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Shift generated Haar spaces on unbounded, closed domains in the complex plane

Maude Giasson, Walter Hengartner, and Gerhard Opfer

Dedicated to the memory of Dieter Gaier

Abstract. Some of the known Haar spaces are linear hulls of shifts of a single function G on $\mathbb{C}\backslash\{0\}$. We study N-dimensional and universal analytic Haar space generators for some closed sets F of $\mathbb{C}$ (in the sense that an arbitrary finite number of shifts generates Haar spaces by forming linear hulls). The suitable function space for our investigation is $\mathrm{C}^\circ(F)$, the space of all complex valued, continuous f on F with the defining property $\lim_{z\in F, z\to\infty} f(z) = 0$. In many cases we are able to characterize universal Haar space generators. We show, in addition, that in $\mathrm{C}^\circ(F)$ a best approximation by elements of finitely dimensional spaces V is unique if and only if V is a Haar space.

Keywords. Complex Haar spaces, shift generated spaces, approximation on unbounded domains, Haar space generators.

2000 MSC. 30C15, 30E10, 41A45, 41A50, 41A52

1 Introduction

Let V be an N-dimensional linear subspace of $\mathrm{C}(K)$, the normed space of all continuous functions f on a compact set K endowed with the norm $\|f\|_K := \max_{z\in K}|f(z)|$. A function $\hat{h} \in V$ is called a *best approximation* of a given function $f \in \mathrm{C}(K)$ if $\rho_V(f) := \|f - \hat{h}\|_K \le \|f - h\|_K$ holds for all $h \in V$. The existence of a best approximation of $f \in \mathrm{C}(K)$ is easy to show and the uniqueness of the best approximation has been characterized by HAAR[4, 1918] and KOLMOGOROFF[8, 1948] in terms of Haar spaces. We are interested in an analogous study of continuous functions on a closed set F of $\mathbb{C}$, where $\mathbb{C}$ stands for the field of complex numbers. In order to use the properties of a Haar space, we have to restrict ourselves to continuous functions f which in case F is unbounded satisfy the relation

$$\lim_{z\in F, z\to\infty} f(z) = 0. \tag{1}$$

The correspondent linear space endowed with the norm $\|f\|_F := \sup_{z\in F}|f(z)|$ will be denoted by $\mathrm{C}^\circ(F)$. The existence of a best approximation of $f \in \mathrm{C}^\circ(F)$ by functions in the finite dimensional subspace V of $\mathrm{C}^\circ(F)$ can be shown applying the same proof as given by MEINARDUS[10, 1967, p.1] for compact sets.

Let $N \in \mathbb{N}$ be given and $h_1, h_2, \ldots, h_N \in \mathrm{C}^\circ(F)$, where $\mathbb{N} := \{1, 2, \ldots\}$ is the set of natural numbers. By

$$V := \langle h_1, h_2, \ldots, h_N \rangle$$

we understand the intersection of all subspaces of $\mathrm{C}^\circ(F)$ which contain $h_1, h_2, \ldots, h_N$. This space is also called the *linear hull* of $h_1, h_2, \ldots, h_N$. We also say that V is *generated* or *spanned* by $h_1, h_2, \ldots, h_N$. The space V has dimension N if and only if the N functions $h_1, h_2, \ldots, h_N$ are linearly independent.

For the uniqueness of the best approximation, we need the following definition.

Definition 1.1 HAAR[4, 1918]. Let $h_j \in \mathrm{C}^\circ(F)$, $j = 1, 2, \ldots, N$ and $V = \langle h_1, h_2, \ldots, h_N \rangle$ be an N-dimensional linear subspace of $\mathrm{C}^\circ(F)$. We call V a *Haar space for* F if each function $h \in V \backslash \{0\}$ vanishes at most at $N - 1$ points of F.

The above stated so called *Haar condition* is equivalent to each of the following two properties.

1. For any selection of N pairwise distinct points $t_j \in F$ and any set of N numbers $\eta_j \in \mathbb{C}$ the interpolation problem

$$h(t_j) = \eta_j, \quad j = 1, 2, \ldots, N$$

has a unique solution $h \in V$.

2. Let $V := \langle h_1, h_2, \ldots, h_N \rangle$ be the linear hull of the N linearly independent functions $h_j \in \mathrm{C}^\circ(F)$, $j = 1, 2, \ldots, N$. Then, the $(N \times N)$ matrix

$$\mathbf{M} := (h_j(t_k)), \quad j, k = 1, 2, \ldots, N$$

is non-singular for any choice of pairwise distinct points $t_j \in F$, $j = 1, 2, \ldots, N$.

The correspondent characterization of Haar and Kolmogoroff becomes:

Theorem 1.2 *Let V be an N-dimensional linear subspace of $\mathrm{C}^\circ(F)$. Then, for each $f \in \mathrm{C}^\circ(F)$, there exists a unique best approximation $\hat{h}$ in V if and only if V is a Haar space.*

Proof: One readily verifies the proofs of Theorem 16 to Theorem 20, pp. 13-18, in MEINARDUS[10, 1967]. □

Unfortunately, this theorem does not hold for the weaker assumption that f is continuous and bounded on F. This will be shown explicitly in the next example. Before we start the example, we introduce some common notation. By $\mathbb{R}$ we understand the field of real numbers, by $\mathbb{R}^+$ the set of non negative real numbers, and $\Re, \Im$ stand for real and imaginary part, respectively, of a complex number.

