UNIVERSITÄT HAMBURG

Discrete Time Queueing Networks with Product Form Steady State:

Availability and Performance Analysis in an Integrated Model

C. Malchin, H. Daduna

Preprint–No. 2006-02 May 2006

FACHBEREICH MATHEMATIK

SCHWERPUNKT MATHEMATISCHE STATISTIK
UND STOCHASTISCHE PROZESSE

BUNDESSTR. 55, D – 20146 HAMBURG

Discrete Time Queueing Networks with Product Form Steady State. Availability and Performance Analysis in an Integrated Model

Christian Malchin Hans Daduna

Abstract

For a discrete time network of generalised Bernoulli servers with unreliable nodes we derive the steady state probabilities for the joint queue length vector for all nodes and the availability status of the network. This allows to assess the performance behaviour and the reliability, resp. availability of the network in an integrated model. Because our result exhibits a product form for the steady state distribution it opens the path to fast algorithmic evaluation of the desired performance and reliability indices.[1]

1 Introduction

In this note we study the asymptotic and stationary behaviour of complex interacting stochastic systems which evolve on a discrete time scale. Motivation for our study are on the one hand classical and recent applications with systems that show an inherent discrete time scale in their behaviour, e.g. time-sharing computer systems, ALOHA type transmission and communication systems and ATM networks. On the other hand there appeared recently a sequence of papers which construct models for some of these applications and to a certain extend develop a theory of discrete time stochastic networks which constitute a generic class of well suited models for many of these situations.

The theory of stochastic networks and their asymptotic and stationary behaviour in continuous and in discrete time is the basis for Performance Analysis in a general framework. The close connection between stochastic networks and especially Performance Analysis in computer and communications systems is described in the two volumes of Kleinrock [Kle75], [Kle76], and the BCMP-paper [BCMP75]. Various other applications in other fields are dealt with in Kelly's book [Kel79].

Performance analysis of computer and communication systems usually relies on queueing networks and related models. Because it has been in many cases a satisfying assumption that the nodes of the networks are reliable, these investigations yielded good predictions of the networks' behaviour with respect to throughput, overload, system time, loss probabilities due to buffer overflow, etc. There are several monographs on the subject available, see e.g. [CY01, BGdT98, Kel79, Dad01b]. On the other hand, reliability considerations especially for computing systems have been a field of intense research over the last forty years. From around the middle of the eighties emerged a field of research under the label of Performability where the problems of performance analysis and reliability are investigated in a unified system description, for a survey see [HMRT01].

The need for integration of performance analysis and reliability has become more important with the fast development of mobile networks, because control of these networks (e.g., WLANs) poses new problems with respect to reliable and fast access to the networks.

Even more, the emergence of wireless ad hoc networks which are formed of nodes with wireless communication capability without relying on a pre-existing infrastructure introduces strong problems concerning availability. So in these networks the required high Quality-of-Service for speech and data transmission has to be guaranteed on the basis of not necessarily highly reliable

[1] This research is part of the DFG-research project "Stochastische Netzwerke in diskreter Zeit: Analyse von Leistung und Verfügbarkeit" (DA774/1-1)

network nodes [SV03]. Furthermore there are many new problems, e.g. with network control and routing protocols in case of link failures in IP networks [MKC03, NSB$^+$03] or with the handling of catastrophic events in mobile cellular networks [KN03].

A different area where queueing network models are successfully used to describe the behaviour of complex networks is production control in Flexible Manufacturing Systems (FMS), [BSY94]. The investigation of unreliable machines in FMS and the evaluation of system availability has found increasing interest recently.

Successful application of queueing network theory often relies on having access to explicit description of the steady state behaviour of the systems under consideration. This is in almost all cases connected with the emergence of so-called product form equilibrium probabilities. This description of the steady state refers to an asymptotic separability of the global state probabilities into factors concerned with only local information. This property is usually the rationale behind efficient numerical algorithms.

Some first steps towards a calculus similar to the product form network calculus of queueing theory for the joint investigation of performance and availability behaviour in continuous time models are e.g. [SD03a, SD03b, SD04]. Approximation methods using product form approximations which incorporate the reliability behaviour into the parameters of the classical queueing networks (e.g. slowing down the service speed to model breakdowns) are reviewed in [CM96]. Those methods are often applied when explicit solutions for the performance and reliability behaviour are not at hand.

While the work mentioned above is focused on the continuous time setting, Kleinrock already studied discrete time systems, especially time shared computer systems, see e.g. [Kle64], [Kle67].

Compared to the classical queueing network models there is a severe lack of explicitly solvable model classes in the field of performability. An analogue to the celebrated universal product form network calculus in continuous time (Jackson-, Gordon-Newell-, and BCMP-networks) does not exist up to now.

Furthermore, it is well known that queueing theory in discrete time poses more difficulties than the parallel theory in continuous time. This is due to combinatorial complexity which stems from the possibility that simultaneous events (multiple departures and arrivals at the same node or at several nodes) can happen at the same time instant with positive probability. On the other hand there is clearly a need for discrete time queueing models because many real world systems show an inherent discrete time scale as mentioned above. For a survey on discrete time networks see the books [BK93, CMP99, Woo94, Dad01a].

Combining the performance analysis for discrete time network models with the simultaneous investigation of their possibly unreliable behaviour is still a challenging project. We contribute in this paper to this field of integrated performance-availability networks.

Our aim is to describe a class of models where nodes may break down and are bypassed if they are broken down. The up-times of the nodes are random and broken down nodes are repaired, the repair time is random as well. To be more specific:
We present a Markovian description of an open network with unreliable nodes in discrete time. The nodes of this network are generalised Bernoulli servers with state dependent service capacity and unlimited waiting room, they allow immediate feedback of customers. There is an at most countable set M of customer types. Customers of the same type are indistinguishable. As long as all nodes are functioning properly the routing of the customers is deterministic, uniquely determined by the customer's type. So the destination node of a jumping customer is determined by his type and his actual stage on the itinerary.

But the nodes of the network are unreliable and can take two availability status, the up and the down status. A node in up status is working properly and a node in down status is not working at all. There are independent breakdowns and restorations, the up and down periods are geometrically distributed. When nodes are broken down they neither provide service nor do they accept new customers. As a consequence, we have to fix rules for routing the customers in case of breakdowns and the modus of interaction between migrating customers and the randomly changing availability status of the nodes. So the routing of the customers will depend on the global availability status of the network. Consequently, the routing of the customers is no longer deterministic, but can be looked upon as being dependent on a random environment. The environment is determined by the network itself and the routing reacts on this environment. To provide some more details:
During a down period of some node the customers there are not serviced and consequently no departures take place from a node in down status. Customers arriving at a node in down status will bypass this node and will be directed to the next station on their route which is in up status.

The main result of this paper is the steady state distribution of the process, describing jointly the state of the queues and the global availability status of the network. The steady state distribution is of product form and completely in line with the celebrated results for Jackson-, Gordon-Newell-, and BCMP-networks. Therefore, similar to the product form theorems for pure queueing networks in continuous time and the subsequently derived computational algorithms, our result opens the path to jointly investigate the performance and availability of the network and their interdependencies, and to subsequently develop computational algorithms for availability and performance.

The central part of the proof is by time reversal: we define two systems (called forward and backward system), i. e. families of transition operators on subsets of a common state space E. From these families we obtain two stochastic matrices p and $\bar{p}$ that govern the corresponding Markov processes X and $\bar{X}$, which are time reversals of another. This is proved by computing generalised local balance equations for the transition kernel of the network and the suitably defined stochastic matrix of the network under time reversal. They have the same invariant measure and in case of ergodicity the same equilibrium distribution.

Although the principle underlying our proof is not new, it turns out that applying this principle is not obvious because the central problem is to construct a model where the suitably selected local flows can be described sufficiently detailed and on the other hand with a minimal set of details to keep the complexity bounded.

As mentioned above, the problem of simultaneous (multiple) events is a severe burden in discrete time stochastic network theory. To a certain extend we overcome the problem because we concentrate on Aloha-type network protocols. These are well known in the literature as a model for describing shared access to common resources by several clients. The consequence for the network with only reliable nodes is, that simultaneous events annihilate one another.

But for the networks with unreliable nodes simultaneous events reoccur, because breakdowns and repairs may happen concurrently with expiring services and new arrivals. For a discussion of similar problems in the more abstract setting of discrete time generalised semi-Markov processes see [HPTD95]. It should be pointed out that the class of models dealt with here do not fit into the setting there. There are other papers with multiple events where steady state probabilities can be obtained at least in a functional form, for a review see [CMP99] and [Dad01a][Chapter 6] - but these results are obtained under specific side constraints which are not fulfilled by the class of models we are interested in. The main distinguishing feature is the queueing discipline First-Come-First-Served for customers of different types required for the nodes under consideration here.

Related work to these investigations is provided by [Yat90], where networks of generalised

discrete time round robin queues are investigated, for an indepth study of the single nodes of this paper see [Yat94]. Both of these papers are closely related to the work in [DS81], [DS83], and [SD83]. All these investigations are concerned with reliable network nodes, which are companions of Kelly's *symmetric servers* [Kel79] in steady state, and offer an explicit equilibrium analysis.

The case of unreliable geometrical nodes under First-Come-First-Served (which are not symmetric) is studied for linear system topologies in [MD06]. The results are in parallel to those we shall present here, but due to the linear tandem structure of the networks, the proofs are much simpler.

The organisation of the paper is as follows: In Section 2 we describe two completely reliable networks in discrete time which are time reversals of another (forward and backward system). The building nodes of the forward network are generalised Bernoulli servers where immediate feedback of customers may occur.

As an introductory example of our general method we give in full detail the description of an isolated single reliable node (forward and backward in time), including explicit expressions for the transition kernels and the corresponding steady state distribution. Furthermore this example is used to demonstrate the power of the type dependent routing regimes.

Proofs are not given at this stage, although even the reduction of the main theorem to the standard case of reliable nodes includes an important generalisation of the results available in the literature: The arrival processes at the network are state dependent (Corollary 2.1).

In section 3 we extend the class of networks which are accessible with an explicit steady state description considerably by incorporating the features of breakdown and repair, i.e., the nodes are unreliable and there occurs the problem of reduced availability. From a technical point of view the superposition of customers' migration and queueing and availability changes yields concurrent events of complicated structure.

Nevertheless, our main result in Theorem 3.1 in Section 3.3 provides the full steady state description of the network in explicit form.

2 Networks with reliable feedback nodes

In the following we describe a forward and a backward network consisting of a finite number of J nodes (service stations) $Q[1], \ldots, Q[J]$. The nodes of the forward network are geometrical servers which allow immediate feedback and have a state dependent service capacity.

State space Let $M := \underline{m}$ for $m \in \mathbb{N}$ or $M := \mathbb{N}$ be the set of customer types and let $M_0 := M \cup \{0\}$. Further let $S\colon M \to \mathbb{N}$ be a function (called *route length*). An underlined integer $m \in \mathbb{N}$ denotes the set $\underline{m} := \{1, 2, \ldots, m\}$ and $\mathbb{N} := \{1, 2, \ldots\}$ is the set of positive integers.

As local state space for node $Q[j]$ we choose the following subset E_j of $\big((M \times \mathbb{N}) \cup \{(0,0)\}\big)^{\mathbb{N}}$. Elements $x \in E_j$ are of the form

$$\begin{aligned} x &= \big(x_1, x_2, \ldots, x_{n(x)}, (0,0), (0,0), \ldots\big) \\ &= \big((x_{1,1}, x_{1,2}), (x_{2,1}, x_{2,2}), \ldots, (x_{n(x),1}, x_{n(x),2}), (0,0), (0,0), \ldots\big) \end{aligned}$$

where the following two conditions shall hold:

(i) For every $x \in E_j$ exists a unique number $n(x) \geqq 0$ such that for all $k \in \mathbb{N}$:

$$k \leqq n(x) \Rightarrow x_{k,1} \in M \text{ and } k > n(x) \Rightarrow x_k = (0,0) \ .$$

(ii) If $x_k \neq (0,0)$ then $1 \leqq x_{k,2} \leqq S(x_{k,1})$.

The local queue length, i. e. the number of customers present at node $Q[j]$, is given by $n(x)$ (if the local state of this node is x). The situation at place number k is given by $x_k = (x_{k,1}, x_{k,2})$. Places with numbers $1 \leqq k \leqq n(x)$ are occupied. On the place with number k there is a customer of type $x_{k,1}$, at *stage* $x_{k,2}$ of his route. Places with numbers $k > n(x)$ are not occupied, for these places $x_k = (0,0)$ holds.

The common state space for the forward and the backward network with J nodes is a subset E of the product space $\prod_{j=1}^{J} E_j$ of the local state spaces E_j of the nodes $Q[j]$.

Remark: typically the state space E (i. e. the set of all feasible states) of a specific network is a subset of $\prod_{j=1}^{J} E_j$. For example: some types may not occur at specific nodes.

A typical state of the network is $x = (x_1, \ldots, x_J) \in E$, with

$$\begin{aligned} x_j &= \big(x_{j,1}, x_{j,2}, \ldots, x_{j,n(x_j)}, (0,0), (0,0), \ldots\big) \\ &= \big((x_{j,1,1}, x_{j,1,2}), (x_{j,2,1}, x_{j,2,2}), \ldots, (x_{j,n(x_j),1}, x_{j,n(x_j),2}), (0,0), (0,0), \ldots\big) \ . \end{aligned}$$

Here $n(x_j)$ is the local queue length at $Q[j]$.The total number of customers in system is $n(x) := \sum_{j=1}^{J} n(x_j)$ (using the same letter n for the local and the total queue length will not introduce difficulties).

The component (projection) x_j of the state x provides the following information about the state of service station $Q[j]$: The situation at place number $k \in \mathbb{N}$ is given by $x_{j,k} = (x_{j,k,1}, x_{j,k,2})$. Places with numbers $1 \leqq k \leqq n(x_j)$ are occupied. On the place with number k there is a customer of type $x_{j,k,1}$ at stage $x_{j,k,2}$ of his route. Places with numbers $k > n(x_j)$ are not occupied, for these places $x_{j,k} = (0,0)$.

Before proceeding, we introduce the insertion- and deletion-operators, acting on $F := \big(\mathbb{N}_0^2\big)^{\mathbb{N}}$. The insertion operator $I[(z_1, z_2), k] \colon F \to F$ with $(z_1, z_2) \in (M \times \mathbb{N}) \cup \{(0,0)\}$ and $k \in \mathbb{N}$ is defined by

$$\big(I[(z_1, z_2), k](x)\big)_l := \begin{cases} x_l & \text{if } 1 \leqq l \leqq k-1 \\ (z_1, z_2) & \text{if } l = k \\ x_{l-1} & \text{if } k+1 \leqq l \end{cases} \ . \tag{1}$$

The deletion operator $D_k \colon F \to F$, with $k \in \mathbb{N}$ is defined by

$$\big(D_k(x)\big)_l := \begin{cases} x_l & \text{if } 1 \leqq l \leqq k-1 \\ x_{l+1} & \text{if } k \leqq l \end{cases} \ . \tag{2}$$

2.1 Forward system

Consider a network consisting of the nodes $Q[1], \ldots Q[J]$. Every node has a waiting room of unlimited capacity. If there are n customers present at station $Q[j]$, then the customers occupying the first $C_j(n)$ places, called *departure places* or *departure positions*, are in service. That is, the *capacity function* $C_j \colon \mathbb{N}_0 \to \mathbb{N}_0$, where $C_j(0) = 0$, $C_j(n) > 0$ for every $n \in \mathbb{N}$ and $C_j(n(x_j)) \leqq n(x_j)$ holds, encodes the information from which places departures of customers are possible. The places with numbers $k \in \{C_j(n(x_j)), C_j(n(x_j)) + 1, \ldots, n(x_j)\}$ are occupied places where the customers are not in service (*waiting places*). From a waiting place no departure can happen by definition. Customers whose service demand is fulfilled are called *admissible for leaving*.

We allow an immediate feedback of customers: if a customer leaves the station he may move back to the same station. A feedback customer leaves the station, the gap in the queue is closed according to the shift protocol and only thereafter he joins the end of the queue, being now on the next stage on his path. Therefore a feedback customer is treated as a new arrival at the node.

Let the network be in state x. The transformation into the successor state y is determined by the following set of decisions. For all departure places of every node there are independent decisions, whether the customer at this place has admission to leave the station (because his service demand has been fulfilled) or not. These decisions are made according to Bernoulli distributions with node dependent parameter p_j at $Q[j]$, respectively. Moreover there is also a decision whether a new customer from the outside stands by for arrival and if this is the case, there is additionally a decision according to a probability law $\alpha(\cdot)$ on M determining this customer's type. These decisions are independent of another and independent of anything else. The customers from the outside arrive one by one in a state dependent Bernoulli input stream with parameter $b(n(x))$. A customer, standing by for arrival, arrives at his destination node (which is determined by routing scheme, to be described below), only if none of the customers at any departure place has admission to leave, otherwise he is rejected. A customer, admissible to leave, leaves his station only if no other customer at any departure place in the network has admission to leave and no new customer from the outside stands by for arrival. Therefore at any transition simultaneous arrivals and departures or multiple jumps at the same time instant are excluded.

The customers are distinguishable by their types, chosen from an at most countable type set M. Customers of the same type are indistinguishable. A customer of type $m \in M$ follows route

$$w(m)\colon (\underline{S(m)+1}) \cup \{0\} \to \underline{J+1}$$

where the route length function $S\colon M \to \mathbb{N}$ determines the route length $S(m)$ for a customer of type m. The route length $S(m)$ is the number of nodes a customer of type m has to visit during his journey through the network, feedback is counted as a new visit (This is no longer true in the case of unreliable nodes).

