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A MESH-INDEPENDENCE RESULT FOR SEMISMOOTH NEWTON METHODS

MICHAEL HINTERMÜLLER* AND MICHAEL ULBRICH†

Abstract. For a class of semismooth operator equations a mesh independence result for generalized Newton methods is established. The main result states that the continuous and the discrete Newton process, when initialized properly, converge q-linearly with the same rate. The problem class considered in the paper includes MCP-function based reformulations of first order conditions of a class of control constrained optimal control problems for partial differential equations for which a numerical validation of the theoretical results is given.

Key words. Variational Inequalities, function spaces, generalized differential, mesh independence principle, semismooth Newton method.

AMS subject classifications. 65J15, 65K10, 49M25, 90C33.

1. Introduction.

This paper is devoted to the study of (local) convergence properties of Newton type methods applied to discretizations of a class of nonsmooth operator equations

$$G(y) = 0, \quad G : L^2(\Omega) \to L^2(\Omega)\,, \tag{1.1}$$

where the operator G is related to an MCP-function based reformulation of the infinite-dimensional *box-constrained variational inequality problem* (BVIP)

$$y \in Y_{\text{ad}}, \quad (F(y), v - y)_{L^2} \geq 0 \qquad \forall\, v \in Y_{\text{ad}}, \tag{1.2}$$

where the feasible set is given by $Y_{\text{ad}} = \{y \in L^2(\Omega) : \alpha \leq y \leq \beta \text{ a.e. on } \Omega\}$ with $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. Here $\Omega \subset \mathbb{R}^n$ is measurable with finite Lebesgue measure $|\Omega| > 0$, $L^2(\Omega)$ is the Hilbert space of square integrable functions, and $F : L^2(\Omega) \to L^2(\Omega)$ is a linear or nonlinear operator.

It is well known that if $G : Y \to Z$ (Y, Z Banach spaces) is Fréchet differentiable, G' is locally Lipschitz and invertible at a solution $\bar{y}$ of (1.1), then the Newton method

$$y^{k+1} = y^k - G'(y^k)^{-1} G(y^k) \tag{1.3}$$

is locally quadratically convergent to $\bar{y}$; see, e.g., [16]. Moreover, for appropriate discretizations $G_h(y_h) = 0$, with $G_h : Y_h \to Z_h$ and Y_h, Z_h suitable finite dimensional spaces, a local solution $\bar{y}_h$ exists and, when initialized properly, the discrete Newton process

$$y_h^{k+1} = y_h^k - G_h'(y_h^k)^{-1} G_h(y_h^k)\,, \tag{1.4}$$

enjoys the property of mesh independence; see [1]. In [8] the Lipschitz uniformity property of the discretization required in [1] is weakened resulting in an asymptotic version of the mesh independence result. In [2] the concepts of [1] are carried over to the case of generalized equations $G(y) \in T(y)$ with $T : Y \to Z$ a multi-valued mapping. Further, the abstract results are applied to control constrained optimal control problems for ordinary differential equations. In [24] the results of [1] are extended to an augmented Lagrangian-SQP method for solving optimal control problems involving a possibly nonlinear partial differential state equation. In contrast to [2] no constraints are considered. By utilizing Lipschitzian localizations, recently in [9] the mesh independence property of Newton's method, when applied

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to discretized variational inequalities or generalized equations, was established under weaker conditions then those in [2]. In [4, 5] asymptotic mesh independence is proved under reduced requirements on the (Fréchet) derivative of the operator G. But still, like in all of the aforementioned results, the operator G is assumed to be Fréchet differentiable with sufficiently smooth derivative.

In many cases the requirement of G being Fréchet differentiable is not adequate. In fact, returning to the BVIP (1.2), it is well known [20] that (1.2) is equivalent to the mixed complementarity problem (MCP)

$$\alpha \le y \le \beta, \quad (y-\alpha)F(y) \le 0, \quad (y-\beta)F(y) \le 0 \quad \text{a.e. in } \Omega. \tag{1.5}$$

Using the equivalence

$$\alpha \le a \le \beta, \;\; (a-\alpha)b \le 0, \;\; (a-\beta)b \le 0 \iff a - P_{[\alpha,\beta]}(a-\sigma b) = 0,$$

where $\sigma > 0$ is arbitrary and

$$P_{[\alpha,\beta]} : \mathbb{R} \to [\alpha,\beta], \quad P_{[\alpha,\beta]}(t) = \max\{\alpha, \min\{t,\beta\}\}$$

denotes the projection onto $[\alpha,\beta]$, we can rewrite (1.5) (and thus (1.2)) in the form

$$G(y) = y - P_{[\alpha,\beta]}(y - \sigma F(y)) = 0 \quad \text{a.e. in } \Omega. \tag{1.6}$$

Here, the projection is applied pointwise on Ω. The operator equation (1.6) is a special case of (1.1) and obviously G is not Fréchet differentiable. By utilizing weaker types of derivatives and approximations of classes of nondifferentiable operators, in, e.g., [6, 7, 12, 14, 15, 17, 18, 19, 20, 22] local convergence properties of the resulting nonsmooth version of Newton's method are proved. Under a semismoothness assumption on G, the rate of convergence is typically q-superlinear. Compared to finite dimensions, in infinite dimensions the generalization of the derivative is a more delicate issue [7, 12, 20, 22]. In finite dimensions the generalized differentiability concepts rely on Rademacher's theorem, which has no analogue in infinite dimensions. While the max- and min-operator, and thus also the projection $P_{[\alpha,\beta]}$ are strongly semismooth in finite dimensions [10], these operators are *not* semismooth as a mapping $L^p(\Omega) \to L^p(\Omega)$, $1 \le p \le +\infty$. In [12, 22] it is shown that a *two norm discrepancy*, i.e., $\max : L^p(\Omega) \to L^q(\Omega)$ with $1 \le q < p \le +\infty$, is required for max to be semismooth, and the same holds true for min and $P_{[\alpha,\beta]}$. In general, this fact necessitates a smoothing step in the corresponding semismooth Newton method [22] in order to achieve locally superlinear convergence. In [12] it was observed that for particular classes of constrained optimal control problems smoothing steps can be skipped due to the properties of the resulting operator F in (1.5). In this paper we exploit the latter fact in order to avoid the necessity of smoothing steps. As a consequence we henceforth assume that F has the following particular form:

$$F(y) = A(y) + \lambda y, \tag{1.7}$$

with $\lambda > 0$ and a continuously Fréchet differentiable operator $A : L^2(\Omega) \to L^2(\Omega)$. Furthermore, we assume that A maps $L^2(\Omega)$ locally Lipschitz continuously to $L^p(\Omega)$ for some $p \in (2, \infty)$. Thus, (1.6) becomes

$$y - P_{[\alpha,\beta]}(y - \sigma(A(y) + \lambda y)) = 0 \quad \text{a.e. in } \Omega. \tag{1.8}$$

For the rest of the paper, we choose $\sigma = 1/\lambda$ and multiply by λ to obtain the following equivalent reformulation of (1.5)

$$G(y) := \lambda y - P_{[\lambda\alpha,\lambda\beta]}(-A(y)) = 0 \quad \text{a.e. in } \Omega. \tag{1.9}$$

Clarke's generalized differential of $\phi(t) = P_{[\lambda\alpha,\lambda\beta]}(t)$ is given by

$$\partial\phi(t) = \begin{cases} 0 & \text{if } t \notin [\lambda\alpha, \lambda\beta], \\ [0,1] & \text{if } t \in \{\lambda\alpha, \lambda\beta\}, \\ 1 & \text{if } t \in (\lambda\alpha, \lambda\beta). \end{cases}$$

Following [22], we define the generalized differential

$$\partial G : L^2(\Omega) \rightrightarrows L^2(\Omega), \quad \partial G(y) = \{\lambda I + D(y) \cdot A'(y) : D(y) \text{ satisfies (1.11)}\}, \tag{1.10}$$

$$D : L^2(\Omega) \to L^\infty(\Omega), \quad D(y)(x) \begin{cases} = 0 & \text{if } -A(y)(x) \notin [\lambda\alpha, \lambda\beta], \\ \in [0,1] & \text{if } -A(y)(x) \in \{\lambda\alpha, \lambda\beta\}, \\ = 1 & \text{if } -A(y)(x) \in (\lambda\alpha, \lambda\beta). \end{cases} \tag{1.11}$$

It has been shown in [20, 22], see also [12], that the operator G is semismooth in the following sense:

$$\sup_{M\in\partial G(y+s)} \|G(y+s) - G(y) - Ms\|_{L^2} = o(\|s\|_{L^2}) \quad \text{as } \|s\|_{L^2} \to 0. \tag{1.12}$$

This result can be used to prove the local q-superlinear convergence of the following nonsmooth Newton's method:

ALGORITHM 1.1.

0. Choose $y^0 \in L^2(\Omega)$.

For $k = 0, 1, 2, \ldots$:

1. If $G(y^k) = 0$, STOP with result y^k.
2. Choose $M_k \in \partial G(y^k)$.
3. Compute the Newton step $s^k \in L^2(\Omega)$ by solving
$$M_k s^k = -G(y^k)$$
and set $y^{k+1} = y^k + s^k$.

The local convergence analysis requires a regularity assumption, e.g., the uniformly bounded invertibility of the operators $M_k \in \mathcal{L}(L^2, L^2)$.

One way to derive an MCP is related to MCP-function based reformulations of first order optimality conditions of box constrained optimal control problems. In [13] certain mesh independence results for the gradient projection method applied to the latter problem class are proved. From the numerical point of view, the gradient projection method has some drawbacks like rather slow convergence compared to Newton-type methods and possible chattering of active resp. inactive sets close to the solution. Also, the results provided in [13] are different from our mesh independence assertions.

The aim of this paper is to prove a mesh independence result for the discrete analogue of Algorithm 1.1. Our main result states that for any given q-linear rate of convergence θ there exists a sufficiently small mesh size $h' > 0$ of discretization and a radius $\delta > 0$ such that, for all $h \le h'$, the continuous and the discrete Newton processes converge at least at the q-linear rate θ when initialized by y^0, y_h^0 satisfying $\max\{\|y_h^0 - \bar{y}_h\|_{L^2}, \|y^0 - \bar{y}\|_{L^2}\} \le \delta$.

In section 2 we introduce appropriate discretizations of problem (1.8) and the discrete version of Algorithm 1.1. The mesh independence result is presented in section 3. Sufficient conditions ensuring regularity are in the focus of section 4. These conditions are motivated by a class of control constrained optimal control problems for semilinear elliptic differential equations. The latter problem class is addressed in section 5. It is shown that the assumptions for the mesh independence result are satisfied. Finally, in section 6 numerical results are presented.

2. Discretization. We now approximate functions in $L^2(\Omega)$ by a finite element discretization. To this end, let be given a (sufficiently regular) subdivision of Ω (e.g., a regular triangulation) into subdomains $T \in \mathcal{T}_h$:

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T, \quad T_1, T_2 \in \mathcal{T}_h, \ T_1 \neq T_2 \Longrightarrow T_1 \cap T_2 \subset \partial T_1 \cap \partial T_2.$$

Usually, in 2D, $\mathcal{T}_h$ will consist of triangles in the interior and of deformed, boundary-fitted triangles on the boundary. The subscript h is a measure for the maximum diameter of all elements in $\mathcal{T}_h$. Now, we define

$$Y_h = \{y_h : \Omega \mapsto \mathbb{R} : y_h|_{\text{int } T} = \text{constant } \ \forall\, T \in \mathcal{T}_h\}.$$

The space Y_h is equipped with the L^2-norm, i.e., $\|\cdot\|_{Y_h} = \|\cdot\|_{L^2}$. The value of y_h on ∂T, $T \in \mathcal{T}_h$ is not important. Appropriate numerical discretization of (1.5) now yields the discrete mixed complementarity problem

$$\alpha \le y_h \le \beta, \quad (y_h - \alpha) F_h(y_h) \le 0, \quad (y_h - \beta) F_h(y_h) \le 0 \quad \text{a.e. in } \Omega \tag{2.1}$$

with $F_h : Y_h \to Y_h$, $F_h(y_h) = A_h(y_h) + \lambda y_h$, and continuously differentiable operator $A_h : Y_h \to Y_h$.

We reformulate (2.1) as in the infinite-dimensional case in the form

$$G_h(y_h) := \lambda y_h - P_{[\lambda\alpha, \lambda\beta]}(-A_h(y_h)) = 0 \tag{2.2}$$

with an operator $G_h : Y_h \to Y_h$. Note that G_h is piecewise constant on the elements $T \in \mathcal{T}_h$, and thus (2.2) is a finite-dimensional system of equations. Then we define the following generalized differential of G_h:

$$\partial G_h : Y_h \rightrightarrows Y_h, \quad \partial G_h(y_h) = \{\lambda I + D_h(y_h) \cdot A_h'(y_h) : D_h(y_h) \text{ satisfies (2.4)}\}, \tag{2.3}$$

$$D_h(y_h) \in Y_h, \quad D_h(y_h)(x) \begin{cases} = 0 & \text{if } -A_h(y_h)(x) \notin [\lambda\alpha, \lambda\beta], \\ \in [0,1] & \text{if } -A_h(y_h)(x) \in \{\lambda\alpha, \lambda\beta\}, \\ = 1 & \text{if } -A_h(y_h)(x) \in (\lambda\alpha, \lambda\beta). \end{cases} \tag{2.4}$$

Again, $D_h(y_h) \in Y_h$ is constant on each $T \in \mathcal{T}_h$. Furthermore, in analogy to the continuous setting, a nonsmooth Newton's method for the solution of (2.2) can be formulated:

ALGORITHM 2.1.

