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C. Malchin , H. Daduna

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Availability and performance analysis in a discrete time tandem network with product form steady state

Christian Malchin Hans Daduna*
Hamburg University
Department of Mathematics
Bundesstraße 55
20146 Hamburg
Germany

Abstract

For a discrete time tandem queueing network with unreliable nodes we derive the steady state probabilities for the joint queue length vector for all nodes and the availability status of the network. This allows to assess the performance behaviour and the reliability, resp. availability of the network in an integrated model. Because our result exhibits a product form for the steady state distribution it opens the path to fast algorithmic evaluation of the desired performance and reliability indices.[1]

MSC 2000 Subject Classification: Primary 60 K 25; Secondary 60 J 10, 60 K 20

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1 Introduction

Performance analysis of computer and communications systems usually relies on queueing network models. Because it has been in many cases a satisfying assumption that the nodes of the networks are reliable, these investigations yielded good predictions of the networks' behaviour with respect to throughput, overload, system time, loss probabilities due to buffer overflow, etc. There are several monographs on the subject available, see e.g. [5], [1], [9], [7].

On the other hand, reliability considerations especially for computing systems have been a field of intense research over the last forty years. From around the middle of the eighties emerged a field of research under the label of PERFORMABILITY where the problems of performance analysis and reliability are investigated in a unified system description, for a survey see [8].

The need for integration of performance analysis and reliability has become even more important with the fast development of mobile networks, because control of these WLANs poses new problems with respect to reliable and fast access to the networks. From a general point of view the emergence of wireless ad hoc networks which are formed of nodes with wireless communication capability without relying on a pre-existing infrastructure introduces strong problems concerning availability. So in these networks the required Quality-of-Service for speech and data transmission has to be guaranteed on the basis of not necessarily highly reliable network

*Corresponding author: Hamburg University, Department of Mathematics, Bundesstraße 55, 20146 Hamburg, Germany; daduna@math.uni-hamburg.de

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nodes [16]. Furthermore there are many new problems, e.g. with network control and routing protocols in case of link failures in IP networks [11], [12], or with the handling of catastrophic events in mobile cellular networks [10].

Compared to the classical queueing network models there is a severe lack of explicitly solvable model classes in the field of performability. An analogue to the celebrated product form network calculus (Jackson-, Gordon-Newell-, and BCMP-networks) does not exists up to now.
Some first steps towards a calculus similar to the product form network calculus of queueing theory for the joint investigation of performance and availability behaviour in continuous time models are e.g. [13], [14], [15]. Approximation methods which incorporate the reliability behaviour into the parameters of the classical queueing networks (e.g. slowing down the service speed to model breakdowns) are reviewed in [3]. These methods are often applied when explicit solutions for the performance and reliability behaviour are not at hand.

It is well known that queueing theory in discrete time poses more difficulties than the parallel theory in continuous time. This is due to combinatorial complexity which stems from the possibility that simultaneous events (multiple departures and arrivals at the same node or at several nodes) can happen at the same time instant with positive probability. On the other hand there is clearly a need for discrete time queueing models because many real world systems show an inherent discrete time scale: e.g. ALOHA systems, ATM networks. For a survey on discrete time networks see the books [2], [4], [17], [6].
Combining the performance analysis for these discrete time network models with the simultaneous investigation of their possibly unreliable behaviour is still a challenging project. We present in this paper some contributions to this field of integrated performance-availability networks.

Our aim is to describe a class of models where the connection via nodes that are broken down is short circuited. The up-times of the nodes are random and broken down nodes are repaired, the repair time is random as well. We restrict our attention in this paper to the realm of classical open tandem networks, i.e. linearly ordered sequences of nodes that are fed by an external stream which is controlled via the overall population size.
Our main result is an explicit expression for the steady state joint distribution of the queue length vector and the availability status vector. The steady state distribution is of product form and completely in line with the celebrated results for Jackson-, Gordon-Newell-, and BCMP-networks. Although the result is of remarkably simple structure the proof is rather involved and requires careful and tedious structural investigations. The central part of the proof is time reversal: we show that local balance equations hold for the transition kernel of the network and a suitably defined stochastic matrix. This matrix can be looked upon as the one step transition matrix of our network under time reversal.
Similar to the product form theorems for pure queueing networks in continuous time and the subsequently derived computational algorithms our result opens the path to jointly investigate the performance of the network and its availability and their interdependencies and to subsequently develop computational algorithms for availability and performance.

2 Open tandem of unreliable state dependent Bernoulli servers in discrete time

2.1 Description of the system

We give the Markovian description of an open J-stage tandem network with unreliable nodes in discrete time. The nodes of this network are Bernoulli servers with state dependent capacity and unlimited waiting room under first-come-first-served (FCFS). The external input stream of indistinguishable customers is Bernoulli with state dependent intensity $b(n)$, controlled by the total network population (number of customers) n. The nodes of the network are unreliable. There are independent breakdowns and restorations, the up and down periods are geometrically distributed with node specific parameters.

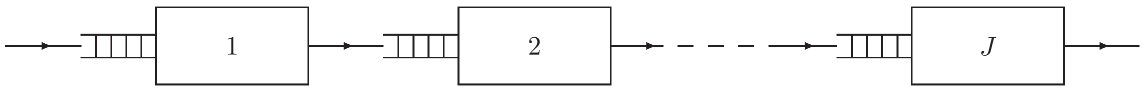

Figure 1: J-stage Tandem of Geo/Geo/1/∞-nodes

Therefore the nodes of the network can take two availability states, the up and down status. A node in up status is working properly and a node in down status is not working at all.
During a down period of a node the customers at this node are not in service and lay dormant, no customer departs from a node in down status. A node in down status is short circuited: Customers arriving at such node jump to the next node in line. This is continued until either a node in up status is reached or the network is left. This routing is sometimes called *skipping regime*. Applying this it follows that customers arriving from outside are directed to the first node in up status, if all nodes are down they are rejected.
State transitions of this network depend on the availability status of the network and the queue lengths: arrivals can only occur at nodes that are in up status, and departures only at nodes that are in up status and occupied as well.