Further $w(m,j) \in \underline{J}$ for $j \in \underline{S(m)}$ and we set $w(m,0) = w\big(m, S(m)+1\big) = J+1$. Here the (node) number $J+1$ represents the outside (exterior) of the network, i.e., source and sink of customers. The family

$$(w(m,1), w(m,2), \ldots, w(m,S(m)))$$

is the ordered list of node numbers the type m customer runs through and $w(m,0) = w\big(m, S(m)+1\big) = J+1$ indicates, that the customer comes from the outside and after he has finished his route departs to the outside.

An arriving customer of type m joins the tail of the queue at his arrival station $Q[w(m,1)]$ (which may be a departure place already), moves through this node according to the service discipline FCFS (first come first served) with state dependent number of service (i. e *departure*) places. The next nodes on his route are $Q[w(m,2)], Q[w(m,3)]$ to $Q[w(m,S(m))]$. Finally, the customer leaves node $Q[w(m,S(m))]$ and jumps to the outside. If a departure of a customer takes place, the gap is closed according to the shift protocol, i. e. by shifting the customers behind the leaving customer one position up (without interchanging their order).

2.2 Backward system

The corresponding backward network, consisting of the nodes $Q[1], \ldots Q[J]$, works in the following way: Customers arrive one by one in a state dependent Bernoulli input stream with parameter $b(n(x))$. Every node has a waiting room of unlimited capacity. At every node $Q[j]$ there is only one departure place: departures only occur from the (occupied) place with highest number. New arrivals at this node take one of the *arrival places* indicated by the capacity function $\bar{C}_j := C_j$.

Given the system is in state y and an arrival occurs at node $Q[j]$, so that local queue length of the successor state x is $n(x_j) = n(y_j) + 1$, the first $C_j(n(y_j) + 1)$ places are the possible arrival places at this station for the new customer.

A description of the system's transitions can be given as follows. Let the system be in state y. The source's behaviour is determined by a decision for or against stand by of a new customer for arrival and, if this decision is positive, a second decision for the new customer's type $m \in M$ according to a probability law $\alpha(\cdot)$ on M. Furthermore the number of the arrival place $k \in \underline{C_i(n(y_i) + 1)}$ at this customer's arrival (destination) node $Q[i]$ must be fixed. The destination node $\overline{Q[i]}$ is determined by the routing, explained below. If the destination node equals the departure node a feedback takes place. If the destination node is a node within the network $(Q[1], \ldots, Q[J])$, the arrival place has to be selected. All these decisions are stochastically independent of another, given y. The decision for the arrival place number k obviously depends on the system's state y by the number of possible arrival places.

Although there is only one departure place at service station $Q[j]$, the departure regime there is determined by a number of $C_j(n(y_j))$ stochastically independent binary decisions, called *departure decisions*. A departure from node $Q[j]$ occurs if and only if no new customer stands by for arrival and exactly one of these $C_j(n(y_j))$ departure decisions is positive (value 1) and all other departure decisions at any other node have the value 0. If this is the case, the customer at position $n(y_j)$ leaves the node $Q[j]$, independent which of the individual departure decisions is positive.
Remark: Obviously it is possible to integrate the set of *departure decisions* into a single decision. But here it turns out that without the more detailed formalism we would not overcome the combinatorial complexity inherent in the later proofs. So the transition operators responsible for departures in the backward system have to be classified not only according to the departure process number but also to the arrival position at the destination node.

Customers in the backward system are classified by the same types as in the forward system, the set of all customer types is M. The direction of the customer's movement is in reversed with respect to his movement in the forward system. To be more specific:
The customer's way through the network is determined by his type with route

$$\bar{w}(m)\colon (\underline{\bar{S}(m) + 1}) \cup \{0\} \to \underline{J + 1},$$

with $\bar{w}(m, j) \in \underline{J}$ for $j \in \underline{\bar{S}(m)}$ and $\bar{w}(m, 0) = \bar{w}\big(m, \bar{S}(m) + 1\big) = J + 1$ by definition. The family $\big(\bar{w}(m, 1), \ldots, \bar{w}(m, \bar{S}(m))\big)$ prescribes the successively visited nodes of a customer of type m and $\bar{w}(m, 0) = \bar{w}\big(m, \bar{S}(m)+1\big) = J+1$ indicates that the customer comes from and goes to the outside. A jumping customer's destination node is completely determined by his type and his current stage on the actual route. We set $\bar{S} := S$ and $\bar{w}(m, i) := w(m, S(m) - i + 1)$. Hence a customer of type m starts his journey at node $Q[\bar{w}(m, 1)] = Q[w(m, S(m))]$, his last node in the forward system and departs from $Q[\bar{w}(m, S(m))] = Q[w(m, 1)]$, his entrance node in the forward system. (In the sequencing from the forward system the customers now move in the direction corresponding to decreasing stages.) A customer who has reached his last stage on his route (i. e. his stage number is one), leaves the system.

An arrival from the outside takes place if and only if a new customer stands by for arrival and all of the $\sum_{j=1}^{J} C_j(n(y_j))$ departure decisions have the result 0. If this is the case, the new customer is inserted at his destination node at the previously determined arrival place (this happens according to the shift protocol, that is, by shifting the customers standing on this place and on the places behind, without interchanging their order).

2.3 Steady state distribution

Now, as a special case of our main Theorem 3.1 (Section 3.3), we get the invariant measure and in case of ergodicity the steady state distribution of the network processes in the framework of completely reliable nodes, for both, the forward and backward system. It should be noticed that even the reduction of the main theorem to the standard case of reliable nodes includes an important generalisation of the results available in the literature: The arrival process at the network is state dependent via the total population size of the network.

Corollary 2.1. *Let X and $\bar{X}$ denote the homogeneous Markov chains with transition matrices p and $\bar{p}$ that describe the evolution over time of the forward and backward system, respectively. If X and $\bar{X}$ are ergodic then*

$$\pi(x) = \frac{1}{K} \left(\prod_{k=1}^{n(x)} \frac{b(k-1)}{1-b(k)} \right) \prod_{l=1}^{J} \left(\prod_{k=1}^{n(x_l)} \frac{\alpha(x_{l,k,1})}{C_l(k)} \right) \frac{q_l^{n(x_l)-C_l(n(x_l))}}{p_l^{n(x_l)}}, \quad x \in E. \tag{3}$$

is the steady state distribution of both X and $\bar{X}$, where K is the normalising constant (α is a distribution an the set of customers $\underline{M}$, describing the frequency of the occurrence of a type-m customer at an arrival instant, see pages 6 and 7).

Proof. Follows from Theorem 3.1. For the case of state independent arrival rates it is proved in [DS83]. □

Remark: [Slotted Aloha–type protocol] The interaction of the arrival and departure decisions which leads to the departure and arrival rules described above yields a slotted Aloha–type protocol. Such protocols are used to regulate usage of shared transmission media in communication systems, [Kle76], section 5.11, [Woo94], section 6.2. If there are $C(n)$ sources of traffic (stations), and the end of a thinktime (service) in position $i \in \{1, \ldots, C(n)\}$ indicates that a message has to be transmitted over the shared medium, then successive transmission is possible if and only if exactly one service ends. If more than one service ends, and more than one message is tried to be send at the same time instant, all those transmission trials are not successful and have to be repeated. A common regime to resolve the conflicts is that the sources retry at random to repeat sending. Due to the memoryless property of the geometrical service time distribution the blocking mechanism according to repetitive service is equivalent to what is known as communication blocking, [Per90], p.455.

2.4 Example: Single Station - Forward system

We have a single station, $Q[1]$, where customers arrive in a state dependent Bernoulli stream of intensity $b(n(x)) > 0$, where $n(x)$ is the number of customers at the station in state x.

The service station has a waiting room of unlimited capacity. We allow immediate feedback of customers: if a customer leaves the station he may reenter immediately.

As was mentioned, the route of a customer through the network is determined by his type. Here in case of a single station this means, for every type $m \in M$ there is a route length $S(m)$, with $S\colon M \to \mathbb{N}$ and a route $w(m)$,

$$w(m)\colon \{0, 1, 2, \ldots, S(m)+1)\} \to \{1, 2\}$$

where $w(m, j) = 1$ for $j \in \{1, 2, \ldots, S(m)\}$ and $w(m, 0) = w\big(m, S(m)+1\big) = 2$ (the outside of the network). The family $(w(m,1), w(m,2), \ldots, w(m, S(m))) = (1, 1, \ldots, 1)$ is the ordered list of node numbers the type m customer runs through. Clearly in case of a single station $S(m)$ is the number of visits, a customer of type m performs at station $Q[1]$, before leaving (and $S(m) - 1$ is just the number of feedbacks).

2.4.1 Transition operators

The definition of the transition operators is more comfortable if one treats the outside (acting as a source or a sink) of the network to some extent as a network node with associated number 2. The local queue length of the sink has the constant value ∞ and its capacity function is $C_2 \colon \bar{\mathbb{N}}_0 \to \{0,1\}$, $C_2(0) := 0$, $C_2(n) := 1$ for $n \in \bar{\mathbb{N}}$.

As transition operators we define (with $D(T)$ the domain of operator T)

(i) $T[m,A] \colon D(T[m,A]) \subset E \to E$ with $m \in M$ and $A \subset \underline{C_1(n(y))}$.

(ii) $T[A] \colon D(T[A]) \subset E \to E$ with $A \subset \underline{C_1(n(y))}$.

The operator $T[m,A]$ indicates that a new customer of type m stands by for arrival and that positive departure decisions are taken for the places with numbers $k \in A$. In an analogous way the operator $T[A]$ indicates that no new customer stands by for arrival and that positive departure decisions are taken for all places with numbers $k \in A$.

A pair (x,T) consisting of a state $x \in E$ and an operator $T \colon D(T) \subset E \to E$, where T stands for one of the operators $T[m,A]$ or $T[A]$, is called *permissible* and the operator T is called *permitted for the state* x, if $x \in D(T)$ holds.

For $x \in E$ let $\mathrm{Per}(x)$ be the set consisting of the operators T permitted for x and

$$\mathrm{Succ}(x) := \bigcup_{T \in \mathrm{Per}(x)} \{T(x)\}$$

be the set of *successor states* of x. The pairs $(x, T[m,A])$ with $m \in M$ and the pairs $(x, T[A])$ are permitted, iff $A \subset \underline{C_1(n(x))}$ holds (i. e. only customers on departure places are in service).

The operators $T[\varnothing]$, $T[A]$ with $|A| \geqq 2$ and $T[m,A]$ with $A \neq \varnothing$ map x onto x (this reflects the Aloha–type protocol of the system). The operators $T[m,\varnothing]$ and $T[\{k\}]$ are defined in the following way.

(i) $T[m,\varnothing]$. Arrival of a customer of type $m \in M$ at the service station. The new customer joins the end of the queue.

$$T[m,\varnothing](x) := (x_1, \ldots, x_{n(x)}, (m,1), (0,0), \ldots).$$

(ii) $T[\{k\}]$. Departure of the customer at place $k \in \underline{C_1(n(x))}$.

(a) Customer leaves the network to the outside: The destination node of the jumping customer from place k is $w(x_{k,1}, x_{k,2}+1) = 2$.

$$T[\{k\}](x) := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n(x)}, (0,0), (0,0), (0,0), \ldots).$$

This describes the case that no further feedback occurs.

(b) Feedback: The destination node of the jumping customer from place k is $w(x_{k,1}, x_{k,2}+1) = 1$

$$T[\{k\}](x) := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n(x)}, (x_{k,1}, x_{k,2}+1), (0,0), (0,0), \ldots).$$

This describes the case, that a further feedback occurs. The customer at place k leaves his place, the gap is closed according to the shift protocol, the customer's stage is increased by one and the customer joins the end of the queue. Thus a feedback customer enters the station according to the general entrance rules.

2.4.2 Transition operator probabilities

The Markovian description of the service station is now constructed in two steps. First, the probabilities $\hat{p}(x,T) \geqq 0$ for the application of permitted operators $T \in \mathrm{Per}(x)$ to state x are defined. The probabilities $\hat{p}(x,T)$ fulfil the condition

$$\sum_{T\in \mathrm{Per}(x)} \hat{p}(x,T) = 1.$$

Secondly, the one step transition matrix p on E is defined by

$$p(x,y) := \sum_{T\in \mathrm{Per}(x)} \hat{p}(x,T)\mathbf{1}[T(x)=y].$$

Remarks:

(1) The probabilities $\hat{p}$ reflect the internal structure of the (observable) transitions, p sums over all possible such "internal causes" (here observable refers to a comparison of predecessor and successor state).

(2) For $y \notin \mathrm{Succ}(x)$ there is no operator $T \in \mathrm{Per}(x)$ such that $T(x) = y$ holds, hence $p(x,y) = 0$.

(3) p is a stochastic matrix:

$$\sum_{y\in E} p(x,y) = \sum_{y\in E}\sum_{T\in \mathrm{Per}(x)} \hat{p}(x,T)\mathbf{1}[T(x)=y] = \sum_{T\in \mathrm{Per}(x)} \hat{p}(x,T) = 1 \ .$$

The operator transition probabilities $\hat{p}$ are for $x \in E$, $A \subset \underline{C_1(n(x))}$ and $m \in M$

$$\hat{p}(x,T[A]) := c(n(x))\Big(\prod_{i\in A} p\Big)\Big(\prod_{i\in \underline{C_1(n(x))}\setminus A} q\Big) = c(n(x))p^{|A|}q^{C_1(n(x))-|A|}, \tag{4}$$

$$\hat{p}(x,T[m,A]) := b(n(x))\alpha(m)\Big(\prod_{i\in A} p\Big)\Big(\prod_{i\in \underline{C_1(n(x))}\setminus A} q\Big) = b(n(x))\alpha(m)p^{|A|}q^{C_1(n(x))-|A|} \ , \tag{5}$$

where $|A|$ is the cardinality of A and $c(n(x)) := 1 - b(n(x))$.

The transition probabilities are built according to the rules described above. For the readers convenience we recall the rules for the present case: there are independent decisions for every departure place, whether the customer there has permission to leave (because his service demand has been accomplished) in accordance to Bernoulli distributions with the same parameter p. Furthermore there are independent decisions whether a new customer stands by for arrival and if so, of what type m the new customer is, in accordance to a Bernoulli distribution with state dependent parameter $b(n(x))$, given the network is in state x and the distribution α on the set M of customer types.

For every $x \in E$

$$\begin{aligned}
&\sum_{T\in \mathrm{Per}(x)} \hat{p}(x,T) \\
&= \sum_{A\in\mathcal{P}(\underline{C_1(n(x))})} \hat{p}(x,T[A]) + \sum_{m\in M}\sum_{A\in\mathcal{P}(\underline{C_1(n(x))})} \hat{p}(x,T[m,A]) \\
&= \sum_{A\in\mathcal{P}(\underline{C_1(n(x))})} c(n(x))p^{|A|}q^{C_1(n(x))-|A|} + \sum_{m\in M}\sum_{A\in\mathcal{P}(\underline{C_1(n(x))})} b(n(x))\alpha(m)p^{|A|}q^{C_1(n(x))-|A|} \\
&= \big(b(n(x))+c(n(x))\big)\sum_{l=0}^{C_1(n(x))}\binom{C_1(n(x))}{l}p^l q^{C_1(n(x))-l} = 1 \ .
\end{aligned}$$

2.4.3 One step transition matrix

The one step transition matrix p of the state dependent Bernoulli server with feedback ("forward system") is for $x, y \in E$

$$p(x,y) = \sum_{T\in\mathrm{Per}(x)} \hat{p}(x,T)\mathbf{1}[T(x)=y] \ .$$

To compute the probability $p(x,y)$ for a pair $(x,y) \in E^2$ one has to find all operators $T \in \mathrm{Per}(x)$ with $T(x) = y$ and to add their corresponding probabilities $\hat{p}(x,T)$. If $y \notin \mathrm{Succ}(x)$, then $p(x,y) = 0$. If $y \in \mathrm{Succ}(x)$ then it is possible to classify transitions with respect to the queue length. There are four different types of transitions named fi, $i \in \underline{4}$, where the "f" indicates that the transition takes place in the forward system.

(f1) Let $n(y) = n(x)+1$. Then an arrival has occurred. This implies $y = T[y_{n(y),1}, \varnothing](x)$, because the new customer's type m is determined by $m = y_{n(y),1}$ (new arrivals join the end of the queue). Hence in this case

$$p(x,y) = \alpha(y_{n(y),1})b(n(x))q^{C_1(n(x))} \ .$$

(f2) Let $n(y) = n(x)-1$. Then a departure to the outside of the network has occurred. This implies $y = T[\{k\}](x)$ for some $k \in \underline{C_1(n(x))}$, where the number k of the actual departure place in general is not determined uniquely. Let $G_{\mathrm{Depart}}(x,y)$ denote the set of all possible departure places $k \in \underline{C_1(n(x))}$, which can trigger a transition $x \to y$. That is, $k \in G_{\mathrm{Depart}}(x,y) \Leftrightarrow T[\{k\}](x) = y$. Using the deletion operators, see (2) on page 5, on has

$$G_{\mathrm{Depart}}(x,y) = \big\{k \in \underline{C_1(x)} \mid D_k(x) = D_{n(x)}(y)\big\} \ . \tag{6}$$

We denote the set on the rhs of (6) for any pair $(x,y) \in E^2$ as

$$G(x,y) := \big\{k \in \underline{C_1(x)} \mid D_k(x) = D_{n(x)}(y)\big\}. \tag{7}$$

Consequently,

$$p(x,y) = c(n(x))g(x,y)pq^{C_1(n(x))-1} \ ,$$

where $g(x,y) := |G(x,y)|$ is the number of relevant departure places.

