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# A FINITE-VOLUME PARTICLE METHOD FOR CONSERVATION LAWS ON MOVING DOMAINS

DELIA TELEAGA\* AND JENS STRUCKMEIER†

**Abstract.** The paper deals with the Finite-Volume particle method (FVPM), a relatively new method for solving hyperbolic systems of conservation laws. A general formulation of the method for bounded and moving domains is presented. Furthermore, an approximation property of the reconstruction formula is proved. Then, based on a two-dimensional test problem posed on a moving domain, a special Ansatz for the movement of the particles is proposed. The numerical results obtained indicate that this method is well-suited for such problems and thus a first step to apply the FVPM to real industrial problems involving free boundaries or fluid-structure interaction is taken. Finally, we perform a numerical convergence study for a shock tube problem and a simple linear advection equation.

**Key words.** hyperbolic systems, mesh-free discretization, moving boundary

**AMS subject classifications.**

**1. Introduction.** The Finite-Volume Particle method (FVPM) is a relatively new meshless method for solving hyperbolic systems of conservation laws, firstly developed in 1998 by Hietel, Steiner and Struckmeier in [6]. There the FVPM is formulated for a system of conservation laws in the spatial domain  $\Omega = \mathbb{R}^d$  and shown to be conservative. Later on Junk and Struckmeier [9] proposed a more stable discretization and proved a Lax-Wendroff-type consistency result for scalar conservation laws in  $\Omega = \mathbb{R}$ . The motivation to develop this new scheme was to unify the advantages of Finite-Volume methods and particle methods in one scheme. The FVPM combines the generic features of a Finite-Volume scheme and a particle method, namely the concept of a numerical flux function and the flow description using moving particles.

This method was the subject of concern also in [8, 10, 11, 15, 17, 18, 19]. The extension of the projection technique for incompressible flows to the FVPM is done in [10, 11]. Isotropic and anisotropic adaptive strategies are investigated in [15]. Moreover, a coupling approach between the classical FVM and the FVPM is proposed in [19]. In [12] an application of the FVPM to a problem with moving boundary in one space dimension is given. Thus the scheme was mainly tested on one- and two-dimensional problems posed on fixed domains, problems where also the mesh-based methods work. But, being a mesh-free method, the FVPM is intended for problems where a mesh-based method may fail, like e.g. problems with moving or free boundaries or fluid-structure interaction problems.

Here we want to take a step forward in applying the FVPM to real industrial problems involving free boundaries or fluid-structure interaction. Therefore, after deriving the FVPM on fixed and bounded domains, we formulate it on moving domains and apply it to a two-dimensional test problem with moving boundary. With this example we want also to exploit an advantage of the FVPM over SPH methods, namely the fact that in FVPM the particles may move in a non-Lagrangian way.

We end up with a numerical convergence study of the FVPM for a shock tube problem and a simple linear advection equation. The convergence results show one

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restriction of the method, namely only first order convergence.

**1.1. Derivation of FVPM on fixed domains.** We follow here the derivation made in [9], but for a system of conservation laws on a bounded domain  $\Omega \subset R^d$ .

We consider conservation laws written in the form

$$\partial_t \mathbf{u} + \nabla \cdot \mathcal{F}(\mathbf{u}) = \mathbf{0}, \quad \forall \mathbf{x} \in \Omega \subset R^d, t > 0 \quad (1.1)$$

with initial conditions  $\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x})$ ,  $\forall \mathbf{x} \in \Omega$ , and with suitable boundary conditions, where  $\Omega \subset R^d$  is a bounded domain,  $\mathbf{u}(\mathbf{x}, t) \in R^m$ ,  $m > 0$  denotes the vector of conservative quantities, and  $\mathcal{F}(\mathbf{u}(\mathbf{x}, t))$  denotes the flux function of the conservation law.

A natural approach to discretize conservation laws is to evaluate the weak formulation of (1.1) with a discrete set of test functions  $\psi_i$ ,  $i = 1, \dots, N$ . In classical Finite-Volume methods (FVM) [13], the test functions are taken as the characteristic functions  $I_{\Omega_i}(\mathbf{x})$  of the control volumes  $\Omega_i$ . The discrete quantities are obtained from cell averages. Note that the characteristic functions form a partition of unity, i.e.  $\sum_{i=1}^N I_{\Omega_i}(\mathbf{x}) = 1$ ,  $\forall \mathbf{x} \in \Omega$ .

A similar approach is used in the following, but we introduce a different set of test functions. Since we want to derive a mesh-free method, we should not make use of a mesh. Therefore the conservative variables are approximated at each time step by a finite set of particles located in the spatial domain  $\Omega$ . From this point of view, the FVPM is a particle method with particle positions  $\mathbf{x}_i(t)$ , which may be irregularly spaced and moving in time. To each position  $\mathbf{x}_i(t)$  we associate a function  $\psi_i(\mathbf{x}, t)$ , what we call *a particle*. As in the Finite-Volume approach, let  $\{\psi_i : i = 1, \dots, N\}$  be a partition of unity, but the supports of the functions should overlap. More exactly, we assume that the particles are smooth functions localized around the particle positions  $\mathbf{x}_i(t)$  and satisfy

$$\sum_{i=1}^N \psi_i(\mathbf{x}, t) = 1, \quad \forall \mathbf{x} \in \Omega, t \geq 0. \quad (1.2)$$

We construct this partition of unity as follows:

Taking a Lipschitz-continuous function  $W : R \rightarrow R_+$  with compact support (otherwise one has to consider long-range interactions between particles), we define

$$\psi_i(\mathbf{x}, t) = \frac{W_i(\mathbf{x}, t)}{\sigma(\mathbf{x}, t)}, \quad (1.3)$$

where  $\sigma(\mathbf{x}, t) = \sum_{i=1}^N W_i(\mathbf{x}, t)$ ,  $W_i(\mathbf{x}, t) = W(\mathbf{x} - \mathbf{x}_i(t))$ ,  $i = 1, \dots, N$ . Such a partition of unity used in FVPM is shown in Fig. 1.1.