State space. The local state of node $Q[i]$, $i \in \underline{J}$, is determined by $(x_{1,i}, x_{2,i}) \in \{0,1\} \times \mathbb{N}_0$. The first coordinate $x_{1,i}$ gives the availability status of node $Q[i]$ ($x_{1,i} = 0$ corresponds to the up status and $x_{1,i} = 1$ to the down status), the second coordinate $x_{2,i}$ gives the local queue length (number of customers present).
For easier understanding the dynamics of the system we introduce an artificial node $Q[J+1]$ that represents the outside of the network (source and sink). We call the outside of the network simply source and fix that (by definition) the source is always in up status (availability of $Q[J+1]$ is 0) and the number of customers present at $Q[J+1]$ is ∞.
Therefore the local state of the source $Q[J+1]$, is $(x_{1,J+1}, x_{2,J+1}) = (0, \infty)$ and does never change (it turns out that this formalism is well tailored to cover the case of closed networks, too).
The state space of our system is therefore a subset of $E = \{0,1\}^{J+1} \times \bar{\mathbb{N}}_0^{J+1}$, where $\bar{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$. To be a *feasible* state, $x \in E$ must fulfil the following conditions. (i) $(x_{1,J+1}, x_{2,J+1}) = (0, \infty)$, (ii) $x_{1,i} \in \mathbb{N}_0$ for all $i \in \underline{J}$. That is, feasible states x are of the form

$$x = (x_1, x_2) = \big((x_{1,1}, x_{1,2}, \dots, x_{1,J}, 0), (x_{2,1}, x_{2,2}, \dots, x_{2,J}, \infty)\big).$$

We denote the system's state space by

$$E_z := \{x \in E \mid (x_{1,J+1}, x_{2,J+1}) = (0, \infty),\ x_{1,i} \in \mathbb{N}_0\ \forall i \in \underline{J}\}.$$

It is a reasonable assumption that using E_z as state space we obtain a Markovian description of the system's development over time. This will be discussed below and the necessary details will be provided. Before doing this we present our main result. The following details are needed:

$n(x)$ is the total network population if the network is in state x, so $n \colon E \to \mathbb{Z}$, $n(x) := \sum_{j=1}^{J} x_{2,j}$.

$\alpha_j(k,l)$ is the probability that the availability of node $Q[j]$ changes $k \mapsto l$, $b(k)$ is the arrival probability if the total network population equals k, and $p_j(k)$ is the state dependent probability for service completion at node $Q[j]$. We abbreviate $c(k) := 1 - b(k)$, $q_j(k) = 1 - p_j(k)$, and assume that $b(k) \in (0,1)$ for all $k \in \mathbb{N}_0$, $p_j(k) \in (0,1)$ for all $k \in \mathbb{N}$ and $p_j(0) = 0$.
We can think of the transition regime for the occurrence of simultaneous events at some node (that is, simultaneous arrival and departure at the same node) to be according to the well known late-arrival-departure-first

(LA-DF) regime, see [6] for details. Our explicit transition probabilities do not rely on such a sequencing and encompasses other interpretations as well.

Theorem 2.1. *The Markov chain X, corresponding to the open J-stage tandem with unreliable state dependent Bernoulli servers has the invariant measure ϕ on E_z, given by*

$$\phi(x) = \left(\prod_{k=1}^{n(x)} \frac{b(k-1)}{c(k)}\right) \left(\prod_{j=1}^{J} \prod_{k=1}^{x_{2,j}} \frac{q_j(k-1)}{p_j(k)}\right) \left(\prod_{j=1}^{J} \alpha_j(x_{1,j} \oplus 1, x_{1,j})\right), \quad x \in E_z$$

The system is ergodic, if the normalising constant

$$K := \sum_{x \in E_z} \left(\prod_{k=1}^{n(x)} \frac{b(k-1)}{c(k)}\right) \left(\prod_{j=1}^{J} \prod_{k=1}^{x_{2,j}} \frac{q_j(k-1)}{p_j(k)}\right) \left(\prod_{j=1}^{J} \alpha_j(x_{1,j} \oplus 1, x_{1,j})\right)$$

is finite. Then π, given by

$$\pi(x) = \frac{1}{K} \left(\prod_{k=1}^{n(x)} \frac{b(k-1)}{c(k)}\right) \left(\prod_{j=1}^{J} \prod_{k=1}^{x_{2,j}} \frac{q_j(k-1)}{p_j(k)}\right) \left(\prod_{j=1}^{J} \alpha_j(x_{1,j} \oplus 1, x_{1,j})\right), \quad x \in E_z,$$

is the unique steady state distribution of X on the state space E_z.

Remarks:
(1) It is obvious that π has the structure one would like to find when searching for explicit product form steady state distributions. The formulae show that asymptotically and in steady state our network nodes behave as if they are independent servers driven by a suitably rescaled arrival stream. As in the continuous time setting this is only an *as if* statement because the nodes do interact - the interaction forces are carried by the travelling customers. In our system there occurs an even stronger interaction, because the jumping customers have to react to the global availability status of the network.
Therefore it is even more astonishing that (asymptotically and in steady state) the performance behaviour and the availability behaviour formally decouple as well - expressed by the product structure of π. Clearly, in equilibrium the customers' queueing behaviour is pathwise strongly determined by the availability status of the network.
(2) Recall that all states have $J+1$ coordinates, but only the first J coordinates describing the local status of the real network nodes contribute to ϕ and π.