(f3) Let $n(y) = n(x)$ and $x \neq y$. Then a feedback has occurred. As it was the case in situation (f2), the number k of the actual departure place in general is not determined uniquely. Let $G_{\mathrm{Feed}}(x,y)$ denote the set of all possible departure places $k \in \underline{C_1(n(x))}$, which can trigger

a transition $x \to y$. That is, $k \in G_{\mathrm{Feed}}(x,y) \Leftrightarrow T[\{k\}](x) = y$. Note that $G_{\mathrm{Depart}}(x,y)$ and $G_{\mathrm{Feed}}(x,y)$ are defined only for special pairs $(x,y) \in E^2$. But using the deletion operators also in this case one has

$$G_{\mathrm{Feed}}(x,y) = \{k \in \underline{C_1(x)} \mid D_k(x) = D_{n(x)}(y)\} = G(x,y) \ . \tag{8}$$

Then, as in the (f2) situation

$$p(x,y) = c(n(x))g(x,y)pq^{C_1(n(x))-1}.$$

(f4) Finally $p(x,x)$ is given by

$$\begin{aligned}
& p(x,x) \\
&= \hat{p}(x,T[\varnothing]) + \sum_{\substack{A \subset \underline{C_1(n(x))} \\ |A| \geqq 2}} \hat{p}(x,T[A]) + \sum_{m \in M} \sum_{\substack{A \subset \underline{C_1(n(x))} \\ A \neq \varnothing}} \hat{p}(x,T[m,A]) \\
&= \hat{p}(x,T[\varnothing]) + \sum_{l=2}^{C_1(n(x))} \sum_{\substack{A \subset \underline{C_1(n(x))} \\ |A|=l}} \hat{p}(x,T[A]) + \sum_{m \in M} \sum_{l=1}^{C_1(n(x))} \sum_{\substack{A \subset \underline{C_1(n(x))} \\ |A|=l}} \hat{p}(x,T[m,A]) \\
&= c(n(x))q^{C_1(n(x))} + c(n(x)) \sum_{l=2}^{C_1(n(x))} \binom{C_1(n(x))}{l} p^l q^{C_1(n(x))-l} \\
&\qquad + b(n(x)) \sum_{l=1}^{C_1(n(x))} \binom{C_1(n(x))}{l} p^l q^{C_1(n(x))-l} \\
&= c(n(x))q^{C_1(n(x))} + c(n(x)) \left(1 - q^{C_1(n(x))} - C_1(n(x))pq^{C_1(n(x))-1}\right) \\
&\qquad + b(n(x)) \left(1 - q^{C_1(n(x))}\right) \\
&= 1 - b(n(x))q^{C_1(n(x))} - c(n(x))C_1(n(x))pq^{C_1(n(x))-1} \ .
\end{aligned}$$

2.5 Example continued: Single Station - Backward System

2.5.1 Transition operators

As transition operators of the backward system we define

(i) $\bar{T}[m,k,A]\colon D(\bar{T}[m,k,A]) \subset E \to E$, with $m \in M$, $k \in \underline{C_1(n(y)+1)}$ and $A \subset \underline{C_1(n(y))}$.

(ii) $\bar{T}[A]\colon D(\bar{T}[A]) \subset E \to E$ with $A \subset \underline{C_1(n(y))}$ and $|A| \neq 1$.

(iii) $\bar{T}[\{i\},k]\colon D(\bar{T}[\{i\},k]) \subset E \to E$ with $i \in \underline{C_1(n(y))}$ and $k \in \underline{C_j(n(y_j)+1-\delta(1,j))}$.

The operator $\bar{T}[m,k,A]$ indicates that a new customer of type m stands by for arrival at position k at $Q[1]$ and that the departure decisions (with numbers) $i \in A$ are positive.

The operator $\bar{T}[A]$ indicates that no new customer stands by for arrival and that the departure decisions $i \in A$ are positive.

The operator $\bar{T}[\{i\},k]$ indicates that no new customer stands by for arrival and that the departure decision i is positive and the arrival place at the destination node has number k.

A pair $(y, \bar{T})$ consisting of a state $y \in E$ and an operator $\bar{T}$: $D(\bar{T}) \subset E \to E$, where T stands for one of the above operators, is called permissible and the operator $\bar{T}$ is called permitted for the state y, if $y \in D(\bar{T})$ holds.

For $y \in E$ let $\overline{\mathrm{Per}}(y)$ be the set consisting of the operators $\bar{T}$ permitted for y and

$$\overline{\mathrm{Succ}}(y) := \bigcup_{\bar{T} \in \overline{\mathrm{Per}}(x)} \{\bar{T}(x)\}$$

be the set of successor states of y with respect to the backward system.

The transition operators are defined as follows.

(i)
$$\bar{T}[m, k, A](y) := y \text{ for } A \neq \varnothing \ .$$

Application of $\bar{T}[m, k, \varnothing]$ results in an arrival of a type m customer from the outside. The new customer occupies place k at his destination node, his stage is $s = S(m)$:

$$\bar{T}[m, k, \varnothing](y) := (y_1, \ldots, y_{k-1}, (m, S(m)), y_k, \ldots, y_{n(y)}, (0, 0), \ldots) \ .$$

(ii) $\bar{T}[A]$ where $A \subset \underline{C_1(n(y))}$ and $|A| \neq 1$. These operators indicate that no new customer stands by for arrival and that the departure decisions with numbers $i \in A$ are positive. They are defined by
$$\bar{T}[A](y) := y$$
(and therefore specification of arrival places is not necessary).

(iii) $\bar{T}[\{i\}, k]$ where $i \in \underline{C_1(n(y))}$ and $k \in \underline{C_j(n(y_j) + 1 - \delta(1, j))}$. Here $j = w(y_{n(y),1}, y_{n(y),2} - 1)$ is the number of the jumping customer's destination node, that is $Q[2]$ (jump to the outside) or $Q[1]$ in case of feedback of the jumping customer.

(a) Customer leaves the network. Because of the definition of the route $\bar{w}$ a jump to the outside occurs if and only if $y_{n(y),2} = 1$ holds. That is, the jumping customer has reached his last stage. If the operator $\bar{T}[\{i\}, k]$ is permitted for y, $k \in \underline{C_2(\infty)}$ must hold, hence $k = 1$. That is, only the operators $\bar{T}[\{i\}, 1]$, $i \in \underline{C_1(n(y))}$ effect a jump of a customer to the outside.
$$\bar{T}[\{i\}, 1](y) := (y_1, \ldots, y_{n(y)-1}, (0, 0), (0, 0), (0, 0), \ldots).$$

(b) Feedback. Let $y_{n(y),2} > 1$. In this case the customer occupying position $n(y)$ has not reached the last stage on his route and his destination node is inside the network. In the special case of a single station a feedback occurs. In particular, if the successor state is designated as x, $n(x) = n(y)$ and $x \neq y$ holds (because exactly one stage is diminished by one).
$$\bar{T}[\{i\}, k](y) := \big(y_1, \ldots, y_{k-1}, (y_{n(y),1}, y_{n(y),2} - 1), y_k, \ldots, y_{n(y)-1}, (0, 0), (0, 0), \ldots\big).$$

2.5.2 Transition operator probabilities

Let $y \in E$, $A \subset \underline{C_1(n(y))}$, $m \in M$ and $k \in \underline{C_1(n(y) + 1)}$. The transition operator probabilities are

$$\hat{\bar{p}}(y, \bar{T}[m, k, A]) := b(n(y))\alpha(m)\frac{1}{C_1(n(y) + 1)} p^{|A|} q^{C_1(n(y))-|A|} \ ,$$

and for $A \subset \underline{C_1(n(y))}$ and $|A| \neq 1$

$$\hat{\bar{p}}(y, T[A]) := c(n(y)) p^{|A|} q^{C_1(n(x))-|A|}.$$

If the customer occupying position $n(y)$ has reached the last stage, $y_{n(y),2} = 1$, then

$$\hat{\bar{p}}(y, \bar{T}[\{i\}, 1]) := c(n(y))pq^{C_1(n(y))-1},$$

where $i \in \underline{C_1(n(y))}$ must hold.
If the customer occupying position $n(y)$ has not reached the last stage, $y_{n(y),2} > 1$, then

$$\hat{\bar{p}}(y, \bar{T}[\{i\}, k]) := c(n(y))\frac{1}{C_1(n(y))}pq^{C_1(n(y))-1},$$

where $i \in \underline{C_1(n(y))}$ and $k \in \underline{C_1(n(y))}$ must hold. Obviously $\hat{\bar{p}} \geqq 0$ holds. If the customer on position $n(y)$ has reached his last stage (departure to the outside), then

$$\begin{aligned}
\sum_{\bar{T}\in\overline{\mathrm{Per}}(y)} \hat{\bar{p}}(y, \bar{T}) &= \sum_{m\in M} \sum_{k\in\underline{C_1(n(y)+1)}} \sum_{A\in\mathcal{P}(\underline{C_1(n(y))})} \hat{\bar{p}}(y, \bar{T}[m, k, A]) + \sum_{\substack{A\in\mathcal{P}(\underline{C_1(n(y))}) \\ |A|\neq 1}} \hat{\bar{p}}(y, \bar{T}[A]) \\
&= \quad + \sum_{\substack{A\in\mathcal{P}(\underline{C_1(n(y))}) \\ |A|=1}} \hat{\bar{p}}(y, \bar{T}[A, 1]) \\
&= \sum_{m\in M} \sum_{k\in\underline{C_1(n(y)+1)}} \sum_{A\in\mathcal{P}(\underline{C_1(n(y))})} b(n(y))\alpha(m)\frac{1}{C_1(n(y)+1)}p^{|A|}q^{C_1(n(y))-|A|} \\
&\quad + \sum_{\substack{A\in\mathcal{P}(\underline{C_1(n(y))}) \\ |A|\neq 1}} c(n(y))p^{|A|}q^{C_1(n(x))-|A|} + \sum_{i=1}^{C_1(n(y))} c(n(y))pq^{C_1(n(y))-1} \\
&= \big(b(n(y)) + c(n(y))\big) \sum_{l=0}^{C_1(n(y))} \binom{C_1(n(y))}{l} p^l q^{C_1(n(y))-l} = 1.
\end{aligned}$$

If the customer on position $n(y)$ has not reached his last stage (feedback), then

$$\begin{aligned}
&\sum_{\bar{T}\in\overline{\mathrm{Per}}(y)} \hat{\bar{p}}(y, \bar{T}) \\
&= \sum_{m\in M} \sum_{k\in\underline{C_1(n(y)+1)}} \sum_{A\in\mathcal{P}(\underline{C_1(n(y))})} \hat{\bar{p}}(y, \bar{T}[m, k, A]) + \sum_{\substack{A\in\mathcal{P}(\underline{C_1(n(y))}) \\ |A|\neq 1}} \hat{\bar{p}}(y, \bar{T}[A]) \\
&\quad + \sum_{\substack{A\in\mathcal{P}(\underline{C_1(n(y))}) \\ |A|=1}} \sum_{k\in\underline{C_1(n(y)+1)}} \hat{\bar{p}}(y, \bar{T}[A, 1]) \\
&= \sum_{m\in M} \sum_{k\in\underline{C_1(n(y)+1)}} \sum_{A\in\mathcal{P}(\underline{C_1(n(y))})} b(n(y))\alpha(m)\frac{1}{C_1(n(y)+1)}p^{|A|}q^{C_1(n(y))-|A|} \\
&\quad + \sum_{\substack{A\in\mathcal{P}(\underline{C_1(n(y))}) \\ |A|\neq 1}} c(n(y))p^{|A|}q^{C_1(n(x))-|A|} + \sum_{i=1}^{C_1(n(y))} \sum_{k\in\underline{C_1(n(y))}} \frac{c(n(y))}{C_1(n(y))}pq^{C_1(n(y))-1} \\
&= \big(b(n(y)) + c(n(y))\big) \sum_{l=0}^{C_1(n(y))} \binom{C_1(n(y))}{l} p^l q^{C_1(n(y))-l} = 1 \ .
\end{aligned}$$

2.5.3 One step transition matrix

Let $x, y \in E$. The one step transition matrix $\bar{p}$ is

$$\bar{p}(y,x) := \sum_{\bar{T} \in \overline{\mathrm{Per}}(y)} \hat{\bar{p}}(y,\bar{T}) \mathbf{1}[\bar{T}(y) = x].$$

There are four different types of transitions named bi, $i \in \underline{4}$, where the "b" indicates that the transition takes place in the backward system.

If $x \notin \overline{\mathrm{Succ}}(y)$, then $\bar{p}(y,x) = 0$ holds. Let $x \in \overline{\mathrm{Succ}}(y)$ and $x \neq y$. Then either $n(x) = n(y)+1$ or $n(x) = n(y)-1$ holds, or $n(x) = n(y)$ holds, the latter is only possible in the case of feedback.

(b1) Let $n(x) = n(y) - 1$. Then a departure to the outside has happened. Then $x = \bar{T}[\{i\}](y)$ holds for every $i \in \underline{C_1(n(y))}$, and therefore

$$\bar{p}(y,x) = c(n(y)) C_1(n(y)) p q^{C_1(n(y))-1}.$$

(b2) Let $n(x) = n(y) + 1$. Then an arrival from the outside has happened. The type m of the arriving customer can be determined by comparing the states y and x (the difference of the number of customers of a given type with respect to states y and x is one for type m and 0 for all other types). Necessarily $x = \bar{T}[m,k,\varnothing](y)$ holds for some $k \in \underline{C_1(n(y)+1)}$. Denote the set of possible arrival place numbers by $\bar{G}_{\mathrm{Arriv}}(y,x)$. That is, $k \in \bar{G}_{\mathrm{Arriv}}(y,x)$ iff y is transformed into x by inserting the pair $(m, S(m))$ at position $k \in \underline{C_1(n(y)+1)}$ of y.

We define for $(z_1,z_2) \in (M \times \mathbb{N}) \cup \{(0,0)\}$ and all pairs $(y,x) \in E^2$ the set

$$\bar{G}_{(z_1,z_2)}(y,x) := \left\{ k \in \underline{C_1(n(x))} \mid I((z_1,z_2),k)(y) = x \right\} , \tag{9}$$

where the insertion-operator $I((z_1,z_2),k)$ is defined by (1) on page 5.

For a pair (y,x) corresponding to a (b2)-transition obviously

$$\bar{G}_{\mathrm{Arriv}}(y,x) = \bar{G}_{(m,S(m))}(y,x).$$

But also

$$\bar{G}_{\mathrm{Arriv}}(y,x) = \left\{ k \in \underline{C_1(x)} \mid D_k(x) = D_{n(x)}(y) \right\} = G(x,y)$$

holds for such a pair, and therefore $|\bar{G}_{\mathrm{Arriv}}(y,x)| = g(x,y)$. Hence

$$\bar{p}(y,x) = b(n(y)) \alpha(m) \frac{g(x,y)}{C_1(n(x))} q^{C_1(n(y))}.$$

(b3) Let $n(x) = n(y)$ and $x \neq y$. Then a feedback has occurred and $x = \bar{T}[\{i\},k](y)$ holds for every $i \in \underline{C_1(n(y))}$ and exactly these arrival place numbers k for which y is transformed into x by the feedback operation $\bar{T}[\{i\},k]$. Let $\bar{G}_{\mathrm{Feed}}(y,x)$ be the set of possible arrival place numbers, that is

$$\bar{G}_{\mathrm{Feed}}(y,x) := \{ k \in \underline{C_1(x)} \mid I((y_{n(y),1}, y_{n(y),2} - 1), k)(D_{n(y)}(y)) = x \},$$

using the insertion-operator. For a pair (y,x) corresponding to a (b3)-transition

$$\bar{G}_{\mathrm{Feed}}(y,x) = \{ k \in \underline{C_1(x)} \mid D_k(x) = D_{n(y)}(y) \} = G(x,y) \tag{10}$$

holds, and therefore

$$\begin{aligned} \bar{p}(y,x) &= \sum_{i \in \underline{C_1(n(y))}} \sum_{k \in \bar{G}_{\mathrm{Feed}}(y,x)} c(n(y)) \frac{1}{C_1(n(y))} p q^{C_1(n(y))-1} \\ &= c(n(y)) g(x,y) p q^{C_1(n(y))-1}. \end{aligned}$$

(b4) Finally $\bar{p}(y,y)$ is defined by

$$\begin{aligned}
\bar{p}(y,y) &:= \sum_{m\in M}\sum_{k=1}^{C_1(n(y)+1)}\sum_{A\subset\overline{\substack{C_1(n(y))\\ A\neq\varnothing}}} b(n(y))\alpha(m)\frac{1}{C_1(n(y)+1)}p^{|A|}q^{C_1(n(y))-|A|} \\
&\quad + \sum_{A\subset\overline{\substack{C_1(n(y))\\ |A|\neq 1}}} c(n(y))p^{|A|}q^{C_1(n(y))-|A|} \\
&= b(n(y))\sum_{l=1}^{C_1(n(y))}\binom{C_1(n(y))}{l}p^l q^{C_1(n(y))-l} \\
&= + c(n(y))\left(q^{C_1(n(y))} + \sum_{l=2}^{C_1(n(y))}\binom{C_1(n(y))}{l}p^l q^{C_1(n(y))-l}\right) \\
&= 1 - b(n(y))q^{C_1(n(y))} - c(n(y))C_1(n(y))pq^{C_1(n(y))-1}.
\end{aligned}$$

Steady state distribution

Corollary 2.2. *([Dad01a], Theorem 5.8) Let X and $\bar{X}$ denote the homogeneous Markov chains corresponding to the transition matrices p and $\bar{p}$, respectively. Then ψ, given by*

$$\psi(x) := \left(\prod_{k=1}^{n(x)}\frac{b(k-1)\alpha(x_{k,1})}{c(k)C_1(k)}\right)\frac{q^{n(x)-C_1(n(x))}}{p^{n(x)}}, \tag{11}$$

is an invariant measure of both chains.

Remark: If the normalising constant

$$G := \sum_{x\in E}\psi(x)$$

is finite, then the probability distribution $\pi := G^{-1}\psi$ is the unique stationary and limiting distribution of X and $\bar{X}$. Taking π as initial distribution the processes are time reversals of another. This also shows that the depicted systems are time reversals of another.

