcal{D}$ of service rates satisfying $\overline{\mathcal{D}} = \mathcal{L}$. For any finite set $\mathcal{L}'$ of service rates and a network $\mathcal{C}'$ with service rates exactly those in $\mathcal{L}'$, we say $\mathcal{C}'$ has support $\mathcal{L}'$.

Lemma 3.19 *Suppose $\alpha \in (0, \infty]$ and θ is a positive real number such that $\theta < \upsilon(\alpha)$. Let $\mathcal{L}''$ be any finite subset of $\mathcal{D}$. Given any $\epsilon > 0$, we can find a cyclic network $\mathcal{C}'$ whose support $\mathcal{L}'$ is a finite set of positive service rates and such that*

(1) it has at least $1/\epsilon$ nodes with service rate μ for each $\mu \in \mathcal{L}'$;

(2) its service-rate measure β' satisfies $\rho(\beta', \beta) < \epsilon$, where ρ is the Prohorov metric (see Billingsley([Bil68], page 237));

(3) for each $\mu' \in \mathcal{L}'$, there exists $\mu \in \mathcal{D}$ such that $\rho(\delta_\mu, \delta_{\mu'}) < \epsilon$;

(4) for each $\mu \in \mathcal{L}''$, there exists $\mu' \in \mathcal{L}'$ such that $\rho(\delta_\mu, \delta_{\mu'}) < \epsilon$;

(5) its customer-to-node ratio α' satisfies

$$|\alpha' - \alpha| < \epsilon \quad \textit{if} \quad \alpha < \infty, \ \textit{and}$$

$$\alpha' > 1/\epsilon \quad \textit{if} \quad \alpha = \infty;$$

and

(6) its throughput T' satisfies $T' > \theta$.

Proof. Consider first the case when α is finite. The definition of υ and the hypothesis on θ imply that $g_\beta(\theta) < \alpha$. In particular,

$$g_\beta(\theta) = \int m(\mu, \theta) d\beta(\mu) < \infty.$$

Therefore, if

$$S_M = \{\mu : \mu \in \mathcal{L} \ \text{and} \ m(\mu, \theta) \leq M\},$$

we have $\lim_{M \to \infty} \beta(S_M) = 1$. Thus for all sufficiently large M,

$$\frac{1}{\beta(S_M)} \int_{S_M} m(\mu, \theta) d\beta(\mu) < \alpha. \tag{3.17}$$

For M such that $\beta(S_M) > 0$, let β^M be a probability on $\mathcal{L}$ defined by

$$\beta^M(A) = \frac{\beta(A \cap S_M)}{\beta(S_M)}$$

for measurable subsets A of $\mathcal{L}$. Since (β^M) converges to β weakly, we can find M such that relation (3.17) is satisfied and

$$\rho(\beta, \beta^M) < \frac{\epsilon}{4}. \tag{3.18}$$

Notice that the left side of the inequality (3.17) may be viewed as the integral, with respect to β^M restricted to S_M, of the bounded continuous function $m(\mu, \theta)$ on S_M. Since S_M is a separable and closed subspace of $\mathcal{L}$ (see Proposition 2.10), we can find a probability measure β_1, whose support is a finite set $\mathcal{L}_1$ such that $\mathcal{L}_1 \subseteq S_M$, and such that

$$\int m(\mu, \theta) d\beta_1(\mu) < \alpha \tag{3.19}$$

and

$$\rho(\beta^M, \beta_1) < \frac{\epsilon}{4}. \tag{3.20}$$

Since β_1 has support a finite subset of $\mathcal{L}$, we can find, using Proposition 2.10, a probability measure β_2 with rational values and with support a finite subset $\mathcal{L}_2$ of $\mathcal{D}$, such that