The rest of the paper is mainly the proof of theorem 2.1.
In this Section we first introduce the rather detailed notation needed. The details are unavoidable due to the complicated interaction of availability and queueing in the system. There is an index of notation at the end of Section 4.
In a second part we finish the Section with an explicit expression for the one step transition matrix p of X. The explicit form of p indicates why a direct solution of the global balance equations is complicated.
We therefore use time reversal: In Section 2.2 we define an appropriate stochastic matrix $\bar{p}$ which can be regarded as the one step transition matrix of an open J-stage tandem of unreliable Bernoulli servers, with customers moving in the opposite direction and suitably defined availability changes (details will follow below). This network is considered as our required time reversed system of the original one.
Finally in Section 3 we show that the local balance equations $\pi(x)p(x,y) = \pi(y)\bar{p}(y,x)$, $x, y \in E_z$ are satisfied which is sufficient to prove the theorem, see [6][Appendix 7.1].

The node set $\underline{J+1}$ will be considered with the natural order, $1 < 2 < \cdots < J+1$, and will be equipped with an addition $\oplus$ according to

$\oplus$	1	2	3	$\cdots$	$J+1$
1	2	3	4	$\cdots$	1
2	3	4	5	$\cdots$	2
3	4	5	6	$\cdots$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
$J+1$	1	2	3	$\cdots$	$J+1$

The set $\{0,1\}$ of local availability states will be equipped with the addition modulo one, also denoted by $\oplus$. Finally on the set $\{0,1\}^{J+1}$ of global availability states $\oplus$ is defined coordinatewise by the addition modulo one.

Let $A(x) := \{i \in \underline{J+1} \mid x_{1,i} = 0\}$ be the set of numbers of available nodes and

$$A_{\mathrm{OCC}}(x) := \{i \in \underline{J+1} \mid x_{1,i} = 0 \text{ and } x_{2,i} > 0\}$$

be the set of numbers of available nodes having customers in service.

Let $z \in \{0,1\}^{J+1}$. Define the function $f(z,\bullet)\colon \underline{J+1} \to \underline{J+1}$ as follows: $f(z,i) := k$ is the least number in $\underline{J+1}$, which is greater or equal to $i \oplus 1$, such that $z_k = 0$ holds. If there is no such k, let $f(z,i) := J+1$.

Then define transition operators $T_{B,\mathrm{ac}}$, $B \subset \underline{J+1}$, $\mathrm{ac} \in \{0,1\}^{J+1}$, on E by

$$T_{B,\mathrm{ac}}(x) := \Big(x_1 \oplus \mathrm{ac}, x_2 + \sum_{i\in B}\big(e_{f(x_1,i)} - e_i\big)\Big).$$

Here *ac* stands for *availability change*. The set B contains the information about the customer's migration: if $i \in B$, a departure from node $Q[i]$ occurs. The destination node of the departing customer is the next node in up status if any, otherwise this customer leaves the network. The information about the node's availability changes is given by the vector ac: the availability of node $Q[i]$ changes, if $\mathrm{ac}_i = 1$ (ac_i denotes the i-th coordinate of ac).

Let $x \in E_z$, $B \in \mathcal{P}(\underline{J+1})$ and $\mathrm{ac} \in \{0,1\}^{J+1}$. Then call operator $T_{B,\mathrm{ac}}$ *proper with respect to* x, if $B \subset A_{\mathrm{OCC}}(x)$ and $\mathrm{ac}_{J+1} = 0$ holds. That is, operator $T_{B,\mathrm{ac}}$ is proper with respect to x, if and only if departures occur only at occupied and working nodes and, additionally the availability status of the source is not changed.

One step transition matrix Let

$$S(x) := \bigcup_{\substack{B\in\mathcal{P}(\underline{J+1})\colon\, B\subset A_{\mathrm{OCC}}(x)\\ \mathrm{ac}\in\{0,1\}^{J+1}\colon\, \mathrm{ac}_{J+1}=0}} \{T_{B,\mathrm{ac}}(x)\}$$

be the set of successor states of state $x \in E_z$. That is, $y \in E_z$ is a successor of $x \in E_z$, if a proper operator T with respect to x exist, such that $T(x) = y$ holds.
Now let $x \in E_z$ be a state, $B \in \mathcal{P}(\underline{J+1})$ with $B \subset A_{\mathrm{OCC}}(x)$ and $\mathrm{ac} \in \{0,1\}^{J+1}$ with $\mathrm{ac}_{J+1} = 0$ (that is, $T_{\mathrm{ac},B}$ is proper with respect to x). Define

$$\begin{aligned}\tilde{p}(x, T_{B,\mathrm{ac}}) \;&:=\; \prod_{j\in\underline{J}}\big(\alpha_j(x_{1,j}, x_{1,j}\oplus \mathrm{ac}_j)\big)\\ &\quad\cdot\left(b\big(n(x)\big)^{\mathbf{1}[J+1\in B]}\prod_{j\in B\setminus\{J+1\}} p_j(x_{2,j})\right)\left(c\big(n(x)\big)^{\mathbf{1}[J+1\notin B]}\prod_{j\in(A_{\mathrm{OCC}}(x)\setminus\{J+1\})\setminus B} q_j(x_{2,j})\right).\end{aligned}$$