0. Choose $y_h^0 \in Y_h$.

   For $k = 0, 1, 2, \ldots$:

1. If $G_h(y_h^k) = 0$, STOP with result y_h^k.
2. Choose $M_{hk} \in \partial G_h(y_h^k)$.
3. Compute the Newton step $s_h^k \in Y_h$ by solving

$$M_{hk} s_h^k = -G_h(y_h^k)$$

   and set $y_h^{k+1} = y_h^k + s_h^k$.

3. Mesh-Independence. We prove that Algorithm 2.1 is mesh independent in the sense that, for any linear rate of convergence θ, there exists a radius $\rho > 0$ such that, for all h sufficiently small, the regions on which Algorithm 1.1 and Algorithm 2.1 converge with at least linear q-rate θ contain the ρ-balls about the respective solutions.

For the proof we need a preparatory result on the mesh independence of

$$\sup_{M_h \in \partial G_h(\bar{y}_h + s_h)} \|G_h(\bar{y}_h + s_h) - G_h(\bar{y}_h) - M_h s_h\|_{Y_h},$$

which requires several assumptions.

Let $\bar{y} \in L^2(\Omega)$ be a solution of (1.5) and assume that strict complementarity holds:

ASSUMPTION 3.1 (Strict complementarity).

$$|\{\min\{\bar{y} - \alpha, \beta - \bar{y}\} + |F(\bar{y})| = 0\}| = 0. \tag{3.1}$$

Since $|\Omega| < \infty$ and

$$\{\min\{\bar{y} - \alpha, \beta - \bar{y}\} + |F(\bar{y})| < \varepsilon\} \downarrow \{\min\{\bar{y} - \alpha, \beta - \bar{y}\} + |F(\bar{y})| = 0\} \quad \text{as } \varepsilon \to 0^+$$

we conclude that

$$\lim_{\varepsilon \to 0^+} |\{\min\{\bar{y} - \alpha, \beta - \bar{y}\} + |F(\bar{y})| < \varepsilon\}| = 0. \tag{3.2}$$

Furthermore, for any h, let be given a solution $\bar{y}_h \in Y_h$ of (2.1). We work under the following

ASSUMPTION 3.2.

1.

$$\lim_{h \to 0^+} \|\bar{y}_h - \bar{y}\|_{L^2} = 0, \tag{3.3}$$

$$\lim_{h \to 0^+} \|A_h(\bar{y}_h) - A(\bar{y})\|_{L^p} = 0. \tag{3.4}$$

2. The discretization family is *locally Lipschitz uniform*, i.e., there exist $h_0 > 0$, $\delta_0 > 0$, and $L > 0$ such that

$$\begin{aligned} \left\|A(y^2) - A(y^1)\right\|_{L^p} &\leq L \left\|y^2 - y^1\right\|_{L^2}, \quad \forall\, y^i \in L^2(\Omega),\ \left\|y^i - \bar{y}\right\|_{L^2} \leq \delta_0, \\ \left\|A_h(y_h^2) - A_h(y_h^1)\right\|_{L^p} &\leq L \left\|y_h^2 - y_h^1\right\|_{L^2} \quad \forall\, y_h^i \in Y_h,\ \left\|y_h^i - \bar{y}_h\right\|_{Y_h} \leq \delta_0,\ h \leq h_0. \end{aligned}$$

3. The discretization family has the *uniform linear approximation property*, i.e., A and A_h, $h \leq h_0$, are Fréchet differentiable in a neighborhood of $\bar{y}$ and $\bar{y}_h$, respectively, and there exists a function $\rho : [0, \delta_0) \to [0, \infty)$ such that

$$\lim_{t \to 0^+} \frac{\rho(t)}{t} = 0, \tag{3.5}$$

$$\|A(y) - A(\bar{y}) - A'(y)(y - \bar{y})\|_{L^2} \leq \rho(\|y - \bar{y}\|_{L^2}) \quad \forall y \in L^2(\Omega),\ \|y - \bar{y}\|_{L^2} \leq \delta_0, \tag{3.6}$$

$$\|A_h(y_h) - A_h(\bar{y}_h) - A_h'(y_h)(y_h - \bar{y}_h)\|_{Y_h} \leq \rho(\|y_h - \bar{y}_h\|_{Y_h}) \quad \forall y_h \in Y_h,\ \|y_h - \bar{y}_h\|_{Y_h} \leq \delta_0,\ h \leq h_0. \tag{3.7}$$

Now let $\gamma \in (0,1)$ be given. Then, due to the semismoothness of G, there exists $\delta' \in (0, \delta_0]$ such that

$$\sup_{M \in \partial G(\bar{y}+s)} \|G(\bar{y}+s) - G(\bar{y}) - Ms\|_{L^2} \leq \gamma \|s\|_{L^2} \quad \forall\, s \in L^2(\Omega),\ \|s\|_{L^2} \leq \delta'.$$

Our aim is to prove the following uniform semismoothness result, which will enable us to show the mesh independence of the semismooth Newton's method 2.1.

THEOREM 3.3. *Under the Assumptions* 3.1 *and* 3.2*, for all* $\gamma \in (0,1)$*, there exist* $\delta' \in (0, \delta_0]$ *and* $h' \in (0, h_0]$ *such that the following holds true:*

$$\sup_{M_h \in \partial G_h(\bar{y}_h+s_h)} \|G_h(\bar{y}_h + s_h) - G_h(\bar{y}_h) - M_h s_h\|_{Y_h} \leq \gamma \|s_h\|_{Y_h} \tag{3.8}$$
$$\forall\, s_h \in Y_h,\ \|s_h\|_{Y_h} \leq \delta',\ \ h \leq h'.$$

$$\sup_{M \in \partial G(\bar{y}+s)} \|G(\bar{y}+s) - G(\bar{y}) - Ms\|_{L^2} \leq \gamma \|s\|_{L^2} \qquad \forall\, s \in L^2(\Omega),\ \|s\|_{L^2} \leq \delta'. \tag{3.9}$$

REMARK 3.4. The existence of a radius $\delta' > 0$ such that (3.9) holds follows from the semismoothness (1.12) of G. Nevertheless, we enclose the proof of (3.9), without any additional work, by defining $Y_0 = L^2(\Omega)$, $G_0 = G$, $\partial G_0 = \partial G$, and $\bar{y}_0 = \bar{y}$. Then for $h = 0$ the Assumption 3.2 obviously holds. Therefore, we can use these assumptions for all $h \geq 0$, and thus can concentrate on proving (3.8) for all $0 \leq h \leq h'$.

3.1. Proof of Theorem 3.3. We define

$$\begin{aligned} c(\bar{y}) &= \min\{\bar{y} - \alpha, \beta - \bar{y}\} + |F(\bar{y})| \geq 0, \\ c_h(\bar{y}_h) &= \min\{\bar{y}_h - \alpha, \beta - \bar{y}_h\} + |F_h(\bar{y}_h)| \geq 0, \\ \Omega(\varepsilon) &= \{c(\bar{y}) < \varepsilon\}, \quad \Omega_h(\varepsilon) = \{c_h(\bar{y}_h) < \varepsilon\}. \end{aligned}$$

The strict complementarity assumption implies (3.2), and thus, for any $\mu > 0$, there exists $\varepsilon = \varepsilon(\mu) > 0$ such that

$$|\Omega(2\varepsilon)| \leq \frac{\mu}{2}.$$

We now use that for all $s, t \in [\alpha, \beta]$ the following holds:

$$|\min\{t - \alpha, \beta - t\} - \min\{s - \alpha, \beta - s\}| \leq |t - s| \quad \forall\, s, t \in [\alpha, \beta]. \tag{3.10}$$

To prove this, we can, without restriction, assume that

$$m_t := \min\{t - \alpha, \beta - t\} \geq \min\{s - \alpha, \beta - s\} =: m_s,$$

otherwise we simply would exchange the roles of m_s and m_t. If $m_s = s - \alpha$, we use $m_t \leq t - \alpha$ to obtain

$$|m_t - m_s| = m_t - m_s \leq (t - \alpha) - (s - \alpha) = t - s \leq |t - s|.$$

Similarly, if $m_s = \beta - s$, we can use $m_t \leq \beta - t$ to derive

$$|m_t - m_s| = m_t - m_s \leq (\beta - t) - (\beta - s) = s - t \leq |t - s|.$$

Hence, by (3.10) and Assumption 3.2.1, we have

$$\begin{aligned} \|c(\bar{y}) - c_h(\bar{y}_h)\|_{L^2} &\leq \|\min\{\bar{y} - \alpha, \beta - \bar{y}\} - \min\{\bar{y}_h - \alpha, \beta - \bar{y}_h\}\|_{L^2} + \|F(\bar{y}) - F_h(\bar{y}_h)\|_{L^2} \\ &\leq (1 + \lambda) \|\bar{y} - \bar{y}_h\|_{L^2} + \|A(\bar{y}) - A_h(\bar{y}_h)\|_{L^2} \to 0 \quad \text{as } h \to 0. \end{aligned}$$

Next, observe that

$$|\Omega_h(\varepsilon)| \le |\Omega(2\varepsilon)| + |\{c(\bar{y}) - c_h(\bar{y}_h) > \varepsilon\}|\,.$$

In the sequel, we will repeatedly use the following estimate: For all $\eta > 0$, all $q \in [1, \infty)$ and all $v \in L^q$

$$|\{|v| \ge \eta\}| = \frac{1}{\eta^q} \int_{\{|v|\ge\eta\}} \eta^q \, dx \le \frac{1}{\eta^q} \int_{\{|v|\ge\eta\}} |v(x)|^q \, dx \le \frac{\|v\|_{L^q}^q}{\eta^q}. \tag{3.11}$$

Applying this inequality, we obtain

$$|\{c(\bar{y}) - c_h(\bar{y}_h) > \varepsilon\}| \le \frac{1}{\varepsilon^2} \|c(\bar{y}) - c_h(\bar{y}_h)\|_{L^2}^2 \to 0 \quad \text{as } h \to 0,$$

and from this we see that we can find $h_1 = h_1(\varepsilon) \in (0, h_0]$ such that

$$|\Omega_h(\varepsilon)| \le \mu \quad \forall\, h \le h_1. \tag{3.12}$$

Next, we show that for $\eta = \eta(\varepsilon) := \min\{1, \lambda\}\varepsilon$ and

$$\Omega_h^1(\varepsilon) := \{|A_h(y_h) - A_h(\bar{y}_h)| < \eta\} \setminus \Omega_h(\varepsilon) \tag{3.13}$$

we have the inclusion

$$\Omega_h^1(\varepsilon) \subset \{|A_h(y_h) - A_h(\bar{y}_h)| < \eta,\ |A_h(\bar{y}_h) + \lambda\alpha| \ge \eta,\ |A_h(\bar{y}_h) + \lambda\beta| \ge \eta\}. \tag{3.14}$$

Since $\bar{y}_h$ solves (2.1), for a.a. $x \notin \Omega_h(\varepsilon)$ one of the following cases occurs

$$\bar{y}_h(x) = \alpha, \quad F_h(\bar{y}_h)(x) = A_h(\bar{y}_h)(x) + \lambda\alpha \ge \varepsilon, \tag{3.15}$$
$$\bar{y}_h(x) = \beta, \quad F_h(\bar{y}_h)(x) = A_h(\bar{y}_h)(x) + \lambda\beta \le -\varepsilon, \tag{3.16}$$
$$\bar{y}_h(x) \in [\alpha+\varepsilon, \beta-\varepsilon], \quad F_h(\bar{y}_h)(x) = A_h(\bar{y}_h)(x) + \lambda\bar{y}_h(x) = 0. \tag{3.17}$$

We have the implications

$$\begin{aligned}
(3.15) &\implies A_h(\bar{y}_h)(x) + \lambda\beta \ge A_h(\bar{y}_h)(x) + \lambda\alpha \ge \varepsilon,\\
(3.16) &\implies A_h(\bar{y}_h)(x) + \lambda\alpha \le A_h(\bar{y}_h)(x) + \lambda\beta \le -\varepsilon,\\
(3.17) &\implies \begin{cases} A_h(\bar{y}_h)(x) + \lambda\alpha = \lambda\alpha - \lambda\bar{y}_h(x) \le -\lambda\varepsilon,\\ A_h(\bar{y}_h)(x) + \lambda\beta = \lambda\beta - \lambda\bar{y}_h(x) \ge \lambda\varepsilon. \end{cases}
\end{aligned}$$

Taking all three cases together, we have shown that

$$x \notin \Omega_h(\varepsilon) \implies x \in \{|A_h(\bar{y}_h) + \lambda\alpha| \ge \eta,\ |A_h(\bar{y}_h) + \lambda\beta| \ge \eta\}.$$

This implies (3.14).

For the estimation of the remainder term

$$R_h(y_h) = G_h(y_h) - G_h(\bar{y}_h) - M_h(y_h - \bar{y}_h)$$

occurring in (3.8) with $M_h \in \partial G_h(y_h)$ we use the splitting (see (3.14)) $\Omega = \Omega_h^1(\varepsilon) \cup \Omega_h^2(\varepsilon)$ with $\Omega_h^1(\varepsilon)$ defined in (3.13) and

$$\Omega_h^2(\varepsilon) = \Omega_h(\varepsilon) \cup \{|A_h(y_h) - A_h(\bar{y}_h)| \ge \eta\}.$$

1. Estimate on $\Omega_h^1(\varepsilon)$:

Let $x \in \Omega_h^1(\varepsilon)$ be arbitrary. Then we have $|A_h(y_h)(x) - A_h(\bar{y}_h)(x)| < \eta$.

