Hamburger Beiträge
zur Angewandten Mathematik

Segmentation of Boundaries into Convex and Concave Parts

Ulrich Eckhardt

Reihe A
Preprint 190
October 2005

Hamburger Beiträge zur Angewandten Mathematik

Reihe A	Preprints
Reihe B	Berichte
Reihe C	Mathematische Modelle und Simulation
Reihe D	Elektrische Netzwerke und Bauelemente
Reihe E	Scientific Computing
Reihe F	Computational Fluid Dynamics and Data Analysis

SEGMENTATION OF BOUNDARIES INTO CONVEX AND CONCAVE PARTS

Ulrich Eckhardt
University of Hamburg, Faculty of Mathematics
Bundesstrasse 55, D–20 146 Hamburg
Eckhardt@math.uni-hamburg.de

Abstract Representing a set by its boundary means a considerable losslesss data reduction by decrease of dimensionality. The price for this reduction is that usually boundaries are topologically much more complex than the sets described by them.

The aim of this paper is to present an approach for handling the boundary of a set. This approach does not make use of differential geometry. It is shown that it is indeed possible to derive important structural properties of a set by inspecting only its boundary.

Keywords: Plane sets, boundaries, convex and concave parts

Introduction

A set in the plane (or in higher dimensional space) can be described efficiently by means of its boundary (or surface) whenever the boundary has the Jordan property which means that it separates the plane into two connected components, the interior and the exterior. In this paper the following question is investigated: What can be said about a set if its boundary is probed in a finite number of points?

When investigating the boundary of a set, it is appropriate to impose 'tameness' assumptions on it. In the context of image processing, polygonal or even discrete sets are considered, so the usual tools of differential geometry are not adequate. Consequently, a 'differentialless' geometry in the sense defined by Latecki und Rosenfeld [16] should be adapted. A very efficient description of a set can be found by attributing its boundary with predicates as convex or concave parts [15]. This representation is closely related to the 'Curvature Scale Space' (see e.g. [19]). In the latter approach the curvature zero crossing points were used as signatures in scale space. These curvature crossing points correspond to points where convex and concave parts of the boundary curve overlap. This means that the description by means of convex and concave parts is more general than the curvature zero crossings since the former does not make use of any concept from differential geometry. Moreover, by labeling parts of the boundary between curvature crossing points as convex or concave parts, ambiguities can be avoided. This is of major importance for shape coding algorithms [11].

The mapping associating to each linear functional the set of its local maximizers on a nonempty compact convex set is upper semi–continuous (this concept will be defined later, see Definition 5.1). Moreover, an upper semi–continuous inverse for this map can be found. The study of this map yields insights into the structure of the boundary of a given set. The mapping can be deformed to yield a homeomorphism from the unit sphere onto the boundary of the set. Moreover, by a finite number of boundary points and tangent directions one can find a directionally convex set containing the given set.

In the general case, things become very complicated. It becomes necessary to rule out 'wild' boundary parts. A minimal requirement is that the boundary of the set under consideration should have the Jordan property. However, some more structure has to be imposed in order to get practical results. A certain regularity condition is stated which is theoretically tractable and practically acceptable.

In 1987 Scherl ([22], see also [7]) constructed a document processing system. In this system, a set (e.g. a letter, a word, a text line ...) is represented by a rather small subset of boundary points together with tangent information. By means of this representation the amount of data can be reduced very efficiently while retaining sufficient information to perform typical pattern recognition tasks like segmentation of letters, words, text lines, or a classification of different document components (text, structuring elements, pictures), or classification of specific letter styles in the text (serifs, slanted letters).

The aim of this paper is to give a theoretical framework for the concepts mentioned above which is able to cover also the discrete case. First, some known theoretical results about properties of boundaries of sets are given. Under a certain regularity condition the boundary consists of finitely many convex and concave parts which can be used for describing the boundary. In the second part it is shown that the boundary of a convex set in $\mathbb{R}^d$ can be mapped "almost homeomorphic" onto the sphere $\mathfrak{S}_{d-1}$. The generalization to the nonconvex case is indicated. It was not intended here to provide algorithmic details, this is partially done in Helene Dörksen's Thesis [4].

1. Sets and Surfaces in $\mathbb{R}^d$

We consider sets in $\mathbb{R}^d$. Denote by $\langle \cdot, \cdot \rangle$ the ordinary scalar product and by $\| \cdot \|$ the Euclidean norm in $\mathbb{R}^d$. The natural topology of $\mathbb{R}^d$ is generated by declaring the sets (open balls)

$$B_\varepsilon(x) = \left\{ y \in \mathbb{R}^d \mid \|y - x\| < \varepsilon \right\}$$

(for $\varepsilon > 0$) to be open sets. If the specific value of $\varepsilon > 0$ does not matter we write $B(x)$ instead of $B_\varepsilon(x)$.

Let $S \subseteq \mathbb{R}^d$ be a bounded set. Denote by cl S its topological closure, by int S its interior and by $\Gamma = \text{bd } S$ the boundary of S.

In $\mathbb{R}^d$ the sphere $\mathfrak{S}_{d-1}$ is defined by

$$\mathfrak{S}_{d-1} = \left\{ x^* \in \mathbb{R}^d \mid \|x^*\| = 1 \right\}.$$

The elements x^* in $\mathfrak{S}_{d-1}$ are also termed *directions*.

Definition 1.1 *A (closed)* surface *in* $\mathbb{R}^d$ *is a set* Γ *which is homeomorphic to the sphere* $\mathfrak{S}_{d-1}$. *A surface in* $\mathbb{R}^2$ *is also termed a (closed)* curve.

One important tool of our investigations will be convexity theory (see the books of Eggleston [8], Valentine [29] or Rockafellar [21]).

Definition 1.2 *A set* $S \subseteq \mathbb{R}^d$ *is said to be* convex *if* $x, y \in S$ *and* $0 \leq \lambda \leq 1$ *together imply* $(1 - \lambda)x + \lambda y \in S$.

Definition 1.3 *Given any set* $S \subseteq \mathbb{R}^d$. *The* convex hull *of* S *is the smallest convex set which contains* S. *It is denoted by* conv S.

Since the intersection of any system of convex sets is always convex, the concept of the convex hull is well defined.