Now, we test the conservation law (1.1) against the new set of test functions  $\psi_i(\mathbf{x}, t)$ , i.e. we consider the weak form given by

$$\int_{\Omega} \left( \partial_t \mathbf{u} + \nabla \cdot \mathcal{F}(\mathbf{u}) \right) \psi_i(\mathbf{x}, t) d\mathbf{x} = 0, \quad i = 1, \dots, N, \quad (1.4)$$

which yields the equations

$$\frac{d}{dt} \int_{\Omega} \mathbf{u} \psi_i d\mathbf{x} = \int_{\Omega} (\mathcal{F}(\mathbf{u}) \cdot \nabla \psi_i + \mathbf{u} \partial_t \psi_i) d\mathbf{x} - \int_{\partial\Omega} \psi_i \mathcal{F}(\mathbf{u}) \cdot \mathbf{n} d\sigma. \quad (1.5)$$

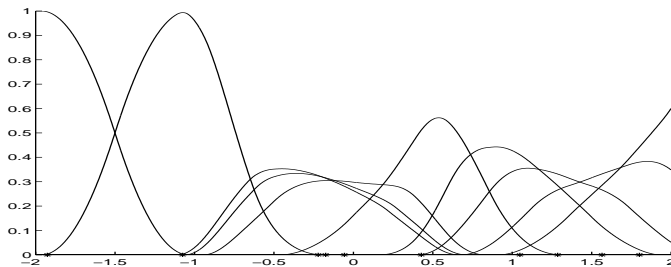


FIG. 1.1. A partition of unity used in FVPM

Note that the boundary term appears only for particles  $i$  which are close to the boundary, i.e.  $\text{supp } \psi_i \cap \partial\Omega \neq \emptyset$ .

Similarly to Finite-Volume methods, one associates to each particle a local average  $\mathbf{u}_i(t)$  through the relation

$$\mathbf{u}_i(t) = \frac{1}{V_i(t)} \int_{\Omega} \mathbf{u}(\mathbf{x}, t) \psi_i(\mathbf{x}, t) d\mathbf{x}, \quad (1.6)$$

where  $V_i(t)$  is the time-dependent volume of a particle  $i$  given by

$$V_i(t) = \int_{\Omega} \psi_i(\mathbf{x}, t) d\mathbf{x}. \quad (1.7)$$

Using definition (1.3), one may obtain [6, 9] from (1.5) the following equations:

$$\frac{d}{dt}(V_i \mathbf{u}_i) = \sum_{j=1}^N \int_{\Omega} \left( (\mathcal{F}(\mathbf{u}) - \mathbf{u} \dot{\mathbf{x}}_i) \Gamma_{ji} - (\mathcal{F}(\mathbf{u}) - \mathbf{u} \dot{\mathbf{x}}_j) \Gamma_{ij} \right) d\mathbf{x} - \mathcal{B}_i, \quad (1.8)$$

where  $\Gamma_{ij}$  is a function localized on the intersection of the supports of particle  $i$  and particle  $j$  given by

$$\Gamma_{ij}(\mathbf{x}, t) = \frac{\psi_i(\mathbf{x}, t)}{\sigma(\mathbf{x}, t)} \nabla W_j(\mathbf{x}, t), \quad \forall i, j = 1, \dots, N, \quad (1.9)$$

and  $\mathcal{B}_i = \int_{\partial\Omega} \psi_i \mathcal{F}(\mathbf{u}) \cdot \mathbf{n} d\sigma$  denotes the boundary term. The discretization of the boundary term  $\mathcal{B}_i$  is explained in [18].

For abbreviation, we introduce the modified flux

$$\mathcal{G}(t, \dot{\mathbf{x}}, \mathbf{u}) = \mathcal{F}(\mathbf{u}) - \mathbf{u} \cdot \dot{\mathbf{x}}, \quad (1.10)$$

where the particle movement  $\dot{\mathbf{x}}$  is given by  $\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}, t)$ ,  $\mathbf{a}(\mathbf{x}, t) \in C^0(R_+, C^1(R^d))$  being a given velocity field. The modified flux  $\mathcal{G}$  consists of the flux of the given conservation law, as well of a contribution  $\mathbf{u} \cdot \mathbf{a}$  due to the particle movement with velocity  $\mathbf{a}$ .

*Remark:* The velocity  $\mathbf{a}$  of the particles should be in such a way that during the movement of the particles their supports always cover the domain completely.

Now, assuming that  $\mathbf{u}$  varies only slightly around  $\bar{\mathbf{u}}$  on the intersection of the supports of  $\psi_i$  and  $\psi_j$ , and that  $\dot{\mathbf{x}}_i \approx \dot{\mathbf{x}}_j := \bar{\dot{\mathbf{x}}}$ , with, e.g.,  $\bar{\dot{\mathbf{x}}} = \frac{\dot{\mathbf{x}}_i + \dot{\mathbf{x}}_j}{2}$ , we have

$$\begin{aligned} \frac{d}{dt}(V_i \mathbf{u}_i) &\approx - \sum_{j=1}^N (\mathcal{F}(\bar{\mathbf{u}}) - \bar{\mathbf{u}} \bar{\dot{\mathbf{x}}}) \int_{\Omega} (\Gamma_{ij} - \Gamma_{ji}) d\mathbf{x} - \mathcal{B}_i \\ &= - \sum_{j=1}^N |\beta_{ij}| (\mathcal{F}(\bar{\mathbf{u}}) - \bar{\mathbf{u}} \bar{\dot{\mathbf{x}}}) \cdot \mathbf{n}_{ij} - \mathcal{B}_i, \end{aligned}$$

where

$$\beta_{ij}(t) = \gamma_{ij}(t) - \gamma_{ji}(t) \text{ and } \mathbf{n}_{ij} = \frac{\beta_{ij}}{|\beta_{ij}|}, \quad (1.11)$$

with  $\gamma_{ij}(t) = \int_{\Omega} \Gamma_{ij}(\mathbf{x}, t) d\mathbf{x}$ .