Case 1: $-A_h(\bar{y}_h)(x) \le \lambda\alpha - \eta$.
We then obtain $-A_h(y_h)(x) < \lambda\alpha$ and thus

$$R_h(y_h)(x) = (\lambda y_h - \lambda\alpha - \lambda\bar{y}_h + \lambda\alpha - \lambda(y_h - \bar{y}_h))(x) = 0$$

Case 2: $-A_h(\bar{y}_h)(x) \ge \lambda\beta + \eta$.
We then obtain $-A_h(y_h)(x) > \lambda\beta$ and thus

$$R_h(y_h)(x) = (\lambda y_h - \lambda\beta - \lambda\bar{y}_h + \lambda\beta - \lambda(y_h - \bar{y}_h))(x) = 0$$

Case 3: $x \in \Omega_h^{1\,\prime}(\varepsilon) = \{|A_h(y_h) - A_h(\bar{y}_h)| < \eta, \ -A_h(\bar{y}_h) \in [\lambda\alpha + \eta, \lambda\beta - \eta]\}$.
Then $A_h(y_h)(x) \in (\lambda\alpha, \lambda\beta)$ and thus

$$\begin{aligned} R_h(y_h)(x) &= (\lambda y_h + A_h(y_h) - \lambda\bar{y}_h - A_h(\bar{y}_h) - (\lambda I + A_h'(y_h))(y_h - \bar{y}_h))(x) \\ &= A_h(y_h) - A_h(\bar{y}_h) - A_h'(y_h)(y_h - \bar{y}_h))(x). \end{aligned}$$

This implies for all $h \le h_0$ and all $y_h \in Y_h$, $\|y_h - \bar{y}_h\| \le \delta_0$:

$$\begin{aligned} \|R_h(y_h)\|_{L^2(\Omega_h^{1\,\prime}(\varepsilon))} &= \|A_h(y_h) - A_h(\bar{y}_h) - A_h'(y_h)(y_h - \bar{y}_h)\|_{L^2(\Omega_h^{1\,\prime}(\varepsilon))} \\ &\le \|A_h(y_h) - A_h(\bar{y}_h) - A_h'(y_h)(y_h - \bar{y}_h)\|_{Y_h} \le \rho(\|y_h - \bar{y}_h\|_{Y_h}), \end{aligned}$$

where we have used (3.7). Now let $\delta_2 = \delta_2(\gamma) > 0$ be so small that

$$\rho(t) \le \frac{\gamma}{2} t$$

for all $t \le \delta_2$, which is possible by (3.5). Then

$$\|R_h(y_h)\|_{L^2(\Omega_h^1(\varepsilon))} = \|R_h(y_h)\|_{L^2(\Omega_h^{1\,\prime}(\varepsilon))} \le \rho(\|y_h - \bar{y}_h\|_{L^2}) \le \frac{\gamma}{2} \|y_h - \bar{y}_h\|_{L^2}$$

for all $y_h \in Y_h$, $\|y_h - \bar{y}_h\| \le \delta_2$.

2. Estimate on $\Omega_h^2(\varepsilon)$:

We already have shown the estimate (3.12) for the measure of $\Omega_h(\varepsilon)$. To estimate the measure of the second set, we use (3.11) and obtain that, for all $h \le h_0$ and all $y_h \in Y_h$, $\|y_h - \bar{y}_h\|_{L^2} \le \delta_0$,

$$|\{|A_h(y_h) - A_h(\bar{y}_h)| \ge \eta\}| \le \eta^{-p} \|A_h(y_h) - A_h(\bar{y}_h)\|_{L^p}^p \le \eta^{-p} L^p \|y_h - \bar{y}_h\|_{L^2}^p .$$

Thus, choosing

$$\delta_3 = \delta_3(\mu, \varepsilon) = \min\left\{\delta_0, \frac{\eta}{L}\mu^{1/p}\right\},$$

we obtain

$$|\{|A_h(y_h) - A_h(\bar{y}_h)| \ge \eta\}| \le \mu$$

for all $y_h \in Y_h$, $\|y_h - \bar{y}_h\|_{L^2} \le \delta_3$ and all $h \le h_0$. Therefore, we arrive at the estimate

$$\left|\Omega_h^2(\varepsilon)\right| \le 2\mu$$

for all $y_h \in Y_h$, $\|y_h - \bar{y}_h\|_{L^2} \le \delta_3$ and all $h \le h_1$.

From

$$R_h(y_h) = -P_{[\lambda\alpha,\lambda\beta]}(-A_h(y_h)) + P_{[\lambda\alpha,\lambda\beta]}(-A_h(\bar{y}_h)) - D_h(y_h)A_h'(y_h)(y_h - \bar{y}_h)$$

and $|P_{[\lambda\alpha,\lambda\beta]}(t) - P_{[\lambda\alpha,\lambda\beta]}(s)| \le |t-s|$ it follows that

$$\|R_h(y_h)\|_{L^p} \le \|A_h(y_h) - A_h(\bar{y}_h)\|_{L^p} + \|A_h'(y_h)(y_h - \bar{y}_h)\|_{L^p} \le 2L\,\|y_h - \bar{y}_h\|_{L^2}\,.$$

Here, we have used that, for all $t > 0$,

$$d_h(t) = \frac{1}{t}(A_h(y_h + t(y_h - \bar{y}_h)) - A_h(y_h))$$

converges to $A_h'(y_h)(y_h - \bar{y}_h)$ as $t \to 0$ and is uniformly bounded in L^p; in fact,

$$\|d_h(t)\|_{L^p} = \frac{1}{t}\,\|A_h(y_h + t(y_h - \bar{y}_h)) - A_h(y_h)\|_{L^p} \le L\,\|y_h - \bar{y}_h\|_{L^2}\,.$$

Therefore,

$$\|A_h'(y_h)(y_h - \bar{y}_h)\|_{L^p} \le L\,\|y_h - \bar{y}_h\|_{L^2}\,.$$

We now can estimate (see [23, Lemma 2.1])

$$\begin{aligned}\|R_h\|_{L^2(\Omega_h^2(\varepsilon))} &\le \left|\Omega_h^2(\varepsilon)\right|^{\frac{1}{2}-\frac{1}{p}}\,\|R_h\|_{L^p(\Omega_h^2(\varepsilon))}\\ &\le (2\mu)^{\frac{p-2}{2p}}2L\,\|y_h - \bar{y}_h\|_{L^2}\,.\end{aligned}$$

Now we can proceed as follows:

Choose (in this order)

$$\mu = \frac{1}{2}\left(\frac{\gamma}{4L}\right)^{\frac{2p}{p-2}}, \quad \varepsilon = \varepsilon(\mu), \quad h_1 = h_1(\varepsilon), \quad \delta_2 = \delta_2(\gamma), \quad \delta_3 = \delta_3(\mu,\varepsilon)$$

and set

$$\delta' = \min\{\delta_2, \delta_3\}, \quad h' = h_1.$$

Then we obtain for all $h \le h_1$ and all $y_h \in Y_h$, $\|y_h - \bar{y}_h\|_{L^2} \le \delta'$:

$$\begin{aligned}\|R_h\|_{Y_h} &\le \|R_h\|_{L^2(\Omega_h^1(\varepsilon))} + \|R_h\|_{L^2(\Omega_h^2(\varepsilon))}\\ &\le \frac{\gamma}{2}\,\|y_h - \bar{y}_h\|_{L^2} + \frac{\gamma}{2}\,\|y_h - \bar{y}_h\|_{L^2} = \gamma\,\|y_h - \bar{y}_h\|_{Y_h}\,.\end{aligned}$$

□

The mesh independence result is established next. For its formulation we introduce arbitrary, nonempty sets $S(\bar{y}+s) \subset \partial G(\bar{y}+s)$ and $S_h(\bar{y}_h + s_h) \subset \partial G_h(\bar{y}_h + s_h)$ for s and s_h with $\|s\|_{L^2} \le \delta_2$ and $\|s_h\|_{L^2} \le \delta_2'$, respectively. Furthermore, we use the notation

$$\bar{B}_\delta(\bar{y}) = \left\{y \in L^2(\Omega) : \|y - \bar{y}\|_{L^2} \le \delta\right\}, \quad \delta > 0.$$

In the proof of Theorem 3.6 we utilize the following attraction theorem for Newton's method.

THEOREM 3.5. *Assume that there exists* $\bar{y} \in L^2(\Omega)$ *with* $G(\bar{y}) = 0$ *and* $\delta_2 > 0$ *such that*

$$\sup\left\{\|M^{-1}\|_{L^2,L^2} : M \in S(\bar{y}+s),\ \|s\|_{L^2} \le \delta_2\right\} \le \kappa$$

for some constant $\kappa > 0$. *Let* $\theta \in (0,1)$ *be given, and let* $\gamma \in (0,1)$ *satisfy* $\gamma\kappa \le \theta$. *Further let* G *satisfy*

$$\sup_{M\in S(\bar{y}+s)} \|G(\bar{y}+s) - G(\bar{y}) - Ms\|_{L^2} \le \gamma \|s\|_{L^2} \quad \forall\, s \in L^2(\Omega),\ \|s\|_{L^2} \le \delta_1$$

for some $0 < \delta_1 \le \delta_2$. *Then, for any* $y^0 \in \bar{B}_{\delta_1}(\bar{y})$, *the generalized Newton's method converges in* $\bar{B}_{\delta_1}(\bar{y})$, *and the iterates satisfy*

$$\|y^{k+1} - \bar{y}\|_{L^2} \le \theta \|y^k - \bar{y}\|_{L^2} \quad \text{for } k = 0, 1, \dots. \tag{3.18}$$

Proof. For $y \in \bar{B}_{\delta_1}(\bar{y})$ and $\mathcal{N}(y) = y - M^{-1}G(y)$, $M \in S(y)$, we obtain

$$\begin{aligned}\|\mathcal{N}(y) - \bar{y}\|_{L^2} &= \|M^{-1}(G(y) - G(\bar{y}) - M(y-\bar{y}))\|_{L^2}\\ &\le \kappa\|G(y) - G(\bar{y}) - M(y-\bar{y})\|_{L^2} \le \theta\|y-\bar{y}\|_{L^2}\end{aligned}$$

Since $y^{k+1} = \mathcal{N}(y^k)$ and $\theta < 1$, this proves the q-linear convergence with rate θ toward $\bar{y}$. □

Note that Theorem 3.5 has an immediate analogue in the discretized setting of Algorithm 2.1.

THEOREM 3.6. *Let* $G : L^2(\Omega) \to L^2(\Omega)$ *be semismooth, and assume that there exist* $\bar{y} \in L^2(\Omega)$ *with* $G(\bar{y}) = 0$ *and* $\bar{y}_h \in Y_h$ *with* $G_h(\bar{y}_h) = 0$ *which satisfy Assumptions 3.1–3.2. Further suppose that there exist* $\delta_2, \delta_2' > 0$, $\kappa, \kappa' > 0$ *and* $h_2' \le h_0$ *such that*

$$\begin{aligned}&\sup\left\{\|M^{-1}\|_{L^2,L^2} : M \in S(\bar{y}+s), \|s\|_{L^2} \le \delta_2\right\} \le \kappa,\\ &\sup\left\{\|M_h^{-1}\|_{L^2,L^2} : M_h \in S_h(\bar{y}_h+s_h), \|s_h\|_{L^2} \le \delta_2'\right\} \le \kappa'\end{aligned}$$

for all $0 < h \le h_2'$. *Then, for arbitrarily fixed* $\theta \in (0,1)$, *there exist* $\bar{\delta} > 0$ *and* $\bar{h} > 0$ *such that for all* $0 < h \le \bar{h}$

$$\|y^{k+1} - \bar{y}\|_{L^2} \le \theta\|y^k - \bar{y}\|_{L^2}, \tag{3.19}$$

$$\|y_h^{k+1} - \bar{y}_h\|_{L^2} \le \theta\|y_h^k - \bar{y}_h\|_{L^2} \tag{3.20}$$

whenever $\max\{\|y^0 - \bar{y}\|_{L^2}, \|y_h^0 - \bar{y}_h\|_{L^2}\} \le \bar{\delta}$.

Proof. Let $\theta \in (0,1)$ be given. For $\gamma \in (0,1)$ with $\gamma\kappa \le \theta < 1$ there exists $\delta_1 \in (0, \delta_2]$ such that

$$\sup_{M\in S(\bar{y}+s)} \|G(\bar{y}+s) - G(\bar{y}) - Ms\|_{L^2} \le \gamma \|s\|_{L^2} \quad \forall\, s \in L^2(\Omega),\ \|s\|_{L^2} \le \delta_1$$

by the semismoothness of G. Theorem 3.5 then yields that $\{y^k\}$, the sequence of iterates of Algorithm 1.1 initialized by $y^0 \in \bar{B}_{\delta_1}(\bar{y})$, converges to $\bar{y}$ q-linearly with rate θ. Now, if necessary, reduce γ (and, thus, δ_1) such that

$$\gamma \max(\kappa, \kappa') \le \theta < 1.$$

From (3.8) we obtain that there exists $\delta_1' \in (0, \delta_2']$ and $h_1' \in (0, h_2']$ such that

$$\sup_{M_h \in S_h(\bar{y}_h + s_h)} \|G_h(\bar{y}_h + s_h) - G_h(\bar{y}_h) - M_h s_h\|_{L^2} \le \gamma \|s_h\|_{L^2}$$
$$\forall\, s_h \in Y_h,\ \|s_h\|_{L^2} \le \delta_1',\ \ h \le h_1'.$$

Like before, Theorem 3.5 yields that $\{y_h^k\}$, the sequence of iterates of Algorithm 2.1 initialized by $y_h^0 \in \bar{B}_{\delta_1'}(\bar{y}_h)$, converges to $\bar{y}_h$ q-linearly with rate θ.