$$\int m(\mu,\theta)d\beta_2(\mu) < \alpha \tag{3.21}$$

and

$$\rho(\beta_1,\beta_2) < \frac{\epsilon}{4}. \tag{3.22}$$

For any service rate ν in $\mathcal{L}''$ satisfying $m(\nu,\theta)=\infty$, if ν_n denotes a positive service rate satisfying

$$|\nu_n(h)-\nu(h)| < \frac{1}{n} \quad \text{for} \quad h<n$$

and $\nu_n^* = \infty$, then (ν_n) converges to ν pointwise. Equivalently,

$$\rho(\delta_{\nu_n},\delta_\nu) \longrightarrow 0.$$

Also, $m(\nu_n,\theta)<\infty$ for all n. We can therefore find a set $\mathcal{L}'$ of positive service rates for which (3) and (4) are satisfied and also find a probability β with rational values and with support $\mathcal{L}'$ such that

$$\int m(\mu,\theta)d\beta'(\mu) < \infty$$

and

$$\rho(\beta_2,\beta') < \frac{\epsilon}{4}. \tag{3.23}$$

¿From (3.18), (3.20), (3.22), and (3.23), it follows that $\rho(\beta,\beta')<\epsilon$. Consider the sequence of networks $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ where for each N, $\mathcal{C}(N)$ has support $\mathcal{L}'$, so that for each service rate $\mu \in \mathcal{L}'$, the number of nodes in $\mathcal{C}(N)$ with service rate μ equals $N'\cdot N\cdot\beta'(\{\mu\})$, where N' is large enough so that $N'\cdot\beta'(\{\mu\})$ is a positive integer, and where the number of customers in $\mathcal{C}(N)$ is

$$k(N) = \left\lceil \alpha \cdot \sum_{\mu\in\mathcal{L}'} N'\cdot N\cdot \beta'(\{\mu\}) \right\rceil.$$

Then, for each N, the service-rate measure for $\mathcal{C}(N)$ is β', and for the sequence of networks, the asymptotic customer density equals α. Therefore, by Remark 3.15,

$$\lim_{N\to\infty} T(N) = g_{\beta'}^{-1}(\alpha).$$

Since (3.21) can be rewritten as

$$g_{\beta'}(\theta) < \alpha,$$

we have

$$\theta < g_{\beta'}^{-1}(\alpha).$$

Choose N_0 large enough so that $T(N_0)>\theta$, so that $|\alpha(N_0)-\alpha|<\epsilon$, and so that $\mathcal{C}(N_0)$ has at least $1/\epsilon$ nodes with service rate μ, for each $\mu \in \mathcal{L}(N_0)$. Then $\mathcal{C}' = \mathcal{C}(N_0)$ satisfies all of the required properties.

Now examine the case when $\alpha = \infty$. Here, $v(\alpha) = \infty$. As in the previous case, we can find a set $\mathcal{L}'$ of positive service rates, with $\mu^* = \infty$ for each $\mu \in \mathcal{L}'$, and a probability measure β' with support $\mathcal{L}'$ and taking rational values such that (1), (2), (3), and (4) are satisfied.

Consider the sequence of networks $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ where for each N, $\mathcal{C}(N)$ has support $\mathcal{L}'$, for each service rate μ in $\mathcal{L}'$, has $N' \cdot N \cdot \beta'(\{\mu\})$ nodes with service rate μ, where N' is large enough so that $N' \cdot \beta'(\{\mu\})$ is integer-valued for each μ, and where the number of customers in $\mathcal{C}(N)$ is

$$k(N) = N \cdot \sum_{\mu \in \mathcal{L}'} N' \cdot N \cdot \beta'(\{\mu\})$$

Then, for each N, the service-rate measure for $\mathcal{C}(N)$ is β', and for the sequence of networks, the asymptotic customer density equals ∞. By Remark 3.15,

$$\lim_{N \to \infty} T(N) = \infty.$$

We can now choose N_0 large enough so that $\mathcal{C}' = \mathcal{C}(N_0)$ satisfies all the conditions. □