Define the one step transition matrix $p\colon E_z \times E_z \to [0,1]$ by

$$p(x,y) \sum_{\substack{B\in\mathcal{P}(\underline{J+1}):\ B\subset A_{\mathrm{OCC}}(x)\\ \mathrm{ac}\in\{0,1\}^{J+1}:\ \mathrm{ac}_{J+1}=0}} \tilde{p}(x,T_{B,\mathrm{ac}})\mathbf{1}[T_{B,\mathrm{ac}}(x)=y].$$

If y is not a successor of x, that is $y \notin S(x)$, then $p(x,y)=0$ by definition.
To get an explicit expression for the one step transition matrix, classify the states $x \in E_z$ as follows. Let $\mathrm{OCC}(x)$ be the set of numbers of occupied nodes (not necessarily available): $\mathrm{OCC}(x) := \{i \in \underline{J+1} \mid x_{2,i} > 0\}$.

(I) Call $x \in E_z$ type I-state, iff $A(x)\setminus\{J+1\} \not\subset \mathrm{OCC}(x)\setminus\{J+1\}$ (equivalently $A(x) \not\subset \mathrm{OCC}(x)$).
(II) Call $x \in E_z$ type II-state, iff $A(x)\setminus\{J+1\} \subset \mathrm{OCC}(x)\setminus\{J+1\}$ (equivalently $A(x) \subset \mathrm{OCC}(x)$).

Let $x, y \in E_z$ be states with $y \in S(x)$. Then the network's (global) availability change $x_1 \mapsto y_1$ is determined by the vector $\mathrm{ac} := x_1 \oplus y_1$.

The following holds:
(i) Let $y \in S(x)$ and $x_2 \neq y_2$. Then there is exactly one operator $T_{B,\mathrm{ac}}$, which is proper with respect to x and with $T_{B,\mathrm{ac}}(x) = y$. Necessarily $\mathrm{ac} = x_1 \oplus y_1$ holds (the set B of departure nodes (numbers) is uniquely determined).

(ii) Let $y \in S(x)$ be a type I-state with $x_2 = y_2$ Then there is exactly one operator $T_{B,\mathrm{ac}}$, which is proper with respect to x and with $T_{B,\mathrm{ac}}(x) = y$ (take $B = \varnothing$ and $\mathrm{ac} = x_1 \oplus y_1$).

(iii) Let $y \in S(x)$ be a type II-state with $x_2 = y_2$ Then exactly two operators $T_{B,\mathrm{ac}}$ are proper with respect to x and with $T_{B,\mathrm{ac}}(x) = y$ (take $B = \varnothing$ or $B = A(x)$ and $\mathrm{ac} = x_1 \oplus y_1$).

The one step transition matrix is therefore

$$p(x,y) = \begin{cases} \left(b\big(n(x)\big)^{\mathbf{1}[J+1\in B]} \prod\limits_{j\in B\setminus\{J+1\}} p_j(x_{2,j})\right)\left(c\big(n(x)\big)^{\mathbf{1}[J+1\notin B]} \prod\limits_{j\in (A_{\mathrm{OCC}}(x)\setminus\{J+1\})\setminus B} q_j(x_{2,j})\right) \\ \quad \cdot \prod\limits_{j\in\underline{J}} \big(\alpha_j(x_{1,j},y_{1,j})\big), \quad \text{if } y\in S(x) \text{ and } x_2\neq y_2; \\ \left(c\big(n(x)\big) \prod\limits_{j\in A_{\mathrm{OCC}}(x)\setminus\{J+1\}} q_j(x_{2,j})\right) \prod\limits_{j\in\underline{J}} \big(\alpha_j(x_{1,j},y_{1,j})\big), \\ \text{if } y\in S(x),\ x_2=y_2 \text{ and } x \text{ is of type I}; \\ \left(b\big(n(x)\big) \prod\limits_{j\in A_{\mathrm{OCC}}(x)\setminus\{J+1\}} p_j(x_{2,j}) + c\big(n(x)\big) \prod\limits_{j\in A_{\mathrm{OCC}}(x)\setminus\{J+1\}} q_j(x_{2,j})\right) \prod\limits_{j\in\underline{J}} \big(\alpha_j(x_{1,j},y_{1,j})\big), \\ \text{if } y\in S(x),\ x_2=y_2 \text{ and } x \text{ is of type II}; \\ 0, \quad \text{if } y\notin S(x). \end{cases}.$$

In the case $y \in S(x)$, $x_2 \neq y_2$ the set B is the (uniquely determined) set of departure node numbers. Note that $A(x) = A_{\mathrm{OCC}}(x)$ for type II-states and that $\{J+1\} \in A_{\mathrm{OCC}}(x) \neq \varnothing$ for all $x \in E_z$.

2.2 The time reversed system

Consider a system of J state dependent Bernoulli servers $Q[1],\ldots,Q[J]$ in series, all with FCFS queueing discipline, fed by a Bernoulli input stream from an external source, with intensity $b(k)$ controlled by the total network population k. The stations are unreliable. The state dependent service probabilities p_j at station $Q[j]$, $j \in \underline{J}$ are the same as for the system described in Section 2.1.