Now define $\bar{\delta} = \min(\delta_1, \delta_1')$ and $\bar{h} = h_1'$. Then the assertion follows. □

4. Sufficient Conditions for Regularity. An important class of complementarity problems results from reformulations of control constrained optimal control problems of tracking type for elliptic partial differential equations; see [12, 20, 21]. This problem class satisfies the structural assumption on F. Frequently, in practice when computing the generalized derivative of G, a particular choice of D (see (1.11)) is used. The following result utilizes these two properties to establish the regularity requirement for Theorem 3.6.

THEOREM 4.1. *Assume that the Fréchet derivative F' of $F : L^2(\Omega) \to L^2(\Omega)$, $F = A + \lambda I$, is continuous at $\bar{y} \in L^2(\Omega)$ and satisfies*

$$(v, F'(\bar{y})v)_{L^2} \ge \gamma \|v\|_{L^2}^2 \quad \forall\, v \in L^2(\Omega).$$

for some $\gamma > 0$. Further, let $S(y) \subset \partial G(y)$ satisfy

$$S(y) = \{\lambda I + D(y) \cdot A'(y) : D(y) \text{ satisfies (1.11) with } D(y)(x) \in \{0,1\} \text{ if } A(y)(x) \in \{\lambda\alpha, \lambda\beta\}\}.$$

Then there exist $\delta > 0$ and $\kappa > 0$ such that

$$M \text{ is invertible and } \left\|M^{-1}\right\|_{L^2} \le \kappa \quad \text{for all } M \in S(y),\ \ y \in L^2(\Omega),\ \|y - \bar{y}\|_{L^2} \le \delta.$$

Proof. In the sequel, for all measurable sets $\mathcal{J} \subset \Omega$ let $E_{\mathcal{J}}$ denote the extension-by-zero operator from $\mathcal{J}$ to Ω. Its adjoint $E_{\mathcal{J}}^*$ is a corresponding restriction operator. By $z_{\mathcal{J}}$ we denote the restriction of z to $\mathcal{J}$. Now let $\delta > 0$ be so small that

$$\kappa_{A'} := \sup\left\{\|A'(y)\|_{L^2,L^2} : \|y - \bar{y}\|_{L^2} \le \delta\right\} \tag{4.1}$$

is finite and, in addition,

$$(v, F'(y)v)_{L^2} \ge \frac{\gamma}{2} \|v\|_{L^2}^2 \quad \forall\, v, y \in L^2(\Omega),\ \|y - \bar{y}\|_{L^2} \le \delta.$$

For any measurable set $\mathcal{J} \subset \Omega$, define $F'(y)_{\mathcal{J}\mathcal{J}} = E_{\mathcal{J}}^* F'(y) E_{\mathcal{J}}$ and observe that

$$(v_{\mathcal{J}}, F'(y)_{\mathcal{J}\mathcal{J}} v_{\mathcal{J}})_{L^2} = (E_{\mathcal{J}} v, F'(y) E_{\mathcal{J}} v)_{L^2} \ge \frac{\gamma}{2} \|v_{\mathcal{J}}\|_{L^2}^2$$

for all $v, y \in L^2(\Omega)$, $\|y - \bar{y}\|_{L^2} \le \delta$. Hence,

$$\left\|F'(y)_{\mathcal{J}\mathcal{J}}^{-1}\right\|_{L^2} \le \frac{2}{\gamma} \quad \forall\, y \in L^2(\Omega),\ \ \|y - \bar{y}\|_{L^2} \le \delta, \tag{4.2}$$

holds for all measurable sets $\mathcal{J} \subset \Omega$ with $|\mathcal{J}| > 0$.

Now let $w \in L^2(\Omega)$ be arbitrary and consider the linear equation

$$Mv = w \quad \Longleftrightarrow \quad \lambda v + D(y)A'(y)v = w. \tag{4.3}$$

Introducing the sets $\mathcal{A} = \{x : D(y)(x) = 0\}$ and $\mathcal{I} = \{x : D(y)(x) = 1\}$, we have

$$\lambda(E_{\mathcal{I}}v_{\mathcal{I}} + E_{\mathcal{A}}v_{\mathcal{A}}) + D(y)A'(y)(E_{\mathcal{I}}v_{\mathcal{I}} + E_{\mathcal{A}}v_{\mathcal{A}}) = E_{\mathcal{I}}w_{\mathcal{I}} + E_{\mathcal{A}}w_{\mathcal{A}}. \tag{4.4}$$

Note that $\mathcal{A} \cup \mathcal{I}$ is a disjoint partition of Ω. Applying $E_{\mathcal{A}}^*$ and considering $(E_{\mathcal{A}}^* E_{\mathcal{I}})v_{\mathcal{I}} = 0$ and $(E_{\mathcal{A}}^* E_{\mathcal{A}})v_{\mathcal{A}} = v_{\mathcal{A}}$ in (4.4) yields

$$v_{\mathcal{A}} = \frac{1}{\lambda} w_{\mathcal{A}}.$$

Here we also utilized the fact $E_{\mathcal{A}}^* D(y)v = 0$. Applying $E_{\mathcal{I}}^*$ to (4.4) gives

$$\lambda v_{\mathcal{I}} + E_{\mathcal{I}}^* A'(y)E_{\mathcal{I}}v_{\mathcal{I}} + E_{\mathcal{I}}^* A'(y)E_{\mathcal{A}}v_{\mathcal{A}} = w_{\mathcal{I}}. \tag{4.5}$$

Define the operators

$$\begin{aligned} F'(y)_{\mathcal{I}\mathcal{I}} &:= \lambda E_{\mathcal{I}}^* E_{\mathcal{I}} + E_{\mathcal{I}}^* A'(y)E_{\mathcal{I}} = \lambda I + E_{\mathcal{I}}^* A'(y)E_{\mathcal{I}}, \\ A'(y)_{\mathcal{I}\mathcal{A}} &:= E_{\mathcal{I}}^* A'(y)E_{\mathcal{A}}. \end{aligned}$$

Then equation (4.5) can be rewritten as

$$F'(y)_{\mathcal{I}\mathcal{I}}v_{\mathcal{I}} + A'(y)_{\mathcal{I}\mathcal{A}}v_{\mathcal{A}} = w_{\mathcal{I}},$$

If $|\mathcal{I}| = 0$, we have

$$v = v_{\mathcal{A}} = \frac{1}{\lambda} w_{\mathcal{A}} = \frac{1}{\lambda} w$$

and thus $\left\|M^{-1}\right\|_{L^2} \le 1/\lambda$.

Now consider the case $|\mathcal{I}| > 0$. Then,

$$\begin{aligned} &\|v_{\mathcal{A}}\|_{L^2} = \frac{1}{\lambda}\|w_{\mathcal{A}}\|_{L^2}, \\ &\|v_{\mathcal{I}}\|_{L^2} \le \left\|F'(y)_{\mathcal{I}\mathcal{I}}^{-1}\right\|_{L^2} \|w_{\mathcal{I}} - A'(y)_{\mathcal{I}\mathcal{A}}v_{\mathcal{A}}\|_{L^2} \le \frac{2}{\gamma}(\|w_{\mathcal{I}}\|_{L^2} + \|A'(y)\|_{L^2}\|v_{\mathcal{A}}\|_{L^2}). \end{aligned}$$

This shows

$$\begin{aligned} \|v\|_{L^2} &\le \|v_{\mathcal{A}}\|_{L^2} + \|v_{\mathcal{I}}\|_{L^2} \le \frac{1}{\lambda}\|w_{\mathcal{A}}\|_{L^2} + \frac{2}{\gamma}\|w_{\mathcal{I}}\|_{L^2} + \frac{2}{\gamma}\cdot\kappa_{A'}\cdot\frac{1}{\lambda}\|w_{\mathcal{A}}\|_{L^2} \\ &\le \max\left\{\frac{2}{\gamma}, \frac{1}{\lambda} + \frac{2\kappa_{A'}}{\gamma\lambda}\right\}(\|w_{\mathcal{I}}\|_{L^2} + \|w_{\mathcal{A}}\|_{L^2}) \\ &\le \sqrt{2}\max\left\{\frac{2}{\gamma}, \frac{1}{\lambda} + \frac{2\kappa_{A'}}{\gamma\lambda}\right\}\|w\|_{L^2} =: \kappa\|w\|_{L^2}. \end{aligned}$$

□

In the same way, regularity of the discrete generalized differential ∂G_h can be proved. Furthermore, if we can find $\gamma > 0$, $\bar{h} > 0$, and $\delta > 0$ such that conditions of the form (4.1) and (4.2) can be ensured for F, A' and F_h, A'_h, $0 < h \le \bar{h}$ with constants independent of h, then the bound κ for the norm of the inverses can be chosen independently of h.

Theorem 4.1 covers a wide range of practically relevant control constrained optimal control problems for partial differential equations; for more details we refer to section 5. In this case the semismooth operator equation corresponding to an MCP-function based reformulation of the first order optimality system involves a nonlinear, Fréchet differentiable operator

A. However, in many applications A is a linear operator, which maps $L^2(\Omega)$ to $L^p(\Omega)$ for some $p > 2$, and which frequently is related to (inverses of) linear elliptic differential operators. Then the regularity result of Theorem 4.1 can be made more concrete. As an example consider the simple control constrained optimal control problem

$$\text{(4.6)} \qquad \begin{cases} \underset{u\in H_0^1(\Omega), y\in L^2(\Omega)}{\text{minimize}} & J(u,y) := \frac{1}{2}\|u-u_d\|_{L^2}^2 + \frac{\lambda}{2}\|y\|_{L^2}^2 \\ \text{subject to} & -\Delta u = y \quad \text{in } \Omega, \\ & \alpha \le y \le \beta \quad \text{a.e. in } \Omega, \end{cases}$$

where $u_d \in L^2(\Omega)$, $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$. This problem admits a unique solution. It is easy to verify that $F(y)$ for this model problem becomes

$$F(y) = B^{-1}j^*(jB^{-1}y - u_d) + \lambda y,$$

where $B \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ represents $-\Delta$ with homogeneous Dirichlet boundary conditions and $j : H_0^1(\Omega) \to L^2(\Omega)$ is the linear embedding operator. Thus, we have $A(y) = B^{-1}j^*(jB^{-1}y - u_d)$, which, by the Sobolev embedding theorem, for $n = 1, 2, 3$ maps $L^2(\Omega)$ to $L^p(\Omega)$ for appropriate $p \in (2, \infty)$.

More general, we relate $B \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ to a linear elliptic second order differential operator which is invertible. Moreover we assume that B is selfadjoint. Then F is continuous at arbitrary $y \in L^2(\Omega)$. Further, for $v \in L^2(\Omega)$ we have

$$(v, F'(y)v)_{L^2} = (v, B^{-1}j^*jB^{-1}v)_{L^2} + \lambda\|v\|_{L^2}^2 = (B^{-1}v, B^{-1}v)_{L^2} + \lambda\|v\|_{L^2}^2 \ge \gamma\|v\|_{L^2}^2$$

for some $\gamma \ge \lambda > 0$. As a consequence we obtain the following corollary to Theorem 4.1.

COROLLARY 4.2. *Assume that* $F : L^2(\Omega) \to L^2(\Omega)$, $F = A + \lambda I$, *with* $A = B^{-1}j^*jB^{-1}$, *where* $B \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ *is a linear elliptic second order differential operator. Further, assume that* $D(y)(x) \in \{0, 1\}$ *whenever* $(Ay)(x) \in \{\lambda\alpha, \lambda\beta\}$. *Then there exists* $\kappa > 0$ *such that*

$$M \text{ is invertible and } \left\|M^{-1}\right\|_{L^2} \le \kappa \quad \text{for all } M \in \partial G(y), \;\; y \in L^2(\Omega).$$

In the discrete setting we obtain for $v_h \in Y_h$

$$(v_h, F_h'(y_h)v_h)_{L^2} \ge \lambda\|v_h\|_{L^2}^2.$$

Thus, for $\gamma = \lambda$, which is independent of h, the L^2-ellipticity of the bilinear forms associated with F_h' and F', respectively, follows. As a consequence (4.2) and its discrete analogue are satisfied with $\gamma = \lambda$ uniformly in h. The boundedness of A' in (4.1) follows from the boundedness of B^{-1}. Note that these results are independent of δ since A does not depend on y. Depending on the norms of appropriate injection operators (for details we refer to Remark 5.6 below) essentially the same bound applies to the discrete operator A_h'. This proves that (4.1) and its discrete analogue are satisfied with a common uniform bound $\kappa_{A'}$. Consequently, from the definition of κ in the proof of Theorem 4.1 we infer that κ can be chosen independently of h and Corollary 4.2 also applies when $L^2(\Omega)$, F, G, and M are replaced by their discrete counterparts Y_h, F_h, G_h, and M_h.