Proof of Theorem 3.17. Towards inductively constructing a sequence of cyclic networks with the desired properties, begin by selecting

(i) a sequence $(\mu_N : N \in \mathbb{N}^+)$ of service rates in $\mathcal{L}$ such that $\mathcal{D} = \{\mu_n : n \in \mathbb{N}^+\}$ is a dense subset of $\mathcal{L}$;

(ii) a sequence $(\sigma_N : N \in \mathbb{N}^+)$ of service rates in $\mathcal{L}$ such that for each N,

$$\sigma_N(h) < \lambda^*_{\mathcal{L}} + \frac{1}{N} \quad \text{for all } h \leq N;$$

and

(iii) an increasing sequence $(\theta_N : N \in \mathbb{N}^+)$ of positive real numbers satisfying

$$\lim_{N \to \infty} \theta_N = v(\alpha). \tag{3.24}$$

Let $\mathcal{C}(1)$ be a network $\mathcal{C}'$ obtained by applying Lemma 3.19 with $\theta = \theta_1$, $\epsilon = 1$, and $\mathcal{L}'' = \{\mu_1\}$. We form $\mathcal{C}(2)$ by adding to $\mathcal{C}(1)$ a node with service rate $\sigma_{k(1)}$ (with no other changes). Now assume we have constructed up through $\mathcal{C}(2N-1)$, $\mathcal{C}(2N)$. To construct $\mathcal{C}(2N+1)$, apply Lemma 3.19 with $\theta = \theta_{N+1}$, $\epsilon = 1/(N+1)$ and $\mathcal{L}'' = \{\mu_n : 1 \leq n \leq N+1\}$. To obtain $\mathcal{C}(2N+2)$, add to $\mathcal{C}(2N+1)$ a node with service rate $\sigma_{k(2N+1)}$. (Make no other changes.) Because of the properties derived from repeated application of Lemma 3.19, every element of $\mathcal{D}$ is persistent, and every persistent rate must be an element of $\overline{\mathcal{D}}$. Furthermore, the asymptotic customer density for the sequence is α and the limiting service-rate measure is β. Thus the constructed sequence $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ has asymptotic profile $(\mathcal{L}, \alpha, \beta)$. Moreover,

$$\limsup_{N \to \infty} T(N) \geq \limsup_{N \to \infty} T(2N-1) \geq \lim_{N \to \infty} \theta_N = v(\alpha)$$

and

$$\liminf_{N \to \infty} T(N) \leq \liminf_{N \to \infty} T(2N) \leq \liminf_{N \to \infty} \left(\lambda^*_{\mathcal{L}} + \frac{1}{k(2N)} \right) = \lambda^*_{\mathcal{L}},$$

with equality holding throughout because of (3.15). □

3.4 Asymptotic queue lengths and limiting throughput

We shall use the characterization theorems of subsection 3.2 to obtain conditions under which the individual queue-length distributions will approach a well-defined limit.

In the following theorem, we shall assume that the sequence $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ of cyclic networks satisfies either the hypotheses of Theorem 3.11 or of Theorem 3.16. Therefore the limiting throughput will exist, and we shall denote it by θ^*.

Theorem 3.20 *Let $(\mathcal{C}(N) : N \in \mathbb{N}^+)$ be a sequence of cyclic networks having asymptotic profile $(\mathcal{L}, \alpha, \beta)$. Suppose that either*

$$\lim_{N\to\infty} \lambda^*_{\mathcal{L}(N)} = \lambda^*_{\mathcal{L}} \tag{3.25}$$

or

$$\ell(\alpha) = \upsilon(\alpha),$$

where $\ell(\alpha)$ and $\upsilon(\alpha)$ are defined in Proposition 3.9. Let $\mu \in \mathcal{L}$, let (N_i) be any strictly-increasing sequence of positive integers and let $(\mu_i : i \geq 1)$ be service rates such that $\mu_i \in \mathcal{L}(N_i)$ for each i and (μ_i) converges pointwise to μ. Let X_i denote the queue length at a node in $C(N_i)$ with service rate μ_i. Then the following hold:

(a) *If the limiting throughput θ^* equals 0, then*

$$\lim_{i\to\infty} P[X_i \leq b] = 1, \tag{3.26}$$

where $b = ULI(\mu)$.