Example 1.3 Let $F := \{x \in \mathbb{R} : x \geq 1\}$, $h := 1/x$ on F and $V := \langle h \rangle$. Then, V is a Haar space over $\mathbb{C}$ of dimension one on F. By $\mathrm{C}^b(F)$ we understand the space of continuous and bounded functions on F. Certainly, $V \subset \mathrm{C}^b(F)$. We will see that the constant function $f := 1$ which belongs to $\mathrm{C}^b(F)$ but not to $\mathrm{C}^\circ(F)$ admits several best approximations in V. We have
$\rho_V(f) := \inf_{v \in V} \sup_{x \geq 1} |1 - v(x)| = 1$, since

$$\sup_{x \geq 1} \left|1 - \frac{a}{x}\right| = \begin{cases} 1 & \text{for } |1-a| \leq 1,\ a \in \mathbb{C}, \\ |1-a| > 1 & \text{otherwise.} \end{cases}$$

Thus, all $v(x) := \dfrac{a}{x}$ with $|1-a| \leq 1$ and $a \in \mathbb{C}$ are best approximations of f.

For further details on uniqueness of uniform approximation and Haar spaces see e.g. [2], [3], [5], [9], [11].

Our next definition is concerned with Haar space generators.

Definition 1.4

1. Let $N \in \mathbb{N}$ be a fixed natural number. A function G defined on $\mathbb{C}\backslash\{0\}$ with values in $\mathbb{C}$ will be called an *N-dimensional Haar space generator for F*, if for each set of N pairwise distinct points $t_1, t_2, \ldots, t_N \in \mathbb{C}\backslash F$ the functions h_j defined by $h_j(z) := G(z - t_j)$, $j = 1, 2, \ldots, N$, are linearly independent and span an N-dimensional Haar space on F for $z \in F$.

2. The function G is called a *universal Haar space generator for F* if G is an N-dimensional Haar space generator for F for all $N \in \mathbb{N}$.

In this paper we deal only with *analytic* Haar space generators. By $H(S)$ we understand the space of all analytic functions defined on the open, non empty set $S \subset \mathbb{C}$. We always suppose that $G \in H(\mathbb{C}\backslash\{0\})$ which means that G is defined everywhere in $\mathbb{C}$ with the (possible) exception of the origin and represents an analytic function. Furthermore, in order that the theory makes sense, we assume always (tacitly) that F is closed and $F \neq \mathbb{C}$. This implies that the open set $\mathbb{C}\backslash F$ is not empty and contains infinitely many points.

Example 1.5 Let

$$G(z) := \frac{\mathrm{e}^{Az+B}}{z}, \quad \|\mathrm{e}^{Az}\|_F < \infty, \quad A,\ B \in \mathbb{C}, \quad z \neq 0. \tag{2}$$

Then, $G \in H(\mathbb{C}\backslash\{0\})$ and G is a universal Haar space generator for all closed subsets F of $\mathbb{C}$. Indeed, for $t_1, t_2, \ldots, t_N$, the function $h(z) = \sum_{k=1}^N \mu_k G(z - t_k)$ belongs to $\mathrm{C}^\circ(F)$ and can have at most $(n-1)$ zeros in $\mathbb{C}$ and hence in F. Furthermore, $\|\mathrm{e}^{Az}\|_F = |\mathrm{e}^{At}| \|\mathrm{e}^{A(z-t)}\|_F < \infty$ implies that for each fixed $t \in \mathbb{C}\backslash F$, the function $G(z-t) \in \mathrm{C}^\circ(F)$. We are aware of the fact that the convenient notation $\|\mathrm{e}^{Az}\|_F$ is not quite accurate. One could introduce, say $\eta_A(z) := \mathrm{e}^{Az}$ and then write $||\eta_A||_F$ instead.

A suitable choice of values for the constant A to have the property $||e^{Az}||_F < \infty$ is contained in the following table for various domains F.

Table 1.6 Values of A for various examples of domains F.

$F =$	set under consideration	$\Rightarrow$ condition for A
$F =$	$\mathbb{R}^+$	$\Rightarrow \Re A \le 0,$
$F =$	$\mathbb{R}$	$\Rightarrow \Re A = 0,$
$F =$	$\mathrm{i}\,\mathbb{R}^+$	$\Rightarrow \Im A \ge 0,$
$F =$	$\mathrm{i}\,\mathbb{R}$	$\Rightarrow \Im A = 0,$
$F =$	$\mathbb{R}^+ \cup \mathrm{i}\,\mathbb{R}^+$	$\Rightarrow \Re A \le 0, \Im A \ge 0,$
$F =$	$\mathbb{R} \cup \mathrm{i}\,\mathbb{R}^+$	$\Rightarrow \Re A = 0, \Im A \ge 0,$
$F =$	$\mathbb{R}^+ \cup \mathrm{i}\,\mathbb{R}$	$\Rightarrow \Re A \le 0, \Im A = 0,$
$F =$	$\mathbb{R} \cup \mathrm{i}\,\mathbb{R}$	$\Rightarrow A = 0,$
$F =$	$\{z \in \mathbb{C} : \lvert z\rvert \ge 1\}$	$\Rightarrow A = 0,$
$F =$	$\{z = x + \mathrm{i}\,y : y = x - 2\}$	$\Rightarrow A = a + \mathrm{i}\,a, a \in \mathbb{R}.$