Definition 1.4 *Let* $\mathfrak{D} \subseteq \mathfrak{S}_{d-1}$ *be a set of directions.*

The set S *is* $\mathfrak{D}$*–convex if for all* $x^* \in \mathfrak{D}$ *and for all* $x \in \mathbb{R}^d$ *the set* $S \cap \{x + tx^* \mid t \in \mathbb{R}\}$ *is convex (i.e. an interval).*

By this Definition, a $\mathfrak{D}$–convex set needs not be connected, in contrast to convex sets. The intersection of any system of $\mathfrak{D}$–convex sets is always $\mathfrak{D}$–convex. However, the intersection of connected $\mathfrak{D}$–convex sets is not necessarily connected (see Example 1.1). By taking the intersection of all $\mathfrak{D}$–convex sets containing a given set S, we obtain the $\mathfrak{D}$*–convex hull* of S which is the smallest $\mathfrak{D}$–convex set containig S. For properties of $\mathfrak{D}$–convex sets see the book of Fink and Wood [9].

Example 1.1 *In Figure 1 an example of two sets is given which are convex with respect to two directions in the plane and connected whose intersection, however, is* $\mathfrak{D}$*–convex but not connected.*

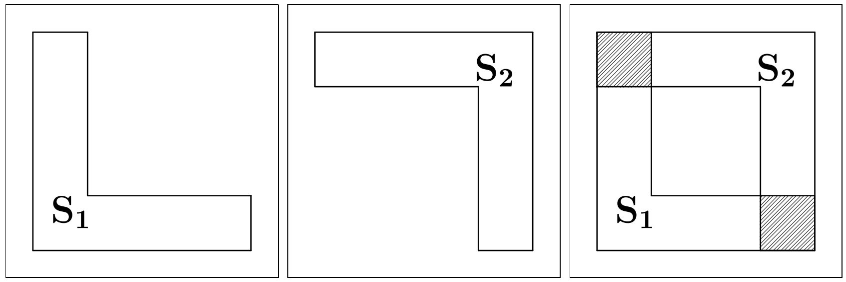

Figure 1. Intersection of two $\mathfrak{D}$–convex sets.

The sets S_1 and S_2 are both connected and $\mathfrak{D}$–convex with respect to $\mathfrak{D} = \{(0,1),(1,0)\}$. Their intersection (shaded area) is $\mathfrak{D}$–convex but not connected.

2. Convex and Concave Points

Given a set $S \subseteq \mathbb{R}^d$ with boundary Γ. Generally we assume that S is a compact set.

DEFINITION 2.1 *$x \in \Gamma$ is a* convex point *of S if there is a neighborhood $B(x)$ such that $B(x) \cap S$ is convex. Denote by $\mathcal{T}_0 \subseteq \Gamma$ the set of all convex points of S.*

$x \in \Gamma$ is a concave point *of S if there is a neighborhood $B(x)$ such that $B(x) \cap \complement S$ is convex ($\complement S = \mathbb{R}^d \setminus S$ is the complement of S). Denote by $\mathcal{S}_0 \subseteq \Gamma$ the set of all concave points of S.*

In Figure 3 below one can find examples for convex (pictures T, L) and concave points (S, L) as well as a point which is neither convex nor concave (picture I). Generally it is possible that a dense subset of the boundary consists of points of the latter type (see the discussion at the begin of Section 4 below). We have to impose regularity conditions in order to rule out such situations.

From the Separation Theorem for Convex Sets [29, Part II] we conclude:

If x is a convex point of S then

C1 there exists a neighborhood $B(x)$ and a direction $x^* \in \mathfrak{S}_{d-1}$ such that $z \in B(x)$ and $\langle z, x^* \rangle > \langle x, x^* \rangle$ imply $z \notin S$.

If x is a concave point of S then

C2 there exists a neighborhood $B(x)$ and a direction $x^* \in \mathfrak{S}_{d-1}$ such that $z \in B(x)$ and $\langle z, x^* \rangle < \langle x, x^* \rangle$ imply $z \in S$.

This observation leads to the Definition

DEFINITION 2.2 *$x \in \Gamma$ is a* T–point *('Top point') of S if there is a neighborhood $B(x)$ and a direction $x^* \in \mathfrak{S}_{d-1}$ such that $z \in B(x)$ and $\langle z, x^* \rangle > \langle x, x^* \rangle$ imply $z \notin S$. Denote by $\mathcal{T} \subseteq \Gamma$ the set of all* T*–points of S.*

$x \in \Gamma$ is an S–point *('Saddle point') of S if there is a neighborhood $B(x)$ and a direction $x^* \in \mathfrak{S}_{d-1}$ such that $z \in B(x)$ and $\langle z, x^* \rangle < \langle x, x^* \rangle$ imply $z \in S$. Denote by $\mathcal{S} \subseteq \Gamma$ the set of all* S*–points of S.*

REMARK 2.1 *The notations '*T*–point' and '*S*–point' are due to Scherl [22] who introduced these concepts in the context of document analysis.*

For deciding whether a given boundary point is a T*– or* S*–point only information from the boundary is needed together with an indication on which side of the boundary the set is situated. In contrast, for deciding whether a boundary point is convex or not, information has to be gathered from its neighborhood which contains points not on the boundary. Therefore, the concept of* T*– or* S*–points is more well–suited for practical applications than the concept of convex and concave points.*

REMARK 2.2 *There exist examples of nonconvex sets whose boundaries consist only of* T*–points. Tietze [25] gave a condition guaranteeing that this situation cannot happen (see condition* **A** *in Section 3 below and [29, Theorem 4.4]).*

We can associate to each T– or S–point of Γ a direction $x^* \in \mathfrak{S}_{d-1}$ such that **C1** or **C2**, respectively, holds for this point. This lead Scherl [22] to the following Definition:

DEFINITION 2.3 *The pair (x, x^*) is a* T *or a x^*–*T *(*S *or x^*–*S*)* descriptor *of S if $x \in \Gamma$ is a* T*– (*S*–) point and if* **C1** *(***C2***) with direction x^* holds in x.*

There are points on the boundary which are T–points as well as S–points. For convenience we give them an extra name:

DEFINITION 2.4 *$x \in \Gamma$ is an* L–point *('Line point') of S if x is as well a* T*– as an* S*–point. Denote by $\mathcal{L} \subseteq \Gamma$ the set of all* L*–points of S.*

We define further:

DEFINITION 2.5 *A point $x \in \Gamma$ is an* extreme T–point *if there exists a direction $x^* \in \mathfrak{S}_{d-1}$ such that $\langle z, x^* \rangle > \langle x, x^* \rangle$ implies $z \notin S$. Denote by $\mathcal{ET} \subseteq \Gamma$ the set of all extreme* T*–points of S.*

A point $x \in \Gamma$ is an extreme S–point *if there exists a direction $x^* \in \mathfrak{S}_{d-1}$ such that $\langle z, x^* \rangle > \langle x, x^* \rangle$ implies $z \notin S$. Denote by $\mathcal{ES} \subseteq \Gamma$ the set of all extreme S–points of S.*

We state some rather simple topological properties which follow directly from the definitions.