(b) *If $0 < \theta^* < \mu^*$, then the queue length distribution for X_i converges in total variation norm to π_{μ,θ^*} (notation from (2.1)). Also, for each positive integer r,*

$$\lim_{i\to\infty} E[(X_i)^r] = \sum_{x=0}^{\infty} x^r \pi_{\mu,\theta^*}(x). \tag{3.27}$$

(c) *If $0 < \theta^* = \mu^*$, then the queue length distribution diverges in the following sense:*

$$\lim_{i\to\infty} P[X_i \geq M] = 1 \quad \text{for each } M \in \mathbb{N}^+ \tag{3.28}$$

and furthermore,

$$\lim_{i\to\infty} E(X_i) = \infty. \tag{3.29}$$

Remark 3.21 *In conclusion (a), the behavior of the queue-length distribution as $i \to \infty$ can depend on the actual value of α. Also, conclusion (c) is relevant only when $\alpha \geq g_\beta(\lambda^*_{\mathcal{L}})$, since otherwise, μ must satisfy $\theta^* < \mu^*$.*

Example 3.22 *(continuation of Example 3.13) In the case when $\alpha < g_\beta(1/2)$, we have $\ell(\alpha) = \upsilon(\alpha)$, and Theorem 3.20 can be applied with $\mu = \mu^1$ to yield the limiting queue-length distribution. However, in this example, if the service rates are relativized as in the work of Malyshev and Yakovlev [MY96], so that the slowest service rate always equals 1, then Assumption W of [MY96] will not be satisfied because their counterparts to $\beta(N)$ (called I_N there) do not converge. Thus, with such a relativization, the limiting queue-length distribution cannot be conveniently identified.*

Proof of Theorem 3.20. (a) Suppose $\theta^* = 0$. Now for each i,

$$T(N_i) \geq P[X_i \geq b+1] \cdot \mu_i(b+1),$$

and hence

$$\lim_{i\to\infty} P[X_i \geq b+1] \leq \lim_{i\to\infty} \frac{T(N_i)}{\mu_i(b+1)} = 0.$$

This finishes the proof of (a).

Towards proving (b) and (c), first observe that if $\theta^* > 0$ then $\alpha > 0$ by Corollary 3.4. Hence it follows that $k(N_i) \to \infty$ and that for each i and for each y such that $0 \leq y < k(N_i)$, we have (as in (3.2))

$$\frac{P[X_i = y+1]}{P[X_i = y]} = \frac{1}{\mu_i(y+1)} T^{-\mu_i}_{k(N_i)-y}, \tag{3.30}$$

where $T^{-\mu_i}_{k(N_i)-y}$ denotes the throughput of the network obtained from $\mathcal{C}(N_i)$ by deleting one node with service rate μ_i and removing y customers.

For the proof of (b), suppose $0 < \theta^* < \mu^*$ and, for each i, let $C^-(N_i)$ denote the network obtained from $C(N_i)$ by deleting one node with service rate μ_i and removing v customers (where v is a fixed non-negative integer). Then, for the sequence of networks $(C^-(N_i) : i \in \mathbb{N}^+)$, the asymptotic profile $(\mathcal{L}', \alpha', \beta')$ and the parameters $\lambda^*_{\mathcal{L}'}$ and $\Phi'_{\beta'}$ must satisfy $\mathcal{L}' \subseteq \mathcal{L}$, $\alpha' = \alpha$, $\beta' = \beta$, $\lambda^*_{\mathcal{L}'} \geq \lambda^*_{\mathcal{L}}$, and $\Phi'_{\beta'} = \Phi_\beta$. Thus, if the hypothesis $\ell(\alpha) = v(\alpha)$ is satisfied for the sequence $(C(N) : N \in \mathbb{N}^+)$, then the same hypothesis is satisfied for $(C^-(N) : N \in \mathbb{N}^+)$. On the other hand, if $\ell(\alpha) < v(\alpha)$, then $\alpha \geq g_\beta(\lambda^*_{\mathcal{L}})$ and hypothesis (3.25) must be true. Then $\theta^* = \lambda^*_{\mathcal{L}}$ and (3.25) still holds for the modified sequence of networks since $\theta^* < \mu^*$. Thus by Theorems 3.11 and 3.16, in all cases we have