If F is a compact subset of $\mathbb{C}$, then, $G \in H(\mathbb{C}\backslash\{0\})$ is a one dimensional Haar space generator for F if and only if G does not vanish on $\mathbb{C}\backslash\{0\}$. In Section 2 we give an example of a one dimensional Haar space generator for the real axis which vanishes at infinitely many points. Furthermore, we collect in Section 2 some preliminary results concerning Haar space generators. In Section 3, we study universal Haar space generators for closed sets F with the property that $\overline{F^\circ}$ is a compact subset of $\mathbb{C}$, where F° denotes the interior of F, and $\overline{F^\circ}$ denotes the closure of the interior of F. In Section 4, we assume that F contains a neighborhood of infinity.

2 Some elementary properties of Haar space generators and auxiliary lemmata

Hengartner & Opfer, [7, 2003] have already shown that in the case where F is compact there is no inclusion property of N-dimensional Haar space generators neither with respect to the dimension N nor with respect to the inclusion $F_1 \subset F_2$ of two sets. Two simple, but useful properties of Haar space generators are contained in the following two lemmata.

Lemma 2.1 Let F be a closed subset of $\mathbb{C}$, and define $F_1 := aF + b$ where $a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C}$. Then, G is an N-dimensional Haar space generator for F if and only if G_1 defined by $G_1(z) := G(z/a)$ is an N-dimensional Haar space generator for F_1.

Proof: Let $z \in F$ and $t \in \mathbb{C}\backslash F$. Then, $\zeta := az + b \in F_1, \tau := at + b \in \mathbb{C}\backslash F_1$ and we have $G_1(\zeta - \tau) = G_1(az - at) = G(z - t)$. $\square$

Lemma 2.2 Suppose that G is an analytic N-dimensional Haar space generator for a closed set $F \subset \mathbb{C}$. Then G_1 defined by

$$G_1(z) := \mathrm{e}^{Az+B} G(z), \quad \|\mathrm{e}^{Az}\|_F < \infty, \quad A,\ B \in \mathbb{C},$$

is also an analytic N-dimensional Haar space generator for F.

Proof: The two (outer) linear combinations

$$\sum_{k=1}^{n} \lambda_k G_1(z - t_k) = \mathrm{e}^{Az} \sum_{k=1}^{n} \lambda_k \mathrm{e}^{-At_k + B} G(z - t_k) = \mathrm{e}^{Az} \sum_{k=1}^{n} \mu_k G(z - t_k)$$

vanish simultaneously and belong to $\mathrm{C}^{\circ}(F)$. Moreover, we have $\|\mathrm{e}^{A(z-t)}\|_F = |\mathrm{e}^{-At}|\|\mathrm{e}^{Az}\|_F < \infty$ for any fixed $t \in \mathbb{C}\backslash F$. $\square$

For example, G defined by $G(z) := z^{n-1}\mathrm{e}^{Az+B}$, $n \in \mathbb{N}$, $\Re A < 0$, is an n-dimensional Haar space generator for $\mathbb{R}^+$, but it is never a j-dimensional Haar space generator for any closed set F if $j > n$. It is also a one-dimensional Haar space generator.

In the next example, we consider analytic one dimensional Haar space generators. By definition, $G \in H(\mathbb{C}\backslash\{0\})$ is a one dimensional Haar space generator for F, if for all $t \in \mathbb{C}\backslash F$, the function $G(z - t) \neq 0$ on F and $\lim_{z \in F, z \to \infty} G(z - t) = 0$. If F is compact, then, G is a one dimensional Haar space generator for F if and only if G does not vanish on $\mathbb{C}\backslash\{0\}$. See Hengartner & Opfer, [6, 2002], [7, 2003]. Such a result does not hold in general for arbitrary closed sets.

Example 2.3 The function G defined by $G(z) = \mathrm{e}^{-z^2}\cos(z)$ is an analytic one dimensional Haar space generator for the real axis $\mathbb{R}$ and vanishes at infinitely many points there. Indeed, fix $t \in \mathbb{C}\backslash\mathbb{R}$. Then, we have for $x \in \mathbb{R}$, $\lim_{x\to\infty} G(x - t) = 0$, and $G(z - t) = 0$ implies that the roots $z = t + (1 + 2k)\pi/2, k \in \mathbb{Z}$ are not in $\mathbb{R}$.

Hence, besides the definition, we do not know of any other universal criterion for G to be an analytic one dimensional Haar space generator for closed, unbounded sets F. The criterion given for compact sets is only sufficient but not necessary. Let G be a one dimensional Haar space generator for F. By definition, $G(z - s) \neq 0$ for all $z \in F$ and all $s \in \complement F$, where $\complement$ stands for *complement*. Let $\Delta F := F - \complement F := \{u \in \mathbb{C} : u = z - s,\ z \in F,\ s \in \complement F\}$ be the algebraic difference between F and $\complement F$. If $\Delta F = \mathbb{C}\backslash\{0\}$ then, it follows that $G(z) \neq 0$ for all $z \neq 0$. For unbounded F however, in general $\Delta F \neq \mathbb{C}\backslash\{0\}$.