The differences are: customers proceed in the reversed direction and the availability switches before the customers move and the queues are rearranged. In detail:
At a network's state transition, the change of the availability status takes place first. Then, with respect to the new availability status of the system, the customers' movements are performed in accordance to the rules given in Section 2.1.
This implies that a station in down status at time $t-1$ is able to accept a new arrival at time t if and only if the availability of this station changes at the same time instant t and similarly a departure from this station is then possible (provided the station is occupied at time instant $t-1$).
If a station in up status at time $t-1$ switches to down status at time t no departures and no arrivals are possible. Customers arriving at a node in down status jump to the next node with respect to decreasing node numbers in line. The customers' movements now can be described by the subtraction $\ominus$, defined by

$\ominus$	1	2	$\cdots$	$J+1$
1	$J+1$	J	$\cdots$	1
2	1	$J+1$	$\cdots$	2
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
$J+1$	J	$J-2$	$\cdots$	$J+1$

Let $z \in \{0,1\}^{J+1}$ and let $g(z,\bullet)\colon \underline{J+1} \to \underline{J+1}$ defined as greatest integer $g(z,i) := k$ in $\underline{J+1}$, which is smaller or equal to $i \ominus 1$ and such that $z_k = 0$ holds. If there is no such k, let $g(z,i) := J+1$.
The transition operators for the time reversed system are $\bar{T}_{B,\mathrm{ac}}$, $B \subset \underline{J+1}$, $\mathrm{ac} \in \{0,1\}^{J+1}$. They are defined on E by

$$\bar{T}_{B,\mathrm{ac}}(x) := \Big(x_1 \oplus \mathrm{ac}, x_2 + \sum_{i \in B} \Big(e_{g(x_1 \oplus \mathrm{ac}, i)} - e_i\Big)\Big), \quad x \in E.$$

Call $\bar{T}_{B,\mathrm{ac}}$ proper with respect to x, iff $\mathrm{ac}_{J+1} = 0$ and $B \subset A_{\mathrm{OCC}}(x_1 \oplus \mathrm{ac}, x_2)$ holds. Let $x \in E_z$ and let $\bar{T}_{B,\mathrm{ac}}$ be proper with respect to x. Then define

$$\begin{aligned}\tilde{\bar{p}}(x, \bar{T}_{B,\mathrm{ac}}) \quad := \quad & \prod_{j \in \underline{J}} \left(\alpha_j(x_{1,j}, x_{1,j} \oplus \mathrm{ac}_j)\right) \\ & \cdot \left(b\big(n(x)\big)^{\mathbf{1}[J+1 \in B]} \prod_{j \in B \setminus \{J+1\}} p_j(x_{2,j})\right) \left(c\big(n(x)\big)^{\mathbf{1}[J+1 \notin B]} \prod_{j \in (A_{\mathrm{OCC}}(x_1 \oplus \mathrm{ac}, x_2) \setminus \{J+1\}) \setminus B} q_j(x_{2,j})\right).\end{aligned}$$

Denote the set of successor states of $x \in E_z$ in the time reversed system by $\bar{S}(x)$:

$$\bar{S}(x) := \bigcup_{\substack{B \in \mathcal{P}(\underline{J+1}) \colon B \subset A_{\mathrm{OCC}}(x_1 \oplus \mathrm{ac}, x_2) \\ \mathrm{ac} \in \{0,1\}^{J+1} \colon \mathrm{ac}_{J+1} = 0}} \{\bar{T}_{B,\mathrm{ac}}(x)\}.$$

In order to determine the one step transition matrix $\bar{p}\colon E_z \times E_z \to [0,1]$ of the time reversed system, which is defined by

$$\bar{p}(x,y) \sum_{\substack{B \in \mathcal{P}(\underline{J+1}) \colon B \subset A_{\mathrm{OCC}}(y_1, x_2) \\ \mathrm{ac} \in \{0,1\}^{J+1} \colon \mathrm{ac}_{J+1} = 0}} \tilde{p}(x, \bar{T}_{B,\mathrm{ac}}) \mathbf{1}[\bar{T}_{B,\mathrm{ac}}(x) = y],$$

we introduce the following classification.
(Ia) Call a pair (ac, x) a type Ia-state, iff $A(x_1 \oplus \mathrm{ac}, x_2) \setminus \{J+1\} \not\subset \mathrm{OCC}(x_1 \oplus \mathrm{ac}, x_2) \setminus \{J+1\}$.
(IIa) Call a pair (ac, x) type IIa-state, iff $A(x_1 \oplus \mathrm{ac}, x_2) \setminus \{J+1\} \subset \mathrm{OCC}(x_1 \oplus \mathrm{ac}, x_2) \setminus \{J+1\}$.

Then the one step transition matrix of the time reversed system is

$$\bar{p}(x,y)=\begin{cases}\left(b\big(n(x)\big)^{\mathbf{1}[J+1\in B]}\prod_{j\in B\setminus\{J+1\}}p_j(x_{2,j})\right)\left(c\big(n(x)\big)^{\mathbf{1}[J+1\notin B]}\prod_{j\in(A_{\mathrm{OCC}}(y_1,x_2)\setminus\{J+1\})\setminus B}q_j(x_{2,j})\right)\\ \quad\cdot\prod_{j\in\underline{J}}\big(\alpha_j(x_{1,j},y_{1,j})\big),\quad \text{if } y\in\bar{S}(x)\text{ and } x_2\neq y_2;\\ \left(c\big(n(x)\big)\prod_{j\in A_{\mathrm{OCC}}(y_1,x_2)\setminus\{J+1\}}q_j(x_{2,j})\right)\prod_{j\in\underline{J}}\big(\alpha_j(x_{1,j},y_{1,j})\big),\\ \text{if } y\in\bar{S}(x),\ x_2=y_2\text{ and }(x_1\oplus y_1,x)\text{ is of type Ia;}\\ \left(b\big(n(x)\big)\prod_{j\in A_{\mathrm{OCC}}(y_1,x_2)\setminus\{J+1\}}p_j(x_{2,j})+c\big(n(x)\big)\prod_{j\in A_{\mathrm{OCC}}(y_1,x_2)\setminus\{J+1\}}q_j(x_{2,j})\right)\prod_{j\in\underline{J}}\big(\alpha_j(x_{1,j},y_{1,j})\big),\\ \text{if } y\in\bar{S}(x),\ x_2=y_2\text{ and }(x_1\oplus y_1,x)\text{ is of type IIa;}\\ 0,\quad\text{if } y\notin\bar{S}(x).\end{cases}$$

where in the case $y\in\bar{S}(x)$ and $x_2\neq y_2$ the set B is the uniquely determined set of departure node numbers.