3 Networks with unreliable feedback nodes

For the networks of section 2 we consider the situation that nodes are unreliable and may break down. The nodes take two availability status (AS) 0 and 1, indicating a perfect functioning node or complete failure of the node, respectively. We call the availability status 0 *up status* and the availability status 1 *down status*.

The nodes may fail to work from time to time randomly. The up and down periods are geometrically distributed with node specific parameters. The customers' routing remains deterministic in the sense, that it is determined by a type dependent route. But the destination node of a jumping customer depends also on the the current state of the network via the availability of the nodes at the time instant when the jump occurs. The destination node of a jumping customer is the next node on his route which is in up status. Therefore the observed route of a customer depends on the network's evolution during his presence in the network.

Also we have to specify rules how the nodes change their availability status.

For every time instant there is for every node a decisions whether the availability status of that node changes or not. These decisions are independent of another and independent of the service and arrival decisions and distributed according to node specific Bernoulli distributions. Thus the holding times of a node in a given availability status are geometrically distributed.

The details of the availability change and the resulting effects on the organisation of queues and the customer's movement are described in the following sections. In particular, simultaneous breakdowns or repairs of different nodes are possible. The network's state incorporates the information about the network's availability status.

Availability status The network's availability status is given by the availability status of the individual nodes. Therefore the network's availability status x is an element of $\{0,1\}^J$ where x_i is the availability status of node $Q[i]$, $i \in \underline{J}$. The nodes change their availability in the following way. For a network state transition there is for every node a decisions whether this node does change his availability status. These decisions are independent and independent from anything else (queue length, customer types etc.). The probabilities for a availability change of node $Q[i]$ are $a_i(0,1)$ for the transition *up to down*, $a_i(0,0)$ for the transition *up to up*, $a_i(1,0)$ for the transition *down to up* and $a_i(1,1)$ for the transition *down to down*. We have $a_i(0,0) + a_i(0,1) = 1$, $a_i(1,0) + a_i(1,1) = 1$ and assume either $a_i(\cdot,\cdot) \in (0,1)$, or $a_i(0,0) = 1$. If $a_i(0,0) = 1$ we have the case of a node which does not break down if the up status has been reached. Clearly this has an effect on the irreducibility of the state space. The breakdown and repair probabilities are node dependent. Let $\mathrm{ac} \in \{0,1\}^J$ be the vector of the availability change decisions: if $\mathrm{ac}_i = 1$ the availability status of $Q[i]$ is changed, otherwise if $\mathrm{ac}_i = 0$ the availability status remains the same.

We denote the addition modulo 2 on the set $\{0,1\}$ as well as the componentwise defined addition modulo 2 on the set $\{0,1\}^J$ by $\oplus$.

Then, given a network availability status x and an availability change $\mathrm{ac} \in \{0,1\}^J$, the new network availability status is given by $y = x \oplus \mathrm{ac}$. Note that $x \oplus x = 0$ for all $x \in \{0,1\}^J$.

State space The network availability status is incorporated into the state space by defining network states as elements

$$x = ({}^1x, {}^2x) = \big(({}^1x_1, \ldots, {}^1x_J), ({}^2x_1, \ldots, {}^2x_J)\big)$$

from the set

$$\hat{E} := \left(\prod_{j=1}^{J} E_j\right) \times \{0,1\}^J = E \times \{0,1\}^J \ , \tag{12}$$

where $\prod_{j=1}^{J} E_j$ is defined on page 5. The second component ${}^2x = ({}^2x_1, \ldots, {}^2x_J)$ of x is the network availability status. Node $Q[i]$ is in availability status ${}^2x_i$. Note that $\hat{E}$ is in general not the minimal state space.

In detail: a typical state $x = ({}^1x, {}^2x) \in \hat{E}$ of the network now is of the following form.

$${}^1x = ({}^1x_1, \ldots, {}^1x_J) = \prod_{j=1}^{J} E_j, \tag{13}$$

$$\begin{aligned} {}^1x_j &= \big({}^1x_{j,1}, {}^1x_{j,2}, \ldots, {}^1x_{j,n(x_j)}, (0,0), (0,0), \ldots\big) \\ &= \big(({}^1x_{j,1,1}, {}^1x_{j,1,2}), ({}^1x_{j,2,1}, {}^1x_{j,2,2}), \ldots, ({}^1x_{j,n(x_j),1}, {}^1x_{j,n(x_j),2}), (0,0), (0,0), \ldots\big) \ . \end{aligned}$$

and

$$^2x = (^2x_1, \ldots, ^2x_J) \in E = \{0,1\}^J. \tag{14}$$

3.1 Forward system

Assume the network is in state x at time t. For determining the successor state y at time $t+1$ we need the following data.

(1) The (values of) the departure decisions for the departure places.

(2) The type of the new customer if some customer stands by for arrival (note that the destination nodes are determined by the customer's route, specified below).

(3) The availability change vector ac.

The state transitions are according to the following rules.

(1) If a node is in down status at time instant t neither arrivals nor departures are possible at this node.

(2) Breakdown of some nodes may happen jointly with an arrival to or a departure from this node. If a node breaks down (the current availability status at time instant t is 0 and a availability status change happens) an arrival or a departure at the same time instant is still possible. After reorganisation of the queue the state of this node is frozen (and the stored information is preserved).

(3) If a node is in down status at time instant t and a availability status change happens, neither arrivals nor departures are possible.

Thus reorganisation of the queues depends only on the current state x and is independent of the availability status *change* ac. We can look upon the transition as first reorganising the queue and then changing the availability.

The route of a customer through the network is determined by its type. For a customer of type m there is a route

$$w(m)\colon \underline{(S(m)+1)} \cup \{0\} \to \underline{J+1},$$

with $S(m) \in \mathbb{N}$ and $w(m,j) \in \underline{J}$ for every $j \in \underline{S(m)}$ and $w(m,0) = w\big(m, S(m)+1\big) = J+1$. The family $(w(m,1), w(m,2), \ldots, w(m,S(m)))$ is the ordered list of nodes the customer of type m wants to visit successively. But the following rule has to be obeyed additionally: if a customer of type-stage description (m,s) leaves node $Q[w(m,s)]$, his destination node is the next node (with respect to increasing stage numbers) on his route, which is in up status. Nodes in down status on higher stages than s are skipped (*skipping rule*). By $w(m,0) = w(m,S(m)+1) = J+1$ it is ensured that customers come from the outside and after traversing their route leave the net to the outside. The outside, represented by node $Q[J+1]$, cannot break down by definition. The pair (m,s), consisting of customer type $m \in M$ and stage number $s \in \underline{S(m)}$ is called *generalised type*.

For $x \in E$ and a pair (m,s), with $m \in M$ and $s \in \{0,1,\ldots,S(m)\}$ let

$$A(x,(m,s)) := \min\{t \mid s < t \leqq S(m), {}^2x_{w(m,t)} = 0\}$$

if $\{t \mid s < t \leqq S(m), {}^2x_{w(m,t)} = 0\} \neq \varnothing$ and $A(x,(m,s)) := S(m)+1$ otherwise. If the network is in state x, the destination node of a type m customer on stage s, $s \in \underline{S(m)}$, leaving node $Q[w(m,s)]$ is

$w\big(m, A(x,(m,s))\big)$. If $A(x,(m,s)) \leqq S(m)$, the customer joins the tail of the queue at his destination node as a type m customer on stage $A(x,(m,s))$. Otherwise if $A(x,(m,s)) = S(m)+1$ he leaves the network. The gap caused by a departing customer is closed according to the shift-protocol. A type m customer arriving from the outside joins the tail of the queue at his destination node $Q[w(m, A(x,(m,0)))]$. It is possible, that none of the nodes on his route $w(m)|_{\underline{J}}$ is in up status (with respect to the current network state x). In this case the new customer is rejected. This is the case iff ${}^2x_{w(m,s)} = 1$ for all $s \in \underline{S(m)}$. Then $A(x,(m,0)) = S(m)+1$ holds and consequently $w\big(m, A(x,(m,0))\big) = J+1$.

For network state x, type $m \in M$ and stage $s \in \{0,1,\ldots,S(m)\}$ let

$$U\big(x,(m,s)\big) := \{i \in \underline{J} \mid i = w(m,t) \text{ for some } t \in \{s+1,\ldots,S(m)\} \text{ and } {}^2x_i = 0\}$$

be the set of nodes on the route of type m customer at stages higher than s, which in state x are in up status;

$$\mathrm{REJ}(x) := \{m \in M \mid U\big(x,(m,0)\big) = \varnothing\},$$

be the set of customers which are rejected when finding the network in state x at their arrival;

$$\mathrm{UP}(x) := \{i \in \underline{J} \mid {}^2x_i = 0\}$$

be the set of numbers of working nodes (nodes in up status) and

$$\mathrm{UP}_{\mathrm{OCC}}(x) := \{i \in \underline{J} \mid n(x_i) > 0,\ {}^2x_i = 0\}$$

be the set of numbers of occupied and working nodes.

3.1.1 Transition operators

The network transition operators are

(i) Arrival of type-m customer at the network jointly with availability status change ac.

$$\hat{T}[m,\mathrm{ac}]\colon \hat{E} \to \hat{E}, \qquad T[m]\colon \hat{E} \to E,$$
$$\hat{T}[m,\mathrm{ac}](x) := \big(T[m](x), {}^2x \oplus \mathrm{ac}\big),$$

The operator $T[m]$ is componentwise for $l \in \underline{J}$ defined as follows. If $l \neq w(m, A(x,(m,0)))$, then

$$\big(T[m](x)\big)_l := {}^1x_l. \tag{15}$$

If $l = w(m, A(x,(m,0)))$, then

$$\big(T[m](x)\big)_l := \big({}^1x_{l,1}, {}^1x_{l,2}, \ldots, {}^1x_{l,n(x_l)}, (m, A(x,(m,0))), (0,0), (0,0), \ldots\big). \tag{16}$$

The domain of the operator $\hat{T}[m,\mathrm{ac}]$ is $\hat{E}$, that is, every pair $(x, \hat{T}[m,\mathrm{ac}])$ is permitted, where $x \in \hat{E}$, $m \in M$ and $\mathrm{ac} \in \{0,1\}^J$. Note that the case ${}^1(\hat{T}[m,\mathrm{ac}](x)) = T[m](x) = {}^1x$ is possible. This happens, iff ${}^2x_{w(m,s)} = 1$ for all $s \in \underline{S(m)}$: the new customer of type m is rejected, because all nodes $Q[w(m,s)]$, $s \in \underline{S(m)}$, on his route are in down status in network state x.

(ii) Departure of the customer from departure place k of working node $Q[i]$ jointly with availability status change ac.

Three cases are possible: (a1) the jumping customer leaves the network, (a2) the (uniquely determined) destination node is inside the network and differs from the departure node, (b) a feedback occurs.

In any of these cases the number of the destination node is $w({}^1x_{i,k,1}, A(x, ({}^1x_{i,k})))$.

$$\hat{T}[i,k,\mathrm{ac}]\colon D(T[i,k,\mathrm{ac}]) \subset \hat{E} \to \hat{E}, \qquad T[i,k]\colon D(T[i,k]) \subset \hat{E} \to E,$$

$$\hat{T}[i,k,\mathrm{ac}](x) := \big(T[i,k](x), {}^2x \oplus \mathrm{ac}\big).$$

The pair $(x, \hat{T}[i,k,\mathrm{ac}])$ is permitted if $i \in \mathrm{UP}_{\mathrm{OCC}}(x)$ and $k \in \underline{C_i(n(x_i))}$ holds. The operators $T[i,k]$ are defined through their components $(T[i,k](x))_l$ for $\overline{l \in \underline{J}}$ in the following way.

(a) Let $j = w({}^1x_{i,k,1}, A(x, ({}^1x_{i,k}))) \neq i$. Departure of customer at departure place k of node $Q[i]$, destination node of the jumping customer is $Q[j] \neq Q[i]$. Then

$$(T[i,k](x))_l = \begin{cases} {}^1x_l & \text{if } l \neq i \text{ and } l \neq j, \\ ({}^1x_{i,1}, \ldots, {}^1x_{i,k-1}, {}^1x_{i,k+1}, \ldots, {}^1x_{i,n(x_i)}, (0,0), \ldots) & \text{if } l = i. \end{cases}$$

And

$$(T[i,k](x))_j = ({}^1x_{j,1}, \ldots, {}^1x_{j,n(x_l)}, ({}^1x_{i,k,1}, A(x, ({}^1x_{i,k}))), (0,0), \ldots)\ . \tag{17}$$

(b) Let $i = w({}^1x_{i,k,1}, A(x, ({}^1x_{i,k})))$. Departure of customer at departure place k of node $Q[i]$, destination node of the jumping customer is $Q[i]$ (feedback). Then for $l \neq i$

$$(T[i,k](x))_l := {}^1x_l\ , \tag{18}$$

and

$$(T[i,k](x))_i := ({}^1x_{i,1}, \ldots, {}^1x_{i,k-1}, {}^1x_{i,k+1}, \ldots, {}^1x_{i,n(x_i)}, ({}^1x_{i,k,1}, A(x, ({}^1x_{i,k}))), (0,0), \ldots) \tag{19}$$

(Here shortcutting some broken down intermediate nodes is covered.

(iii) Availability status change ac, no arrival, no departure. This operator is defined on $\hat{E}$ by

$$\hat{T}[\mathrm{ac}](x) := \big({}^1x, {}^2x \oplus \mathrm{ac}\big), \qquad \mathrm{ac} \in \{0,1\}^J.$$

3.1.2 Transition operator probabilities

The transition operator probabilities are

$$\hat{p}(x, \hat{T}[m,\mathrm{ac}]) := b(n(x))\alpha(m) \left(\prod_{l \in \mathrm{UP}(x)} q_l^{C_l(n(x_l))} \right) \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2x_l \oplus \mathrm{ac}_l) \right) \tag{20}$$

for $m \in M$ and $\mathrm{ac} \in \{0,1\}^J$, and

$$\hat{p}(x, \hat{T}[i,k,\mathrm{ac}]) := c(n(x))\frac{p_i}{q_i} \left(\prod_{l \in \mathrm{UP}(x)} q_l^{C_l(n(x_l))} \right) \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2x_l \oplus \mathrm{ac}_l) \right) \tag{21}$$

for $i \in \mathrm{UP}_{\mathrm{OCC}}(x)$, $k \in \underline{C_i(n(x_i))}$ and $\mathrm{ac} \in \{0,1\}^J$.

Note that if node $\overline{Q[i]}$ is in up status and the local queue length is 0, then no departure is possible. This is reflected by $C_i(0) = 0$ and therefore we may take the product of the departure

possibilities over the set $\mathrm{UP}(x)$ of working or the set $\mathrm{UP}_{\mathrm{OCC}}(x)$ of working and occupied nodes.

Finally $\hat{p}(x, \hat{T}[\mathrm{ac}](x))$ is defined as follows. Let $\mathrm{UP}(x) = \{r(1), \ldots, r(j)\}$ be the numbers of working nodes, if the network's state is x. Let

$$W := \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2x_l \oplus \mathrm{ac}_l) \right).$$

be the probability of the availability change ac, then

$$\begin{aligned}
&\hat{p}(x, \hat{T}[\mathrm{ac}]) \\
&:= W\Bigg(\sum_{\substack{(A_{r(1)},\ldots,A_{r(j)}) \in \prod_{k=1}^{j} \mathcal{P}(\underline{C_{r(k)}(n(x_{r(k)}))}) \\ |A_{r(1)} \cup \cdots \cup A_{r(j)}| \neq 0}} b(n(x)) \prod_{j \in \mathrm{UP}(x)} p_j^{|A_j|} q_j^{C_j(n(x_j)) - |A_j|} \\
&\qquad + \sum_{\substack{(A_{r(1)},\ldots,A_{r(j)}) \in \prod_{k=1}^{j} \mathcal{P}(\underline{C_{r(k)}(n(x_{r(k)}))}) \\ |A_{r(1)} \cup \cdots \cup A_{r(j)}| \neq 1}} c(n(x)) \prod_{j \in \mathrm{UP}(x)} p_j^{|A_j|} q_j^{C_j(n(x_j)) - |A_j|} \Bigg) \\
&= W\Bigg(\sum_{(A_{r(1)},\ldots,A_{r(j)}) \in \prod_{k=1}^{j} \mathcal{P}(\underline{C_{r(k)}(n(x_{r(k)}))})} b(n(x)) \prod_{j \in \mathrm{UP}(x)} p_j^{|A_j|} q_j^{C_j(n(x_j)) - |A_j|} \\
&\qquad + \sum_{(A_{r(1)},\ldots,A_{r(j)}) \in \prod_{k=1}^{j} \mathcal{P}(\underline{C_{r(k)}(n(x_{r(k)}))})} c(n(x)) \prod_{j \in \mathrm{UP}(x)} p_j^{|A_j|} q_j^{C_j(n(x_j)) - |A_j|} \\
&\qquad - b(n(x)) \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} - \sum_{i \in \mathrm{UP}_{\mathrm{OCC}}(x)} c(n(x)) C_i(n(x_i)) \frac{p_i}{q_i} \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} \Bigg) \\
&= W\Bigg(\big(b(n(x)) + c(n(x))\big) \prod_{j \in \mathrm{UP}(x)} \sum_{A_j \in \mathcal{P}(\underline{C_j(n(x_j))})} p_j^{|A_j|} q_j^{C_j(n(x_j)) - |A_j|} \\
&\qquad - b(n(x)) \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} - c(n(x)) \sum_{i \in \mathrm{UP}_{\mathrm{OCC}}(x)} C_i(n(x_i)) \frac{p_i}{q_i} \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} \Bigg) \\
&= W\Bigg(1 - b(n(x)) \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} - c(n(x)) \sum_{i \in \mathrm{UP}_{\mathrm{OCC}}(x)} C_i(n(x_i)) \frac{p_i}{q_i} \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} \Bigg). \quad (22)
\end{aligned}$$

Remarks:

(1) Clearly $\hat{p}$ is a stochastic matrix.