To mention an example, let $F := \{z \in \mathbb{C} : \Re z \leq 0\}$ be the closed, left half-plane in $\mathbb{C}$. Then, $\complement F = \{z \in \mathbb{C} : \Re z > 0\}$ is the open right half plane and $\Delta F = \{z \in \mathbb{C} : \Re z < 0\} \neq \mathbb{C}\backslash\{0\}$. Another class of examples is obtained if F is any straight line in $\mathbb{C}$. But if we add to the straight line any compact set which contains points outside that line, then $\Delta F = \mathbb{C}\backslash\{0\}$ and G is nonvanishing on $\mathbb{C}\backslash\{0\}$. Another case will be treated in Lemma 4.2.

In what follows, we deal with analytic two dimensional Haar space generators. Instead of assuming, that G is also a one dimensional Haar space generator, we assume that $G(z) \neq 0$ for $z \in \mathbb{C}\backslash\{0\}$ satisfying $\lim_{z \in F, z \to \infty} G(z-t) = 0$. Recall that we always assume that $F \neq \mathbb{C}$, which implies that $\mathbb{C}\backslash F$ contains infinitely many points. We shall use the following notation.

Definition 2.4 Let $f(x_1, x_2, ..., x_n)$ be any function with $n > 1$ variables. If we consider f as a function of the first $m < n$ variables $x_1, x_2, ..., x_m$ alone, keeping the remaining variables $x_{m+1}, x_{m+2}, ..., x_n$ fixed, then, we shall write $f(x_1, x_2, ..., x_m | x_{m+1}, x_{m+2}, ..., x_n)$.

Our next lemma is an immediate consequence of the definition of a two dimensional Haar space generator.

Lemma 2.5 *Let F be an infinite closed subset of $\mathbb{C}, F \neq \mathbb{C}$, containing the two distinct points z_1 and z_2. Let $G \in H(\mathbb{C}\backslash\{0\})$ and let $t \in \mathbb{C}\backslash F$, $G(z) \neq 0$ on $\mathbb{C}\backslash\{0\}$. Furthermore, suppose that $\lim_{z \in F, z \to \infty} G(z-t) = 0$ for all $t \in \mathbb{C}\backslash F$. Then, G is a two dimensional analytic Haar space generator for F if and only if*

$$Q(t|z_1, z_2) := \frac{G(z_1 - t)}{G(z_2 - t)}, \quad z_1, z_2 \in F, \ t \in \mathbb{C}\backslash F \tag{3}$$

is univalent in $\mathbb{C}\backslash F$, i.e. $Q(s|z_1, z_2) = Q(t|z_1, z_2)$ implies $s = t$.

Proof: Suppose $Q(s|z_1, z_2) = Q(t|z_1, z_2)$, $s \neq t$, $s, t \in \mathbb{C}\backslash F$. This is equivalent to

$$\frac{G(z_1 - t)}{G(z_1 - s)} = \frac{G(z_2 - t)}{G(z_2 - s)} =: \lambda$$

for some $\lambda \in \mathbb{C}\backslash\{0\}$. Then, it follows that

$$\begin{aligned} G(z_1 - t) - \lambda G(z_1 - s) &= 0, \\ G(z_2 - t) - \lambda G(z_2 - s) &= 0 \end{aligned}$$

which contradicts the fact that G is a two dimensional Haar space generator for F. $\square$

The next lemma sharpens Lemma 2.5. The proof of it is exactly the same as the proof in Lemma 2.4 in HENGARTNER & OPFER, [7, 2003] which is the equivalent statement for compact sets F.

Lemma 2.6 *Let F be an infinite closed subset of $\mathbb{C}, F \neq \mathbb{C}$, containing the two distinct points z_1 and z_2. Suppose that $G \in H(\mathbb{C}\backslash\{0\}), G(z) \neq 0$ on $\mathbb{C}\backslash\{0\}$ and that $\lim_{z\in F, z\to\infty} G(z-t) = 0$ for all $t \in \mathbb{C}\backslash F$. Then, G is a two dimensional analytic Haar space generator for F if and only if $Q(t|z_1, z_2)$, defined in (3) is univalent in $\mathbb{C}\backslash[\overline{F^\circ} \cup \{z_1\} \cup \{z_2\}]$.*

3 The case where F° is bounded

Our main result of the section is the following theorem.

Theorem 3.1 *Suppose that the set F and the function G satisfy the following properties:*

1. *F is a closed subset of $\mathbb{C}$, $F \neq \mathbb{C}$,*
2. *F contains at least one cluster point in $\mathbb{C}$,*
3. *F° is bounded and $F \neq \overline{F^\circ}$,*
4. *$G \in H(\mathbb{C}\backslash\{0\})$,*
5. *$G(z) \neq 0$ on $\mathbb{C}\backslash\{0\}$.*

Then, G is a universal Haar space generator for F if and only if G is of the form

$$G(z) := \frac{\mathrm{e}^{Az+B}}{z}, \quad \|\mathrm{e}^{Az}\|_F < \infty, \quad A,\ B \in \mathbb{C}, \quad z \neq 0.$$