The flux is approximated in terms of the discrete values with the help of a numerical flux function  $\mathbf{g}_{ij} = \mathbf{g}(t, \mathbf{x}_i, \mathbf{u}_i, \mathbf{x}_j, \mathbf{u}_j, \mathbf{n}_{ij})$  of the modified flux function  $\mathcal{G}(t, \dot{\mathbf{x}}, \mathbf{u})$ , i.e. we use the approximation

$$(\mathcal{F}(\bar{\mathbf{u}}) - \bar{\mathbf{u}} \bar{\dot{\mathbf{x}}}) \cdot \mathbf{n}_{ij} \approx \mathbf{g}_{ij}.$$

*Remark:* The numerical flux function  $\mathbf{g}$  can be any numerical flux function used in FVM, but it has to be consistent with the modified flux function  $\mathcal{G}$ , not with  $\mathcal{F}$ . In the case considered above, if  $\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}, t)$ , i.e.  $\dot{\mathbf{x}}(t) \neq \mathbf{a}(\mathbf{x}, t, \mathbf{u})$ , one can easily modify a numerical flux function consistent with  $\mathcal{F}$  to generate a flux function consistent with  $\mathcal{G}(t, \dot{\mathbf{x}}, \mathbf{u}) = \mathcal{F}(\mathbf{u}) - \mathbf{u} \cdot \dot{\mathbf{x}}$ .

One ends up with the following system of ODEs

$$\frac{d}{dt}(V_i \mathbf{u}_i) = - \sum_{j=1}^N |\beta_{ij}| \mathbf{g}_{ij} - \mathcal{B}_i, \quad (1.12)$$

with the initial condition

$$\mathbf{u}_i(0) = \frac{1}{V_i(0)} \int_{\Omega} \mathbf{u}_0(\mathbf{x}) \psi_i(\mathbf{x}, 0) d\mathbf{x}. \quad (1.13)$$

Using an explicit Euler discretization of the time derivative, the resulting scheme has a similar structure to a FVM scheme, namely

$$V_i^{n+1} \mathbf{u}_i^{n+1} = V_i^n \mathbf{u}_i^n - \Delta t \sum_{j \in N(i)} |\beta_{ij}^n| \mathbf{g}_{ij}^n - \mathcal{B}_i, \quad (1.14)$$

with  $\mathbf{u}_i^0 = \frac{1}{V_i^0} \int_{\Omega} \mathbf{u}_0(\mathbf{x}) \psi_i(\mathbf{x}, 0) d\mathbf{x}$ . The set  $N(i)$  denotes the neighbours of a particle  $i$ , i.e. the particles  $j$  with the property that  $\text{supp } \psi_j \cap \text{supp } \psi_i \neq \emptyset$ .

A natural reconstruction of a function from the discrete values is given by

$$\tilde{\mathbf{u}}(\mathbf{x}, t) = \sum_{i=1}^N \mathbf{u}_i^n \psi_i(\mathbf{x}, t) I_{[t_n, t_{n+1})}(t), \quad \mathbf{x} \in \Omega, t \in [0, T]. \quad (1.15)$$

Finally one should notice that (1.14) contains two unknowns at the time level  $n+1$ , namely  $\mathbf{u}_i^{n+1}$  and  $V_i^{n+1}$ . Thus we need an additional equation for the volumes

of particles. There are two approaches to find  $V_i^{n+1}$ . One may apply numerical integration to compute  $V_i^{n+1}$  from definition (1.7), or one may obtain the additional equation for  $V_i^{n+1}$  by differentiating equation (1.7) with respect to  $t$ , namely

$$\dot{V}_i(t) = \sum_{j \in N(i)} (\gamma_{ij} \dot{\mathbf{x}}_j - \gamma_{ji} \dot{\mathbf{x}}_i). \quad (1.16)$$

**1.2. Some properties of the FVPM.** As one can notice from the previous derivation of the FVPM, this scheme is defined by the following factors: the position of the particles  $\mathbf{x}_i$ , the smoothing kernel  $W$ , the smoothing length  $h$ , the velocity of the particles  $\mathbf{a}$  and the numerical flux function  $\mathbf{g}$ . Geometrical information about the particles and about their relative position is carried by the coefficients  $\beta_{ij}$ . Therefore they are also called *geometrical coefficients*. Only from formula (1.14), one would say that they play an important role in the scheme. Indeed, they have a significant influence on the properties of the method. Their heuristic interpretation (see e.g. [10]) shows the similarity of the FVPM with the classical Finite-Volume approach.

Therefore, before presenting the properties of the scheme (1.12), we remember some important properties of the geometrical coefficients  $\beta_{ij}$ .

PROPOSITION 1.1. *The coefficients  $\beta_{ij}$  satisfy*

$$\beta_{ij} = -\beta_{ji}, \quad \forall \quad i, j = 1, \dots, N \quad (1.17)$$

$$\beta_{ij} = 0, \quad \text{if} \quad \text{supp } \psi_i \cap \text{supp } \psi_j = \emptyset \quad (1.18)$$

$$\beta_{ii} = 0, \quad \forall \quad i = 1, \dots, N \quad (1.19)$$

and

$$\sum_j \beta_{ij} = \begin{cases} 0, & \text{if } \text{supp } \psi_i \cap \partial\Omega = \emptyset \\ -\int_{\partial\Omega} \psi_i \mathbf{n} \, d\sigma, & \text{if } \text{supp } \psi_i \cap \partial\Omega \neq \emptyset. \end{cases} \quad (1.20)$$

COROLLARY 1.2. *In the non-moving case, i.e. when  $\dot{\mathbf{x}} = 0$ , property (1.20) is sufficient to preserve constant states.*

Proofs of the above properties may be found in [18].

The properties of the scheme (1.12) are mainly proved for the one-dimensional case and a scalar conservation law, like a Lax-Wendroff-type consistency result in [9]. Monotonicity of the scheme, under a CFL-like condition, and a  $L^\infty$ -stability for finite times of the approximate solution  $\tilde{\mathbf{u}}$  was obtained in [17], also for one-dimensional, scalar conservation laws. Only the conservativity of the method is proved in the general case.