LEMMA 2.1 *The sets $\mathcal{T}_0$ and $\mathcal{S}_0$ are open subsets of Γ.*
$\mathcal{L} = \mathcal{T}_0 \cap \mathcal{S}_0 = \mathcal{T} \cap \mathcal{S}$ is open.
The sets $\mathcal{ET}$ and $\mathcal{ES}$ are closed subsets of Γ.

DEFINITION 2.6 *Given a set $S \subseteq \mathbb{R}^d$ with boundary Γ. The subset Γ_0 of Γ is termed a* convex patch of the boundary *if* conv $\Gamma_0 \subseteq S \cup \Gamma$.
Γ_0 *is termed a* concave patch of the boundary *if* conv $\Gamma_0 \subseteq \mathbb{C}S \cup \Gamma$, *where $\mathbb{C}S = \mathbb{R}^d \setminus S$ is the complement of S.*

Clearly, Γ_0 is a concave patch of the boundary of S if and only if it is a convex patch of the (dlosure of) boundary of $\mathbb{C}S$.
In the following we will investigate questions like these:

- Are boundary patches consisting entirely of convex (concave) points convex (concave) patches?
- If $\Gamma = \text{bd } S$ is the union of finitely many convex patches, does this imply that S is convex?
- What can be said about points of Γ which are neither convex nor concave points?
- How many convex or concave points do exist on the boundary of a — say closed, connected, bounded — set? More mathematically: is the set $\mathcal{T} \cup \mathcal{S}$ dense on Γ?

The first two questions were answered by a couple of Theorems due to Heinrich Tietze (see [25–28] and [29, Part IV]). The last two questions are not easy to answer. We need very deep results from topology or else very strong assumptions (e.g. the requirement that all boundary points are 'tame' [16]).

3. S–Points and $\mathfrak{D}$–Convexity

We now are going to answer the first two questions above concerning convex and concave points or patches, respectively. It should be remarked here, however, that all results proved in this section hold only in the plane $\mathbb{R}^2$.
We introduce the following assumption:

A The interior int S of the set S is connected and S is *regular closed*, i.e. $S = \text{cl int } S$.

LEMMA 3.1 *Let $S \subseteq \mathbb{R}^2$ be a set fulfilling condition* **A**.
Assume that there exists a direction $x^* \in \mathfrak{S}_1$ and a number α such that the set $\{x \in S \mid \ \langle x, x^* \rangle = \alpha\}$ is not connected.
Then there exists an S–point $x \in \Gamma$. More precisely, either the pair (x, x^*) or the pair $(x, -x^*)$ is an S–descriptor pair of S.

The assertion of the Lemma is essentially the assertion of the Theorem of Léja and Wilkosz [18] (see [29, Theorem 4.8]). The assumption involving direction $x^* \in \mathfrak{S}_1$ is equivalent to the assumption that S is not convex.

Lemma 3.1 may be sharpened as follows:

COROLLARY 3.1 *Let $\mathfrak{D} \subseteq \mathfrak{S}_1$.*
If a set $S \subseteq \mathbb{R}^2$ fulfilling condition **A** *does not contain any $\pm x^*$–S–descriptor points with $x^* \in \mathfrak{D}$ then it is $\mathfrak{D}$–convex.*

The contrary of the Corollary is not necessarily true as it is illustrated in Figure 2. In order to prove the converse of the Corollary we need a nondegeneracy assumption.

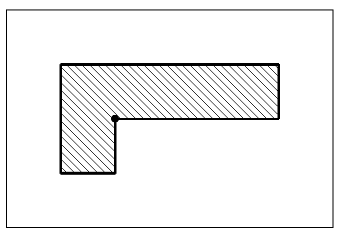

Figure 2. Example of a $\mathfrak{D}$–convex set.

The set S in the picture is $\mathfrak{D}$–convex with respect to the set $\mathfrak{D}$ containing the horizontal and the vertical directions. However, there is a vertex point (marked $\bullet$) which is an S–point of S.

DEFINITION 3.1 *Let (x, x^*) be an S–descriptor of a set $S \subseteq \mathbb{R}^d$. x is termed a* strict S–point *with respect to $x^* \in \mathfrak{S}_{d-1}$ if for any line ℓ through x perpendicular to x^*, the component of $\ell \cap S$ containing x is closed.*

REMARK 3.1 *The requirement that a point is a strict S–point is not a 'local' one since one has to check in each case when an S–point is encountered which is adjacent to a component of $\mathcal{L}$, whether the point on* *the other end of the component is also an* S*–point. This is the reason why the concept of* $\mathfrak{D}$*–convexity, which is a very natural concept in the framework of digital geometry (see [9]), is 'harder' to handle than ordinary convexity (see [2, 3, 5, 6]).*

Lemma 3.2 *Given a set* $S \subseteq \mathbb{R}^2$ *fulfilling condition* $\mathbf{A}$*. Let* $\mathfrak{D} \subseteq \mathfrak{S}_1$*.*
If Γ *contains a strict* S*–descriptor* (x, x^*) *with* $x^* \in \mathfrak{D}$ *then* S *is not* $\mathfrak{D}$*–convex.*

The proof of this Lemma is an extension of the proof of Léja and Wilkosz' Theorem [29, Theorem 4.8].
We now are able to state a Characterization Theorem for $\mathfrak{D}$–convexity.