3 Proof of the theorem

Proof. It suffices to show

$$\pi(x)p(x,y)=\pi(y)\bar{p}(y,x)\tag{1}$$

for all pairs $(x,y)\in E_z^2$. We call the system described in Section 2.1 *original system* in contrast to the *reversed system*, described in Section 2.2. Note, that both systems are time reversals of another, but they are obviously not reversible.

Let $x\in E_z$, $y\in S(x)$ and $x_2\neq y_2$. Then there is exactly one operator $T_{B,\mathrm{ac}}$ (proper with respect to x), transforming x into y and at most one operator $\bar{T}_{\bar{B},\bar{\mathrm{ac}}}$ (proper with respect to y), transforming y into x. Clearly $\mathrm{ac}=\bar{\mathrm{ac}}=x_1\oplus y_1$ must hold. The destination node of a jumping customer departing from node $i\in B$ is $Q[f(x_1,i)]$. Necessarily $x_{1,f(x_1,i)}=0$ must hold. This implies $f(x_1,i)\in A(y_1\oplus\mathrm{ac},y_2)=A(x_1,y_2)$. Furthermore $y_{2,f(x_1,i)}>0$ holds because of the arrival at node $Q[f(x_1,i)]$. A simultaneous departure at this node is only possible if $x_{2,f(x_1,i)}\geqq 1$, hence $f(x_1,i)\in A_{\mathrm{OCC}}(y_1\oplus\mathrm{ac},y_2)=A_{\mathrm{OCC}}(x_1,y_2)$. Take $\bar{B}:=\bigcup_{i\in B}\{f(x_1,i)\}$. The destination node number of a customer in the time reversed system departing from node $Q[f(x_1,i)]$ is given by $g(y_1\oplus\mathrm{ac},f(x_1,i))=g(x_1,f(x_1,i))=i$. Saying it the other way round, $\bar{T}_{\bar{B},\mathrm{ac}}$ and $T_{B,\mathrm{ac}}$ are inverse in the following sense:

$$\begin{aligned}\bar{T}_{\bar{B},\mathrm{ac}}\big(T_{B,\mathrm{ac}}(x)\big)&=\bar{T}_{\bar{B},\mathrm{ac}}\Big(x_1\oplus\mathrm{ac},x_2+\sum_{i\in B}\big(e_{f(x_1,i)}-e_i\big)\Big)\\&=\Big(x_1\oplus\mathrm{ac}\oplus\mathrm{ac},x_2+\sum_{i\in B}\big(e_{f(x_1,i)}-e_i\big)+\sum_{j\in\bar{B}}\big(e_{g(x_1\oplus\mathrm{ac}\oplus\mathrm{ac},j)}-e_j\big)\Big)\\&=\Big(x_1,x_2+\sum_{i\in B}\big(e_{f(x_1,i)}-e_i\big)+\sum_{i\in B}\big(e_{g(x_1,f(x_1,i))}-e_{f(x_1,i)}\big)\Big)=x.\end{aligned}$$

The local balance equation (1) takes on the form

$$
\begin{aligned}
&\left(\prod_{k=1}^{n(x)} \frac{b(k-1)}{c(k)}\right) \prod_{j=1}^{J} \left(\alpha_j(x_{1,j} \oplus 1, x_{1,j}) \alpha_j(x_{1,j}, y_{1,j}) \prod_{k=1}^{x_{2,j}} \frac{q_j(k-1)}{p_j(k)}\right) \\
&\cdot \left(b\big(n(x)\big)^{\mathbf{1}[J+1 \in B]} \prod_{j \in B \setminus \{J+1\}} p_j(x_{2,j})\right) \left(c\big(n(x)\big)^{\mathbf{1}[J+1 \notin B]} \prod_{j \in (A_{\mathrm{OCC}}(x) \setminus \{J+1\}) \setminus B} q_j(x_{2,j})\right) \\
= &\left(\prod_{k=1}^{n(y)} \frac{b(k-1)}{c(k)}\right) \prod_{j=1}^{J} \left(\alpha_j(y_{1,j} \oplus 1, y_{1,j}) \alpha_j(y_{1,j}, x_{1,j}) \prod_{k=1}^{y_{2,j}} \frac{q_j(k-1)}{p_j(k)}\right) \qquad (2) \\
&\cdot \left(b\big(n(y)\big)^{\mathbf{1}[J+1 \in \bar{B}]} \prod_{j \in \bar{B} \setminus \{J+1\}} p_j(y_{2,j})\right) \left(c\big(n(y)\big)^{\mathbf{1}[J+1 \notin \bar{B}]} \prod_{j \in (A_{\mathrm{OCC}}(x_1, y_2) \setminus \{J+1\}) \setminus \bar{B}} q_j(y_{2,j})\right).
\end{aligned}
$$

The availability status change is described by

$$
\prod_{j=1}^{J} \alpha_j(x_{1,j} \oplus 1, x_{1,j}) \alpha_j(x_{1,j}, y_{1,j}) = \prod_{j=1}^{J} \alpha_j(y_{1,j} \oplus 1, y_{1,j}) \alpha_j(y_{1,j}, x_{1,j}).
$$