Provided that the coefficients  $\beta_{ij}$  satisfy the skew symmetry condition (1.17) and that the numerical flux function  $\mathbf{g}$  is conservative, scheme (1.12) is *conservative* in the sense of a classical Finite-Volume method, i.e.

$$\frac{d}{dt} \left( \sum_{i=1}^N V_i \mathbf{u}_i \right) = - \int_{\partial\Omega} \mathcal{F}(\mathbf{u}) \cdot \mathbf{n} \, d\sigma. \quad (1.21)$$

Here we want to give an approximation property of the reconstruction formula (1.15) for a scalar conservation law on a two-dimensional domain  $\Omega \subset R^2$ . One may study in which sense, for example,  $u^0$  is approximated by  $\tilde{u}(\mathbf{x}, 0) = \sum_i u_i(0) \psi_i(\mathbf{x}, 0)$ ,

where  $u_i(0)$  are the local averages  $\frac{1}{V_i(0)} \int u^0(\mathbf{x}) \psi_i(\mathbf{x}, 0) d\mathbf{x}$ . A similar approximation result is also given in [9], but for another norm,  $x \in R$  and for  $u \in C_0^1(\mathbb{R} \times R^+)$ . Here we apply the convergence results obtained by I. Babuska and J.M. Melenk in [1] for the Partition of unity method (PUM).

**THEOREM 1.3.** (*Approximation property*) *Let  $\Omega \subset R^2$  be an open set,  $\{\Omega_i\}$  be an open cover of  $\Omega$  satisfying a pointwise overlap condition*

$$\exists M \in \mathbb{N} : \forall \mathbf{x} \in \Omega, \#\{i | \mathbf{x} \in \Omega_i\} \leq M, \quad (1.22)$$

*and  $\{\psi_i\}$  be a partition of unity subordinate to the cover  $\{\Omega_i\}$ , i.e.  $\text{supp } \psi_i \subset \bar{\Omega}_i, \forall i$ , satisfying*

$$\|\psi_i\|_{L^\infty} \leq C_\infty, \quad (1.23)$$

*where  $C_\infty$  is a constant. Let  $u \in H^1(\Omega)$  be the function to be approximated and  $u_i := \frac{1}{V_i} \int_{\Omega_i} u(\mathbf{x}) \psi_i(\mathbf{x}) d\mathbf{x}$  be the local averages, where  $V_i = \int_{\Omega_i} \psi_i(\mathbf{x}) d\mathbf{x}$ . Then  $\tilde{u}(\mathbf{x}) := \sum_i u_i \psi_i(\mathbf{x})$  satisfies*

$$\|u - \tilde{u}\|_{L^2(\Omega)} \leq MC_\infty^2 C h \|\nabla u\|_{L^2(\Omega)}, \quad (1.24)$$

*where  $h = \max_i \{\text{diam}(\Omega_i)\}$  is a typical space scale called smoothing length and  $C$  is a constant independent of  $h$ .*

*Proof.* The proof of this theorem follows very closely reference [1]. First we state the following lemma.

**LEMMA 1.4.** [1] *Let  $\Omega$  be an open set,  $\{\Omega_i\}$  be an open cover of  $\Omega$  satisfying the pointwise overlap condition (1.22). Let  $u, u_i \in H^1(\Omega)$  be such that  $\text{supp } u_i \subset \bar{\Omega}_i \cup \Omega$ . Then*

$$\sum_i \|u\|_{L^2(\Omega_i)}^2 \leq M \|u\|_{L^2(\Omega)}^2, \quad (1.25)$$

$$\left\| \sum_i u_i \right\|_{L^2(\Omega)}^2 \leq M \sum_i \|u_i\|_{L^2(\Omega_i)}^2. \quad (1.26)$$

Using the fact that  $\sum_i \psi_i \equiv 1$  and formulas (1.23) and (1.26), we can write

$$\begin{aligned} \|u - \tilde{u}\|_{L^2(\Omega)}^2 &= \left\| \sum_i \psi_i (u - u_i) \right\|_{L^2(\Omega)}^2 \leq M \sum_i \|\psi_i (u - u_i)\|_{L^2(\Omega_i)}^2 \\ &\leq MC_\infty^2 \sum_i \|u - u_i\|_{L^2(\Omega_i)}^2. \end{aligned} \quad (1.27)$$

Now we need to estimate  $\|u - u_i\|_{L^2(\Omega_i)}$ . For this we use the following two lemmas which are similar to the Lemmas 7.12 and 7.16 in [5].

**LEMMA 1.5.** *With the notations and under the assumptions from Theorem 1.3, we have*

$$|u(x) - u_i| \leq \frac{C_\infty h^2}{2V_i} \int_{\Omega_i} \frac{1}{|x-y|} |\nabla u(y)| dy \quad \text{a.e. on } \Omega_i. \quad (1.28)$$

*Proof.* The proof of this lemma is very similar to that of Lemma 7.16 from [5], where  $\psi_i \equiv 1$ .  $\square$



LEMMA 1.6. [5] Let  $\mu \in (0, 1)$  and  $V_\mu$  be an operator on  $L^1(\Omega)$  defined by

$$(V_\mu f)(x) = \int_{\Omega} |x - y|^{2(\mu-1)} f(y) dy. \quad (1.29)$$

The operator  $V_\mu$  maps  $L^p(\Omega)$  continuously into  $L^q(\Omega)$  for any  $q$ ,  $1 \leq q \leq \infty$  satisfying  $0 \leq \delta = \delta(p, q) = p^{-1} - q^{-1} < \mu$ . Furthermore, for any  $f \in L^p(\Omega)$ ,

$$\|V_\mu f\|_q \leq \left( \frac{1 - \delta}{\mu - \delta} \right)^{1-\delta} (2\pi)^{1-\mu} |\Omega|^{\mu-\delta} \|f\|_p. \quad (1.30)$$

Now, applying Lemma 1.6 with  $f = |\nabla u| \in L^2(\Omega_i)$ , since  $u \in H^1(\Omega_i)$ ,  $p = 2$ ,  $\mu = \frac{1}{2} \in (0, 1)$ ,  $q = 2$  and, hence,  $\delta = 0$ , we obtain

$$\left\| \int_{\Omega_i} \frac{1}{|y - x|} |\nabla u(y)| dy \right\|_{L^2(\Omega_i)} \leq 2\sqrt{2\pi} |\Omega_i|^{1/2} \|\nabla u\|_{L^2(\Omega_i)}. \quad (1.31)$$