Theorem 3.1 *Given a set* $S \subseteq \mathbb{R}^2$ *with property* $\mathbf{A}$*. Let* $\mathfrak{D} \subseteq \mathfrak{S}_1$*.*
S *is* $\mathfrak{D}$*–convex if and only if its boundary* Γ *contains no strict* S*–descriptors* (x, x^*) *with* $x^* \in \mathfrak{D}$*.*

The proof of this Theorem follows immediately from Lemma 3.2 and Lemma 3.1.
For a subset $\mathfrak{D}$ of directions we define $\mathfrak{D}$*–convex* and $\mathfrak{D}$*–concave patches* of the boundary as in Definition 2.6 by replacing the convex hull by the $\mathfrak{D}$–convex hull.
We define a *strict* T*–descriptor* of the set S to be a strict S–descriptor of the (closure of the) +complement of S.
With these definitions we state:

Theorem 3.2 *Let* $S \subseteq \mathbb{R}^2$ *be a set with boundary* Γ *fulfilling condition* $\mathbf{A}$*.*

1 *A connected subset* Γ_0 *of* Γ *is a convex (concave) patch of* Γ *if and only if it consists entirely of* T*– (*S*–) points.*

2 *A connected subset* Γ_0 *of* Γ *is a* $\mathfrak{D}$*–convex (*$\mathfrak{D}$*–concave) patch of* Γ *if and only if does not contain any strict* S*–descriptors (strict* T*–descriptors)* (x, x^*) *with* $x \in \Gamma_0$ *and* $x^* \in \mathfrak{D}$*.*

4. Regular Boundaries

In order to answer the questions concerning $\mathcal{T}$ and $\mathcal{S}$ from the end of Section 2 we need concepts from topology. The last one of these questions is indeed very deep and it cannot be treated here. There are indeed sets having nontrivial parts of the boundary consisting entirely of points which are neither T– nor S–points. An example of such a set (the 'Warsaw circle') is given in [10]. In this example, however, the boundary is not a Jordan curve. Based on a characterization of Jordan curves in the plane by Schönflies ([23, 24], see [20, §40]), Kaufmann [12] was able to prove that for a Jordan boundary the set $\mathcal{T} \cup \mathcal{S}$ is dense on the boundary. The famous von Koch curve [14] is an example of a Jordan curve with the property that all three sets $\mathcal{T}$, $\mathcal{S}$ and also the complement of these both sets are dense on the curve. We need a strong regularity condition to rule out such situations. Generally we assume that the boundary Γ of a set $S \subseteq \mathbb{R}^d$ is a surface, i.e. that it is a homeomorphic image of $\mathfrak{S}_{d-1}$.

DEFINITION 4.1 *$x \in \Gamma$ is a* regular point *of S if there is a neighborhood $B(x)$ such that both $B(x) \cap \mathcal{T}$ and $B(x) \cap \mathcal{S}$ consist of at most finitely many connected components.*

The boundary of a set is called a regular boundary *if it consists only of regular boundary points.*

REMARK 4.1 *In $\mathbb{R}^2$ a point x is regular if and only if there is a neighborhood $B(x)$ such that either*

1 $B(x) \cap (\mathcal{T} \cup \mathcal{S})$ consists of exactly one connected component, or else

2 $(B(x) \cap \Gamma) \setminus \{x\}$ consists of two connected components. Each such component is completely contained in $\mathcal{T}$ or $\mathcal{S}$.

There remains one class of boundary points on regular boundaries:

DEFINITION 4.2 *$x \in \Gamma$ is an* I–point *('Indifferent point') of S if x is regular and neither a T–point nor an S–point. Denote by $\mathcal{I} \subseteq \Gamma$ the set of all I–points of S.*

By definition, I–points are always isolated points on the (regular) boundary. They separate components of $\mathcal{T}$ and $\mathcal{S}$. In Figure 3 examples for all types of regular points in the plane $\mathbb{R}^2$ are shown.

5. Upper Semi–Continuous Mappings

For analyzing the boundary of a set wee need some results from the theory of set–valued mappings (see [1]).

DEFINITION 5.1 *Let E and F be two topological spaces and denote by $\mathcal{P}(F)$ the collection of all non–empty subsets of F. A point–to–set mapping $f : E \longrightarrow \mathcal{P}(F)$ is said to be* upper semi–continuous *if, for any point $x_0 \in E$ and any open set $U \subseteq F$ with $f(x_0) \subseteq U$, there exists a neighborhood V of x_0 such that $f(x) \subseteq U$ for all $x \in V$.*

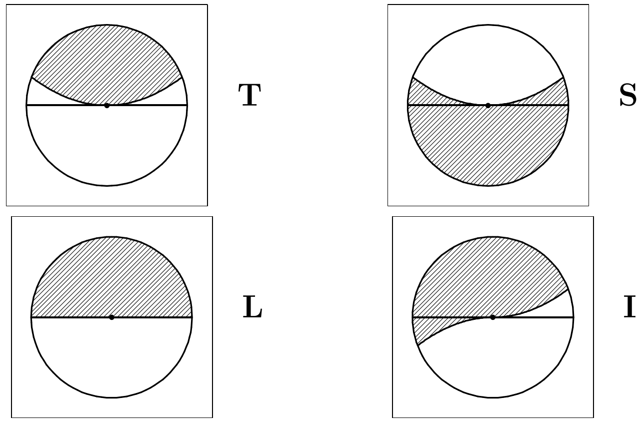

Figure 3. Types of regular points in the plane.

The set S is shaded, the boundary point x under consideration is marked $\bullet$. The line through the point indicates the line perpendicular to x^*.

Consequences

- Every continuous function is upper semi–continuous.
- Every upper semi–continuous point–to point mapping is continuous
- The composition of upper semi–continuous mappings is upper semi–continuous.

DEFINITION 5.2 *For a set–valued mapping $f : E \longrightarrow \mathcal{P}(F)$ we define a set–valued inverse mapping. For $y \in F$ let*

$$f^{-1}(y) = \{x \in E \mid y \in f(x)\}.$$

The assertions of the following two Theorems follow directly from the definition of upper semi–continuity:

THEOREM 5.1 *Assume that the set–valued mapping $f : E \longrightarrow \mathcal{P}(F)$ is upper semi–continuous and in addition assume that $f(x)$ is connected for all $x \in E$.*

Then for each connected subset $S \subseteq E$ the image $f(S)$ is also connected.