By direct verification we can show for all pairs $(k, l) \in \{0, 1\}$

$$
\alpha_j(k \oplus 1, k) \alpha_j(k, l)\big) = \alpha_j(l \oplus 1, l) \alpha_j(l, k). \qquad (3)
$$

Now consider nodes $Q[i]$ with $i \in \complement A(x)$. Here the queue length is not changed by the transition $x \mapsto y$ in the original system: $y_{2,i} = x_{2,i}$; in the reversed system the queue length at nodes $Q[i]$ with $i \in \complement A(y_1 \oplus \mathrm{ac}, y_2)$ is not changed by the transition $y \mapsto x$. Because of $\complement A(y_1 \oplus \mathrm{ac}, y_2) = \complement A(x)$ the equality of the factors on the left and right hand side of (1) corresponding to those nodes follows.
Let

$$
\begin{aligned}
U &:= \{i \in A(x) \mid i \notin B \ \wedge \ g(x_1, i) \in B\} \\
E_1 &:= \{i \in A(x) \mid i \in B \ \wedge \ g(x_1, i) \in B)\} \\
E_0 &:= \{i \in A(x) \mid i \notin B \ \wedge \ g(x_1, i) \notin B\} \\
D &:= \{i \in A(x) \mid i \in B \ \wedge \ g(x_1, i) \notin B\}
\end{aligned}
$$

and

$$
\begin{aligned}
U_r &:= \{i \in A(y_1 \oplus \mathrm{ac}, y_2) \mid i \notin \bar{B} \ \wedge \ f(y_1 \oplus \mathrm{ac}, i) \in \bar{B}\} \\
E_{r,1} &:= \{i \in A(y_1 \oplus \mathrm{ac}, y_2) \mid i \in \bar{B} \ \wedge \ f(y_1 \oplus \mathrm{ac}, i) \in \bar{B})\} \\
E_{r,0} &:= \{i \in A(y_1 \oplus \mathrm{ac}, y_2) \mid i \notin \bar{B} \ \wedge \ f(y_1 \oplus \mathrm{ac}, i) \notin \bar{B}\} \\
D_r &:= \{i \in A(y_1 \oplus \mathrm{ac}, y_2) \mid i \in \bar{B} \ \wedge \ f(y_1 \oplus \mathrm{ac}, i) \notin \bar{B}\}.
\end{aligned}
$$

Here U is the set of node numbers where the queue length is increased by one during the transition $x \mapsto T_{\mathrm{ac},B}(x)$ (the U indicates *up*). The sets E_1 and E_0 correspond to those nodes in availability status up, where the queue length remains unchanged, in the former case because of simultaneous arrival and departure, in the latter case because of no arrival and departure (the E indicates *equal*). Finally D corresponds to the nodes, where the queue length is decreased by one (the D indicates *down*).

The sets with the additional subscript r are the analogously defined sets with respect to transition $y \mapsto \bar{T}_{\mathrm{ac},\bar{B}}(y)$ in the reversed system.
Obviously $i \in B \Rightarrow f(x_1, i) \in \bar{B}$. The restriction of $f(x_1, \bullet)$ on $A_{\mathrm{OCC}}(x)$ is injective, which implies $i \notin B \Rightarrow f(x_1, i) \notin \bar{B}$ for $i \in A_{\mathrm{OCC}}(x)$. That is, $U = D_r$. In an analogous way one shows $D = U_r$, $E_1 = E_{r,1}$ and $E_0 = E_{r,0}$. The equalities for these sets yield

$$\prod_{j\in U}\left(\prod_{k=1}^{x_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)q_j(x_{2,j}) = \prod_{j\in D_r}\left(\prod_{k=1}^{y_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)\frac{p_j(y_{2,j})}{q_j(y_{2,j}-1)}q_j(x_{2,j}) = \prod_{j\in D_r}\left(\prod_{k=1}^{y_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)p_j(y_{2,j}), \tag{4}$$

$$\prod_{j\in E_1}\left(\prod_{k=1}^{x_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)p_j(x_{2,j}) = \prod_{j\in E_{r,1}}\left(\prod_{k=1}^{y_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)p_j(y_{2,j}), \tag{5}$$

$$\prod_{j\in E_0}\left(\prod_{k=1}^{x_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)q_j(x_{2,j}) = \prod_{j\in E_{r,0}}\left(\prod_{k=1}^{y_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)q_j(y_{2,j}) \tag{6}$$

and

$$\prod_{j\in D}\left(\prod_{k=1}^{x_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)p_j(x_{2,j}) = \prod_{j\in U_r}\left(\prod_{k=1}^{y_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)\frac{q_j(y_{2,j})}{p_j(y_{2,j}+1)}p_j(x_{2,j}) = \prod_{j\in D_r}\left(\prod_{k=1}^{y_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)q_j(y_{2,j}). \tag{7}$$