Using (1.28) and (1.31), we obtain

$$\|u - u_i\|_{L^2(\Omega_i)} \leq \frac{C_\infty h^2}{V_i} \sqrt{2\pi} h \|\nabla u\|_{L^2(\Omega_i)}. \quad (1.32)$$

Moreover, from (1.27) we get

$$\begin{aligned} \|u - \tilde{u}\|_{L^2(\Omega)}^2 &\leq MC_\infty^2 \sum_i C_\infty^2 \left( \frac{h^2}{V_i} \right)^2 2\pi h^2 \|\nabla u\|_{L^2(\Omega_i)}^2 \\ &\leq MC_\infty^4 C^2 h^2 \sum_i \|\nabla u\|_{L^2(\Omega_i)}^2 \leq M^2 C_\infty^4 C^2 h^2 \|\nabla u\|_{L^2(\Omega)}^2, \end{aligned}$$

where we used (1.25) and the fact that  $V_i = O(h^2)$ . Thus, estimate (1.24) is obtained, which completes the proof of Theorem 1.3.  $\square$

**2. FVPM with moving boundaries.** In the following we will formulate the method in the general case when the domain  $\Omega$  depends on time, i.e.  $\Omega = \Omega(t) \subset R^d$ . Then a general discussion about how the particles may move in FVPM will be used further on in §3.1. Here will be also explained why a correction procedure for the computation of the geometrical coefficients is needed and a simple correction procedure is proposed.

**2.1. A modified FVPM on moving domains.** When the domain  $\Omega$  is moving, i.e. its shape is changing with time and thus  $\Omega$  depends on time, i.e.  $\Omega = \Omega(t)$ , we have to take into account a contribution due to the moving boundary.

Let us assume that the boundary  $\Gamma(t)$  of the domain  $\Omega(t)$  is moving with a velocity  $\mathbf{b}(\mathbf{x}, t)$ , i.e. we can write

$$\Gamma(t) = \left\{ \mathbf{x}(t) \mid \mathbf{x}(t) = \mathbf{x}_0 + \int_0^t \mathbf{b}(\mathbf{x}(\tau), \tau) d\tau, \mathbf{x}_0 \in \Gamma_0 \right\}, \quad (2.1)$$

where  $\Gamma_0$  is the initial boundary. Then, since the integration volume  $\Omega(t)$  changes in time, in formula (1.5) appears a new term:

$$\begin{aligned} \frac{d}{dt} \int_{\Omega(t)} \mathbf{u} \psi_i d\mathbf{x} &= \int_{\Omega(t)} \frac{d}{dt} (\mathbf{u} \psi_i) d\mathbf{x} + \underbrace{\int_{\partial\Omega(t)} \psi_i \mathbf{u} \cdot \mathbf{b} \cdot \mathbf{n} d\sigma}_{\text{the new term}} \\ &= \int_{\Omega(t)} (\mathcal{F}(\mathbf{u}) \cdot \nabla \psi_i + \mathbf{u} \partial_t \psi_i) d\mathbf{x} - \int_{\partial\Omega(t)} \psi_i \left( \mathcal{F}(\mathbf{u}) - \underbrace{\mathbf{u} \cdot \mathbf{b}} \right) \cdot \mathbf{n} d\sigma. \end{aligned}$$

The derivation of the system of ODE's for  $\mathbf{u}_i$  can be continued exactly like in §1.1 such that we obtain the following system

$$\frac{d}{dt} (V_i \mathbf{u}_i) = - \sum_{j \in N(i)} |\beta_{ij}| \mathbf{g}_{ij} - \int_{\partial\Omega(t)} \psi_i (\mathcal{F}(\mathbf{u}) - \mathbf{u} \cdot \mathbf{b}) \cdot \mathbf{n} d\sigma, \quad (2.2)$$

which differs from (1.12) only in the boundary term. Even equation (1.16) for the volume  $V_i(t)$  changes in this case to

$$\dot{V}_i(t) = \sum_{j \in N(i)} (\gamma_{ij} \dot{\mathbf{x}}_j - \gamma_{ji} \dot{\mathbf{x}}_i) + \int_{\partial\Omega(t)} \psi_i \mathbf{b} \cdot \mathbf{n} d\sigma. \quad (2.3)$$

However, one may still use equation (1.7) for  $V_i(t)$ .

**2.2. How particles should move.** In formula (2.2) there are incorporated two movements:  $\mathbf{a}$ , the movement of the particles (through the numerical flux function  $\mathbf{g}$ ), and  $\mathbf{b}$ , the movement of the boundary. Now we have to answer the question: let the velocity field  $\mathbf{b}$  be given, how should the particles move?

In contrast to Lagrangian methods, as e.g. the Smoothed Particle Hydrodynamics (SPH) method [14], where particles are moved with the velocity of the fluid, in FVPM the particles are allowed to move along an almost arbitrary velocity field. Here we want to exploit this advantage of FVPM, since there are cases when pure Lagrangian particles lead to severe difficulties, as e.g. when solving problems where shocks are developed [15, 18]. Recognizing that shock waves are generic solutions of conservation laws, we are looking for non-Lagrangian particle motions.

As noted previously, in FVPM the particles should be moved in such a way that they always cover the domain. One may also imagine that the particles should be quite homogeneously distributed in the domain. Otherwise, when at the same time there are regions with very high and very low density of particles, the smoothing length  $h$  is either too small (and then holes are developed), or too large (and then some particles will have a high number of neighbours). In [15], where an adaptive smoothing length is used, it is shown that large variations in the smoothing length may lead to instabilities. Hence, beginning with homogeneously distributed particles, we want to have a smoothly varying particle distribution.

Based on a two-dimensional test problem posed on a moving domain, a special Ansatz for the movement of the particles will be proposed.