In the following Theorem 4.3 we restate the mesh independence result of Theorem 3.6 under the requirements of Theorem 4.1. This is interesting since it covers the semismooth Newton methods in [12, 22] which utilize the particular choice of $D(y)$. In section 6 the mesh independent behavior of these algorithms is demonstrated.

THEOREM 4.3. *Let $G : L^2(\Omega) \to L^2(\Omega)$ be semismooth, and assume that there exist $\bar y \in L^2(\Omega)$ with $G(\bar y) = 0$ and $\bar y_h \in Y_h$ with $G_h(\bar y_h) = 0$ which satisfy Assumptions 3.1–3.2. Further suppose that the assumptions of Theorem 4.1 are satisfied and there exist $\delta_2' > 0$, $\kappa' > 0$ and $h_2' \le h_0$ such that*

$$\sup\left\{\|M_h^{-1}\|_{L^2,L^2} : M_h \in S_h(\bar y_h + s_h),\ \|s_h\|_{L^2} \le \delta_2'\right\} \le \kappa' \quad \forall\, h \le h_2'.$$

Then, for arbitrarily fixed $\theta \in (0,1)$, there exist $\bar\delta > 0$ and $\bar h > 0$ such that for all $0 < h \le \bar h$

$$\|y^{k+1} - \bar y\|_{L^2} \le \theta \|y^k - \bar y\|_{L^2}, \tag{4.7}$$

$$\|y_h^{k+1} - \bar y_h\|_{L^2} \le \theta \|y_h^k - \bar y_h\|_{L^2} \tag{4.8}$$

with $\max\{\|y^0 - \bar y\|_{L^2}, \|y_h^0 - \bar y_h\|_{L^2}\} \le \bar\delta$.

Proof. The proof essentially follows the lines of the proof of Theorem 3.6 with possibly smaller δ_1 due to the result of Theorem 4.1. □

Note that in the case of the linear-quadratic control problem (4.6) the boundedness assumption on $\{\|M_h^{-1}\|_{L^2,L^2} : M_h \in S_h(\bar y_h + s_h),\ \|s_h\|_{L^2} \le \delta_2'\}$ follows from Corollary 4.2 and the discussion thereafter.

5. Application to constrained optimal control problems. In this section we apply the mesh independence result of Theorem 4.3 to control constrained semilinear elliptic optimal control problems; see, e.g., [3]. We consider the following problem:

$$\begin{array}{ll} \text{minimize} & J(u,y) = \frac{1}{2}\|u - u_d\|_{L^2}^2 + \frac{\lambda}{2}\|y\|_{L^2}^2 \\ \text{subject to} & (u,y) \in H^1(\Omega) \times L^2(\Omega), \\ & Cu + f(u) = y \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma = \partial\Omega, \\ & y \in Y_{\text{ad}} = \{y \in L^2(\Omega) \mid \alpha \le y(x) \le \beta \text{ for a.a. } x \text{ in } \Omega\}, \end{array} \tag{5.1}$$

where $u_d \in L^4(\Omega)$, $\lambda > 0$, $\alpha \in \mathbb{R}$, and C denotes a second-order elliptic operator of the form

$$Cu(x) = -\sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i}(x))_{x_j}.$$

The coefficients are supposed to be Lipschitz continuous functions in $\bar\Omega$ satisfying the ellipticity condition

$$\sum_{i,j=2}^{n} a_{ij}(x)\xi_i\xi_j \ge \gamma_a \|\xi\|^2 \quad \text{for all } (\xi, x) \in \mathbb{R}^n \times \bar\Omega, \quad \gamma_a > 0.$$

It is assumed throughout that $\Omega \subset \mathbb{R}^n$, with $n = 2, 3$, is convex and bounded with sufficiently smooth boundary Γ. We point out that with respect to the objective functional more general cases can be considered; see [3]. However, in order to avoid additional technicalities we restrict ourselves to the class of tracking-type objective functionals as stated in problem (5.1). The function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be of class C^3, and f' is nonnegative. This implies the assumptions posed in [3]: For all $\kappa > 0$ there exists $\gamma_\kappa > 0$ such that

$$|f(u)| + |f'(u)| + |f''(u)| \le \gamma_\kappa,$$
$$|f''(u^2) - f''(u^1)| \le \gamma_\kappa |u^2 - u^1|$$

for all $(u, u^1, u^2) \in [-\kappa, \kappa]^3$. In addition, we require that there exist constants c_1, c_2 such that

$$|f'''(u)| \le c_1 + c_2 |u|^{\frac{p-6}{2}} \quad \forall\, u \in \mathbb{R}.$$

Here, we fix $p \in [6, \infty)$ for $n = 2$ and $p = 6$ for $n = 3$. Then we have the continuous embedding

$$H_0^1(\Omega) \subset L^p(\Omega).$$

REMARK 5.1. The function f could also be a Carathéodory function that depends on x and u.

REMARK 5.2. Iterated application of Lemma A.1 shows that all the growth conditions stated in Theorem A.2 are satisfied for $q = 2$. Therefore, we have all assertions of Theorem A.2 available. In particular, the superposition operator $u \in H_0^1(\Omega) \subset L^p(\Omega) \mapsto f(u) \in L^2(\Omega)$ is twice continuously Fréchet differentiable.

It is known (see below) that under the above assumptions the semilinear elliptic PDE

$$Cu + f(u) = y \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Gamma \tag{5.2}$$

admits a unique solution $u(y) \in H_0^1(\Omega)$ for every $y \in L^2(\Omega)$ and that $u(y)$ enjoys the additional regularity $u(y) \in H^2(\Omega)$; see the appendix in [3]. Further, by classical arguments, one can show that (5.1) admits at least one solution.

To obtain a finite-dimensional approximation of (5.1), the discrete control space $Y_h \subset L^2(\Omega)$ is chosen as described in section 2. Y_h is equipped with the inner product $(\cdot,\cdot)_{Y_h} = (\cdot,\cdot)_{L^2}$ and it is identified with its dual, i.e., $Y_h^* = Y_h$. The discrete state space $U_h \subset H_0^1(\Omega)$ consists of piecewise linear finite elements and is equipped with the same norm as $H_0^1(\Omega)$, namely $\|\cdot\|_{H^1}$, see [3] for details. Using these spaces, we formulate the discrete control problem

$$\begin{aligned} &\text{minimize} && J(u_h, y_h) \\ &\text{subject to} && (u_h, y_h) \in Y_h \times U_h, \\ &&& \langle Cu_h + f(u_h), \phi_h \rangle_{H^{-1}, H_0^1} = (y_h, \phi_h)_{L^2} \quad \forall\, \phi_h \in U_h, \\ &&& y_h \in Y_{\text{ad}} \cap Y_h. \end{aligned} \tag{5.3}$$

For any $y_h \in Y_h$, the discrete state equation possesses a unique solution $u_h(y_h) \in U_h$. Furthermore, the problem (5.3) possesses at least one solution, see [3].

We now analyze the differential operator

$$E : H_0^1(\Omega) \to H^{-1}(\Omega), \quad E(u) = Cu + f(u) \tag{5.4}$$

and its discretization

$$E_h : U_h \to U_h^*, \quad \langle E_h(u^h), \phi_h \rangle_{U_h^*, U_h} = \langle Cu^h + f(u_h), \phi_h \rangle_{H^{-1}, H_0^1} \quad \forall\, \phi_h \in U_h. \tag{5.5}$$

Defining the natural injection $j_h : u_h \in U_h \mapsto u_h \in H_0^1(\Omega)$, which is linear and continuous with $\|j_h\|_{U_h, H_0^1} = 1$, we can write

$$E_h = j_h^* \circ E \circ j_h.$$

Since the injection j_h acts like the identity, we will omit it in the sequel. The adjoint operator j_h^*, which is the projection from the space $H^{-1}(\Omega)$ of bounded linear forms on $H_0^1(\Omega)$ onto the space U_h^* of bounded linear forms on U_h, however, is important.

By the Sobolev embedding theorem, there exists a constant $k_p > 0$ such that

$$\|\cdot\|_{L^2} \leq \|\cdot\|_{H^1}, \quad \|\cdot\|_{H^{-1}} \leq \|\cdot\|_{L^2}, \quad \|\cdot\|_{L^p} \leq k_p \|\cdot\|_{H^1}.$$

We proceed by defining the linear injection operator $i_h \in \mathcal{L}(Y_h, L^2(\Omega))$. The adjoint of i_h is the averaging operator $i_h^* : L^2(\Omega) \to Y_h$ given by the explicit formula $i_h^* v = \Pi_h v$ with

$$(\Pi_h v)|_T = \frac{1}{|T|}\int_T v(x)\,dx \quad \forall\, T \in \mathcal{T}_h. \tag{5.6}$$

Furthermore, since $\|i_h\|_{Y_h,L^2} = 1$, we also have $\|i_h^*\|_{L^2,Y_h} = 1$. For the purpose of abbreviation, let us finally define $e_h \in \mathcal{L}(U_h, Y_h)$, $e_h = i_h^* j j_h$. Then $\|e_h\|_{U_h,Y_h} = \|e_h^*\|_{Y_h,U_h^*} \le 1$.

The state equation and the discrete state equation, respectively, can be written in the form

$$E(u) = y, \tag{5.7}$$

$$E_h(u_h) = e_h^* y_h. \tag{5.8}$$

THEOREM 5.3. *The operators E and E_h, $h > 0$, defined in* (5.4) *and* (5.5)*, respectively have the following properties:*

a) *E and E_h, $h > 0$, are twice continuously Fréchet differentiable with*

$$E'(u)v = Cv + f'(u)v, \quad E''(u)(v^1, v^2) = f''(u)v^1v^2,$$
$$E_h'(u_h)v_h = j_h^*(Cv_h + f'(u_h)v_h), \quad E_h''(u_h)(v_h^1, v_h^2) = j_h^*(f''(u_h)v_h^1v_h^2).$$

b) *E and E_h, $h > 0$, are strongly monotone. More precisely, there exists $\nu > 0$ such that*

$$\langle E(u^2) - E(u^1), u^2 - u^1\rangle_{H^{-1},H_0^1} \ge \nu \left\|u^2 - u^1\right\|_{H^1}^2 \quad \forall\, u^1, u^2 \in H_0^1(\Omega),$$
$$\langle E_h(u_h^2) - E(u_h^1), u_h^2 - u_h^1\rangle_{U_h^*,U_h} \ge \nu \left\|u_h^2 - u_h^1\right\|_{U_h}^2 \quad \forall\, u_h^1, u_h^2 \in U_h.$$

c) *E and E_h, $h > 0$, are invertible and their inverses are Lipschitz continuous, i.e., with ν as in* b)*,*

$$\left\|E^{-1}(v^2) - E^{-1}(v^1)\right\|_{H^1} \le \nu^{-1}\left\|v^2 - v^1\right\|_{H^{-1}} \quad \forall\, v^1, v^2 \in H^{-1}(\Omega),$$
$$\left\|E_h^{-1}(v_h^2) - E_h^{-1}(v_h^1)\right\|_{U_h} \le \nu^{-1}\left\|v_h^2 - v_h^1\right\|_{U_h^*} \quad \forall\, v_h^1, v_h^2 \in U_h^*.$$

d) *For all $u \in L^2(\Omega)$ and all $u_h \in U_h$, $h > 0$, the linear operators $E'(u) \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ and $E_h'(u_h) \in \mathcal{L}(U_h, U_h^*)$ are continuously invertible with*

$$\left\|E'(u)^{-1}\right\|_{H^{-1},H_0^1} \le \nu^{-1}, \quad \left\|E_h'(u_h)^{-1}\right\|_{U_h^*,U_h} \le \nu^{-1}.$$

Proof.

a)

By Lemma A.1 and Theorem A.2, under the stated assumptions on f the superposition operator

$$S_f : u \in L^p(\Omega) \mapsto f(u) \in L^2(\Omega)$$

is twice continuously differentiable with derivatives

$$S_f'(u)v = f'(u)v, \quad S_f''(u)(v^1, v^2) = f''(u)v^1v^2.$$

Now the assertions on E follow from the continuous embedding $H_0^1(\Omega) \subset L^p(\Omega)$. Furthermore, by the continuity and linearity of j_h, E_h is twice continuously differentiable as well with

$$E_h'(u_h)v_h = j_h^*(E'(j_hu_h)(j_hv_h)) = j_h^*(E'(u_h)v_h) = j_h^*(Cv_h + f'(u_h)v_h),$$
$$E_h''(u_h)(v_h^1, v_h^2) = j_h^*(E''(j_hu_h)(j_hv_h^1, j_hv_h^2)) = j_h^*(f''(u_h)v_h^1v_h^2).$$

b)

For all $u^1, u^2 \in L^2(\Omega)$ we obtain with $\delta u = u^2 - u^1$

$$\int_\Omega (f(u^2(x)) - f(u^1(x)))\delta u(x)\, dx = \int_\Omega \int_0^1 f'(u^1(x) + t\delta u(x))\delta u(x)\, dt\, \delta u(x)\, dx$$
$$= \int_\Omega \int_0^1 f'(u^1(x) + t\delta u(x))\, dt\, \delta u(x)^2\, dx \geq 0.$$

Hence, by the ellipticity of C, there exists $\nu > 0$ with

$$\langle E(u^2) - E(u^1), \delta u\rangle_{H^{-1},H_0^1} \geq \langle C\delta u, \delta u\rangle_{H^{-1},H_0^1} \geq \nu \|\delta u\|_{H^1}^2 .$$