Proof: The case where F is bounded is proved in Hengartner & Opfer, [7, 2003]. Hence, suppose that F is unbounded. Fix $z_1 \in F\backslash\overline{F^\circ}$ and $z_2 \in F, z_1 \neq z_2$. By Lemma 2.6, $Q(t|z_1, z_2)$ (cf. (3)) defines an analytic nonvanishing function on $\mathbb{C}\backslash[\{z_1\} \cup \{z_2\}]$ and is univalent on $\mathbb{C}\backslash[\overline{F^\circ} \cup \{z_1\} \cup \{z_2\}]$. Therefore, there is an $m \in \{-1, 0, 1\}$ such that the three limits

1. $\lim_{t\to z_1} \left(\frac{t-z_1}{t-z_2}\right)^m Q(t|z_1, z_2)$,
2. $\lim_{t\to z_2} \left(\frac{t-z_1}{t-z_2}\right)^m Q(t|z_1, z_2)$,
3. $\lim_{t\to \infty} \left(\frac{t-z_1}{t-z_2}\right)^m Q(t|z_1, z_2)$

exist and are in $\mathbb{C}\backslash\{0\}$. Item 2 follows from the special form of Q and the analyticity. Nonvanishing at infinity follows from the argument principle applied to $Q(t|z_1, z_2)$ and the domain bounded by the three circles $\{z : |z| = R\}$, $\{z : |z - z_1| = \rho\}$ and $\{z : |z - z_2| = \rho\}$ where $R > 0$ is large and $\rho > 0$ is small.[1] The case $m = 0$ is excluded since $Q(t|z_1, z_2)$ would be a bounded nonvanishing entire function and hence, by Liouville's theorem, a constant which is impossible. Applying Liouville's theorem to $\left(\frac{t-z_1}{t-z_2}\right)^m Q(t|z_1, z_2)$ for $m = \pm 1$ yields that $Q(t|z_1, z_2)$ is a Möbius transformation of the form

$$Q(t|z_1, z_2) = c(z_1, z_2)\frac{z_1 - t}{z_2 - t} \quad \text{or} \tag{4}$$

$$Q(t|z_1, z_2) = c(z_1, z_2)\frac{z_2 - t}{z_1 - t}. \tag{5}$$

Since F contains at least one cluster point in $\mathbb{C}$, we conclude by the identity principle that

$$c(z_1|z_2) = \frac{G(z_1 - t)}{G(z_2 - t)}\left(\frac{z_2 - t}{z_1 - t}\right)^m, m \in \{-1, 1\} \tag{6}$$

admits as a function of z_1 an analytic continuation onto $\mathbb{C}$ which is independent of t. Differentiation of $\log c(z_1, z_2)$ with respect to t yields

$$\frac{G'(z_1 - t)}{G(z_1 - t)} \pm \left[\frac{1}{z_1 - t}\right] = \frac{G'(z_2 - t)}{G(z_2 - t)} \pm \left[\frac{1}{z_2 - t}\right] = A \tag{7}$$

for all t, where A is independent of z_1. Fix $t \in \mathbb{C}\backslash F$ and substitute $\zeta = z_1 - t$. The integration of relation (7) with respect to ζ implies

$$G(\zeta) := \zeta e^{A\zeta + B}, \quad \lim_{z \in F, z \to \infty} G(z) = 0, \quad A,\ B \in \mathbb{C} \quad \text{or}$$

$$G(\zeta) := \frac{e^{A\zeta + B}}{\zeta}, \quad \|e^{Az}\|_F < \infty, \quad A,\ B \in \mathbb{C}.$$

The first case is excluded because there is no closed set F for which G is a three dimensional Haar space generator. $\square$

As an immediate corollary of our theory we obtain:

Theorem 3.2 *Suppose that the set F and the function G satisfy the conditions of Theorem 3.1. If G is a one, two and three dimensional Haar space generator for F, then, G is a universal Haar space generator for F.*

Examples of sets $F \subset \mathbb{C}$ satisfying the three properties of the foregoing two theorems can be constructed easily.

[1] In this sense large and small means *sufficiently* large and small. This will also be applied to complex numbers z in the sense that $|z| > 0$ and $|z|$ is large or small.

Example 3.3 Define

$$F := \{z \in \mathbb{C} : |z| \leq 1\} \cup \{z \in \mathbb{C} : z = x + \mathrm{i}\, y,\ x \geq 1,\ y = 0\}.$$

Then, F is closed, different from $\mathbb{C}$, unbounded, contains cluster points, $F^\circ = \{z \in \mathbb{C} : |z| < 1\}$ is bounded and $\overline{F^\circ} = \{z \in \mathbb{C} : |z| \leq 1\} \neq F$. Another class of examples is obtained by choosing a closed, unbounded F with cluster points in $\mathbb{C}$ such that $F^\circ = \emptyset$.

4 The case where F contains $\{z : |z| > R\}$ for some $R > 0$

First, let us remark that the requirement $\lim_{z \in F, z \to \infty} G(z-t) = 0$ for all $t \in \mathbb{C} \backslash F$, reduces to the condition $G(\infty) = 0$.