THEOREM 5.2 *The set–valued mapping $f : E \longrightarrow \mathcal{P}(F)$ is upper semi–continuous if and only if for each closed set $S \subseteq F$ the set $f^{-1}(S)$ is closed.*

Remark 5.1 *As a consequence of the Theorem, if f maps E into the system of all closed subsets of F then f^{-1} maps F into the system of all closed subsets of E.*

6. Tangent Mappings

Definition 6.1 *Let S be a nonempty compact subset of $\mathbb{R}^d$. The function $\mu : \mathfrak{S}_{d-1} \longrightarrow \mathbb{R}$ which is defined by*

$$\mu(x^*) = \max_{x \in S} \langle x, x^* \rangle$$

is called the support functional *of S (see [13, Section 2.3.]).*

The set–valued mapping $\Psi : \mathfrak{S}_{d-1} \longrightarrow \mathcal{P}(S)$ with

$$\Psi(x^*) = \arg \max_{x \in S} \langle x, x^* \rangle$$

is called the tangent mapping *of S.*

It is well–known (see e.g. [1]) that μ is continuous and that Ψ is upper semi–continuous whenever S is a nonempty compact subset of $\mathbb{R}^d$.

Remark 6.1 *For nonempty compact $S \subseteq \mathbb{R}^d$ all sets $\Psi(x^*)$ are closed since*

$$\Psi(x^*) = S \cap \{x \in \mathbb{R}^d \mid \langle x, x^* \rangle = \mu(x^*)\}.$$

Theorem 6.1 *Let S be a nonempty closed convex subset of $\mathbb{R}^d$.*

The set–valued mapping $\Psi^{-1} : S \longrightarrow \mathcal{P}(\mathfrak{S}_{d-1})$ with

$$\Psi^{-1}(x) = \{x^* \in \mathfrak{S}_{d-1} \mid x \in \Psi(x^*)\}$$

is upper semi–continuous.

Lemma 6.1 *Assume that the conditions of Theorem 6.1 hold. Let $y \in$ bd S and x_1^*, x_2^* be two directions in $\Psi^{-1}(y)$.*

Then all directions from the set

$$\Big\{x \in \mathbb{R}^d \mid x = \lambda_1 x_1^* + \lambda_2 x_2^*, \lambda_1, \lambda_2 \geq 0\Big\} \;\cap\; \mathfrak{S}_{d-1}$$

belong to $\Psi^{-1}(y)$.

Remark 6.2 *Lemma 6.1 states that all directions in $\Psi^{-1}(y)$ can be obtained as the intersection of a convex cone with vertex Θ_d (= origin of $\mathbb{R}^d$) and $\mathfrak{S}_{d-1}$. This implies that $\Psi^{-1}(y)$ is always a connected subset of $\mathfrak{S}_{d-1}$.*

7. Structure of Regular Boundaries — The Convex Case

Ψ^{-1}, the inverse of the tangent mapping Ψ, is only a well–defined mapping on Γ if S is a convex set. Whenever S is convex, Ψ^{-1} is even an upper semi–continuous mapping. Hence, the pair (Ψ, Ψ^{-1}) is in the convex case 'nearly' a homeomorphism. We sketch here, how this can be shown rigourosly. First we need a concept from convexity theory:

Definition 7.1 *Let S be a closed convex set and $x \in bd\ S$. The direction $x^* \in \mathfrak{S}_{d-1}$ is termed the* normal vector *of a* supporting hyperplane *at S in x whenever $\langle z, x^* \rangle > \langle x, x^* \rangle$ implies $z \notin S$ (see* **C1***).*

The Separation Theorem for Convex Sets [29, Part II] guarantees that a convex set has at least one supporting hyperplane in each of its boundary points.

Definition 7.2 *Let S be a nonempty bounded closed convex set in $\mathbb{R}^d$.*

S is smooth *if for each boundary point of S the supporting hyperplane is uniquely determined.*

S is strictly convex *if all supporting hyperplanes meet the boundary of S in exactly one point.*

If a set S is strictly convex then the mapping Ψ is a point–to–point map from $\mathfrak{S}_{d-1}$ to the boundary Γ of S. If S is smooth then the inverse map Ψ^{-1} is a point–to–point map. Hence, if S is smooth and strictly convex, then both Ψ and Ψ^{-1} are continuous and inverse to each other, consequently, $\Psi : \mathfrak{S}_{d-1} \longrightarrow \Gamma$ is a homeomorphism.

Theorem 7.1 *Given a closed bounded convex set S with nonempty interior and a real number $\varepsilon > 0$.*

Then there exists a closed bounded convex set S_ε which is smooth and strictly convex such that $S \subseteq S_\varepsilon$ and $d_H(S, S_\varepsilon) < \varepsilon$.

Here, d_H denotes the Hausdorff distance *for sets, this means in this context ($S \subseteq S_\varepsilon$) that for each $x_\varepsilon \in S_\varepsilon$ there is an $x \in S$ such that $\|x - x_\varepsilon\| < \varepsilon$.*

For a proof of this Theorem see [8, Theorem 34].

For any closed bounded convex set S with boundary Γ which has nonempty interior we can construct an ε–homeomorphism in the following way:

- Construct S_ε as in Theorem 7.1.
- The mapping $\Psi_\varepsilon : \mathfrak{S}_{d-1} \longrightarrow$ bd S_ε is a homeomorphism.

- Select any point $x_0 \in \text{int } S$. The central projection $\Pi_{x_0} : \text{bd } S_\varepsilon \longrightarrow \Gamma$ with center x_0 is

$$\Pi_{x_0}(x) = \{x_0 + \lambda(x - x_0) \mid \lambda > 0 \text{ maximal such that } x_0 + \lambda(x - x_0) \in S\}.$$

  $\Pi_{x_0}(x)$ exists for any $x \neq x_0$ by compactness of S and is a homeomorphism.

- Let $\Psi : \mathfrak{S}_{d-1} \longrightarrow \mathcal{P}(\Gamma)$ be the — generally set–valued — map as defined above. The composite mapping $\Pi_{x_0} \circ \Psi_\varepsilon : \mathfrak{S}_{d-1} \longrightarrow \Gamma$ is a homeomorphism with the property that for given $\varepsilon > 0$ there exists a number C such that for each $x^* \in \mathfrak{S}_{d-1}$ and each $x \in \Psi(x^*)$ there exists an $x_\varepsilon = \Pi_{x_0}\Psi_\varepsilon(x^*)$ such that $\|x - x_\varepsilon\| < C \cdot \varepsilon$ and for each each $x_\varepsilon = \Pi_{x_0}\Psi_\varepsilon(x^*)$ there exists an $x \in \Psi(x^*)$ such that $\|x - x_\varepsilon\| < C \cdot \varepsilon$.