Furthermore, with $\delta u_h = u_h^2 - u_h^1$,

$$\langle E_h(u_h^2) - E_h(u_h^1), \delta u_h\rangle_{U_h^*,U_h} = \langle E(u_h^2) - E(u_h^1), \delta u_h\rangle_{H^{-1},H_0^1} \geq \nu \|\delta u_h\|_{H^1}^2 = \nu \|\delta u_h\|_{U_h}^2 .$$

c)

From the Browder-Minty theorem on monotone operators we obtain that E and E_h are surjective. Now let $E(u^1) = v^1$ and $E(u^2) = v^2$. Then by b)

$$\nu \left\|u^2 - u^1\right\|_{H^1}^2 \leq \langle E(u^2) - E(u^1), u^2 - u^1\rangle_{H^{-1},H_0^1} = \langle v^2 - v^1, u^2 - u^1\rangle_{H^{-1},H_0^1}$$
$$\leq \left\|v^2 - v^1\right\|_{H^{-1}} \left\|u^2 - u^1\right\|_{H^1} .$$

This proves the injectivity of E, thus its invertibility, and the Lipschitz continuity of E^{-1}. The assertion on E_h can be proved in exactly the same way.

d)

For all $v \in H_0^1(\Omega)$, there holds

$$\langle E'(u)v, v\rangle_{H^{-1},H_0^1} = \langle Cv, v\rangle_{H^{-1},H_0^1} + \int_\Omega f'(u(x))v(x)^2\, dx \geq \langle Cv, v\rangle_{H^{-1},H_0^1} \geq \nu \|v\|_{H^1}^2 .$$

Therefore, the linear operator $E'(u)$ is strongly monotone and thus, as in c), we obtain the invertibility of $E'(u)$ and the bound on its inverse. In the same way we obtain the assertion on E_h, since

$$\langle E_h'(u_h)v_h, v_h\rangle_{U_h^*,U_h} = \langle E'(u_h)v_h, v_h\rangle_{H^{-1},H_0^1} \geq \nu \|v_h\|_{H^1}^2 = \nu \|v_h\|_{U_h}^2 .$$

□

In the following we consider the reduced version of problem (5.1) given by

$$\text{(5.9)} \qquad \begin{array}{ll} \text{minimize} & \hat{J}(y) = \frac{1}{2}\|u(y) - u_d\|_{L^2}^2 + \frac{\lambda}{2}\|y\|_{L^2}^2 \\ \text{subject to} & y \in Y_{\text{ad}}, \end{array}$$

where $u(y) \in H_0^1(\Omega)$ denotes the unique solution of (5.2) for given $y \in L^2(\Omega)$. Doing the same with the discrete problem (5.3), we obtain the discrete reduced problem

$$\text{(5.10)} \qquad \begin{array}{ll} \text{minimize} & \hat{J}_h(y_h) = \frac{1}{2}\|u_h(y_h) - u_d\|_{L^2}^2 + \frac{\lambda}{2}\|y_h\|_{L^2}^2 \\ \text{subject to} & y_h \in Y_{\text{ad}} \cap Y_h \end{array}$$

with $u_h(y_h) \in U_h$ denoting the unique solution of the discrete state equation.

To avoid redundant argumentations, we introduce $U_0 = H_0^1(\Omega)$, $Y_0 = L^2(\Omega)$, $i_0 : y \in L^2(\Omega) \mapsto y \in L^2(\Omega)$, $j_0 : u \in H_0^1(\Omega) \mapsto u \in H_0^1(\Omega)$, and $e_0 : u \in H_0^1(\Omega) \mapsto u \in L^2(\Omega)$.

Then the continuous control problem (5.1) equals the problem (5.3) with $h = 0$ and the state equation (5.7) coincides with (5.8), $h = 0$. Furthermore, the reduced control problem (5.9) is identical to (5.10) with $h = 0$.

THEOREM 5.4. *The operators*

$$y \in L^2(\Omega) \mapsto u(y) \in H_0^1(\Omega) \quad \textit{and} \quad y_h \in Y_h \mapsto u_h(y_h) \in U_h$$

as well as the reduced objective functions $\hat{J}$ and $\hat{J}_h$ are twice continuously Fréchet differentiable.

Proof. Let $h \geq 0$ (this includes the continuous case $h = 0$). Then we have $u_h(y_h) = E_h^{-1}(e_h y_h)$ and, by Theorem 5.3, the inverse function theorem can be applied to E_h and yields that E_h^{-1} is twice continuously Fréchet differentiable. Since the quadratic functional J is smooth, the function $\hat{J}(y) = J(u_h(y_h), y_h)$ is twice continuously differentiable, too. □

The first order optimality conditions for (5.9) are given by

$$\bar{y} \in Y_{\text{ad}}, \quad (\nabla\hat{J}(\bar{y}), y - \bar{y})_{L^2} \geq 0 \quad \text{for all } y \in Y_{\text{ad}}. \tag{5.11}$$

This is a problem of the form (1.2). Let us characterize $\nabla\hat{J}(\bar{y})$. In fact, we have

$$(\nabla\hat{J}(\bar{y}), v)_{L^2} = (\bar{u}(\bar{y}) - u_d, u'(\bar{y})v)_{L^2} + \lambda(\bar{y}, v)_{L^2}, \tag{5.12}$$

where u' denotes the derivative of $u(y)$ with respect to y. In order to derive a computable expression for $u'(\bar{y})$ we use the *adjoint method*. For this purpose we define the adjoint state $w = w(y) \in H_0^1(\Omega)$ as the solution of the adjoint equation

$$E'(u(y))^* w = \nabla_u J(u(y), y),$$

which in detail reads

$$C^* w + f'(u(y))w = u(y) - u_d \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \Gamma. \tag{5.13}$$

By Theorem 5.3, (5.13) admits a unique solution $w(y) \in H_0^1(\Omega)$ and elliptic regularity results imply $w(y) \in H^2(\Omega) \cap C^{0,1}(\bar{\Omega})$. The adjoint gradient representation is then given by

$$\nabla\hat{J}(y) = w(y) + \lambda y. \tag{5.14}$$

Alternatively, we may write

$$\nabla\hat{J}(y) = A(y) + \lambda y,$$

where $A : L^2(\Omega) \mapsto w(y) \in L^p(\Omega)$ (note the embedding $H_0^1 \subset L^p$) is realized for given y by first solving (5.2) for u and then solving (5.13) for w. Therefore, $F(y) := \nabla\hat{J}(y)$ meets the structural requirement (1.7).

The same adjoint calculus can be carried out for the discrete problem and results in the discrete adjoint equation

$$E_h'(u_h(y_h))^* w_h = j_h^* \nabla_u J(u_h(y_h), y_h),$$

which uniquely specifies the adjoint state $w_h = w_h(u_h) \in U_h$. In detail, the adjoint equation reads

$$\langle C^* w_h + f'(u_h(y_h))w_h, \phi_h\rangle_{H^{-1}, H_0^1} = (u_h(y_h) - u_d, \phi_h)_{L^2} \quad \forall\, \phi_h \in U_h.$$

We obtain the discrete adjoint gradient representation

$$\nabla \hat{J}(y_h) = e_h w_h + \lambda y_h = i_h^* w_h + \lambda y_h.$$

Setting $F_h(y_h) = \nabla \hat{J}(y_h) = A_h(y_h) + \lambda y_h$ with $A_h(y_h) = e_h w_h(y_h)$, the discrete control problem is equivalent to (2.1) and F_h has the required structure.

Next, we state error estimates. For the proofs we refer to the recent paper [3].

THEOREM 5.5. *Denote by* $(\bar{y}_h)_{h>0}$ *a sequence of solutions to* (5.3) *that converges to a solution* $\bar{y}$ *of* (5.1). *Then, for sufficiently small* $h > 0$, *we have*

$$\|u(\bar{y}) - u_h(\bar{y}_h)\|_{H^1} + \|w(\bar{y}) - w_h(\bar{y}_h)\|_{H^1} \le c(h + \|\bar{y} - \bar{y}_h\|_{L^2}), \tag{5.15}$$

$$\|u(\bar{y}) - u_h(\bar{y}_h)\|_{L^2} + \|w(\bar{y}) - w_h(\bar{y}_h)\|_{L^2} \le c(h^2 + \|\bar{y} - \bar{y}_h\|_{L^2}), \tag{5.16}$$

$$\|u(\bar{y}) - u_h(\bar{y}_h)\|_{L^\infty} + \|w(\bar{y}) - w_h(\bar{y}_h)\|_{L^\infty} \le c(h^{2-\frac{n}{2}} + \|\bar{y} - \bar{y}_h\|_{L^2}), \tag{5.17}$$

$$\|\bar{y} - \bar{y}_h\|_{L^2} \le ch. \tag{5.18}$$

Now we can verify Assumption 3.2.1: The first requirement (3.3) follows immediately from (5.18). For the second requirement we need the inequality

$$\|\Pi_h v\|_{L^q} \le \|v\|_{L^q} \quad \forall\, v \in L^q(\Omega), \quad q \in [2, \infty], \tag{5.19}$$

where Π_h is defined by (5.6). For $q = \infty$ this is obvious. To establish (5.19) for $2 \le q < \infty$, let $v \in L^q(\Omega)$ be arbitrary. Then

$$\|\Pi_h v\|_{L^q}^q = \sum_{T \in \mathcal{T}_h} \left\| \frac{1}{|T|} \int_T v(x)\, dx \right\|_{L^q(T)}^q = \sum_{T \in \mathcal{T}_h} \frac{1}{|T|^{q-1}} \left| \int_T v(x)\, dx \right|^q .$$

Now, by Hölder's inequality,

$$\left| \int_T v(x)\, dx \right| \le \int_T |v(x)|\, dx \le \left(\int_T dx \right)^{\frac{q-1}{q}} \left(\int_T |v(x)|^q\, dx \right)^{\frac{1}{q}} = |T|^{\frac{q-1}{q}} \|v|_T\|_{L^q(T)} .$$

Hence,

$$\|\Pi_h v\|_{L^q}^q = \sum_{T \in \mathcal{T}_h} \frac{1}{|T|^{q-1}} \left| \int_T v(x)\, dx \right|^q \le \sum_{T \in \mathcal{T}_h} \frac{1}{|T|^{q-1}} |T|^{q-1} \|v|_T\|_{L^q(T)}^q = \|v\|_{L^q}^q .$$

Furthermore, for the regular triangulations under consideration, it can be shown that (see [11])

$$\|w(\bar{y}) - \Pi_h w(\bar{y})\|_{L^2} \le ch \|w(\bar{y})\|_{H^1} . \tag{5.20}$$

Now we can prove (3.4) by invoking (5.15), (5.17), (5.18), (5.19), and (5.20):

$$\begin{aligned}
\|A(\bar{y}) - A_h(\bar{y}_h)\|_{L^p} &\le \|w(\bar{y}) - e_h w_h(\bar{y}_h)\|_{L^p} = \|w(\bar{y}) - i_h^* w_h(\bar{y}_h)\|_{L^p} \\
&\le \|\Pi_h(w(\bar{y}) - w_h(\bar{y}_h))\|_{L^p} + \|w(\bar{y}) - \Pi_h w(\bar{y})\|_{L^p} \\
&\le \|w(\bar{y}) - w_h(\bar{y}_h)\|_{L^p} + \|w(\bar{y}) - \Pi_h w(\bar{y})\|_{L^\infty}^{\frac{p-2}{p}} \|w(\bar{y}) - \Pi_h w(\bar{y})\|_{L^2}^{\frac{2}{p}} \\
&\le k_p ch + (2\|w(\bar{y})\|_{L^\infty})^{\frac{p-2}{p}} (ch \|w(\bar{y})\|_{H^1})^{\frac{2}{p}} \to 0 \quad \text{as } h \to 0.
\end{aligned}$$

It remains to prove the Assumptions 3.2.2 and 3. The nonlinearity of the state equation makes this task lengthy. For the reader who wants to see an immediate result, we first consider the linear quadratic case in the next remark, for which the remaining assumptions can be verified very quickly.

REMARK 5.6. Consider the special case (4.6), i.e., $f \equiv 0$ and $C = -\Delta$. Then with the notation introduced in the discussion of problem (4.6), we have $E(u) = Bu$ and $E_h(u_h) = B_h u_h$ with $B_h = j_h^* B j_h$. Furthermore,

$$A(y) = B^{-1} j^* (jB^{-1} y - u_d), \quad A_h(y_h) = e_h B_h^{-1} j_h^* j^* (j j_h B_h^{-1} e_h^* y_h - j_h^* u_d).$$

We obtain

$$\begin{aligned}
&\|A(y) - A(\bar{y})\|_{L^p} \le k_p \left\|B^{-1} j^* j B^{-1} (y - \bar{y})\right\|_{H^1} \le k_p \nu^{-1} \left\|B^{-1}(y - \bar{y})\right\|_{H^{-1}} \\
&\qquad \le k_p \nu^{-1} \left\|B^{-1}(y - \bar{y})\right\|_{H^1} \le k_p \nu^{-2} \|y - \bar{y}\|_{L^2}, \\
&\|A_h(y_h) - A_h(\bar{y}_h)\|_{L^p} \le k_p \left\|e_h B_h^{-1} j_h^* j^* j j_h B_h^{-1} e_h^* (y_h - \bar{y}_h)\right\|_{U_h} \le k_p \nu^{-2} \|y_h - \bar{y}_h\|_{Y_h}.
\end{aligned}$$

This implies Assumption 3.2.2 with $L = k_p \nu^{-2}$.