**2.3. A new correction procedure.** Since the coefficients  $\beta_{ij}$  are given by integral quantities, see (1.11), one needs to introduce a numerical integration technique to compute the geometrical coefficients and this leads to additional errors, such that

conditions (1.17) - (1.20) may not be satisfied exactly. But the properties of the FVPM heavily rely on the conditions (1.17) - (1.20) of the  $\beta_{ij}$ 's. Furthermore, if (1.20) is not exactly satisfied, then constant states will be no longer preserved (see Corollary 1.2).

Hence, applying a numerical quadrature to compute the geometrical coefficients requires in the sequel a correction procedure to ensure that the conditions (1.17) - (1.20) are satisfied exactly. See §3.2 or [10, 17, 18] for comparing numerical results obtained with and without correction procedures. First of all one should notice that if one uses formula (1.11), then some properties of  $\beta_{ij}$ 's still hold, like  $\beta_{ii} = \mathbf{0}$ ,  $\beta_{ij} = -\beta_{ji}$  and  $\beta_{ij} = \mathbf{0}$  for non-overlapping particles.

Let us discuss in the following how to ensure condition (1.20). The basic idea consists in firstly computing the coefficients  $\tilde{\beta}_{ij}$  by rough numerical integration in order to keep the computation costs as low as possible, and then to modify them by an appropriate correction term  $\tilde{\beta}_{ij}$ .

A correction procedure given in [17], based on solving an undetermined linear system, works in general only for one-dimensional problems, since in multidimensional computations it is too expensive. A second approach, proposed by Keck [10], consists of a fast pairwise correction, where the error is successively shifted from the first to the last particle.

Here we will use the following approach. Instead of correcting the coefficients  $\tilde{\beta}_{ij}$  obtained by a numerical quadrature, we add for all interior particles  $i$  the following term to the right hand side of scheme (1.14):

$$-\Delta t \mathcal{G}(\mathbf{u}_i^n) \cdot \tilde{\beta}_{ii}^n, \quad (2.4)$$

where  $\tilde{\beta}_{ii}^n := -\sum_j \tilde{\beta}_{ij}^n$  is the defect for the particle  $i$  and  $\mathcal{G}$  is the modified flux (1.10). Then scheme (1.14) becomes

$$V_i^{n+1} \mathbf{u}_i^{n+1} = V_i^n \mathbf{u}_i^n - \Delta t \sum_{j \in N(i)} |\tilde{\beta}_{ij}^n| \mathbf{g}_{ij}^n - \Delta t \mathcal{G}(\mathbf{u}_i^n) \cdot \tilde{\beta}_{ii}^n. \quad (2.5)$$

Remember that  $\mathbf{g}$  is a numerical flux function consistent with  $\mathcal{G}$ , i.e.  $\mathbf{g}(t, \mathbf{x}, \mathbf{u}, \mathbf{x}, \mathbf{u}, \mathbf{n}) = \mathcal{G}(\mathbf{u}) \cdot \mathbf{n}$ . In this way, using the scheme (2.5), the constant states are preserved in the non-moving case, even if  $\sum_j \tilde{\beta}_{ij} \neq 0$ .

A disadvantage of this approach is that the conservativity of the scheme is lost by adding this correction term. Indeed, in general one has

$$\sum_{i=1}^N V_i^{n+1} \mathbf{u}_i^{n+1} = \sum_{i=1}^N V_i^n \mathbf{u}_i^n - \Delta t \sum_{i=1}^N \mathcal{G}(\mathbf{u}_i^n) \cdot \tilde{\beta}_{ii}^n \neq \sum_{i=1}^N V_i^n \mathbf{u}_i^n.$$

Because the defect  $\tilde{\beta}_{ii}^n$  is of the order of the integration error, i.e.

$$|\tilde{\beta}_{ii}^n| = \mathcal{O}(h^{s+d-1}), \quad (2.6)$$

where  $s \geq 1$  is the order of the numerical quadrature used to compute  $\tilde{\beta}_{ij}^n$ , we may write

$$\sum_{i=1}^N V_i^{n+1} \mathbf{u}_i^{n+1} = \sum_{i=1}^N V_i^n \mathbf{u}_i^n + \mathcal{O}(h^s),$$

assuming  $\Delta t = \mathcal{O}(h)$ , which follows from a typical CFL-type condition, as well as  $N = \mathcal{O}(\frac{1}{h^d})$ . Despite the fact that scheme (2.5) is only approximately conservative, we will use this modified FVPM scheme in the numerical computations from §3.

**3. Numerical experiments.** In this section we apply the FVPM scheme (2.5) to different test problems in order to validate the method.

To implement a FVPM scheme, one has to start by defining a set of particles in the domain. We are using uniform or non-uniform distributed, and non-moving or moving particles. More precisely, the domain  $\Omega$  is divided firstly into  $N = nx \cdot ny$  rectangular cells of size  $\Delta x \times \Delta y$ , where  $nx = \frac{1}{\Delta x}$ ,  $ny = \frac{1}{\Delta y}$  are the number of cells in the  $x$ -direction and  $y$ -direction, respectively. Then, if we want to use uniform distributed particles, we take a particle  $i$  in the middle of the cell, otherwise we let it be randomly distributed within the cell and we call it a *quasi-random* particle distribution. Since we choose  $\Delta y = \Delta x$ , further on only  $\Delta x$  will appear. If and how particles are moved, will be specified later in the example.

A next step is to compute the geometrical coefficients  $\beta_{ij}$  using (1.11). In the particular case of uniform distributed, non-moving particles and  $h = \Delta x$ , coefficients  $\beta_{ij}$  can be computed exactly [17, 18] and no correction procedure is needed. When using quasi-random particles, firstly we compute the intersection of the supports of pairwise interacting particles, then a Gauss quadrature rule with  $4 \times 4$  points is applied to compute  $\beta_{ij}$ . When particles are moving in time, volumes  $V_i$  are computed in each time step from (1.7) by the same quadrature rule as used for computing  $\beta_{ij}$ .

We use the numerical flux function of Roe [13] for computing the term  $g_{ij}$ . The solution is reconstructed on a uniform grid of size  $\Delta x \times \Delta x$  using formula (1.15).

**3.1. A test problem with moving boundaries.** We concentrate now on simulating a flow around an oscillating circle in a spatial two-dimensional geometry.