Theorem 4.1 *Suppose that the set F and the function G satisfy the following properties:*

1. *F is a closed subset of $\mathbb{C}, F \neq \mathbb{C}$ containing $\{z : |z| > R\}$ for some $R > 0$,*
2. *$F \backslash \overline{F^\circ}$ is nonempty,*
3. *$G \in H(\mathbb{C} \backslash \{0\})$,*
4. *$G(z) \neq 0$ on $\mathbb{C} \backslash \{0\}$.*

Then, G is a universal Haar space generator for F if and only if G is of the form

$$G(z) := \frac{1}{z}.$$

Proof: Fix $z_1 \in F \backslash \overline{F^\circ}$ and $z_2 \in F, z_1 \neq z_2$. By Lemma 2.6, $Q(t|z_1, z_2)$ (cf. (3)) defines an analytic nonvanishing function on $\mathbb{C} \backslash [\{z_1\} \cup \{z_2\}]$ and is univalent on $\mathbb{C} \backslash [\overline{F^\circ} \cup \{z_1\} \cup \{z_2\}]$. Hence, there is an $m \in \{-1, 0, 1\}$ such that the two limits

1. $\lim_{t \to z_1} \left(\dfrac{t - z_1}{t - z_2} \right)^m Q(t|z_1, z_2)$ and
2. $\lim_{t \to z_2} \left(\dfrac{t - z_1}{t - z_2} \right)^m Q(t|z_1, z_2)$

exist and are in $\mathbb{C} \backslash \{0\}$. Define $K(z|t_1, t_2) := \frac{G(z-t_1)}{G(z-t_2)}, \quad t_1, t_2 \in \mathbb{C} \backslash F, \quad t_1 \neq t_2$. Then, the statements on the two limits are equivalent to the existence of the two limits

1. $\lim_{z \to t_1} \left(\dfrac{z-t_1}{z-t_2}\right)^m K(z|t_1,t_2)$ and

2. $\lim_{z \to t_2} \left(\dfrac{z-t_1}{z-t_2}\right)^m K(z|t_1,t_2)$,

which are again in $\mathbb{C}\backslash\{0\}$. Next, $K(z|t_1,t_2)$ is univalent on $\{z : |z| > R\}$. Hence, there is an $M \in \{-1,0,1\}$ such that $K(z|t_1,t_2) = z^M C(z|t_1,t_2)$, where $C(z|t_1,t_2)$ is analytic at infinity and $C(\infty|t_1,t_2) \neq 0$. By the argument principle applied to $K(z|t_1,t_2)$ and the domain bounded by the three circles $\{z : |z| = R\}$, $\{z : |z-t_1| = \rho\}$, and $\{z : |z-t_2| = \rho\}$ where R is large and ρ is small, we conclude that $M = 0$ and that the limit $\lim_{z\to\infty} \left(\frac{z-t_1}{z-t_2}\right)^m K(z|t_1,t_2)$ exists and is in $\mathbb{C}\backslash\{0\}$. Again the case $m = 0$ is excluded since $K(z|t_1,t_2)$ would be a bounded nonvanishing entire function and hence, by Liouville's theorem, a constant which is impossible. Replacing z by t_1, t_1 by z_1, t_2 by z_2 and $K(z|t_1,t_2)$ by $Q(t|z_1,z_2)$ we follow the proof of Theorem 3.1 which leads us to the conclusion of Theorem 4.1. □

We now study the special case $F = \overline{F^\circ}$. We shall establish several lemmata. We have already seen (Example 2.3) that in general one dimensional Haar space generators may have infinitely many zeros. The first lemma shows that this is impossible if F contains $\{z : |z| > R\}$ for some $R > 0$ and $F = \overline{F^\circ}$.

Lemma 4.2 *Let F be a closed subset of $\mathbb{C}$ containing $\{z : |z| > R\}$ for some $R > 0$ which has the property $F = \overline{F^\circ}$. Let $G \in H(\mathbb{C}\backslash\{0\})$ and $G(\infty) = 0$. Then, G is a one dimensional Haar space generator for F if and only if G does not vanish on $\mathbb{C}\backslash\{0\}$.*

Proof: If G does not vanish on $\mathbb{C}\backslash\{0\}$, then, $G(z-t) \neq 0$ for all $z \in F$ and all $t \in \mathbb{C}\backslash F$. Hence, G is a one dimensional Haar space generator. Suppose now that $G(z^\star) = 0$ for some $z^\star \neq 0$. Choose $z_0 \in F^\circ$ and denote by d the straight line $d := \{z : z = z_0 + \mathrm{i}\,\lambda z^\star; \lambda \in \mathbb{R}\}$. Define

$$\mu_1 = \sup\{\mu > 0 : (z_0 + \mu d) \cap \overline{\mathbb{C}\backslash F} \neq \emptyset\}$$

and let t_1 be one of the support points of $z_0 + \mu_1 d$ with respect to $\mathbb{C}\backslash F$. Put $z_1 = t_1 + z^\star$. Then, we have $z_1 \in F^\circ$ and $t_1 \in \partial F$. Since $F = \overline{F^\circ}$, there is a small $a \in \mathbb{C}$ such that $z = z_1 + a \in F$ and $t = t_1 + a \in \mathbb{C}\backslash F$. Since $G(z-t) = G(z^\star) = 0$, we conclude that G is not a one dimensional Haar space generator. □

Since $\overline{\mathbb{C}\backslash F}$ is compact we obtain by Lemma 2.2 Hengartner & Opfer, [7, 2003] immediately the following lemma.