$$\lim_{i\to\infty} T^{-\mu_i}_{k(N_i)-v} = \theta^*. \tag{3.31}$$

Suppose that θ satisfies $\theta^* < \theta < \mu^*$. Then for all sufficiently large i and for $0 \leq y < k(N_i)$,

$$\frac{P[X_i = y+1]}{P[X_i = y]} \leq \frac{\theta}{\mu_i(y+1)}, \tag{3.32}$$

where the last inequality uses Lemma 2.18 and (3.30) and (3.31). Since $\theta^* > 0$, by (3.30) and (3.31) we have

$$\lim_{i\to\infty} \frac{P[X_i = y]}{P[X_i = y+1]} = 0 \quad \text{for} \quad y < b.$$

Hence

$$\lim_{i\to\infty} \frac{P[X_i = y]}{P[X_i = b]} = 0 \quad \text{for} \quad y < b \tag{3.33}$$

and

$$\lim_{i\to\infty} \frac{P[X_i = y+1]}{P[X_i = y]} = \frac{\theta^*}{\mu(y+1)} \quad \text{for} \quad y \geq b.$$

The latter equality then implies that

$$\lim_{i\to\infty} \frac{P[X_i = y]}{P[X_i = b]} = \prod_{h=b+1}^{y} \frac{1}{\mu(h)} \cdot (\theta^*)^{y-b} \quad \text{for} \quad y \geq b+1. \tag{3.34}$$

Next, because of (3.32),

$$\frac{P[X_i = y]}{P[X_i = b]} \leq (\theta)^{y-b} \prod_{h=b+1}^{y} \frac{1}{\mu_i(h)}. \tag{3.35}$$

Then

$$\mu_i(y) \cdot \frac{P[X_i = y]}{P[X_i = b]} \leq \theta \prod_{h=b+1}^{y-1} \frac{\theta}{\mu_i(h)} \quad \text{for} \quad y \geq b+1. \tag{3.36}$$

In view of (3.33), the inequality in (3.36) is valid for all positive y provided i is sufficiently large. Now by definition of $T(N_i)$,

$$\lim_{i\to\infty} \sum_{y=1}^{k(N_i)} \frac{\mu_i(y) \cdot P[X_i = y]}{P[X_i = b]} = \lim_{i\to\infty} \frac{T(N_i)}{P[X_i = b]}. \tag{3.37}$$

Note that the first b terms in the summation in (3.37) converge to 0 because of (3.33) and because $\mu(b) = 0$. According to Lemma 2.11, if θ' satisfies $\theta < \theta' < \mu^*$, then for $y \geq b+1$, the product on the right side of (3.36) is bounded above by

$$L \cdot \left(\frac{\theta}{\theta'}\right)^{y-1-b}, \tag{3.38}$$

where L is a positive real number. Therefore the dominated convergence theorem applies, and the limits in (3.37) must be finite. However, by hypothesis, $\lim_{i\to\infty} T(N_i)$ exists and is finite and positive. Thus $\lim_{i\to\infty} P[X_i = b]$ must also exist and be positive. Then use (3.34) to deduce that $\lim_{i\to\infty} P[X_i = y]$ exists for $y \geq b+1$. However, from (3.33), we have