8. General Plane Sets with Regular Boundaries

If a set $S \subseteq \mathbb{R}^2$ has a regular boundary then its boundary consists of a finite number of components of $\mathcal{T}$ (convex parts of the boundary) and $\mathcal{S}$ (concave parts).

Generally, for the convex hull conv S of a (compact) set, we can apply the results derived above. Specifically, for each $\varepsilon > 0$ there exists an ε–homeomeorphism bd conv $S \longrightarrow \mathfrak{S}_1$ as shown above.

The *convex defect* $S \setminus$ conv S consists by regularity of the boundary of a finite number of components. Each of these components consists of finitely many convex or concave parts. We can construct for each of these parts an ε–homeomeorphism on a new copy of $\mathfrak{S}_1$ so that we finally get an ε–homeomeorphism mapping the boundary of S onto a finite number of $\mathfrak{S}_1$'s. Instead of elaborating this procedure formally, we illustrate it by means of a simple example (see Figure 8).

Along convex or concave parts of a set in $\mathbb{R}^2$ the succession of descriptors is not arbitrary. Since $\mathfrak{S}_1$ is oriented and since there is an ε–homeomeorphism mapping each convex or concave part of the boundary onto $\mathfrak{S}_1$, also the boundary is oriented. Note, however, that the succession of descriptors is reverted by transition from a convex to a concave part and vice versa. Scherl illustrated this effect by means of so–called legal descriptor cycles [22, Figure 5.2.4.], see Figure 4.

9. Applications

It has been shown by Scherl [22] by means of numerous examples and also by a prototype implementation that it is possible to describe

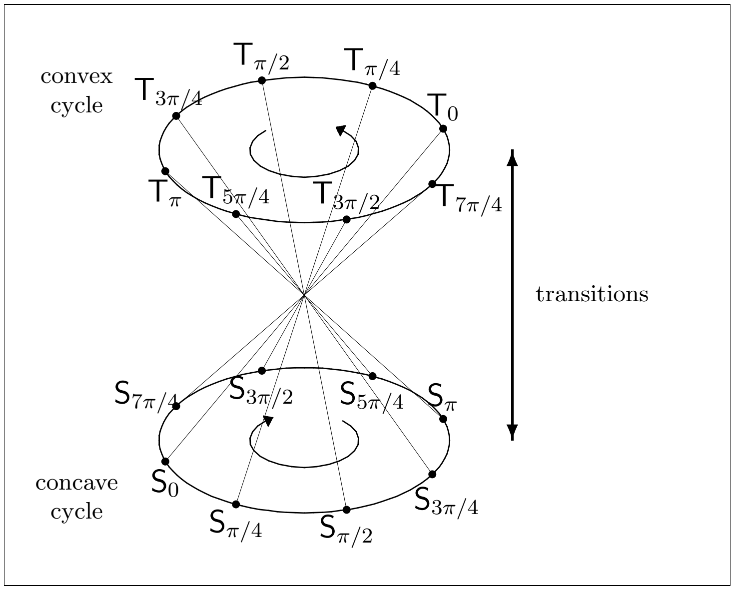

Figure 4. Scheme of descriptor cycles and legal transitions.

Descriptor directions corresponding to multiples of $\frac{\pi}{4}$ are indicated. These directions are the main directions in the digital plane $\mathbb{Z}^2$.

shape by a relatively small number of descriptor points. As in Scherl's experiments we consider the case of plane sets and the set of descriptors belonging to directions $k \cdot \frac{\pi}{4}$ which are well adapted to the digital plane $\mathbb{Z}^2$ which is the set of all points of the plane having integer coordinates.

The extraction of the descriptors can be done very efficiently as a by–product of boundary extraction at virtually no additional cost. The oriented set of all descriptors (points with a label indicating the corresponding tangent direction and also pointers indicating the succession relation of the descriptor points on the oriented boundary) yields a data reduction while retaining the rough shape of the set under consideration.

The boundary and the descriptors may be viewed as a pyramid structure:

- The bottom of the pyramid is the ordered sequence of boundary points of the set which is coded in some appropriate manner (e.g. by means of a chain code).
- Digital sets are subsets of $\mathbb{Z}^d$. Under certain known conditions (which means that some discrete Jordan Theorem holds, i.e. a set

is uniquely determined by its discrete boundary) a digital set can be represented by means of its boundary. Specifically for plane digital sets it is possible to select a subset of the boundary points which can be joined according to the orientation of the boundary as to to yield a *faithful* representation of the set, i.e. the reduced boundary is a polygonal Jordan curve which contains exactly all points of the given set in its interior [5].

- A subset of the boundary of a set – or else of any faithful representation of the boundary of a digital set – is given by the oriented set of all descriptors. The tangents corresponding to the T–descriptors belonging to a set $\mathfrak{D}$ of directions yield the $\mathfrak{D}$–convex hull of the set. By joining any two descriptor points which are immediate successors on the oriented boundary, a closed polygonal curve is obtained which, however, in general needs not to be a simple curve. Nevertheless, these curves can be efficiently used for a rough representation of shape.

- The set of all extreme T–descriptors belonging to a set $\mathfrak{D}$ of directions (here all directions $k \cdot \frac{\pi}{4}$) provide the smallest convex polytope whose sides have directions from $\mathfrak{D}$ (a so–called $\mathfrak{D}$*–polytope*) which contains the set under investigation.

The data structure provided by this pyramid can be used for different pattern recognition tasks. For example, the linear time convexity detection algorithm of I. Debled–Renesson et al. [2] starts at the top of the pyramid with direction set $\mathfrak{D} = \{k \cdot \frac{\pi}{2}\}$. First, the authors verify $\mathfrak{D}$–convexity of the given set. Then the boundary of this set is segmented by means of all descriptor points and descriptor tangents having directions from $\mathfrak{D}$. This results in boundary parts having a very favourable structure considering convexity detection.

An interesting subject is the investigation of this 'Scherl–pyramid' under discrete boundary evolution [15]. Specifically, the information obtained from a faithful representation of a digital set can be used to control the evolution process. This, however, is far beyond the topic of this paper (see [5]).