Lemma 4.3 *Let F be a closed subset of $\mathbb{C}$ containing $\{z : |z| > R\}$ for some $R > 0$ which has the property $F = \overline{F^\circ}$. Suppose that $G \in H(\mathbb{C}\backslash\{0\})$, $G(\infty) = 0$. Then, G is a one dimensional Haar space generator for F if and only if G_1, defined by $G_1(z) := G(-z)$ is a one dimensional Haar space generator for $\overline{\mathbb{C}\backslash F}$.*

In the next lemma we show, that the previous lemma is also true for two dimensional Haar space generators.

Lemma 4.4 *Let F be a closed subset of $\mathbb{C}$ containing $\{z : |z| > R\}$ for some $R > 0$ which has the property $F = \overline{F^\circ}$. Suppose that $G \in H(\mathbb{C}\backslash\{0\})$, $G(\infty) = 0$, and that G does not vanish in $\mathbb{C}\backslash\{0\}$. If G is a two dimensional Haar space generator for F then, G_1, defined by $G_1(z) := G(-z)$ is a two dimensional Haar space generator for $\overline{\mathbb{C}\backslash F}$.*

Proof: Let us first note that under the given conditions for F the closed set $\overline{\mathbb{C}\backslash F}$ can be expressed in the form $\overline{\mathbb{C}\backslash F} = \mathbb{C}\backslash F^\circ$. In the proof of Theorem 4.1 we find that G is a two dimensional Haar space generator for F if and only if

$$K(z|t_1, t_2) \;:=\; \frac{G(z - t_1)}{G(z - t_2)}, \quad z \in F;\; t_1, t_2 \in \mathbb{C}\backslash F,\; t_1 \neq t_2 \tag{8}$$

is univalent. Similarly, G_1 is a two dimensional Haar space generator for $\mathbb{C}\backslash F^\circ$ if and only if

$$L(t|z_1, z_2) \;:=\; \frac{G_1(t - z_1)}{G_1(t - z_2)}, \quad t \in \mathbb{C}\backslash F^\circ;\; z_1, z_2 \in F^\circ,\; z_1 \neq z_2$$

is univalent. We have $\mathbb{C}\backslash F \subset \mathbb{C}\backslash F^\circ$. A point $t \in \mathbb{C}\backslash F^\circ$ is not in $\mathbb{C}\backslash F$ if and only if $t \in \partial F$, where ∂F is the boundary of F. Suppose that G_1 is not a two dimensional Haar space generator for $\mathbb{C}\backslash F^\circ$. Then, the above defined L is not univalent and by using the definition of G_1, points $z_1, z_2 \in F^\circ$, $z_1 \neq z_2$ and $t_1, t_2 \in \mathbb{C}\backslash F^\circ$, $t_1 \neq t_2$ exist, such that the non univalence of L can be written as $K(z_1|t_1, t_2) = K(z_2|t_1, t_2)$, where we allow here that $t_1, t_2 \in \partial F$ which is permissible since $z_1, z_2 \in F^\circ$. We will distinguish three cases:

Case 1: Both points t_1, t_2 are not boundary points of F, i.e. $t_1, t_2 \in \mathbb{C}\backslash F$. Thus, K in (8) is not univalent, and G is not a two dimensional Haar space generator for F.

Case 2: In this case, we suppose that one of the two points t_1, t_2, say $t_2 \notin \mathbb{C}\backslash F$ or equivalently, $t_2 \in \partial F$. We have $K(z_1|t_1, t_2) = K(z_2|t_1, t_2)$. There exists a small complex number a such that $z_1' = z_1 + a \in F^\circ$, $z_2' = z_2 + a \in F^\circ$, $t_1' = t_1 + a \in \mathbb{C}\backslash F$ and $t_2' = t_2 + a \in \mathbb{C}\backslash F$ which leads to $K(z_1'|t_1', t_2') = K(z_2'|t_1', t_2')$. Thus, K defined in (8) is not univalent and G is not a two dimensional Haar space generator for F.

Case 3: Suppose that $t_1, t_2 \in \partial F$. Then $K(z_1|t_1, t_2) = K(z_2|t_1, t_2) := \lambda \neq 0$. It will be shown that an equation of the form $K(\hat{z}_1|\hat{t}_1, t_2) = K(\hat{z}_2|\hat{t}_1, t_2)$ holds for $\hat{z}_1, \hat{z}_2 \in F^\circ$ and $\hat{t}_1 \in \mathbb{C}\backslash F$. This would bring us to Case 2 and the lemma is proved.

Consider the two open disks $D_1 := D(z_1, r)$ and $D_2 := D(z_2, r)$ with center z_1, z_2, respectively and radius $r > 0$. Choose r so small that
(a) $D_1 \cap D_2 = \emptyset$, possible because of $z_1 \neq z_2$,

(b) $D_1, D_2 \subset F^\circ$, possible because of $z_1, z_2 \in F^\circ$,
(c) $E \cap \{0\} = \emptyset$, where $E := K(D_1|t_1,t_2) \cap K(D_2|t_1,t_2)$. Since E is an open set containing $\lambda \neq 0$, (c) is possible.