$$\lim_{i\to\infty} P[X_i = y] = 0 \quad \text{for } y < b.$$

These facts allow us to conclude that the queue-length distribution converges in total variation norm to π_{μ,θ^*} (in the notation of (2.2)). To show convergence of the r^{th} moment $\sum_{y=1}^{\infty} y^r P[X_i = y]$, the reasoning is analogous. Because of (3.35) and Lemma 2.11, we can apply dominated convergence to conclude

$$\lim_{i\to\infty} \sum_{y=1}^{k(N_i)} y^r \frac{P[X_i = y]}{P[X_i = b]} = \sum_{y=1}^{\infty} y^r \frac{\pi_{\mu,\theta^*}(y)}{\pi_{\mu,\theta^*}(b)}.$$

Since

$$\lim_{i\to\infty} P[X_i = b] = \pi_{\mu,\theta^*}(b) > 0,$$

the relation (3.27) follows.

To prove (c), note that if $\alpha < g_\beta(\lambda^*_{\mathcal{L}})$ we must have by Theorem 3.11

$$\theta = g_\beta^{-1}(\alpha) < \lambda^*_{\mathcal{L}} \leq \mu^*.$$

Hence when $0 < \theta^* = \mu^*$, we must have $\alpha \geq g_\beta(\lambda^*_{\mathcal{L}})$. In this case, by Theorems 3.11 and 3.16, we must also have $\theta^* = \lambda^*_{\mathcal{L}}$.

Let u be any positive integer and let θ satisfy $0 < \theta < 1$. Since $\lambda^*_{\mathcal{L}} = \mu^*$, whether or not $\lambda^*_{\mathcal{L}} = \infty$, we may now use Corollary 3.8 to conclude from (3.30) that, for all sufficiently large i, and for all y such that $0 \leq y < u$,

$$\frac{P[X_i = y+1]}{P[X_i = y]} \geq \theta = \frac{\pi^u_{\mu^1,\theta}(y+1)}{\pi^u_{\mu^1,\theta}(y)},$$

where the notations μ^1 and π^u are as in Examples 2.5 and 2.7, respectively. Hence by Lemma 2.8, we have for all y such that $0 \leq y < u$,

$$P[X_i \leq y] \leq \sum_{x=0}^{y} \pi^u_{\mu^1,\theta}(x) = \frac{1-\theta^{y+1}}{1-\theta^{u+1}}.$$

Taking limits, first as $u \to \infty$, and then as θ increases to 1, we can conclude (3.28). Then from (3.28) and Markov's inequality, we obtain for each M,

$$\liminf_{i\to\infty} E(X_i) \geq \liminf_{i\to\infty} (M \cdot P[X_i \geq M]) \geq M.$$

Thus (3.29) holds. $\square$

It is possible to strengthen Lemma 3.19 and adapt the ideas in the proof of Theorem 3.17 to obtain, with some asymptotic profile $(\mathcal{L}, \alpha, \beta)$, a sequence of cyclic networks where, for some rate μ which occurs infinitely often and lies in the support of β, the queue-length distributions at a node with service rate μ fail to converge.

3.5 Extension to Gordon-Newell networks

Our results obtained so far hold for closed networks with general routing (Gordon–Newell networks) as well, where each node is structured like the one described in Section 2.1. The travelling customers all follow the same rules. A customer departing from node i selects with probability $r(i,j)$ node j to be his next destination node and immediately proceeds to that node j. The stochastic routing matrix $R = (r(i,j) : 1 \leq i,j \leq n)$ is assumed to be irreducible, and we denote by $\eta = (\eta(j) : j = 1, \ldots, n)$ the unique solution of the traffic equation

$$\eta = \eta \cdot R \tag{3.39}$$

which satisfies

$$\sum_{j=1}^{n} \eta(j) = n. \tag{3.40}$$

We regard $\eta(j)$ as the (steady-state) visit rate at node j under the routing matrix R. Condition (3.40) stipulates that the average visit rate among the nodes is 1.