(2) It is possible to think of the network to evolve in the following way: At nodes which are in down status the queue is frozen. At the nodes in $\mathrm{UP}(x)$ (the nodes in up status) rules similar to those in section 2.1 are applied. It will turn out that for our further development we do not need to record the internal network decisions as done there.

(3) If $\mathrm{UP}(x) = \varnothing$ for network state x (all nodes in down status), then

$$\hat{p}(x, \hat{T}[\mathrm{ac}]) = \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2x_l \oplus \mathrm{ac}_l) \right) (1 - b(n(x))) \,,$$

holds. Note that here only the availability status is transformed according to the availability change ac. In this situation every new customer is rejected, regardless of his type: $\mathrm{REJ}(x) = M$. If additionally $\mathrm{ac} = (0, \ldots, 0)$ holds, the corresponding transition is $x \mapsto x$.

3.1.3 One step transition matrix

For $x \in \hat{E}$ let $\mathrm{Per}(x)$ be the set consisting of the operators T permitted for x and

$$\mathrm{Succ}(x) := \bigcup_{T \in \mathrm{Per}(x)} \{T(x)\}$$

be the set of *successor states* of x.

Let x and y be states with $y \in \mathrm{Succ(x)}$. The availability change ac which occurs with this transition can always be recovered by $\mathrm{ac} = {}^2x \oplus {}^2y$.

The network transitions can be classified with respect to differences of the local queue lengths into five different types, labeled by f1, f2, f3a, f3b and f4 ("f' indicates **f**orward system).

(f1) Let $n(y) = n(x) + 1$. This is equivalent to $n({}^1y_j) = n({}^1x_j) + 1$ for some $j \in \underline{J}$ and $n({}^1y_l) = n({}^1x_l)$ for all $l \in \underline{J} \setminus \{j\}$. This corresponds to an external arrival on position $n(x_j) + 1$ at destination node $Q[j]$, which is necessarily in up status 0. The arriving customer's type $m = {}^1y_{j,n(y_j),1}$ can be recognised. Necessarily holds $m \notin \mathrm{REJ}(x)$, $A(x,(m,0)) \leqq S(m)$, $j = w\big(m, A(x,(m,0))\big) \in \underline{J}$, ${}^2x_j = 0$ and ${}^1y_{j,n(y_j)} = \big(m, A(x,(m,0))\big)$.

There is exactly one operator, $\hat{T}[m, {}^2x \oplus {}^2y]$, where $m = {}^1y_{j,n(y_j),1}$ and $j = w\big(m, A(x,(m,0))\big)$, producing this transition. Hence

$$p(x,y) = b(n(x))\alpha({}^1y_{j,n(y_j),1}) \left(\prod_{l \in \mathrm{UP}(x)} q_l^{C_l(n(x_l))} \right) \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2y_l) \right).$$

(f2) Let $n(y) = n(x) - 1$. This is equivalent to $n({}^1y_i) = n({}^1x_i) - 1$ for some $i \in \underline{J}$ and $n({}^1y_l) = n({}^1x_l)$ for all $l \in \underline{J} \setminus \{i\}$. This corresponds to a departure of a customer to the outside. The generalised type (m,s) of the departing customer can be determined by comparison of x and y. In general, the departure place is not uniquely determined. The set of possible departure place numbers is

$$G_i({}^1x_i, {}^1y_i) := \{k \in \underline{C_i(n(x_i))} \mid D_k({}^1x_i) = D_{n(x_i)}({}^1y_i)\}. \tag{23}$$

If the departing customer at $Q[i]$ leaves the network, $A(x,(m,s)) = J+1$ must hold, that is, all the nodes on the route of the type m-customer corresponding to stages higher than s are in down status or the jumping customer has already reached his last stage. This is the case, iff

$$s = S(m) \text{ or } {}^2x_i = 1 \quad \forall i \in \{w(m,s+1), \ldots, w(m,S(m))\} \ .$$

Then $\hat{T}[i,k,\mathrm{ac}](x) = y$ iff $k \in G_i({}^1x_i, {}^1y_i)$. Hence with $g_i({}^1x_i, {}^1y_i) = |G_i({}^1x_i, {}^1y_i)|$

$$p(x,y) = c(n(x)) g_i({}^1x_i, {}^1y_i) \frac{p_i}{q_i} \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2y_l) \right) \left(\prod_{l \in \mathrm{UP}(x)} q_l^{C_l(n(x_l))} \right).$$

(f3) Let $n(y) = n(x)$ and ${}^1x \neq {}^1y$. Then we have to distinguish the following two cases.

(f3a) Destination node is not the departure node. This is equivalent to $n({}^1y_i) = n({}^1x_i) - 1$ for some $i \in \underline{J}$ and $n({}^1y_j) = n({}^1x_j) + 1$ for some $j \in \underline{J}$, $i \neq j$ and $n({}^1y_l) = n({}^1x_l)$ for all $l \in \underline{J} \setminus \{i, j\}$. The number i of the departure node can be determined, as well as the deleted generalised type (m, s) by comparison of x and y. But in general it is not possible to determine the departure place. The set of possible departure place numbers is $G_i({}^1x_i, {}^1y_i)$. Also the destination node number j and the additional generalised (m, s') type present at $Q[j]$ (with respect to y) can be determined by comparison of x and y. Necessarily $A(x, (m, s)) \leqq J$ holds, that is, the jumping customer has not reached his last stage and at least one of the nodes on his route corresponding to stages higher than s is in up status (with respect to x), that is

$$s < S(m) \text{ and } {}^2x_i = 0 \text{ for some } i \in \{w(m, s+1), \ldots, w(m, S(m))\}.$$

Then $\hat{T}[i, k, \mathrm{ac}](x) = y$, iff $k \in G_i({}^1x_i, {}^1y_i)$ and $\mathrm{ac} = {}^2x \oplus {}^2y$. Further $w(m, s) = i$, $w(m, A(x, (m, s))) = j$ and ${}^1y_{j,n(y_j)} = (m, A(x, (m, s)))$ holds. Therefore

$$p(x, y) = c(n(x)) g_i({}^1x_i, {}^1y_i) \frac{p_i}{q_i} \left(\prod_{l \in \mathrm{UP}(x)} q_l^{C_l(n(x_l))} \right) \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2y_l) \right).$$

(f3b) Feedback. A feedback takes place iff $n({}^1y_l) = n({}^1x_l)$ for all $l \in \underline{J}$, ${}^1y_i \neq {}^1x_i$ for some $i \in \underline{J}$ and ${}^1y_j = {}^1x_j$ for all $j \in \underline{J} \setminus \{i\}$. Therefore the feedback queue can be determined and one can proceed further as in the single station case. Necessarily $s < S(m)$ (with respect to x), $i = w(m, s) = w(m, A(x, (m, s)))$ and $y_{i,n(y_i)} = (m, A(x, (m, s)))$ holds (the feedback customer is of type m) and one has

$$p(x, y) = c(n(x)) g_i({}^1x_i, {}^1y_i) \frac{p_i}{q_i} \left(\prod_{l \in \mathrm{UP}(x)} q_l^{C_l(n(x_l))} \right) \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2y_l) \right).$$

(f4) Let finally ${}^1y = {}^1x$. The operator $\hat{T}[{}^2x \oplus {}^2y]$ trigger this transition. But if $\mathrm{REJ}(x) \neq \varnothing$, then also the operators $\hat{T}[m, {}^2x \oplus {}^2y]$ with $m \in \mathrm{REJ}(x)$ triggers this transition. Recall that $m \in \mathrm{REJ}(x)$ iff ${}^2x_{w(m,s)} = 1$ for all $s \in \underline{S(m)}$. Then $A(x, (m, 0)) = S(m) + 1$ and $w\big(m, A(x, (m, 0))\big) = J + 1$ holds. This describes the situation that a new customer arriving at the network from the outside is rejected, because all of the nodes on his route are in down status. Therefore

$$\begin{aligned}
&p\big(({}^1x, {}^2x), ({}^1x, {}^2y)\big) \\
&= \left(1 - b(n(x)) \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} - c(n(x)) \sum_{i \in \mathrm{UP}_{\mathrm{OCC}}(x)} C_i(n(x_i)) \frac{p_i}{q_i} \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} \right. \\
&\qquad \left. + \sum_{m \in \mathrm{REJ}(x)} b(n(x)) \alpha(m) \prod_{l \in \mathrm{UP}(x)} q_l^{C_l(n(x_l))} \right) \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2y_l) \right). \\
&= \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2y_l) \right) \left(1 - b(n(x)) \sum_{m \in M \setminus \mathrm{REJ}(x)} \alpha(m) \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} \right. \\
&\qquad \left. - c(n(x)) \sum_{i \in \mathrm{UP}_{\mathrm{OCC}}(x)} C_i(n(x_i)) \frac{p_i}{q_i} \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} \right)
\end{aligned} \tag{24}$$

3.2 Backward system

The service stations $Q[1], \ldots, Q[J]$, when in up status, are working as the backward systems of geometrical feedback Bernoulli servers, described in section 2.2. The following additional rules have to be taken into consideration to determine the system's evolution in time.

The route of a customer through the network depends on the availability status of the network and additionally on the availability change connected with the transition. Consequently, the destination node of a jumping customer depends on the *new* availability status.

The state transitions are in accordance with the following rules.

(1) If a node is in down status at time instant t arrivals or departures are possible iff the availability changes simultaneously.

(2) If a node in up status breaks down neither an arrival nor a departure at the same time instant is possible any more.

Therefore the reorganisation of the queues depends not only on the current state x, but also on the availability status change ac. The transition can therfore be sequentially ordered: first the availability change is performed and then the queue is reorganised.

The routing takes on the following form. Let $\bar{S} = S$ and $\bar{w}(m,i) = w(m, S(m) - i + 1)$. The direction of the customer's movement is reversed with respect to the forward system and the customers move in direction of decreasing stages numbers s. $\bar{w}(m) = \big(\bar{w}(m,1), \ldots, \bar{w}(m, \bar{S}(m))\big)$ determines the nodes, visited by a customer of type m on his route through the net, where the following rule has to be obeyed: if a departure of a type (m,s) customer from node $Q[w(m,s)]$ takes place, the destination node for this customer is the next node in decreasing stage order on his route which in up status after the availability change has been performed. This is the skipping rule for the backward system (an external arrival has to be treated analogously).

For $y \in \hat{E}$ and a pair (m,s), where $m \in M$ and $s \in \{1, \ldots, S(m), S(m)+1\}$, let

$$B(y,(m,s)) := \max\{t \mid 1 \leqq t < s, {}^2y_{w(m,t)} = 0\}$$

if $\max\{t \mid 1 \leqq t < s, {}^2y_{w(m,t)} = 0\} \neq \varnothing$ and $B(y,(m,s)) := 0$ otherwise.

Let the network be in state y. If a jump of a (m,s)-type customer, $s \in \underline{S(m)+1}$, from some node (the source is not excluded), combined with the availability change ac takes place, the new stage of the jumping customer is given by $B(({}^1y, {}^2y \oplus \mathrm{ac}), (m,s))$ and his destination node $Q[i]$ is determined by

$$i = w\big(m, B(({}^1y, {}^2y \oplus \mathrm{ac}), (m,s))\big).$$

Note that $B(({}^1y, {}^2y \oplus \mathrm{ac}), (m,s)) = 0$, iff all nodes on the route of the jumping customer's corresponding to stages $s' < s$ are in down status. Therefore, as for the forward system, it is possible that an external arrival is rejected. This happens, iff all nodes on the external arrival's route are in down status after the availability change has been performed. The relevant information is ${}^2y \oplus \mathrm{ac}$, with the network's availability status ac at the arrival instant and ${}^2y$ the availability change.

For $m \in M$, $s \in \{1, \ldots, S(m), S(m)+1\}$ and $y \in \hat{E}$ let

$$V\big(y,(m,s)\big) := \{i \in \underline{J} \mid i = w(m,t) \text{ for some } t \in \{1, \ldots, s-1\} \text{ and } {}^2y_i = 0\}$$

be the set of numbers of those nodes on the route of an m type customer following stage s and being in up status. Further on let

$$\overline{\mathrm{REJ}}(y) := \{m \in M \mid V\big(y,(m,S(m)+1)\big) = \varnothing\}$$

be the set of customer types, which are rejected when the availability change transforms the network's availability status into ${}^2y$.

3.2.1 Transition operators

The network transition operators are

(i) Arrival of a type-m customer at the network at position k of his destination node jointly with availability status change ac.

The pair $(y,\hat{\bar{T}}[m,k,\mathrm{ac}])$ is permitted, if $\mathrm{ac} \in \{0,1\}^J$, $m \in M$ and $k \in \underline{C_j(n(y_j)+1)}$ holds, where $j = w\big(m, B(({}^1y, {}^2y \oplus \mathrm{ac}),(m,S(m)+1))\big)$.

$$\hat{\bar{T}}[m,k,\mathrm{ac}]\colon \hat{E} \to \hat{E}, \qquad \bar{T}[m,k]\colon \hat{E} \to E$$

$$\hat{\bar{T}}[m,k,\mathrm{ac}](y) := \big(\bar{T}[m,k](y), {}^2y \oplus \mathrm{ac}\big)$$

The components of $\bar{T}[m,k](y)$ are for $l \in \underline{J}$

$$\big(\bar{T}[m,k](y)\big)_l := {}^1y_l, \qquad \text{if } l \neq w(m,B(({}^1y,{}^2y\oplus\mathrm{ac}),(m,S(m)+1)))$$

and for $l = w(m,B(({}^1y,{}^2y\oplus\mathrm{ac}),(m,S(m)+1)))$

$$\big(\bar{T}[m,k](y)\big)_l := ({}^1y_{l,1},\ldots,{}^1y_{l,k-1},(m,s),{}^1y_{l,k},\ldots,{}^1y_{l,n(y_l)},(0,0),\ldots)\ ,$$

where $s = B(({}^1y,{}^2y\oplus\mathrm{ac}),(m,S(m)+1))$. Note that ${}^1\big(\hat{\bar{T}}[m,k,\mathrm{ac}](y)\big) = {}^1y$ if $m \in \overline{\mathrm{REJ}}({}^1y,{}^2y\oplus\mathrm{ac})$ (rejection of an external type m customer arriving from the outside, finding all nodes of his route down after the availability change has been performed) Then $k = 1$, because of $C_{J+1}(\infty) = 1$.

(ii) Departure of a customer at node $Q[i]$, caused by the departure process with number n and arrival at position k of the destination node, jointly with availability status change ac.

The pair $(y,\hat{\bar{T}}[i,n;k,\mathrm{ac}])$ is permitted, if $i \in \mathrm{UP}({}^1y,{}^2y\oplus\mathrm{ac})$ (departure from some node is possible only if this node is in up status after the availability change, i.e. $({}^2y\oplus\mathrm{ac})_i = 0$), $n \in \underline{C_i(n(y_i))}$, and for destination node j holds $k \in \underline{C_j\big(n(y_j)+1-\delta(i,j)\big)}$. Note that $n \in \underline{C_i(n(y_i))}$ implies $n(y_i) \geqq 1$, and therefore $i \in \mathrm{UP}_{\mathrm{OCC}}({}^1y,{}^2y\oplus\mathrm{ac})$. Operators of this form indicate that no new customer stands by for arrival and that overall only one departure decision is positive (namely the departure decision with number n at node $Q[i]$). The customer at position $n(y_i)$ at $Q[i]$ departs, jumps to his destination node $Q[j]$, where $j = w\big(m,B(({}^1y,{}^2y\oplus\mathrm{ac}),({}^1y_{i,n(y_i)}))\big)$ and occupies position k there.

Three cases have to be distinguished.