If only information from a finite number of 'probes' of boundary points is available (together with tangent directions) then the observation that the sequence of descriptors along the boundary is oriented can be used to find inclusions for the missing parts of the boundary if it is assumed that the directions between two successive probes lie within a certain interval. Such 'interpolation' assertions can be easily derived. We give one simple example:

Theorem 9.1 *Given in the plane a set S with oriented boundary Γ. Let (x_1, x_1^*) and (x_2, x_2^*) be two T–descriptors of S. Assume that on the boundary part Γ_{12} which is between (in the sense of the orientation of Γ) x_1 and x_2 there are only descriptors (x, x^*) which are between (in the sense of orientation of $\mathfrak{S}_1$) x_1^* and x_2^*. Then Γ_{12} is completely contained in the parallelogram of all $x \in \mathbb{R}^2$ satisfying the inequalities*

$$\langle x_2, x_1^* \rangle \leq \langle x, x_1^* \rangle \leq \langle x_1, x_1^* \rangle \,,$$
$$\langle x_1, x_2^* \rangle \leq \langle x, x_2^* \rangle \leq \langle x_2, x_2^* \rangle \,.$$

The proof of this assertion follows from investigating the convex hull of Γ_{12}. In Figure 5 an illustrative example is shown. It is possible to derive inclusions for other situations where also S–descriptors are taken into account.

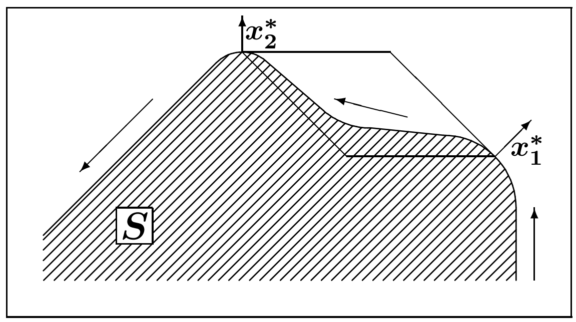

Figure 5. Example for Theorem 9.1.

Long arrows indicate the orientation of the boundary.

We conclude this discussion with a simple example. In Figure 6 a digital set and its boundary is given. The process of finding a stack of $\mathfrak{S}_1$'s such that the convex and concave parts of the (outer) boundary can be homeomorphically mapped on this stack is illustrated in Figure 8. In Figure 7 the smallest convex $\mathfrak{D}$–polygon containing the set as obtained from the extreme T–descriptors is shown as well as the polygonal approximation which is found by joining the descriptor points by line segments.

10. Conclusions

Under suitable conditions it is possible to derive properties of boundaries of sets using only tools from convexity theory without making any differentiability assumptions. It was shown that the boundary of a set can be mapped 'almost' homeomorphically to a stack of spheres.

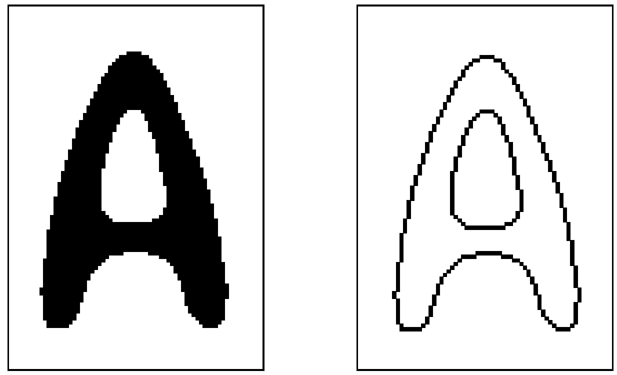

Figure 6. Digitization of letter 'A'.

In the left picture a digital set is given. The right picture shows the boundary of the set. The boundary can be understood to consist of two closed polygonal curves in $\mathbb{R}^2$.

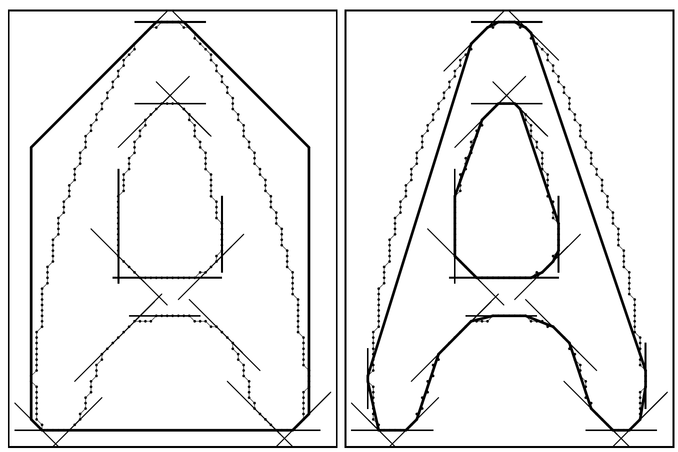

Figure 7. $\mathfrak{D}$–convex hull (left) and descriptor approximation (right).

The set $\mathfrak{D}$ consists of all directions $k \cdot \frac{\pi}{4}$.

By gathering informations from a finite number of points along the boundary one can extract properties which are relevant for the shape of a set. These properties can be arranged in a hierarchical manner as a pyramid structure. The informations obtained in this way can be used for defining convex and concave parts of the boundary and for investigating and controlling discrete evolution of boundary curves.

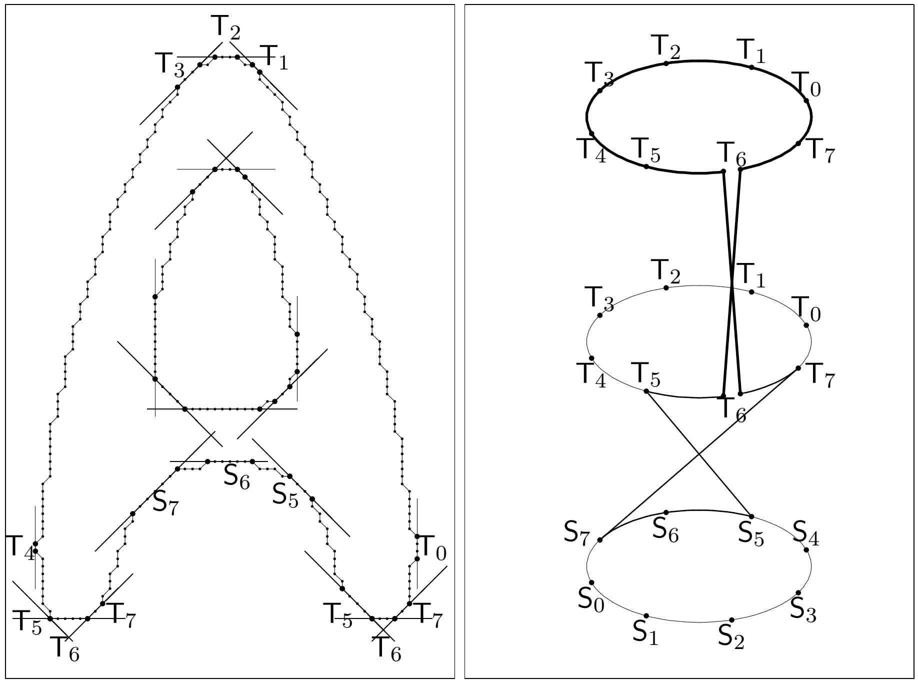

Figure 8. Mapping of the outer boundary of a set on a stack of descriptor cycles.