The joint queue lengths process $X = (X(t) : t \geq 0)$ for the closed network with positive service rates $\mu_j = (\mu_j(h) : h \in \mathbb{N}), j = 1, \ldots, n$ lives on the state space

$$S(k,n) := \{(x_1, \ldots, x_n) : x_j \in \mathbb{N}, j = 1, \ldots, n, x_1 + \cdots + x_n = k\}. \tag{3.41}$$

The analogue of Theorem 2.13 is

Theorem 3.23 *X is ergodic on $S(k,n)$, and its unique equilibrium distribution $\pi = (\pi(x_1,\dots,x_n) : (x_1,\dots,x_n) \in S(k,n))$ is*

$$\pi(x_1,\dots,x_n) = \prod_{j=1}^{n} \prod_{h=1}^{x_j} \left(\frac{\eta(j)\theta}{\mu_j(h)}\right) \cdot \widetilde{G}_\theta(k,n)^{-1}, \tag{3.42}$$

where $\theta > 0$ is any constant and

$$\widetilde{G}_\theta(k,n) = \sum_{(x_1,\dots,x_n)\in S(k,n)} \prod_{j=1}^{n} \prod_{h=1}^{x_j} \left(\frac{\eta(j)\theta}{\mu_j(h)}\right) \tag{3.43}$$

is the normalizing constant. Note that π does not depend on θ.

With throughput defined as in Definition 2.15 we obtain again an explicit simple representation.

Lemma 3.24 *For $1 \le i \le n$ and for each $\theta > 0$ we have*

$$T_{k,n}(i) = \frac{\sum_{(x_1,\dots,x_n)\in S(k,n)} \prod_{j=1}^{n} \prod_{h=1}^{x_j} \left(\frac{\eta(j)\theta}{\mu_j(h)}\right) \cdot \mu_i(x_i)}{\sum_{(x_1,\dots,x_n)\in S(k,n)} \prod_{j=1}^{n} \prod_{h=1}^{x_j} \left(\frac{\eta(j)\theta}{\mu_j(h)}\right)} = \eta(i) \cdot \frac{\widetilde{G}_1(k-1,n)}{\widetilde{G}_1(k,n)}. \tag{3.44}$$

Definition 3.25 *The network's (average) throughput is*

$$T_{k,n} = \frac{1}{n}\sum_{i=1}^{n} T_{k,n}(i) = \frac{1}{n}\sum_{i=1}^{n} \eta(i) \cdot \frac{\widetilde{G}_1(k-1,n)}{\widetilde{G}_1(k,n)} = \frac{\widetilde{G}_1(k-1,n)}{\widetilde{G}_1(k,n)} \tag{3.45}$$

Consider now a closed cycle as described in Section 2.2 with positive service rates

$$\overline{\mu}_j = (\overline{\mu}_j(h) := \mu_j(h) \cdot \eta(j)^{-1} : h \in \mathbb{N}). \tag{3.46}$$

and compute the associated average throughput $\overline{T}_{k,n}$ of the cycle according to Definition 2.15 and Definition 2.17. It is immediately seen that

$$\overline{T}_{k,n} = \frac{\widetilde{G}_1(k-1,n)}{\widetilde{G}_1(k,n)} = T_{k,n}$$

holds, where $T_{k,n}$ is from Definition 3.25.

Consequently we can reduce the problem of determining the asymptotic behavior of throughput in Gordon-Newell networks to that for closed cyclic networks. This is similar to the procedure sketched in [MY96] for the case of networks with state independent service rates.

We believe that an approach analogous to that in the earlier sections will lead to convergence results for *discrete-time cyclic* networks with state-dependent service rates. (A special case with only finitely-many node types was treated in [DPR03].) However, for discrete-time analogs of Gordon-Newell networks, our approach fails since an analog of (3.42) is no longer valid.

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