The assumptions on G imply that K is analytic and thus, (locally) invertible at all points where the derivative is not vanishing. Now, the derivatives can vanish only at isolated points, otherwise K would be constant. Thus, there is a $\lambda' \in E, z_1' \in D_1$ and $z_2' \in D_2$ such that $K(z_1'|t_1,t_2) = K(z_2'|t_1,t_2) = \lambda'$ where the derivatives at z_1', z_2' do not vanish. Fixing the point t_2, we apply the implicit function theorem to K in a neighborhood $U_1 \subset D_1$ of z_1'. In other words, there is a univalent mapping $z_1(t)$ from a neighborhood V_1 of t_1 to a neighborhood U_1 of z_1' such that $K(z_1(t)|t,t_2) = K(z_1'|t_1,t_2) = \lambda'$. Analogous, there is a univalent mapping $z_2(t)$ from a neighborhood V_2 of t_1 to a neighborhood $U_2 \subset D_2$ of z_2' such that $K(z_2(t)|t,t_2) = K(z_2'|t_1,t_2) = \lambda'$. Now choose $\hat{t} \in (V_1 \cap V_2) \cap (\mathbb{C}\setminus F)$ and put $\hat{z}_1 = z_1(\hat{t})$ and $\hat{z}_2 = z_2(\hat{t})$. Then, $K(z_1'|t_1,t_2) = K(\hat{z}_1|\hat{t},t_2) = K(\hat{z}_2|\hat{t},t_2) = K(z_2'|t_1,t_2)$ and we are in the situation of Case 2. □

Hengartner & Opfer, [7, 2003] have shown that one and two dimensional Haar space generators for a compact subset K of $\mathbb{C}$ have to be of the form

$$G(z) := \frac{\mathrm{e}^{Az+B}}{z^m}, \quad m \in \{0, \pm 1 \pm 2\}.$$

The cases $m = 0, -1, -2$ are not 4-dimensional and hence not universal Haar space generators, since any four functions $G(z - t_j), t_j \in \mathbb{C}\setminus K$, are linearly dependent. Furthermore, the condition $G(\infty) = 0$ is equivalent to $A = 0$. Together with Lemma 4.3 and Lemma 4.4, we have proved so far that a universal Haar space generator G for a closed set $F = \overline{F^\circ}$, $\overline{\mathbb{C}\setminus F}$ compact, is of the form

$$G(z) := \frac{1}{z} \quad \text{or} \quad G(z) := \frac{1}{z^2}.$$

Applying Lemma 4.4 to results obtained in Hengartner & Opfer, [7, 2003], we can eliminate the case $m = 2$ for most closed sets of the form $F = \overline{F^\circ}, \overline{\mathbb{C}\setminus F}$ compact. In particular, we obtain the following theorem.

Theorem 4.5 *Let F be a closed subset of $\mathbb{C}$ containing $\{z : |z| > R\}$ with the property that $F = \overline{F^\circ}$. Suppose that $G \in H(\mathbb{C}\setminus\{0\})$, $G(\infty) = 0$, and that G does not vanish in $\mathbb{C}\setminus\{0\}$. Let G be a two and three dimensional Haar space generator for F. Then, G defined by $G(z) := 1/z$ is the only universal Haar space generator for F provided that one of the following conditions holds.*

1. $\overline{\mathbb{C}\backslash F}$ *is not a nonempty convex set.*

2. $\overline{\mathbb{C}\backslash F}$ *is convex and* ∂F *contains a corner of angle less than* π *seen from the outside of* F.

3. $\overline{\mathbb{C}\backslash F}$ *is an ellipse or a disk.*

Example 4.6 Let $F := \{x \in \mathbb{R} : x \geq 1\}$ and $f(x) := \exp(-x)$ for $x \geq 1$. We approximate with functions $v(x) := \sum_{j=1}^{n} \frac{a_j}{x-s_j}$ where $-1 < s_1 < s_2 < \cdots < s_n < 1$. With $n = 3$ and shifts $s = (-0.5, -0.1, 0.1)^{\mathrm{T}}$ we obtained an approximation with coefficients $a = (-5.15678, 8.51799, -3.53861)^{\mathrm{T}}$ resulting in an error of 0.0061. The error curve is depicted in Figure 4.7.

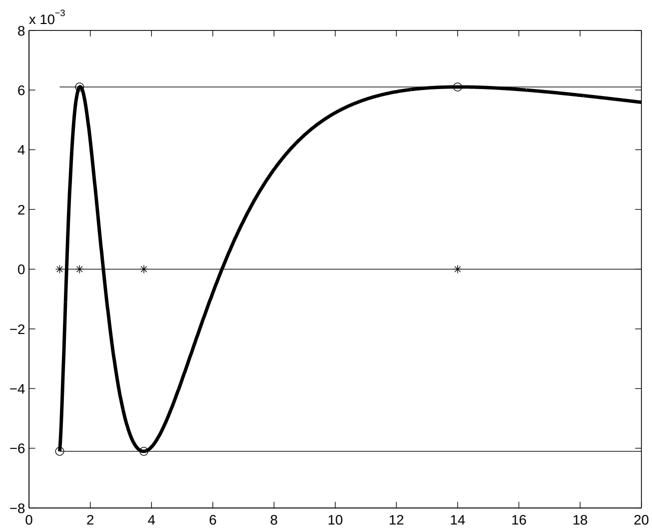

Figure 4.7 Error curve of best approximation of $\exp(-x)$ in $x \geq 1$.

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