Descriptor tangents with angles $k \cdot \frac{\pi}{4}$ are indicated by T_k or S_k, respectively.
The descriptor tangent T_6 at the bottom of the set meets the set in two disjoint components. Therefore the two upper copies of $\mathfrak{S}_1$ are cut up and the points corresponding to T_6 on the upper and the middle copy are identified as indicated to make the mapping of the boundary to the stack of circles biunique.

There are two important topics which are not treated here. One of them is the extension to higher dimensions. It is possible to derive properties of higher dimensional sets by investigating two–dimensional sections of them. The second problem not treated here is much more difficult. Usually in applications sets are given in a discrete manner as ‘digital sets’. Therefore it is desirable to have a completely discrete theory. However, it turns out that the discrete case is much more complicated than the continuous one [2–6].

References

[1] Aubin, J.–P. and Frankowska, H. (1990). *Set–Valued Analysis*. Boston, Mass.: Birkhäuser-Verlag.

[2] Debled–Rennesson, I., Rémy, J.–L. and Rouyer–Degli, J. (2000). Detection of the discrete convexity of polyominoes. *Lecture Notes in Computer Science 1953*, 491–504, Berlin: Springer–Verlag.

[3] Debled–Rennesson, I. and Reveillès, J.–P. (1995) A linear algorithm for segmentation of digital curves. *International Journal of Pattern Recognition and Artificial Intelligence*, 9:635–662.

[4] Dörksen, H. (2004). Shape Representations of Digital Sets based on Convexity Properties. PhD Thesis, Universität Hamburg, Fachbereich Mathematik, Juni 2004.

[5] Eckhardt, U. and Reiter, H. (2003). Polygonal representations of digital sets. *Algorithmica*, 38(1):5–23.

[6] Eckhardt, U. (2001). Digital lines and digital convexity. *Lecture Notes in Computer Science 2243*, 207–226, Berlin: Springer–Verlag.

[7] Eckhardt, U., Scherl, W. and Yu, Z. (1987). Representation of plane curves by means of descriptors in Hough space. I. Continuous theory. Report of *Rechenzentrum der Universität Hamburg*.

[8] Eggleston, H.G. (1958). *Convexity*, Cambridge: University Press.

[9] Fink, E. and Wood, D. (2004). *Restricted–Orientation Convexity*, Berlin: Springer–Verlag.

[10] Giraldo, A., Gross, A. and Latecki, L.J. (1999) Digitizations preserving shape. *Pattern Recognition*, 32:365–376.

[11] Heuer, J., Kaup, A., Eckhardt, U., Latecki, L.J. and Lakämper, R. (1999). Region localization in MPEG–7 with good coding and retrieval performance. ISO/IEC JTC1/SC29/WG11/M5417, MPEG 99, Maui, USA.

[12] Kaufmann, B. (1931). Über die Konvexitäts– und Konkavitätsstellen auf Jordankurven. *Journal für die reine und angewandte Mathematik*, 164:112–127.

[13] Kindratenko, V.V. (2003) On using functions to describe the shape. *Journal of Mathematical Imaging and Vision*, 18:225–245.

[14] von Koch, H. (1906). Sur une courbe continue sans tangente obtenue par une construction géométrique élémentaire. *Acta Mathematica*, 30:145–174.

[15] Latecki, L.J. and Lakämper, R. (1999). Convexity rule for shape decomposition based on discrete contour evolution. *Computer Vision and Image Understanding*, 73(3):441–454.

[16] Latecki, L.J. and Rosenfeld, A. (1998). Supportedness and tameness. Differentialless geometry of plane curves. *Pattern Recognition*, 31(5):607–622.

[17] Latecki, L.J. (1998). *Discrete Representation of Spatial Objects in Computer Vision*, Dordrecht: Kluwer.

[18] Léja, F. et Wilkosz, W. (1924). Sur une propriété des domaines concaves. *Annales de la Société Polonaise de Mathématique*, 2:222-224.

[19] Mokhtarian, F. (1997). Silhouette–based occluded object recognition through curvature scale space. *Machine Vision and Applications*, 10(3):87–97.

[20] Rinow, W. (1975). *Lehrbuch der Topologie*, Berlin: Deutscher Verlag der Wissenschaften.

[21] Rockafellar, R.T. (1970). *Convex Analysis*, Princeton, NJ: Princeton University Press.

[22] Scherl, W. (1987). *Bildanalyse allgemeiner Dokumente*. Berlin: Springer–Verlag.

[23] Schönflies, A. (1900). Die Entwicklung der Lehre von den Punktmannigfaltigkeiten. *Jber. Deutsch. Math.-Verein.*, VIII(2):1–250.

[24] Schönflies, A. (1908). Die Entwicklung der Lehre von den Punktmannigfaltigkeiten. Bericht erstattet der Deutschen Mathematiker-Vereinigung. Teil II. *Jber. Deutsch. Math.-Verein.*, Ergänzungsband II.

[25] Tietze, H. (1929). Bemerkungen über konvexe und nichtkonvexe Figuren. *Journal für die reine und angewandte Mathematik*, 160:67–69.

[26] Tietze, H. (1928). Über Konvexheit im kleinen und im großen und über gewisse den Punkten einer Menge zugeordnete Dimensionszahlen. *Mathematische Zeitschrift*, 28:697–707.

[27] Tietze, H. (1927). Eine charakteristische Eigenschaft der abgeschlossenen konvexen Punktmengen. *Math. Annalen*, 99:394–398.

[28] Tietze, H. (1927). Über konvexe Figuren. *Journal für die reine und angewandte Mathematik*, 158:168–172.

[29] Valentine, F.A. (1964). *Convex Sets*, New York: McGraw–Hill Book Company.