The Assumption 3.2.3 is trivial, since

$$A(y) - A(\bar{y}) - A'(y)(y - \bar{y}) = 0, \quad A_h(y_h) - A_h(\bar{y}_h) - A_h'(y_h)(y_h - \bar{y}_h) = 0.$$

We now return to the control problem with semilinear state equation.

THEOREM 5.7.

a) *The operators $u(\cdot) : L^2(\Omega) \to H_0^1(\Omega)$ and $u_h(\cdot) : Y_h \to U_h$, $h > 0$, are Lipschitz continuous with modulus ν^{-1} and there holds*

$$\|u(y)\|_{H^1} \le \nu^{-1} \|y\|_{L^2}, \quad \|u_h(y_h)\|_{U_h} \le \nu^{-1} \|y_h\|_{Y_h}.$$

b) *For any bounded set $V \subset L^2(\Omega)$, there exists $L_V > 0$ such that the Fréchet derivatives $u'(\cdot)$ and $u_h'(\cdot)$, $h > 0$, are Lipschitz continuous on V and $V \cap Y_h$, respectively, with modulus L_V. Furthermore, for all $y \in L^2(\Omega)$ and $y_h \in Y_h$, we have the bounds*

$$\|u'(y)\|_{L^2, H^1} \le \nu^{-1}, \quad \|u_h'(y_h)\|_{Y_h, U_h} \le \nu^{-1}.$$

Proof. Throughout the proof, let $h \ge 0$ and $y_h^i \in Y_h$ be arbitrary and set

$$u_h^i = u_h(y_h^i), \quad i = 1, 2, \quad \delta y_h = y_h^2 - y_h^1, \quad \delta u_h = u_h^2 - u_h^1.$$

a)

$$\|\delta u_h\|_{U_h} = \left\|E_h^{-1}(e_h^* y_h^2) - E_h^{-1}(e_h^* y^1)\right\|_{U_h} \le \nu^{-1} \|e_h^* \delta y_h\|_{U_h^*} \le \nu^{-1} \|\delta y_h\|_{Y_h}.$$

The growth estimate follows from $u(0) = 0$ and $u_h(0) = 0$.

b)

Let $r > 0$ be such that $\|y\|_{L^2} \le r$ for all $y \in V$ and consider $y^i \in V$, $y_h^i \in V \cap Y_h$. Then, since $u(0) = 0$ and $u_h(0) = 0$, we have by a) that

$$\left\|u^i\right\|_{H^1} \le \nu^{-1} r, \quad \left\|u_h^i\right\|_{U_h} \le \nu^{-1} r.$$

Since $u_h(\cdot)$ is Lipschitz continuous with modulus ν^{-1}, we conclude

$$\|u_h'(y_h) v_h\|_{U_h} = \lim_{t \to 0} \frac{\|u_h(y_h + t v_h) - u_h(y_h)\|_{U_h}}{t} \le \nu^{-1} \|v_h\|_{Y_h} \quad \forall\, y_h, v_h \in Y_h.$$

Differentiation of the (discrete) state equation (5.8) yields

$$E_h'(u_h^i)u_h'(y_h^i)v_h = j_h^*(Cu_h'(y_h^i)v_h + f'(u_h^i)u_h'(y_h^i)v_h) = e_h^* v_h,$$

Hence,

$$\begin{aligned} &E_h'(u_h^2)(u_h'(y_h^2) - u_h'(y_h^1))v_h = \\ &\quad = E_h'(u_h^2)u_h'(y_h^2)v_h - E_h'(u_h^1)u_h'(y_h^1)v_h + (E_h'(u_h^1) - E_h'(u_h^2))u_h'(y_h^1)v_h \\ &\quad = j_h^*\left((f'(u_h^1) - f'(u_h^2))(u_h'(y_h^1)v_h)\right). \end{aligned}$$

We use Hölder's inequality to estimate

$$\begin{aligned} &\left\| j_h^*\left((f'(u_h^1) - f'(u_h^2))(u_h'(y_h^1)v_h)\right)\right\|_{U_h^*} \\ &\quad \le \left\|(f'(u_h^1) - f'(u_h^2))(u_h'(y_h^1)v_h)\right\|_{L^2} \le \left\|f'(u_h^1) - f'(u_h^2)\right\|_{L^{\frac{2p}{p-2}}} \left\|u_h'(y_h^1)v_h\right\|_{L^p} \\ &\quad \le k_p\nu^{-1}\left\|f'(u_h^1) - f'(u_h^2)\right\|_{L^{\frac{2p}{p-2}}} \|v_h\|_{U_h} \le k_p\nu^{-1}\left\|f'(u_h^1) - f'(u_h^2)\right\|_{L^{\frac{2p}{p-2}}} \|v_h\|_{Y_h}. \end{aligned}$$

According to Theorem 5.7 a) with f, p and q replaced by f', p and $\frac{2p}{p-2}$, respectively, we have that the operator $u \in L^p(\Omega) \mapsto f'(u) \in L^{\frac{2p}{p-2}}(\Omega)$ is Lipschitz continuous on $\left\{\|u\|_{L^p} \le \nu^{-1}r\right\}$ with a constant L_r. Hence,

$$\begin{aligned} \left\|(u_h'(y_h^2) - u_h'(y_h^1))v_h\right\|_{U_h} &\le \left\|E_h'(u_h^2)^{-1}\right\|_{U_h^*,U_h} k_p\nu^{-1}L_r\|\delta u_h\|_{L^p}\|v_h\|_{Y_h} \\ &\le k_p^2\nu^{-2}L_r\|\delta u_h\|_{U_h}\|v_h\|_{Y_h} =: L_V\|\delta u_h\|_{U_h}\|v_h\|_{Y_h}. \end{aligned}$$

The uniform Lipschitz continuity of $u_h'(\cdot)$, $h \ge 0$, on V is proved. □

THEOREM 5.8. *For any bounded set $V \subset L^2(\Omega)$, the following holds:*

a) *The operators $w(\cdot) : L^2(\Omega) \to H_0^1(\Omega)$ and $w_h(\cdot) : Y_h \to U_h$, $h > 0$, are Lipschitz continuous and bounded on V and $V \cap Y_h$, respectively, with Lipschitz constant and bound independent of h.*

b) *The Fréchet derivatives $w'(\cdot)$ and $w_h'(\cdot)$, $h > 0$, exist, and these operators are Lipschitz continuous on V and $V \cap Y_h$, respectively, with a Lipschitz constant independent of h.*

Proof. Let $V \subset L^2(\Omega)$ be bounded and choose $r > 0$ such that $\|y\|_{L^2} \le r$ for all $y \in V$. Now consider any $h \ge 0$. As in the proof of Theorem 5.7 a), there holds

$$\|u_h(y_h)\|_{U_h} \le \nu^{-1}r \quad \forall\, y_h \in V \cap Y_h,\ h \ge 0.$$

a)

Let $y_h^i \in V \cap Y_h$, $i = 1, 2$, be arbitrary and set

$$u_h^i = u_h(y_h^i), \quad w_h^i = w_h(y_h^i), \quad \delta y_h = y_h^2 - y_h^1, \quad \delta u_h = u_h^2 - u_h^1, \quad \delta w_h = w_h^2 - w_h^1.$$

We have

$$E_h'(u_h^i)^* w_h^i = j_h^*(C^* w_h^i + f'(u_h^i)w_h^i) = j_h^*(u_h^i - u_d).$$

Furthermore, we obtain the uniform bound

$$\left\|w_h^i\right\|_{U_h} = \left\|E_h'(u_h^i)^{-1}j_h^*(u_h^i - u_d)\right\|_{U_h} \le \nu^{-1}\left\|u_h^i - u_d\right\|_{H^{-1}} \le \nu^{-1}(\nu^{-1}r + \|u_d\|_{L^2}).$$

Next, we use the adjoint equation to derive

$$\begin{aligned} E_h'(u_h^2)^*\delta w_h &= E_h'(u_h^2)^* w_h^2 - E_h'(u_h^1)^* w_h^1 + (E_h'(u_h^1)^* - E_h'(u_h^2)^*)w_h^1 \\ &= j_h^*\left(\delta u_h + (f'(u_h^1) - f'(u_h^2))w_h^1\right). \end{aligned}$$

Hence,

$$\begin{aligned}\|\delta w_h\|_{U_h} &\le \left\|(E_h'(u_h^2)^*)^{-1}\right\|_{U_h^*,U_h} \left\|j_h^*(\delta u_h + (f'(u_h^1) - f'(u_h^2))w_h^1)\right\|_{U_h^*} \\ &\le \nu^{-1}(\|\delta u_h\|_{L^2} + \left\|f'(u_h^1) - f'(u_h^2)\right\|_{L^{\frac{2p}{p-2}}} \left\|w_h^1\right\|_{L^p}).\end{aligned}$$

Since $f'(\cdot) : L^p(\Omega) \to L^{\frac{2p}{p-2}}(\Omega)$ is Lipschitz continuous on the bounded set $\left\{\|u\|_{L^p} \le \nu^{-1}r\right\}$ with a constant L_r, we obtain

$$\|\delta w_h\|_{U_h} \le \nu^{-1}(1 + k_p L_r \nu^{-1}(\nu^{-1}r + \|u_d\|_{L^2}))\,\|\delta u_h\|_{L^2} =: L_V\,\|\delta u_h\|_{L^2}\,.$$

b)

We consider the adjoint equation

$$E_h'(u_h(y_h))^* w_h - j_h^*(u_h(y_h) - u_d) = 0.$$

The operator on the left is continuously Fréchet differentiable and the partial derivative with respect to w_h is $E_h'(u_h(y_h))^*$. This operator is continuously invertible so that the implicit function theorem can be applied to prove that $y_h \mapsto w_h(y_h)$ is continuously Fréchet differentiable.

Now let $y_h \in V \cap Y_h$ be arbitrary. With $u_h = u_h(y_h)$ and $w_h = w_h(y_h)$ we obtain by differentiation

$$E_h'(u_h)^* w_h'(y_h) + f''(u_h)w_h \cdot u_h'(y_h) - j_h^* u_h'(y_h) = 0.$$

It was shown in Theorem 5.7 and in a) that the operators

$$u_h(\cdot), w_h(\cdot) : Y_h \mapsto U_h \quad \text{and} \quad u_h'(\cdot) : Y_h \mapsto \mathcal{L}(Y_h, U_h)$$

are Lipschitz continuous and bounded on $V \cap Y_h$ with Lipschitz constant and bound independent of h. Furthermore, by Theorem A.2 a), the operator $f''(\cdot) : L^p(\Omega) \to L^{\frac{2p}{p-4}}(\Omega)$ is Lipschitz continuous on $\left\{\|u\|_{L^p} \le \nu^{-1}r\right\}$. Since, by Hölder's inequality,

$$\begin{aligned}\|f''(u_h)w_h(u_h'(y_h)v_h)\|_{U_h^*} &\le \|f''(u_h)w_h(u_h'(y_h)v_h)\|_{L^2} \\ &\le \|f''(u_h)\|_{L^{\frac{2p}{p-4}}} \|w_h\|_{L^p} \|u_h'(y_h)v_h\|_{L^p} \\ &\le k_p^2 \|f''(u_h)\|_{L^{\frac{2p}{p-4}}} \|w_h\|_{U_h} \|u_h'(y_h)v_h\|_{U_h}\,,\end{aligned}$$

we conclude that for $h \ge 0$ the operator

$$f''(u_h(\cdot))w_h(\cdot) \cdot u_h'(\cdot) - j_h^* u_h'(\cdot) : Y_h \mapsto \mathcal{L}(U_h, U_h^*)$$

is Lipschitz continuous and bounded on $V \cap Y_h$ with Lipschitz constant and bound independent of h. It remains to show that the operator

$$y_h \in Y_h \times U_h^* \mapsto (E_h'(u_h(y_h))^*)^{-1} \in \mathcal{L}(U_h^*, U_h)$$

is Lipschitz continuous and bounded on $V \cap Y_h$ with Lipschitz constant and bound independent of $h \ge 0$. This, however, can be done exactly as in part a). □

We are now in a position to verify the remaining Assumptions 3.2.2 and 3.

Since Assumptions 3.2.1 is already shown, we see that we can choose $h_0 > 0$ and $\delta_0 > 0$ and a bounded set $V \subset L^2(\Omega)$ such that $y \in V$ holds for all $y \in Y$ with $\|y - \bar{y}\|_{L^2} \le \delta_0$ and $y_h \in V \cap Y_h$ holds for all $y_h \in Y_h$ with $\|y_h - \bar{y}_h\|_{Y_h} \le \delta_0$, $0 < h < h_0$. From Theorem 5.8

we then obtain a constant $L_V > 0$ such that the following estimates hold: For all $y^i \in L^2(\Omega)$, $\left\|y^i - \bar{y}\right\|_{L^2} \le \delta_0$,

$$\left\|A(y^2) - A(y^1)\right\|_{L^p} \le k_p \left\|w(y^2) - w(y^1)\right\|_{H^1} \le k_p L_V \left\|y^2 - y^1\right\|_{L^2}.$$

Further, for all $0 < h < h_0$ and all $y_h^i \in Y_h$, $\left\|y_h^i - \bar{y}_h\right\|_{Y_h} \le \delta_0$,

$$\begin{aligned}\left\|A_h(y_h^2) - A_h(y_h^1)\right\|_{L^p} &= \left\|i_h^*(w_h(y_h^2) - w_h(y_h^1))\right\|_{L^p} \le k_p \left\|w_h(y_h^2) - w_h(y_h^1)\right\|_{U_h} \\ &\le k_p L_V \left\|y_h^2 - y_h^1\right\|_{Y_h}.\end{aligned}$$

This proves Assumptions 3.2.2.