(a) Destination node $Q[j]$ is the outside (sink) of the network ($j = J+1$). Then $k = 1$ holds. A departing customer of type $(m,s) = ({}^1y_{n(y_i)})$ leaves the network, iff either he has already reached stage $s = 1$, or when $s > 1$ and all nodes on his route, corresponding

to stages $s' < s$ are down after the availability change has been performed (that is $V(({}^1y, {}^2y \oplus \mathrm{ac}), {}^1y_{i,n(y_i)}) = \varnothing$).

$$\hat{\bar{T}}[i,n;1,\mathrm{ac}]\colon \hat{E} \to \hat{E}, \qquad \bar{T}[i,n;1]\colon \hat{E} \to E$$

$$\hat{\bar{T}}[i,n;1,\mathrm{ac}](y) := \big(\bar{T}[i,n;1](y), {}^2y \oplus \mathrm{ac}\big)$$

The components of $\bar{T}[i,n]$ are

$$\big(\bar{T}[i,n;1](y)\big)_l \quad := \quad \begin{cases} {}^1y_l & \text{if } l \neq i \\ ({}^1y_{i,1}, \ldots, {}^1y_{i,n(y_i)-1}, (0,0), \ldots) & \text{if } l = i \end{cases} .$$

(b) Destination node $Q[j]$ lies inside the network and no feedback occurs ($j \neq J+1$, $i \neq j$); the customer remains inside the network and enters position k at $Q[j]$. Recall that the jumping customer remains inside the network, iff $V(({}^1y, {}^2y \oplus \mathrm{ac}), {}^1y_{i,n(y_i)}) \neq \varnothing$ holds. Then $s = B(({}^1y, {}^2y \oplus \mathrm{ac}), {}^1y_{i,n(y_i)})$ is the new stage of the jumping customer and $j = w\big({}^1y_{i,n(y_i),1}, B(({}^1y, {}^2y \oplus \mathrm{ac}), {}^1y_{i,n(y_i)})\big)$ his destination node. Furthermore, $k \in \underline{C_j(n(y_j)+1)}$ holds because arrivals are possible only at arrival places.

$$\hat{\bar{T}}[i,n;k,\mathrm{ac}]\colon \hat{E} \to \hat{E}, \qquad \bar{T}[i,n;k]\colon \hat{E} \to E$$

$$\hat{\bar{T}}[i,n;k,\mathrm{ac}](y) := \big(\bar{T}[i,n;k](y), {}^2y \oplus \mathrm{ac}\big)$$

The components of $\bar{T}[i,n;k]$ are

$$\begin{aligned} \big(\bar{T}[i,n;k](y)\big)_l &:= {}^1y_l, \qquad \text{if } l \neq i \text{ and } l \neq j \\ \big(\bar{T}[i,n;k](y)\big)_i &:= ({}^1y_{i,1}, \ldots, {}^1y_{i,n(y_i)-1}, (0,0), \ldots) \\ \big(\bar{T}[i,n;k](y)\big)_j &:= ({}^1y_{j,1}, \ldots, {}^1y_{j,k-1}, (y^1_{i,n(y_i)}, s), {}^1y_{j,k}, \ldots, {}^1y_{j,n(y_j)}, (0,0), \ldots). \end{aligned}$$

(c) A feedback occurs ($j \neq J+1$, $i = j$); the customer remains inside the network and after leaving the node, then occupies position k at $Q[i]$. His stage is diminished appropriately, leading to $s = B(({}^1y, {}^2y \oplus \mathrm{ac}), {}^1y_{i,n(y_i)})$.
Here $i \in V(({}^1y, {}^2y \oplus \mathrm{ac}), {}^1y_{i,n(y_i)}) \neq \varnothing$, $i = j = w\big({}^1y_{i,n(y_i),1}, B(({}^1y, {}^2y \oplus \mathrm{ac}), {}^1y_{i,n(y_i)})\big)$ and $k \in \underline{C_j(n(y_j))}$ holds.

$$\hat{\bar{T}}[i,n;k,\mathrm{ac}]\colon \hat{E} \to \hat{E}, \qquad \bar{T}[i,n;k]\colon \hat{E} \to E$$

$$\hat{\bar{T}}[i,n;k,\mathrm{ac}](y) := \big(\bar{T}[i,n;k](y), {}^2y \oplus \mathrm{ac}\big)$$

The components of $\bar{T}[i,n;k]$ are

$$\begin{aligned} \big(\bar{T}[i,n;k](y)\big)_l &:= {}^1y_l, \qquad \text{if } l \neq i, \\ \big(\bar{T}[i,n;k](y)\big)_i &:= ({}^1y_{i,1}, \ldots, {}^1y_{i,k-1}, (y^1_{i,n(y_i)}, s), {}^1y_{i,k}, \ldots, {}^1y_{i,n(y_i)-1}, (0,0), \ldots). \end{aligned}$$

(iii) Availability status change ac, no arrival, no departure. This operator is defined on $\hat{E}$ by

$$\bar{T}[\mathrm{ac}](y) := \big({}^1y, {}^2y \oplus \mathrm{ac}\big).$$

3.2.2 Transition operator probabilities

The transition operator probabilities are

$$\hat{\bar{p}}(y,\hat{\bar{T}}[m,k,\mathrm{ac}]) := \frac{b(n(y))\alpha(m)}{C_j(n(y_j)+1)}\left(\prod_{l\in \mathrm{UP}({}^1y,{}^2y\oplus \mathrm{ac})} q_l^{C_l(n(y_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2y_l,{}^2y_l\oplus \mathrm{ac}_l)\right),$$

where $m\in M$, $j=w\big(m,B(({}^1y,{}^2y\oplus \mathrm{ac}),(m,S(m)+1))\big)$, ${}^2y_j\oplus \mathrm{ac}_j=0$ and $k\in \underline{C_j(n(y_j)+1)}$.

$$\hat{\bar{p}}(y,\bar{T}[i,n;k,\mathrm{ac}]) := \frac{c(n(y))}{C_j\big(n(y_j)+1-\delta(i,j)\big)}\frac{p_i}{q_i}\left(\prod_{l\in \mathrm{UP}({}^1y,{}^2y\oplus \mathrm{ac})} q_l^{C_l(n(y_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2y_l,{}^2y_l\oplus \mathrm{ac}_l)\right),$$

where ${}^2y_i\oplus \mathrm{ac}_i=0$, $n(y_i)>0$, $n\in \underline{C_i(n(y_i))}$, $j=w\big({}^1y_{i,n(y_i),1},B(({}^1y,{}^2y\oplus \mathrm{ac}),{}^1y_{i,n(y_i),2})\big)$ and $k\in \underline{C_j(n(y_j)+1-\delta(i,j))}$.

Finally $\hat{\bar{p}}(y,\bar{T}[\mathrm{ac}](y))$ is defined as follows. Let $\mathrm{UP}({}^1y,{}^2y\oplus\mathrm{ac})=\{r(1),\dots,r(j)\}$ be the numbers of nodes in up status under the availability status change to ${}^2y\oplus\mathrm{ac}$. Let W be the probability of the availability status change ac:

$$W := \left(\prod_{l=1}^{J} a_l({}^2y_l,{}^2y_l\oplus \mathrm{ac}_l)\right).$$

Then

$$\begin{aligned}
&\hat{\bar{p}}(y,\hat{\bar{T}}[\mathrm{ac}](y))\\
&:= W\Bigg(\sum_{\substack{(A_{r(1)},\dots,A_{r(j)})\in \prod_{k=1}^{j}\mathcal{P}(\underline{C_k(n(y_k))})\\ |A_1\cup\dots\cup A_J|\neq 0}} b(n(y))\prod_{j\in \mathrm{UP}({}^1y,{}^2y\oplus\mathrm{ac})} p_j^{|A_j|}q_j^{C_j(n(y_j))-|A_j|}\\
&\qquad + \sum_{\substack{(A_{r(1)},\dots,A_{r(j)})\in \prod_{k=1}^{j}\mathcal{P}(\underline{C_k(n(y_k))})\\ |A_1\cup\dots\cup A_J|\neq 1}} c(n(y))\prod_{j\in \mathrm{UP}({}^1y,{}^2y\oplus\mathrm{ac})} p_j^{|A_j|}q_j^{C_j(n(y_j))-|A_j|}\Bigg)\\
&= \left(\prod_{l=1}^{J} a_l({}^2y_l,{}^2y_l\oplus \mathrm{ac}_l)\right)\cdot\Bigg(1-b(n(y))\prod_{j\in \mathrm{UP}({}^1y,{}^2y\oplus\mathrm{ac})} q_j^{C_j(n(y_j))}\\
&\qquad - c(n(y))\sum_{i\in \mathrm{UP}_{\mathrm{OCC}}({}^1y,{}^2y\oplus\mathrm{ac})} C_i(n(y_i))\frac{p_i}{q_i}\prod_{j\in \mathrm{UP}({}^1y,{}^2y\oplus\mathrm{ac})} q_j^{C_j(n(y_j))}\Bigg).
\end{aligned}\tag{25}$$

3.2.3 One step transition matrix

For $y\in E$ let $\overline{\mathrm{Per}}(y)$ be the set consisting of the operators $\bar{T}$ permitted for y and

$$\overline{\mathrm{Succ}}(y) := \bigcup_{\bar{T}\in\overline{\mathrm{Per}}(y)}\{\bar{T}(y)\}$$

be the set of successor states of y under all permitted backward transitions.

Let x and y be states with $x \in \overline{\mathrm{Succ}}(y)$. The availability change ac connected with transition $y \to x$ can always be recovered by $\mathrm{ac} = {}^2y \oplus {}^2x$.

The network transitions can be classified with respect to differences of the local queue lengths into five different types, labeled by b1, b2, b3a, b3b and b4 ("b" indicates **b**ackward system).

(b1) Let $n(x) = n(y) - 1$. This is equivalent to $n(x_i) = n(y_i) - 1$ for some $i \in \underline{J}$ and $n(x_j) = n(y_j)$ for all $j \in \underline{J} \setminus \{i\}$. This corresponds to a departure of a customer at node $Q[i]$, who leaves the network. Necessarily $n(y_i) > 0$ holds. The operators $\hat{\bar{T}}[i, n; 1, {}^2y \oplus {}^2x]$ where $n \in \underline{C_i(n(y_i))}$ map y onto x, hence

$$\bar{p}(y,x) = c(n(y))C_i(n(y_i))\frac{p_i}{q_i}\left(\prod_{l \in \mathrm{UP}({}^1y, {}^2x)} q_l^{C_l(n(y_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2y_l, {}^2x_l)\right).$$

(b2) Let $n(x) = n(y)+1$. This is equivalent to $n(y_j) = n(x_j)+1$ for some $j \in \underline{J}$ and $n(y_j) = n(x_j)$ for all $l \in \underline{J} \setminus \{j\}$. This corresponds to an external arrival, and the arriving customer's destination node is $Q[j]$. The node number j and the type of the new customer can be determined by comparison of y and x. The new customer's stage is $s = B(({}^1y, {}^2x), (m, S(m)+1))$ and $w(m,s) = j$. In general, the arrival position k can not be determined uniquely by comparison of y and x. Let

$$\bar{G}_j({}^1y_j, {}^1x_j) = \{k \in \underline{C_j(n(y_j)+1)} \mid D_k(x_j) = y_j\}$$

be the set of possible arrival positions at $Q[j]$ under this transition. Then

$$\bar{G}_j({}^1y_j, {}^1x_j) = \{k \in \underline{C_j(n(x_j))} \mid D_k(x_j) = y_j\} = G_j({}^1x_j, {}^1y_j)$$

holds. Exactly the operators $\hat{\bar{T}}[m, k; {}^2y \oplus {}^2x])$ with $k \in G_j({}^1x_j, {}^1y_j)$ map y to x and therefore

$$\bar{p}(y,x) = b(n(y))\alpha(m)\frac{g_j({}^1x_j, {}^1y_j)}{C_j(n(y_j)+1)}\left(\prod_{l \in \mathrm{UP}({}^1y, {}^2x)} q_l^{C_l(n(y_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2y_l, {}^2x_l)\right).$$

(b3) Let $n(x) = n(y)$ and ${}^1y \neq {}^1x$. Then two cases have to be distinguished.

(b3a) Destination node is not the departure node. This is equivalent to $n(x_j) = n(y_j) - 1$ for some $j \in \underline{J}$ and $n(x_i) = n(y_i) + 1$ for some $i \in \underline{J}$, $i \neq j$ and $n(y_l) = n(x_l)$ for all $l \in \underline{J} \setminus \{i, j\}$. This corresponds to a jump from node $Q[j]$ to node $Q[i]$. The numbers j and i of departure and arrival node can be determined, as well as the deleted (generalised) type (m, s) at $Q[j]$ and the new (generalised) type $\big(m, B(({}^1y, {}^2x), (m, s))\big)$ present at $Q[i]$, by comparing y and x. The arrival position at $Q[i]$ can not be determined uniquely in general. The set of possible arrival positions is $G_i({}^1x_i, {}^1y_i)$. Then $x = \bar{T}[j, n; k, {}^2y \oplus {}^2x](y)$ holds, iff $n \in \underline{C_j(n(y_j))}$ and $k \in G_i(x_i, y_i)$ and therefore

$$\bar{p}(y,x) = c(n(y))C_j(n(y_j))\frac{g_i({}^1x_i, {}^1y_i)}{C_i(n(y_i)+1)}\frac{p_j}{q_j}\left(\prod_{l \in \mathrm{UP}({}^1y, {}^2x)} q_l^{C_l(n(y_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2y_l, {}^2x_l)\right).$$

(b3b) Feedback. This is equivalent to $n(x_l) = n(y_l)$ for all $l \in \underline{J}$ and ${}^1x_i \neq {}^1y_i$ for one and only one $i \in \underline{J}$. Let $\bar{G}_i^{\mathrm{Feed}}({}^1y_i, {}^1x_i)$ be the set of possible arrival positions k (that is $k \in \bar{G}_i^{\mathrm{Feed}}({}^1y_i, {}^1x_i) \Leftrightarrow \hat{\bar{T}}[i, n; k, {}^2y \oplus {}^2x](y) = x$). Then

$$\begin{aligned}\bar{G}_i^{\mathrm{Feed}}({}^1y_i, {}^1x_i) &= \{k \in \underline{C_i(n(x_i))} \mid I[{}^1(y_{i,n(y_i),1}, s'), k] \circ D_{n(y_i)}({}^1y_i) = {}^1x_i\} \\ &= \{k \in \underline{C_i(n(x_i))} \mid D_k({}^1x_i) = D_{n(y_i)}({}^1y_i)\},\end{aligned}$$

where $s' = B(({}^1y, {}^2x), {}^1y_{i,n(y_i)})$ is the new stage of the feedback customer. Recalling that $n(y_i) = n(x_i)$ we therefore have

$$\bar{G}_i^{\mathrm{Feed}}({}^1y_i, {}^1x_i) = \{k \in \underline{C_i(n(x_i))} \mid D_k({}^1x_i) = D_{n(x_i)}({}^1y_i)\} = G_i({}^1x_i, {}^1y_i),$$

in analogy to the non-feedback case (b3a). The operators I and D_k are defined on page 5. Hence $x = \hat{\bar{T}}[i, n; k, {}^2y \oplus {}^2x](y)$ iff $n \in \underline{C_i(n(y_i))}$ and $k \in G_i({}^1x_i, {}^1y_i)$ holds. Necessarily $w({}^1y_{i,n(y_i)}) = w({}^1y_{i,n(y_i),1}, s') = i$ must hold. We therefore have

$$\bar{p}(y,x) = c(n(y)) g_i({}^1x_i, {}^1y_i) \frac{p_i}{q_i} \left(\prod_{l \in \mathrm{UP}({}^1y, {}^2x)} q_l^{C_l(n(y_l))} \right) \left(\prod_{l=1}^{J} a_l({}^2y_l, {}^2x_l) \right).$$

Remark: denoting the node number of the arrival node by i and the node number of the departure node by j we can cover both cases, (b3a) non-feedback and (b3b) feedback:

$$\bar{p}(y,x) = c(n(y)) C_j(n(y_j)) \frac{g_i({}^1x_i, {}^1y_i)}{C_i(n(y_i) + 1 - \delta(i,j))} \frac{p_j}{q_j} \left(\prod_{l \in \mathrm{UP}({}^1y, {}^2x)} q_l^{C_l(n(y_l))} \right) \left(\prod_{l=1}^{J} a_l({}^2y_l, {}^2x_l) \right).$$

(b4) If ${}^1y = {}^1x$ holds, clearly $\hat{\bar{T}}[{}^2y \oplus {}^2x]$ triggers this transition. If $\overline{\mathrm{REJ}}({}^1y, {}^2x) \neq \varnothing$ then also the operators $\hat{\bar{T}}[m, 1, {}^2y \oplus {}^2x]$, where $m \in \overline{\mathrm{REJ}}({}^1y, {}^2x)$, transform y into x. Recall that a customer of type $m \in \overline{\mathrm{REJ}}({}^1y, {}^2x)$ arriving as a new customer from the outside will find all nodes on his route in down status and is therefore rejected. Hence

$$\begin{aligned} &\bar{p}\big(({}^1y, {}^2y), ({}^1y, {}^2x)\big) \\ &= \left(\prod_{l=1}^{J} a_l({}^2y_l, {}^2x_l) \right) \cdot \Bigg(1 - b(n(y)) \sum_{m \in M \setminus \overline{\mathrm{REJ}}({}^1y, {}^2x)} \alpha(m) \prod_{j \in \mathrm{UP}({}^1y, {}^2x)} q_j^{C_j(n(y_j))} \\ &\qquad - c(n(y)) \sum_{i \in \mathrm{UP}_{\mathrm{OCC}}({}^1y, {}^2x)} C_i(n(y_i)) \frac{p_i}{q_i} \prod_{j \in \mathrm{UP}({}^1y, {}^2x)} q_j^{C_j(n(y_j))} \Bigg). \end{aligned} \tag{26}$$

3.3 Steady state distribution

Theorem 3.1. *Let X and $\bar{X}$ denote the homogeneous Markov chains corresponding to the transition matrices p and $\bar{p}$, respectively. Then $\psi\colon E \to \mathbb{R}_+$, given by*

$$\psi(x) = \left(\prod_{k=1}^{n(x)} \frac{b(k-1)}{c(k)} \right) \prod_{l=1}^{J} \left(\prod_{k=1}^{n(x_l)} \frac{\alpha({}^1x_{l,k,1})}{C_l(k)} \right) \frac{q_l^{n(x_l) - C_l(n(x_l))}}{p_l^{n(x_l)}} a_l({}^2x_l \oplus 1, {}^2x_l), \tag{27}$$

is an invariant measure for both chains.

Proof. From (24) and (26) we have

$$\begin{aligned} p(x,x) \;=\; & \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2x_l) \right) \Bigg(1 - b(n(x)) \sum_{m \in M \setminus \mathrm{REJ}(x)} \alpha(m) \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} \\ & - c(n(x)) \sum_{i \in \mathrm{UP}_{\mathrm{OCC}}(x)} C_i(n(x_i)) \frac{p_i}{q_i} \prod_{j \in \mathrm{UP}(x)} q_j^{C_j(n(x_j))} \Bigg) \end{aligned}$$

and

$$\bar{p}(x,x) = \left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2x_l)\right)\cdot\left(1-b(n(x))\sum_{m\in M\setminus\overline{\mathrm{REJ}}(x)}\alpha(m)\prod_{j\in\mathrm{UP}(x)} q_j^{C_j(n(x_j))}\right.$$
$$\left. - c(n(x))\sum_{i\in\mathrm{UP_{OCC}}(x)} C_i(n(x_i))\frac{p_i}{q_i}\prod_{j\in\mathrm{UP}(x)} q_j^{C_j(n(x_j))}\right).$$

Obviously $\mathrm{REJ}(x)=\overline{\mathrm{REJ}}(x)$ holds and therefore $p(x,x)=\bar{p}(x,x)$ for all $x\in\hat{E}$. We now have to show

$$\psi(x)p(x,y)=\psi(y)\bar{p}(y,x) \tag{28}$$

for all pairs $(x,y)\in\hat{E}\setminus\Delta$ where Δ is the diagonal of $\hat{E}$. We do this by verifying

$$y\in\mathrm{Succ}(x) \text{ and } x\mapsto_{\mathrm{f}j} y \Leftrightarrow x\in\overline{\mathrm{Succ}}(y) \text{ and } y\mapsto_{\mathrm{b}j} x,$$

for $j=1,2,3\mathrm{a},3\mathrm{b},4$ and then by verifying the validity of (28) for these pairs.