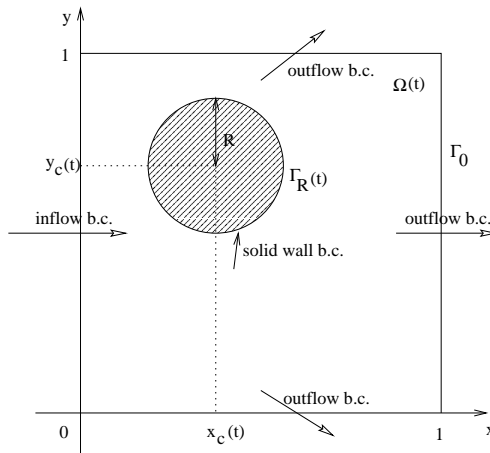


FIG. 3.1. The computational domain of the test problem

The computational domain is given by  $\Omega(t) = [0, 1] \times [0, 1] \setminus B_R(t)$ , where  $B_R(t) = \{(x, y) \in \mathbb{R}^2 : \|x - x_c(t), y - y_c(t)\| \leq R\}$  is the ball of center  $(x_c(t), y_c(t))$  and radius  $R$ . Let us denote the domain's boundary by  $\partial\Omega(t) := \Gamma_0 \cup \Gamma_R(t)$ , where  $\Gamma_0$  is the exterior boundary and  $\Gamma_R(t)$  is the boundary of the moving ball (see Fig. 3.1).

We consider a simple, rigid movement of the ball, although one may consider another types of movements. In our example the ball is oscillating up and down, for example with respect to the following equations:

$$\dot{x}_c(t) = 0, \quad x_c(0) = x_c^0 \quad (3.1)$$

$$\dot{y}_c(t) = A\omega \cos(\omega t), \quad y_c(0) = y_c^0, \quad (3.2)$$

where  $A$  is the amplitude of the motion and  $\omega$  is the frequency.

For the fluid-structure interaction problem which is considered here the effects due to viscosity can be neglected. Hence, the fluid is modeled by Euler's equations for compressible inviscid flow [13] given by

$$\partial_t \mathbf{u}(\mathbf{x}, t) + \nabla \cdot \mathcal{F}(\mathbf{u}(\mathbf{x}, t)) = 0, \quad \mathbf{x} \in \Omega(t), \quad (3.3)$$

where

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}, \quad \mathbf{f}_1 = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(\rho E + p) \end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(\rho E + p) \end{pmatrix}, \quad \mathcal{F} = (\mathbf{f}_1, \mathbf{f}_2),$$

$\rho$ ,  $u$ ,  $v$ ,  $E$  and  $p$  denote the density, velocity in the  $x$  and  $y$  direction, total energy and pressure, respectively. Equation (3.3) is closed by the equation of state for a perfect gas:  $p = (\gamma - 1)\rho \left( E - \frac{u^2 + v^2}{2} \right)$ , with  $\gamma = 1.4$ .

Equation (3.3) is completed by the initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega(t),$$

as well as the following boundary conditions:

- inflow boundary conditions on the left part of  $\Gamma_0$ ,
- outflow boundary conditions on the rest of  $\Gamma_0$ ,
- solid wall boundary conditions on the moving boundary  $\Gamma_R(t)$ .

Following the discussion from §2.2, we assume that the movement of the particles  $\mathbf{a}$  is given by the solution of a Laplace equation with the following boundary conditions: namely *zero* velocity at the fixed boundary  $\Gamma_0$  and the boundary velocity at the moving boundary  $\Gamma_R(t)$ :

$$\begin{cases} \Delta \mathbf{a}(\mathbf{x}, t) = \mathbf{0}, & \Omega(t) \\ \mathbf{a}(\mathbf{x}, t) = \mathbf{0}, & \Gamma_0 \\ \mathbf{a}(\mathbf{x}, t) = \mathbf{b}, & \Gamma_R(t) \end{cases}. \quad (3.4)$$

The boundary velocity  $\mathbf{b}$  is given by the motion of the ball, i.e.  $\mathbf{b} = (\dot{x}_c(t), \dot{y}_c(t))$ , where  $\dot{x}_c(t)$ ,  $\dot{y}_c(t)$  are given by (3.1) and (3.2), respectively.

In this way the velocity field  $\mathbf{a}$  will be smooth and the particles will follow the time-dependent computational domain, since particles near the boundary move with the boundary and do not get out of the domain, because the solution  $\mathbf{a}$  satisfies a maximum principle.

Our approach is very similar with the Arbitrary Lagrange Euler method (ALE-method [4]) and with the spring analogy method [2] used in mesh-based computations on a moving fluid domain. As in FVPM, in the ALE method the coordinates are neither fixed (Eulerian) nor moving with the fluid (Lagrangian), but they can move in an arbitrary way. When the boundary is displaced the mesh is deformed and mesh cells are squeezed and stretched or maybe even inverted. The spring analogy is then used in [3] to restore the mesh to a more regular state. In our example, we may also imagine that the particles are linked between the boundaries through fictitious springs.

*Remark:* In our example, since the movement of the boundary is restricted to a rigid body movement of an isolated object, the whole distribution of particles could

be moved with the boundary. In this way the particles remain rigid, i.e. there is no relative motion between the particles. The advantage of this rigid movement is clear, we do not have to recompute every time the coefficients  $\beta_{ij}$  for example. However, the rigid movement approach is less general than the one proposed here.

In [18] we have also investigated under which conditions on the motion of the circle and the smoothing length of the particles no 'holes' are developed in the domain. By a 'hole' we understand a space which is not covered by the support of any particle.

Now we present some numerical results concerning this test problem.

If the circle moves periodically up and down, like specified in (3.1), (3.2), there exists a periodic solution, i.e. after a few oscillations up and down the flow becomes periodic, with the same period as the circle's movement. To see this, we compute the difference between the solution every time when the circle attains its initial position, moving upwards, i.e. exactly after a complete period:

$$e_k = \sum_{i \in N} |\rho_i^k V_i^k - \rho_i^{k+1} V_i^{k+1}|, \quad k = 0, 1, \dots, k_{max},$$

where  $k_{max} = [T/P]$ ,  $P = 2\pi/\omega$  is the period of the movement,  $T$  is the final time,  $t_0$  is the time when the circle starts to move,  $\rho_i^k = \rho_i(t_0 + kP)$  and  $V_i^k = V_i(t_0 + kP)$ .