We now proceed to Assumptions 3.2.3. By Theorem 5.8, the operators $w'(\cdot)$ and $w_h'(\cdot)$, $h > 0$, are Lipschitz continuous on V and $V \cap Y_h$, respectively, with a common modulus L_V'. Hence, for all $y \in L^2(\Omega)$, $\|y - \bar{y}\|_{L^2} \le \delta_0$, we have with $s = y - \bar{y}$

$$\begin{aligned}&\|A(y) - A(\bar{y}) - A'(y)(y - \bar{y})\|_{L^2} = \left\|\int_0^1 (w'(\bar{y} + ts) - w'(y))s\,dt\right\|_{L^2} \\ &\le \int_0^1 \|(w'(\bar{y} + ts) - w'(y))s\|_{H^1}\,dt \le \int_0^1 L_V'(1-t)\,\|s\|_{L^2}^2\,dt = \frac{L_V'}{2}\,\|s\|_{L^2}^2.\end{aligned}$$

In the same way, for all $0 < h < h_0$ and all $y_h \in Y_h$, $\|y_h - \bar{y}_h\|_{Y_h} \le \delta_0$, we obtain with $s_h = y_h - \bar{y}_h$

$$\begin{aligned}&\|A_h(y_h) - A_h(\bar{y}_h) - A_h'(y_h)(y_h - \bar{y}_h)\|_{Y_h} = \left\|\int_0^1 i_h^*(w_h'(\bar{y}_h + ts_h) - w_h'(y_h))s_h\,dt\right\|_{Y_h} \\ &\le \int_0^1 \|(w_h'(\bar{y}_h + ts_h) - w_h'(y_h))s_h\|_{U_h}\,dt \le \int_0^1 L_V'(1-t)\,\|s_h\|_{Y_h}^2\,dt = \frac{L_V'}{2}\,\|s_h\|_{Y_h}^2.\end{aligned}$$

Hence, (3.5), (3.6), and (3.7) are satisfied with $\rho(t) = \frac{L_V' t}{2}$.

6. Numerical validation. For the numerical validation of our mesh independence result we consider the following optimal control problem with a semilinear governing equation.

$$\begin{aligned}&\text{minimize} && J(u,y) = \tfrac{1}{2}\|u - u_d\|_{L^2}^2 + \tfrac{\lambda}{2}\|y\|_{L^2}^2 \\ &\text{subject to} && (u,y) \in H^1(\Omega) \times L^2(\Omega), \\ & && -\Delta u + u^3 + u = y \text{ in } \Omega, \quad u = 0 \text{ on } \Gamma = \partial\Omega, \\ & && y \in Y_{\text{ad}} = \{y \in L^2(\Omega) \mid -4 \le y(x) \le 0 \text{ for a.a. } x \text{ in } \Omega\},\end{aligned} \tag{6.1}$$

with $\Omega = (0,1)^2$, $u_d = \sin(2\pi x_1)\sin(2\pi x_2)\exp(2x_1)/6$, and $\lambda = 0.001$. For the discretization of (6.1) we use the procedure described in section 2. We initialize Algorithm 2.1 with $y_h^0 = 0$, i.e., the initial control is set to the upper bound. The generalized derivatives are determined according to Corollary 4.2.

For the results reported on in Tables 6.1-6.3 we use the following notation:

$$\begin{aligned}\text{res}_h^k &= \|\lambda y_h^k - P_{[-4\lambda,0]}(-A_h(y_h^k))\|_{L^2}, \\ l_h^k &= \|y_h^k - y_h^*\|_{L^2}, \\ q_h^k &= \|y_h^k - y_h^*\|_{L^2}/\|y_h^{k-1} - y_h^*\|_{L^2}.\end{aligned}$$

Here y_h^* denotes a reference solution computed by a previous run of the algorithm with the same initialization. In all test runs the algorithm terminates as soon as $\text{res}_h^k \le \sqrt{\epsilon_M}$, with

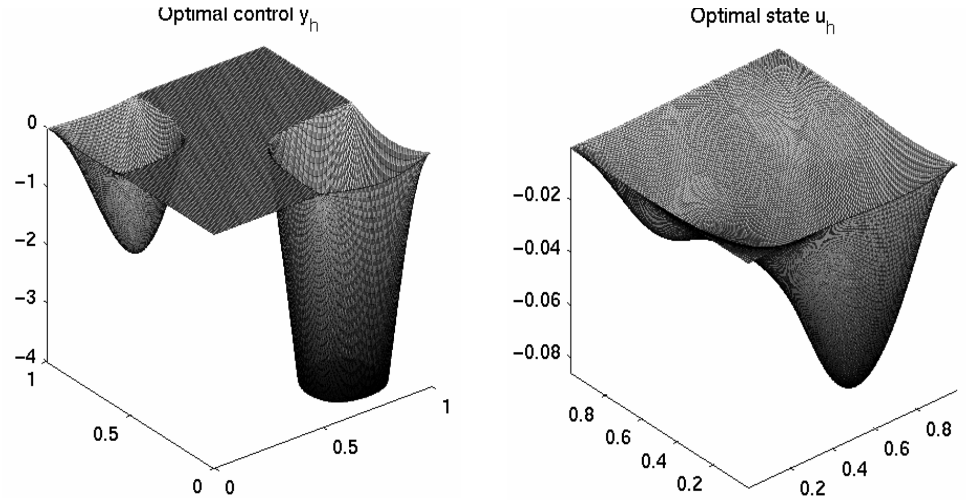

FIG. 6.1. *Optimal control and state for $h = 1/256$.*

ϵ_M the machine precision. Figure 6.1 shows the optimal control y_h and the corresponding optimal state u_h for $h = 1/256$.

In Table 6.1 we provide the convergence behavior of the complementarity residual res_h^k. The changes in the residuals for decreasing h (consider the columns of Table 6.1) are monotonically decreasing. This stabilizing effect clearly indicates an asymptotically mesh independent behavior.

TABLE 6.1
Convergence behavior of $\mathrm{res}_h^k = \|\lambda y_h^k - P_{[-4\lambda,0]}(-A_h(y_h^k))\|_{L^2}$.

h	res_h^k			
	1	2	3	4
1/16	2.464	1.137	0.056	0
1/32	2.575	1.209	0.062	3.853E-5
1/64	2.604	1.237	0.062	4.677E-5
1/128	2.609	1.243	0.062	3.597E-5
1/256	2.611	1.244	0.062	5.045E-5

In the following Table 6.2 we display the quantities l_h^k which are involved in the linear rate of convergence assertions of our mesh independence results Theorem 3.6 resp. Theorem 4.3. Like in the previous table we can observe a certain stabilizing behavior with respect to decreasing mesh-size h. This clearly validates the assertion of Theorem 4.3.

Finally, in Table 6.3 we provide the quotients $q_h^k = \|y_h^k - y_h^*\|_{L^2}/\|y_h^{k-1} - y_h^*\|_{L^2}$. With respect to decreasing h we observe again the stabilizing behavior as before. Each row in Table 6.3 corresponds to the convergence history of Algorithm 2.1 with fixed h. Obviously, the algorithm converges superlinearly for fixed h. Combining the observations of this behavior with the behavior with respect to decreasing h, we infer that the superlinear rate of convergence does not deteriorate with respect to decreasing h. Moreover, independently of the mesh-size h Algorithm 2.1 requires 5 iterations until its successful termination. The latter behavior is known as *strong mesh independence* (see [1]) which numerically augments our theoretical results.

Appendix A. Appendix.

TABLE 6.2
Convergence behavior of $l_h^k = \|y_h^k - y_h^*\|_{L^2}$.

h	l_h^k			
	1	2	3	4
1/16	3.197	1.152	0.056	1.425E-5
1/32	3.318	1.223	0.062	6.112E-5
1/64	3.350	1.251	0.063	6.286E-5
1/128	3.353	1.257	0.063	5.780E-5
1/256	3.355	1.258	0.062	6.693E-5

TABLE 6.3
Convergence behavior of $q_h^k = \|y_h^k - y_h^*\|_{L^2} / \|y_h^{k-1} - y_h^*\|_{L^2}$.

h	q_h^k			
	1	2	3	4
1/16	1.602	0.360	0.048	2.560E-4
1/32	1.628	0.368	0.051	9.794E-4
1/64	1.635	0.374	0.050	1.005E-3
1/128	1.636	0.375	0.050	9.180E-4
1/256	1.636	0.375	0.050	1.071E-3

LEMMA A.1. *Let the continuously differentiable function* $f : \mathbb{R} \to \mathbb{R}$ *satisfy*

$$|f'(u)| \le c_1 + c_2 |u|^q$$

with $c_1, c_2 \ge 0$ *and* $q > 0$. *Then, for all* $u, d \in \mathbb{R}$,

$$|f(u)| \le c_1|u| + \frac{c_2}{q+1}|u|^{q+1} \le c_1 + \left(c_1 + \frac{c_2}{q+1}\right)|u|^{q+1}.$$

Proof.

$$\begin{aligned} |f(u)| \le \int_0^1 |f'(tu)u|\,dt \le |u| \int_0^1 (c_1 + c_2|tu|^q)\,dt \le c_1|u| + |u|^{q+1}\frac{c_2}{q+1}t^{q+1}|_0^1 \\ = c_1|u| + \frac{c_2}{q+1}|u|^{q+1} \le c_1 + \left(c_1 + \frac{c_2\zeta}{q+1}\right)|u|^{q+1}. \end{aligned}$$

□

THEOREM A.2.

a) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuous and assume that there exist constants* $c_1, c_2 \ge 0$ *with*

$$|f(s)| \le c_1 + c_2|s|^{\frac{p}{q}} \quad \forall\, s \in \mathbb{R},$$

where $p, q \in [1, \infty)$. *Then the superposition operator* $S_f : L^p(\Omega) \to L^q(\Omega)$, $S_f(u) = f(u)$ *is continuous with*

$$\|f(u)\|_{L^q} \le c_1 \, |\Omega|^{\frac{1}{q}} + c_2 \, \|u\|_{L^p}^{\frac{p}{q}}.$$

b) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuously differentiable and assume that there exist constants* $c_1, c_2 \ge 0$ *with*

$$|f'(s)| \le c_1 + c_2|s|^{\frac{p-q}{q}} \quad \forall\, s \in \mathbb{R},$$

where $p, q \in [1, \infty)$, $p > q$. *Then the superposition operator* $S_f : L^p(\Omega) \to L^q(\Omega)$, $S_f(u) = f(u)$ *is continuously Fréchet differentiable with derivative*

$$S_f'(u)v = f'(u)v.$$

Furthermore, on any bounded subset $V \subset L^p(\Omega)$, S_f *is Lipschitz continuous.*

c) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be twice continuously differentiable and assume that there exist constants* $c_1, c_2 \geq 0$ *with*

$$|f''(s)| \leq c_1 + c_2|s|^{\frac{p-2q}{q}} \quad \forall\, s \in \mathbb{R},$$

where $p, q \in [1, \infty)$, $p > 2q$. *Then the superposition operator* $S_f : L^p(\Omega) \to L^q(\Omega)$, $S_f(u) = f(u)$ *is twice continuously Fréchet differentiable with derivatives*

$$S_f'(u)v = f'(u)v, \quad S_f''(u)(v^1, v^2) = f''(u)v^1v^2.$$

a) For the continuity, see [25, Prop. 26.6]. We now prove the bound.

$$\|f(u)\|_{L^q}^q \leq \left\|c_1 + c_2|u|^{\frac{p}{q}}\right\|_{L^q} \leq c_1\,|\Omega|^{\frac{1}{q}} + c_2\,\|u\|_{L^p}^{\frac{p}{q}}.$$

b) The continuous differentiability of S_f is proved in, e.g., [20, Appendix].

Now consider $V = \{u : \|u\|_{L^p} \leq r\}$, $r > 0$. Then, for $u^1, u^2 \in V$ and $d = u^2 - u^1$, we can use the bound in a) (applied to f') to derive

$$\begin{aligned}
\left\|S_f(u^2) - S_f(u^1)\right\|_{L^q} &= \left\|\int_0^1 S_f'(u^1 + td)d\,dt\right\|_{L^q} \leq \int_0^1 \left\|f'(u^1 + td)d\right\|_{L^p} dt \\
&\leq \int_0^1 \left\|f'(u^1 + td)\right\|_{L^{\frac{pq}{p-q}}} \|d\|_{L^p}\, dt \\
&\leq \int_0^1 \left(c_1\,|\Omega|^{\frac{p-q}{pq}} + c_2\,\|u + td\|_{L^p}^{\frac{p-q}{q}}\right) dt\, \|d\|_{L^p} \\
&\leq \left(c_1\,|\Omega|^{\frac{p-q}{pq}} + c_2 r^{\frac{p-q}{q}}\right) \|d\|_{L^p} =: L_V\, \|d\|_{L^p}.
\end{aligned}$$

The proofs of c) can be found in [20, Appendix]. □

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