(f1-b1) Let $y\in\mathrm{Succ}(x)$ and $x\mapsto_{\mathrm{f1}} y$ by an external arrival of an m-type customer at $Q[j]$ with $j=w\big(m,A(x,(m,0))\big)\in\underline{J}$. Then $n(y_j)=n(x_j)+1$, $n(y_i)=n(x_i)$ for $i\in\underline{J}\setminus\{j\}$, $n(y)=n(x)+1$ and ${}^1y_{j,n(y_j)}=\big(m,A(x,(m,0))\big)$. The only operator triggering this transition is $\hat{T}[m,{}^2x\oplus{}^2y]$. This implies $y\mapsto_{\mathrm{b1}} x$ (hence $x\in\overline{\mathrm{Succ}}(y)$) by a departure of a m-type customer from $Q[j]$ to the outside. The corresponding operators are $\hat{\hat{T}}[j,n;1,{}^2x\oplus{}^2y]$, where $n\in\underline{C_j(n(y_j))}$. The new stage of the jumping customer is $B(({}^2y,{}^2x),(m,A(x,(m,0))))=0$ and therefore the customer leaves the network. The opposite direction follows in an analogous way. Therefore

$$\begin{aligned}
&\psi(x)p(x,y)\\
&= b(n(x))\alpha({}^1y_{j,n(y_j),1})\left(\prod_{l\in\mathrm{UP}(x)} q_l^{C_l(n(x_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2x_l,{}^2y_l)\right) \qquad (29)\\
&\quad\cdot\left(\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(x_l)-C_l(n(x_l))}}{p_l^{n(x_l)}}a_l({}^2x_l\oplus 1,{}^2x_l),\\
&= b(n(x))\left(\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)\alpha({}^1y_{j,n(y_j),1})\left(\prod_{k=1}^{n(x_j)}\frac{\alpha({}^1x_{j,k,1})}{C_j(k)}\right)\left(\frac{q_j}{p_j}\right)^{n(x_j)}\\
&\quad\cdot\left(\prod_{l\in\mathrm{UP}(x)\setminus\{j\}}\left(\frac{q_l}{p_l}\right)^{n(x_l)}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\left(\prod_{l\in\underline{J}\setminus\mathrm{UP}(x)}\frac{q_l^{n(x_l)-C_l(n(x_l))}}{p_l^{n(x_l)}}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\\
&\quad\cdot\left(\prod_{l=1}^{J}a_l({}^2x_l,{}^2y_l)a_l({}^2x_l\oplus 1,{}^2x_l)\right)
\end{aligned}$$

and

$$
\begin{aligned}
&\psi(y)\bar{p}(y,x)\\
&= c(n(y))C_j(n(y_j))\frac{p_j}{q_j}\Big(\prod_{l\in \mathrm{UP}({}^1y_{,}{}^2x)} q_l^{C_l(n(y_l))}\Big)\Big(\prod_{l=1}^{J} a_l({}^2y_l,{}^2x_l)\Big)\\
&\quad\cdot\left(\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}a_l({}^2y_l\oplus 1,{}^2y_l)\\
&= c(n(y))\left(\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)C_j(n(y_j))\frac{p_j}{q_j}\left(\prod_{k=1}^{n(y_j)}\frac{\alpha({}^1y_{j,k,1})}{C_j(k)}\right)\left(\frac{q_j}{p_j}\right)^{n(y_j)}\\
&\quad\cdot\left(\prod_{l\in \mathrm{UP}({}^1y_{,}{}^2x)\setminus\{j\}}\left(\frac{q_l}{p_l}\right)^{n(y_l)}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\\
&\quad\cdot\left(\prod_{l\in \underline{J}\setminus \mathrm{UP}({}^1y_{,}{}^2x)}\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\\
&\quad\cdot\left(\prod_{l=1}^{J}a_l({}^2y_l,{}^2x_l)a_l({}^2y_l\oplus 1,{}^2y_l)\right).
\end{aligned}
$$

First of all

$$
b(n(x))\left(\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)=c(n(y))\left(\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)
$$

holds, because of $n(y)=n(x)+1$. Recalling that $n(y_j)=n(x_j)+1$ and ${}^1x_{j,k}={}^1y_{j,k}$ and in particular ${}^1x_{j,k,1}={}^1y_{j,k,1}$ for position numbers $k=1,\ldots,n(x_j)$ at $Q[j]$, we have

$$
\alpha({}^1y_{j,n(y_j),1})\left(\prod_{k=1}^{n(x_j)}\frac{\alpha({}^1x_{j,k,1})}{C_j(k)}\right)\left(\frac{q_j}{p_j}\right)^{n(x_j)}=C_j(n(y_j))\frac{p_j}{q_j}\left(\prod_{k=1}^{n(y_j)}\frac{\alpha({}^1y_{j,k,1})}{C_j(k)}\right)\left(\frac{q_j}{p_j}\right)^{n(y_j)}.
$$

Clearly ${}^1x_l={}^1y_l$ for the nodes which are not involved. Furthermore $\mathrm{UP}(x)=\mathrm{UP}({}^1y_{,}{}^2x)$, taking into account that set of nodes in up status depends only on the network's availability state. Therefore

$$
\left(\prod_{l\in \mathrm{UP}(x)\setminus\{j\}}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\left(\frac{q_l}{p_l}\right)^{n(x_l)}=\left(\prod_{l\in \mathrm{UP}({}^1y_{,}{}^2x)\setminus\{j\}}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\left(\frac{q_l}{p_l}\right)^{n(y_l)}
$$

and

$$
\left(\prod_{l\in \underline{J}\setminus \mathrm{UP}(x)}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(x_l)-C_l(n(x_l))}}{p_l^{n(x_l)}}=\left(\prod_{l\in \underline{J}\setminus \mathrm{UP}({}^1y_{,}{}^2x)}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}.
$$

Finally

$$
a_l(x,y)a_l(x\oplus 1,x)=a_l(y,x)a_l(y\oplus 1,y)
$$

holds for $x = y$, $x, y \in \{0,1\}$. And for $x \neq y$

$$a_l(0,1)a_l(0\oplus 1,0) = a_l(0,1)a_l(1,0) = a_l(1,0)a_l(1\oplus 1,1).$$

Thus

$$\left(\prod_{l=1}^{J} a_l({}^2x_l, {}^2y_l)a_l({}^2x_l\oplus 1, {}^2x_l)\right) = \left(\prod_{l=1}^{J} a_l({}^2y_l, {}^2x_l)a_l({}^2y_l\oplus 1, {}^2y_l)\right).$$

and therefore $\psi(x)p(x,y) = \psi(y)\bar{p}(y,x)$ for these corresponding types of state transitions.

(f2-b2) Let $y \in \mathrm{Succ}(x)$ and $x \mapsto_{\mathrm{f2}} y$ by a departure of a m-type customer at the departure position $k \in G_i(x_i, y_i)$ at $Q[i]$ to the outside: $y = \hat{T}[i,k,{}^2x\oplus{}^2y](x)$. Denote by (m,s) type and stage of the departing customer. Then $A(x,(m,s)) = S(m)+1$ holds. This implies $x = \hat{\bar{T}}[m,k',{}^2x\oplus{}^2y](y)$ for $k' \in G_i(x_i,y_i)$. Indeed, from $A(x,(m,s)) = S(m)+1$ the new stage of the arriving customer of type m in the reversed system is $B(({}^2y,{}^2x),(m,S(m)+1)) = s$. This shows $y \mapsto_{\mathrm{b2}} x$ by an arrival of a m-type customer at a arrival position with number $k' \in \bar{G}_i(y_i,x_i)$. Again the opposite direction follows in an analogous way. Therefore

$$\begin{aligned}
&\psi(x)p(x,y)\\
&= c(n(x))g_i({}^1x_i,{}^1y_i)\frac{p_i}{q_i}\left(\prod_{l\in\mathrm{UP}(x)} q_l^{C_l(n(x_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2x_l,{}^2y_l)\right) \qquad (30)\\
&\quad\cdot\left(\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(x_l)-C_l(n(x_l))}}{p_l^{n(x_l)}}a_l({}^2x_l\oplus 1,{}^2x_l)\\
&= \left(c(n(x))\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)\left(g_i({}^1x_i,{}^1y_i)\left(\frac{q_i}{p_i}\right)^{n(x_i)-1}\prod_{k=1}^{n(x_i)}\frac{\alpha({}^1x_{i,k,1})}{C_i(k)}\right)\\
&\quad\cdot\left(\prod_{l\in\mathrm{UP}(x)\setminus\{i\}}\left(\frac{q_l}{p_l}\right)^{n(x_l)}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\\
&\quad\cdot\left(\prod_{l\in J\setminus\mathrm{UP}(x)}\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right) \qquad (31)\\
&\quad\cdot\left(\prod_{l=1}^{J} a_l({}^2x_l,{}^2y_l)a_l({}^2x_l\oplus 1,{}^2x_l)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\psi(y)\bar{p}(y,x)\\
&= b(n(y))\alpha(m)\frac{g_i({}^1x_i,{}^1y_i)}{C_i(n(x_i)+1)}\left(\prod_{l\in\mathrm{UP}({}^1y,{}^2x)} q_l^{C_l(n(y_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2y_l,{}^2x_l)\right)\\
&\quad\cdot\left(\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}a_l({}^2y_l\oplus 1,{}^2y_l)\\
&= \left(b(n(y))\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)\left(\frac{g_i({}^1x_i,{}^1y_i)}{C_i(n(x_i)+1)\alpha(m)}\left(\frac{q_i}{p_i}\right)^{n(y_i)}\prod_{k=1}^{n(y_i)}\frac{\alpha({}^1y_{i,k,1})}{C_i(k)}\right)\\
&\quad\cdot\left(\prod_{l\in\mathrm{UP}({}^1y,{}^2x)\setminus\{i\}}\left(\frac{q_l}{p_l}\right)^{n(y_l)}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\\
&\quad\cdot\left(\prod_{l\in\underline{J}\setminus\mathrm{UP}({}^1y,{}^2x)}\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\\
&\quad\cdot\left(\prod_{l=1}^{J}a_l({}^2y_l,{}^2x_l)a_l({}^2y_l\oplus 1,{}^2y_l)\right).
\end{aligned}$$

Clearly we have

$$c(n(x))\left(\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)=b(n(y))\left(\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)$$

and recalling that the jumping customer is of type m, we have

$$g_i({}^1x_i,{}^1y_i)\left(\frac{q_i}{p_i}\right)^{n(x_i)-1}\prod_{k=1}^{n(x_i)}\frac{\alpha({}^1x_{i,k,1})}{C_i(k)}=\frac{g_i({}^1x_i,{}^1y_i)\alpha(m)}{C_i(n(x_i)+1)}\left(\frac{q_i}{p_i}\right)^{n(y_i)}\prod_{k=1}^{n(y_i)}\frac{\alpha({}^1y_{i,k,1})}{C_i(k)}.$$

(f3a-b3a) Let $y\in\mathrm{Succ}(x)$ and $x\mapsto_{\mathrm{f3a}} y$ by a departure of a customer from $Q[i]$ at a departure position $k\in G_i(x_i,y_i)$ and destination node $Q[j]$ inside the network ($i\neq J+1, i\neq j$). Denote the jumping customer's generalised type by ${}^1x_{i,k}=(m,s)$. Then $y=\hat{T}[i,k;{}^2x\oplus{}^2y](x)$ holds and $s'=A(x,(m,s))$ is the new stage and $y\mapsto_{\mathrm{b3a}} x$ by a departure of the customer at position $n(y_j)$ at $Q[j]$. The new stage of this customer is

$$s''=B(({}^1y,{}^2x),(m,s'))=\max\{t\mid 0\leqq t<s',{}^2x_{w(m,t)}=0\}=s,$$

which implies, that the destination node of the jumping customer is indeed $Q[i]$. Possible arrival positions at $Q[i]$ are $k'\in G_i(x_i,y_i)$. Therefore $x=\hat{\hat{T}}[j,n;k,{}^2x\oplus{}^2y](y)$ for $n\in\underline{C_j(n(y_j))}$, $k'\in G_i(x_i,y_i)$ and furthermore $n(y_i)=n(x_i)-1$, $n(y_j)=n(x_j)+1$ and $n(y_l)=$

$n(x_l)$ for $l \in \underline{J} \setminus \{i,j\}$ (opposite direction analogously). Hence

$$
\begin{aligned}
&\psi(x)p(x,y) \\
&= c(n(x))g_i({}^1x_i,{}^1y_i)\frac{p_i}{q_i}\left(\prod_{l\in \mathrm{UP}(x)} q_l^{C_l(n(x_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2x_l,{}^2y_l)\right) \qquad (32)\\
&\quad\cdot\left(\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(x_l)-C_l(n(x_l))}}{p_l^{n(x_l)}}a_l({}^2x_l\oplus 1,{}^2x_l),\\
&= \left(c(n(x))\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)\left(\prod_{l=1}^{J}a_l({}^2x_l,{}^2y_l)a_l({}^2x_l\oplus 1,{}^2x_l)\right)\\
&\quad\cdot\left(g_i({}^1x_i,{}^1y_i)\left(\frac{q_i}{p_i}\right)^{n(x_i)-1}\prod_{k=1}^{n(x_i)}\frac{\alpha({}^1x_{i,k,1})}{C_i(k)}\right)\left(\left(\frac{q_j}{p_j}\right)^{n(x_j)}\prod_{k=1}^{n(x_j)}\frac{\alpha({}^1x_{j,k,1})}{C_j(k)}\right)\\
&\quad\cdot\left(\prod_{l\in\mathrm{UP}(x)\setminus\{i,j\}}\left(\frac{q_l}{p_l}\right)^{n(x_l)}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\\
&\quad\cdot\left(\prod_{l\in\underline{J}\setminus\mathrm{UP}(x)}\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)
\end{aligned}
$$

This is in analogy to the expression for $\psi(x)p(x,y)$ in the case of a jump to the outside (30). Further on

$$
\begin{aligned}
&\psi(y)\bar{p}(y,x)\\
&= c(n(y))C_j(n(y_j))\frac{g_i({}^1x_i,{}^1y_i)}{C_i(n(y_i)+1)}\frac{p_j}{q_j}\left(\prod_{l\in\mathrm{UP}({}^1y,{}^2x)} q_l^{C_l(n(y_l))}\right)\left(\prod_{l=1}^{J}a_l({}^2y_l,{}^2x_l)\right) \qquad (33)\\
&\quad\cdot\left(\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}a_l({}^2y_l\oplus 1,{}^2y_l),\\
&= \left(c(n(y))\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)\left(\prod_{l=1}^{J}a_l({}^2y_l,{}^2x_l)a_l({}^2y_l\oplus 1,{}^2y_l)\right)\\
&\quad\cdot\left(C_j(n(y_j))\left(\frac{q_j}{p_j}\right)^{n(y_j)-1}\prod_{k=1}^{n(y_j)}\frac{\alpha({}^1y_{j,k,1})}{C_j(k)}\right)\\
&\quad\cdot\left(\frac{g_i({}^1x_i,{}^1y_i)}{C_i(n(y_i)+1)}\left(\frac{q_i}{p_i}\right)^{n(y_i)}\prod_{k=1}^{n(y_i)}\frac{\alpha({}^1y_{i,k,1})}{C_i(k)}\right)\\
&\quad\cdot\left(\prod_{l\in\mathrm{UP}({}^1y,{}^2x)\setminus\{i,j\}}\left(\frac{q_l}{p_l}\right)^{n(y_l)}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\\
&\quad\cdot\left(\prod_{l\in\underline{J}\setminus\mathrm{UP}({}^1y,{}^2x)}\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right).
\end{aligned}
$$

First

$$\left(g_i({}^1x_i,{}^1y_i)\left(\frac{q_i}{p_i}\right)^{n(x_i)-1}\prod_{k=1}^{n(x_i)}\frac{1}{C_i(k)}\right)\left(\left(\frac{q_j}{p_j}\right)^{n(x_j)}\prod_{k=1}^{n(x_j)}\frac{1}{C_j(k)}\right)$$
$$= \left(C_j(n(y_j))\left(\frac{q_j}{p_j}\right)^{n(y_j)-1}\prod_{k=1}^{n(y_j)}\frac{1}{C_j(k)}\right)\left(\frac{g_i({}^1x_i,{}^1y_i)}{C_i(n(y_i)+1)}\left(\frac{q_i}{p_i}\right)^{n(y_i)}\prod_{k=1}^{n(y_i)}\frac{1}{C_i(k)}\right),$$

because of $n(y_i)=n(x_i)-1$ and $n(y_j)=n(x_j)+1$; second

$$\left(\prod_{k=1}^{n(x_i)}\alpha({}^1x_{i,k,1})\right)\left(\prod_{k=1}^{n(x_j)}\alpha({}^1x_{j,k,1})\right)=\left(\prod_{k=1}^{n(y_j)}\alpha({}^1y_{j,k,1})\right)\left(\prod_{k=1}^{n(y_i)}\alpha({}^1y_{i,k,1})\right),$$

because of $y_i=D_k({}^1x_i)$ and $x_j=D_{k'}({}^1y_j)$, where $k\in\underline{C_i(n(x_i))}$ and $k'\in\underline{C_j(n(y_j))}$ and henceforth ${}^1x_{i,k,1}={}^1y_{j,k',1}$ (the customer's type does not change, only his stage changes). This implies $\psi(x)p(x,y)=\psi(y)\bar{p}(y,x)$ for an internal jump.