Note that the relevant information is $i \in B$ or $i \notin B$. If $i \notin B$ holds, it is not necessary to distinguish between $i \in A_{\mathrm{OCC}}(x)$ or $i \in A(x) \setminus A_{\mathrm{OCC}}(x)$, that is $x_{2,i} > 0$ or $x_{2,i} = 0$: because of $q_i(0) = 1$ (by definition) the corresponding factor in both cases is one.
Taking the product of the left and the right hand sides of (4),(5),(6) and (7), respectively one gets

$$\begin{aligned}&\prod_{j\in A(x)}\left(\prod_{k=1}^{x_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)\left(\prod_{j\in B\setminus\{J+1\}}p_j(x_{2,j})\right)\left(\prod_{j\in(A_{\mathrm{OCC}}(x)\setminus\{J+1\})\setminus B}q_j(x_{2,j})\right)\\ &\quad= \prod_{j\in A(x)}\left(\prod_{k=1}^{y_{2,j}}\frac{q_j(k-1)}{p_j(k)}\right)\left(\prod_{j\in\bar{B}\setminus\{J+1\}}p_j(y_{2,j})\right)\left(\prod_{j\in(A_{\mathrm{OCC}}(x_1,y_2)\setminus\{J+1\})\setminus\bar{B}}q_j(y_{2,j})\right).\end{aligned}$$

Finally, with respect to the source: $n(y) = n(x) + 1$ (*external arrival in the original system*) $\Leftrightarrow (J+1 \in B \ \wedge\ g(x_1, J{+}1) \notin B) \Leftrightarrow (J{+}1 \notin \bar{B} \wedge f(x_1, J{+}1) \in \bar{B}) \Leftrightarrow n(x) = n(y){-}1$ (*external departure in the reversed system*) and analogously for the remaining three cases. This yields

$$\left(\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)b\big(n(x)\big)^{\mathbf{1}[J+1\in B]}c\big(n(x)\big)^{\mathbf{1}[J+1\notin B]} = \left(\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)b\big(n(y)\big)^{\mathbf{1}[J+1\in\bar{B}]}c\big(n(y)\big)^{\mathbf{1}[J+1\notin\bar{B}]}$$

and therefore (1) holds for the case $x \in E_z$, $y \in S(x)$ and $x_2 \neq y_2$.
Let now $x \in E_z$, $y \in S(x)$ and $x_2 = y_2$. Then $A(y_1 \oplus x_1 \oplus y_1, y_2) = A(x)$ as well as $A_{\mathrm{OCC}}(y_1 \oplus x_1 \oplus y_1, y_2) = A_{\mathrm{OCC}}(x)$. From this we see, that x is a type I-state if and only if $(x_1 \oplus y_1, y)$ is a type Ia-state and x is a type II-state if and only if $(x_1 \oplus y_1, y)$ is a type IIa-state. Now, together with (3), the validity of the local balance equations follows for this type of transitions. In all other cases the left hand side and the right hand side of (1) are 0. $\square$

4 Conclusion

We have investigated in this paper an example of a discrete time network with unreliable nodes, which in our opinion is an important model because its *reliable* version has found many applications in telecommunication (transmission lines) and production systems (production flow lines). The positive message obviously is that even in the realm of discrete time systems with the additional feature of breakdown and repair successful product form modelling is possible. On the other hand it should be noticed that the shortcutting mechanism in many cases is not realistic for rerouting protocols. Such a drawback is what our model shares with almost all complex network models that exhibit an explicit product form equilibrium and are therefore amenable to algorithmic investigations like Buzen's procedure or (modified) Mean-Value-Analysis. Recall e.g. that Kleinrock's *independence assumption* requires that the length of a message on its path through a transmission network is independently fixed for any subsequent channel. On the other hand it is exactly this assumptions which makes the Jackson network a model for data transmission networks, and it is a common observations that performance predictions with these models work surprisingly well.
Nevertheless, there is still much research to be done. Our ongoing research project is at present concerned with more general topologies for networks which are in a sense of ALOHA-type, and with further rerouting schemes for these networks. We also start simulation studies comparing the product networks' characteristics with more detailed and/or realistic models.

Notations

$$\begin{aligned}
E &= \{0,1\}^{J+1} \times \mathbb{Z}_\infty^{J+1} \\
E_z &= \{x \in E \mid (x_{1,J+1}, x_{2,J+1}) = (0,\infty),\ x_{1,i} \in \mathbb{N}_0\ \forall i \in \underline{J}\} \\
\mathrm{OCC}(x) &= \{i \in \underline{J+1} \mid x_{2,i} > 0\} \\
A(x) &= \{i \in \underline{J+1} \mid x_{1,i} = 0\} \\
A_{\mathrm{OCC}}(x) &= \{i \in \underline{J+1} \mid x_{1,i} = 0 \text{ and } x_{2,i} > 0\} \\
\alpha_j(k,l) & \quad \text{Probability of availability change } k \mapsto l \text{ at node } Q[j] \\
f(z,\bullet) & \quad \text{see page 5} \\
g(z,\bullet) & \quad \text{see page 7} \\
T_{B,\mathrm{ac}} & \quad \text{transition operator of the original system; see page 5} \\
S(x) &= \bigcup_{\substack{B \in \mathcal{P}(\underline{J+1}):\ B \subset A_{\mathrm{OCC}}(x) \\ \mathrm{ac} \in \{0,1\}^{J+1}:\ \mathrm{ac}_{J+1}=0}} \{T_{B,\mathrm{ac}}(x)\} \\
\bar{T}_{B,\mathrm{ac}} & \quad \text{transition operator of the reversed system; see page 7} \\
\bar{S}(x) &= \bigcup_{\substack{B \in \mathcal{P}(\underline{J+1}):\ B \subset A_{\mathrm{OCC}}(x_1 \oplus \mathrm{ac}, x_2) \\ \mathrm{ac} \in \{0,1\}^{J+1}:\ \mathrm{ac}_{J+1}=0}} \{\bar{T}_{B,\mathrm{ac}}(x)\} \\
n\colon E \to \mathbb{Z} & \quad n(x) := \sum_{j=1}^{J} x_{2,j},\ \text{total network population}
\end{aligned}$$

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