(f3b-b3b) Let $y\in\mathrm{Succ}(x)$ and $x\mapsto_{\mathrm{f3b}} y$ by a feedback of a customer from $Q[i]$ at a departure position $k\in G_i(x_i,y_i)$. Denoting the jumping customer's generalised type by ${}^1x_{i,k}=(m,s)$ we have $y=\hat{T}[i,k;{}^2x\oplus{}^2y](x)$ and $s'=A(x,(m,s))$ is his new stage. Then $y\mapsto_{\mathrm{b3b}} x$ by the departure of the customer at position $n(y_i)$ at $Q[i]$. The new stage of this customer is indeed

$$s''=B(({}^1y,{}^2x),(m,s'))=\max\{t\mid 0\leqq t<s',{}^2x_{w(m,t)}=0\}=s.$$

Possible arrival places are those with numbers $k'\in G_i(x_i,y_i)$. Therefore $x=\hat{\bar{T}}[j,n;k',{}^2x\oplus{}^2y](y)$ for $n\in\underline{C_i(n(y_j))}$, $k'\in G_i(x_i,y_i)$ (opposite direction analogously) Hence

$$\begin{aligned}
&\psi(x)p(x,y)\\
&= c(n(x))g_i({}^1x_i,{}^1(y_i))\frac{p_i}{q_i}\left(\prod_{l\in\mathrm{UP}(x)}q_l^{C_l(n(x_l))}\right)\left(\prod_{l=1}^{J}a_l({}^2x_l,{}^2y_l)\right) \qquad (34)\\
&\quad\cdot\left(\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(x_l)-C_l(n(x_l))}}{p_l^{n(x_l)}}a_l({}^2x_l\oplus 1,{}^2x_l),\\
&= \left(c(n(x))\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)\left(\prod_{l=1}^{J}a_l({}^2x_l,{}^2y_l)a_l({}^2x_l\oplus 1,{}^2x_l)\right)\\
&\quad\cdot\left(g_i({}^1x_i,{}^1y_i)\left(\frac{q_i}{p_i}\right)^{n(x_i)-1}\prod_{k=1}^{n(x_i)}\frac{\alpha({}^1x_{i,k,1})}{C_i(k)}\right)\\
&\quad\cdot\left(\prod_{l\in\mathrm{UP}(x)\setminus\{i\}}\left(\frac{q_l}{p_l}\right)^{n(x_l)}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\left(\prod_{l\in\underline{J}\setminus\mathrm{UP}(x)}\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)
\end{aligned}$$

and

$$
\begin{aligned}
&\psi(y)\bar{p}(y,x) \\
&= c(n(y))g_i({}^1x_i,{}^1y_i)\frac{p_i}{q_i}\left(\prod_{l\in \mathrm{UP}({}^1y,{}^2x)} q_l^{C_l(n(y_l))}\right)\left(\prod_{l=1}^{J} a_l({}^2y_l,{}^2x_l)\right) \qquad (35)\\
&\quad\cdot\left(\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}a_l({}^2y_l\oplus 1,{}^2y_l) \\
&= \left(c(n(y))\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)\left(\prod_{l=1}^{J} a_l({}^2y_l,{}^2x_l)a_l({}^2y_l\oplus 1,{}^2y_l)\right) \\
&\quad\cdot\left(g_i({}^1x_i,{}^1y_i)\left(\frac{q_i}{p_i}\right)^{n(y_i)-1}\prod_{k=1}^{n(y_i)}\frac{\alpha({}^1y_{i,k,1})}{C_i(k)}\right) \\
&\quad\cdot\left(\prod_{l\in \mathrm{UP}({}^1y,{}^2x)\setminus\{i\}}\left(\frac{q_l}{p_l}\right)^{n(y_l)}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right) \\
&\quad\cdot\left(\prod_{l\in \underline{J}\setminus \mathrm{UP}({}^1y,{}^2x)}\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right).
\end{aligned}
$$

Clearly

$$
g_i({}^1x_i,{}^1y_i)\left(\frac{q_i}{p_i}\right)^{n(x_i)-1}\prod_{k=1}^{n(x_i)}\frac{\alpha({}^1x_{i,k,1})}{C_i(k)} = g_i({}^1x_i,{}^1y_i)\left(\frac{q_i}{p_i}\right)^{n(y_i)-1}\prod_{k=1}^{n(y_i)}\frac{\alpha({}^1y_{i,k,1})}{C_i(k)},
$$

and we have $\psi(x)p(x,y) = \psi(y)\bar{p}(y,x)$ also in the case of feedback.

(f4-b4) Now let $y \in \mathrm{Succ}(y)$ and $x \mapsto_{\mathrm{f4}} y$ by the availability change $\mathrm{ac} = {}^2x \oplus {}^2y$ only (that is, ${}^1x = {}^1y$). Then clearly $x \in \overline{\mathrm{Succ}}(y)$ and $y \mapsto_{\mathrm{b4}} x$ by the same availability change $\mathrm{ac} = {}^2y \oplus {}^2x$. We have

$$
\begin{aligned}
&\psi(x)p(x,y) \\
&= \left(\prod_{l=1}^{J} a_l({}^2x_l,{}^2y_l)\right)\left(1 - b(n(x))\sum_{m\in M\setminus \mathrm{REJ}(x)}\alpha(m)\prod_{j\in \mathrm{UP}(x)} q_j^{C_j(n(x_j))}\right. \\
&\qquad\qquad\left. - c(n(x))\sum_{i\in \mathrm{UP}_{\mathrm{OCC}}(x)} C_i(n(x_i))\frac{p_i}{q_i}\prod_{j\in \mathrm{UP}(x)} q_j^{C_j(n(x_j))}\right) \\
&\quad\cdot\left(\prod_{k=1}^{n(x)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(x_l)}\frac{\alpha({}^1x_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(x_l)-C_l(n(x_l))}}{p_l^{n(x_l)}}a_l({}^2x_l\oplus 1,{}^2x_l)
\end{aligned}
$$

and

$$\begin{aligned}
&\psi(y)\bar{p}(y,x)\\
&= \left(\prod_{l=1}^{J} a_l({}^2y_l, {}^2x_l)\right)\cdot\left(1-b(n(y))\sum_{m\in M\setminus\overline{\mathrm{REJ}}({}^1y,{}^2x)}\alpha(m)\prod_{j\in\mathrm{UP}({}^1y,{}^2x)} q_j^{C_j(n(y_j))}\right.\\
&\qquad\left.-c(n(y))\sum_{i\in\mathrm{UP}_{\mathrm{OCC}}({}^1y,{}^2x)} C_i(n(y_i))\frac{p_i}{q_i}\prod_{j\in\mathrm{UP}({}^1y,{}^2x)} q_j^{C_j(n(y_j))}\right)\\
&\quad\cdot\left(\prod_{k=1}^{n(y)}\frac{b(k-1)}{c(k)}\right)\prod_{l=1}^{J}\left(\prod_{k=1}^{n(y_l)}\frac{\alpha({}^1y_{l,k,1})}{C_l(k)}\right)\frac{q_l^{n(y_l)-C_l(n(y_l))}}{p_l^{n(y_l)}}a_l({}^2y_l\oplus 1, {}^2y_l).
\end{aligned}$$

From $\mathrm{REJ}(x)=\overline{\mathrm{REJ}}({}^1y,{}^2x)$ we obtain the required identity.

□

4 Conclusion

We have investigated in this paper a discrete time network with unreliable nodes, which in our opinion is an important model because already its *reliable* version has found many applications, e.g., in telecommunication and production systems. Our aim was to find an analogue to the celebrated product form steady state distributions which are well known in the equilibrium theory of queueing networks in continuous time.

The positive message obviously is that even in the realm of discrete time systems with the additional feature of breakdown and repair product form modelling is possible. On the other hand it turned out that the proofs are extremely tedious and generalisations towards more complicated networks seem to be not easy.

Nevertheless, there is still much research to be done. We hope that our forthcoming research in this area will contribute further to the modelling techniques with product form network scenarios.

Part of our ongoing research is concerned with networks with discrete time symmetric servers see [Yat90], [Yat94], [DS81], [DS83], and [SD83] for the case of reliable network nodes.

List of symbols

$\mathbb{N}$	:=	$\{1, 2, \ldots\}$ Positive Integers
$\mathbb{N}_0$	:=	$\{0, 1, 2, \ldots\}$
$\underline{n}$	:=	$\{1, \ldots, n\}, \quad n \in \mathbb{N}$
$\mathcal{P}(A)$		set of all subsets of set A
$n(x),\ n(x_j)$		Queue length, see page 5
$\mathrm{Per}(x),\ \overline{\mathrm{Per}}(x)$		Sets of operators permitted for state x, see pages 9 and 13
$\mathrm{Succ}(x),\ \overline{\mathrm{Succ}}(x)$		Sets of successor states of state x, see pages 9 and 13
$G_i(x_i, y_i)$	:=	$\{k \in \underline{C_i(x)} \mid D_k(x_i) = D_{n(x_i)}(y_i)\}$, see pages 11 and 22
$U\big(x, (m, s)\big)$	:=	$\{i \in \underline{J} \mid i = w(m,t)$ for some $t \in \{s+1, \ldots, S(m)\}$ and ${}^2x_i = 0\}$, see 19
$\mathrm{REJ}(x)$	:=	$\{m \in M \mid U\big(x, (m, 0)\big) = \varnothing\}$, see page 19
$V\big(y, (m, s)\big)$	:=	$\{i \in \underline{J} \mid i = w(m,t)$ for some $t \in \{1, \ldots, s-1\}$ and ${}^2y_i = 0\}$ see 24
$\overline{\mathrm{REJ}}(y)$	:=	$\{m \in M \mid V\big(y, (m, S(m)+1)\big) = \varnothing\}$, see page 25
$\mathrm{UP}(x)$	:=	$\{i \in \underline{J} \mid {}^2x_i = 0\}$, see page 19
$\mathrm{UP}_{\mathrm{OCC}}(x)$	:=	$\{i \in \underline{J} \mid n(x_i) > 0,\ {}^2x_i = 0\}$, see page 19
$S\colon M \to \mathbb{N}$		route length, see pages 4 and 6
$A(x, (m, s))$	:=	$\min\{t \mid s < t \leqq S(m), {}^2x_{w(m,t)} = 0\}$, see page 18
$B(y, (m, s))$	:=	$\max\{t \mid 1 \leqq t < s, {}^2y_{w(m,t)} = 0\}$, see page 24

References

[BCMP75] F. Baskett,M. Chandy,R. Muntz, and F.G. Palacios. Open, closed and mixed networks of queues with different classes of customers. *Journal of the Association of Computing Machinery* 22: pages 248-260, 1975.

[BGdT98] G. Bolch, S. Greiner, H. de Meer, and K. S. Trivedi. *Queueing networks and Markov chains.* John Wiley, New York, 1998.

[BK93] H. Bruneel and Byung G. Kim. *Discrete-Time Models for Communication Systems including ATM.* Kluwer Academic Publications, Boston, 1993.

[BSY94] J. A. Buzacott, J. G. Shanthikumar, and D. D. Yao. Jackson network models of manufacturing systems. In D. D. Yao, editor, *Stochastic Modeling and Analysis of Manufacturing Systems*, Springer Series in Operations Research, chapter 1, pages 1–46. Springer, New York, 1994.

[CM96] R. Chakka and I. Mitrani. Approximate solutions for open networks with breakdowns and repairs. In Kelly. F.P., Zachary. S., and I. Ziedins, editors, *Stochastic networks, Theory and Applications*, volume 4 of *Royal Statistical Society Lecture Notes Series*, chapter 16, pages 267–280. Clarendon Press, Oxford, 1996.

[CMP99] X. Chao, M. Miyazawa, and M. Pinedo. *Queueing Networks – Customers, Signals, and Product Form Solutions.* Wiley, Chichester, 1999.

[CY01] H. Chen and D.D. Yao. *Fundamentals of queueing networks.* Springer, Berlin, 2001.

[Dad01a] H. Daduna. *Queueing Networks with Discrete Time Scale: Explicit Expressions for the Steady State Behavior of Discrete Time Stochastic Networks*, volume 2046 of *Lecture Notes in Computer Science*. Springer, Berlin, 2001.

[Dad01b] H. Daduna. Stochastic networks with product form equilibrium. In D.N. Shanbhag and C.R. Rao, editors, *Stochastic Processes: Theory and Methods*, volume 19 of *Handbook of Statistics*, chapter 11, pages 309–364. Elsevier Science, Amsterdam, 2001.

[DS81] H. Daduna and R. Schassberger. A discrete-time round-robin queue with Bernoulli input and general arithmetic service time distributions. *Acta Informatica* 15: pages 159-175, 1981

[DS83] H. Daduna and R. Schassberger. Networks of queues in discrete time. *Zeitschrift fuer Operations Research ZOR*, 27:159 – 175, 1983.

[HMRT01] B.R. Haverkort, R. Marie, G. Rubino, and K. Trivedi. *Performability Modeling, Technique and Tools*. Wiley, New York, 2001.

[HPTD95] W. Henderson, C.E.M. Pearce, P.G. Taylor, and N.M. van Dijk, Insensitivity in discrete-time generalized Semi-Markov processes allowing multiple events and probabilistic service scheduling. *Annals of Applied Probability* 5: pages 78-96, 1995.

[Kel79] F. P. Kelly. *Reversibility and Stochastic Networks*. John Wiley and Sons, Chichester – New York – Brisbane – Toronto, 1979.

[Kle64] L. Kleinrock. Analysis of a time–shared processor. *Naval Research Logistics Quarterly* 10: pages 59-73, 1964.

[Kle67] L. Kleinrock. Time–shared systems: A theoretical treatment. *Journal of the Association of Computing Machinery* 14: pages 242-261, 1967.

[Kle75] L. Kleinrock. *Queueing Theory*, volume I. John Wiley and Sons, New York, 1975.

[Kle76] L. Kleinrock. *Queueing Theory*, volume II. John Wiley and Sons, New York, 1976.

[KN03] K. Kordybach and S. Nousiainen. Radio resource management in WCDMA-based networks in emergency situations. In J. Charzinski, R. Lehnert, and P. Tran-Gia, editors, *Providing Quality of Service in Heterogeneous Environments*, volume 5a of *Teletraffic Science and Engineering*, pages 331–338, Amsterdam, 2003. Proceedings of the 18th International Teletraffic Congress, Elsevier.

[MD06] C. Malchin and H. Daduna. Availability and Performance Analysis in a Discrete Time Tandem Network with Product Form Steady State. In R. German and A. Heindl *Measuring, Modelling and Evaluation of Computer and Communication Systems*, pages 381-398, Nürnberg, 2006. Proceedings of the 13th GI/ITG Conference, VDE Verlag.

[MKC03] M. Menth, S. Kopf, and J. Charzinski. Impact of resilience requirements on the performance of network admission control methods. Technical report, University of Würzburg, Institute of Computer Science, 2003. No. 309.

[NSB+03] A. Nucci, B. Schroeder, S. Bhattacharyya, N. Taft, , and C. Diot. IGP link weight assignment for transient link failures. In J. Charzinski, R. Lehnert, and P. Tran-Gia, editors, *Providing Quality of Service in Heterogeneous Environments*, volume 5a of *Teletraffic Science and Engineering*, pages 321–330, Amsterdam, 2003. Proceedings of the 18th International Teletraffic Congress, Elsevier.

[Per90] H. G. Perros. Approximation algorithms for open queueing networks with blocking. In H. Takagi, editor, *Stochastic Analysis of Computer and Communication Systems*, pages 451–498. North-Holland, Amsterdam, 1990.

[SD03a] C. Sauer and H. Daduna. Availability formulas and performance measures for separable degradable networks. *Economic Quality Control*, 18:165–194, 2003.

[SD03b] C. Sauer and H. Daduna. Separable networks with unreliable servers. In J. Charzinski, R. Lehnert, and P. Tran-Gia, editors, *Providing QoS in Heterogeneous Environments*, volume 5b of *Teletraffic Science and Engineering*, pages 821–830. Elsevier Science, Amsterdam, 2003.

[SD04] C. Sauer and H. Daduna. Degradable networks with general up and down time distributions. In P. Buchholz, R. Lehnert, and M. Pioro, editors, *Proceedings of the 12th GI/ITG Conference on MMB together with 3rd PGTS*, pages 185–194, Berlin, 2004. VDE-Verlag.

[SD83] H. Daduna and R. Schassberger. A discrete-time technique for solving closed queueing network models of computer systems. In P.J. Kühn and K.M. Schulz (eds.): *Messung, Modellierung und Bewertung von Rechensystemen*, Informatik-Fachberichte 61: pages 122-134, Berlin, 1983.

[SV03] M. Schopp and J. Viallon. Service availability in distributed network architectures: Evaluation and planning issues. In J. Charzinski, R. Lehnert, and P. Tran-Gia, editors, *Providing Quality of Service in Heterogeneous Environments*, volume 5a of *Teletraffic Science and Engineering*, pages 251–260, Amsterdam, 2003. Proceedings of the 18th International Teletraffic Congress, Elsevier.

[Woo94] M.E. Woodward. *Communication and Computer Networks: Modelling with Discrete–Time Queues*. IEEE Computer Society Press, Los Alamitos, CA, 1994.

[Yat90] R. D. Yates. *High speed round-robin queueing networks.* Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, 1990.

[Yat94] R. D. Yates. Analysis of discrete time queues via the reversed process. Queueing Systems and Their Applications, 18: pages 107–116, 1994.