For this computation we choose  $N = 50 \times 50$  uniform distributed particles,  $t_0 = 0$ ,  $\omega = 10\pi$ ,  $A = 0.1$ ,  $P = 2\pi/\omega = 0.2$ , and  $T = 4.05$ . Hence,  $k_{max} = 20$ . As can be seen in Fig. 3.2, after around 10 complete oscillations, the differences  $e_k$  are so small that the flow can be considered to be periodic.

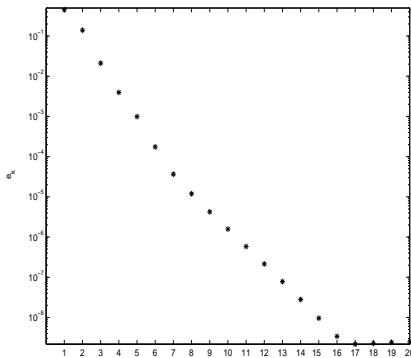


FIG. 3.2. Differences  $e_k$  versus  $k$

Now we choose  $N = 100 \times 100$  quasi-random distributed and moving particles. The movement of the circle is as before, i.e.  $A = 0.1$  and  $\omega = 10\pi$ . The solution at time  $T = 0.55$  is presented in Fig. 3.3 and 3.4. In Fig. 3.3(left) one may see the irregular particle positions together with their corresponding density. The solution reconstructed on a uniform grid is shown in Fig. 3.3(right) (isolines of the density) and Fig. 3.4 (isolines of the velocity components). These results show that the method works also in the case of a time-dependent domain and using irregular distributed and moving particles.

**3.2. A shock tube problem.** Here we apply the FVPM to the two-dimensional shock tube problem from inviscid gas dynamics governed by the Euler equations (3.3)

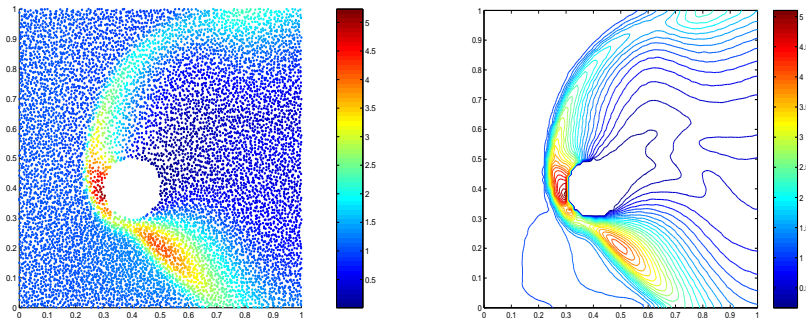


FIG. 3.3.  $N = 100 \times 100$  quasi-random distributed particles and their corresponding density (left) and isolines of the density (right)

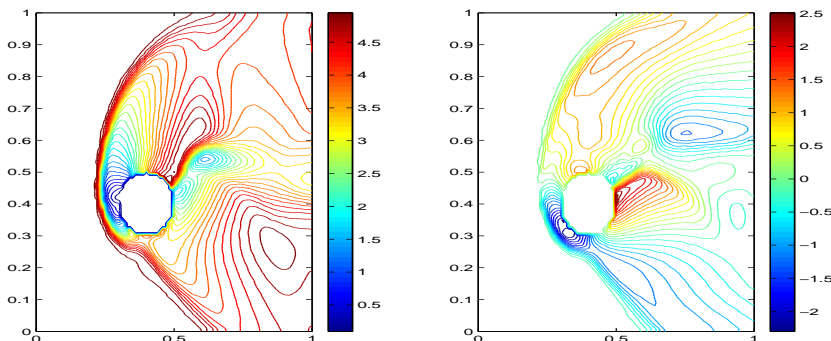


FIG. 3.4. Isolines of  $u$  - (left) and  $v$  - velocity component (right) in the same case as in Fig. 3.3

with the following initial conditions

$$\rho(x, y) = \begin{cases} \rho_L, & x \leq 0.5 \\ \rho_R, & x > 0.5 \end{cases}, \quad p(x, y) = \begin{cases} p_L, & x \leq 0.5 \\ p_R, & x > 0.5 \end{cases}, \\ \mathbf{v}(x, y) = \mathbf{0},$$

for  $(x, y) \in [0, 1] \times [0, 1] \subset \mathbb{R}^2$ , with parameters suggested by Sod [16]

$$\rho_L = 1, \quad \rho_R = 0.125, \quad p_L = 1, \quad p_R = 0.1.$$

This problem can be realized experimentally by the sudden breakdown of a diaphragm in a long one-dimensional tube separating two initial gas states at different pressures and densities. If viscous effects can be neglected along the tube walls and if an infinitely long tube is considered, the exact solution to the 1D Euler equations can be obtained on the basis of simple waves separating regions of uniform conditions [7].

We perform a 2D computation for, in fact, a 1D problem. We expect to obtain a quasi-1D solution for the two-dimensional Euler's equation (3.3), i.e. the solution should vary only in  $x$ -direction.

The computations are done using  $N = 100 \times 100$  or  $N = 50 \times 50$  uniformly or quasi-random distributed particles. The solution is computed at time  $T = 0.15$ . The coefficients  $\beta_{ij}$  are computed either exactly or approximately. In the first case, when  $\beta_{ij}$  are computed exactly, we are using also a layer of ghost cells such that no boundary approximations are made, since on the top and bottom we apply periodic boundary

conditions, and on the left and right part we supply the constant states from the initial condition. This can be made as long as the shock or rarefaction wave does not reach the boundary, and this is not the case at  $T = 0.15$ . In the case when  $\beta_{ij}$  can not be computed exactly, we apply the boundary treatment and the correction procedure explained before. On the top and bottom part of the domain we apply the solid wall boundary condition and on the left and right part of the boundary we apply outlet boundary condition. No particular measures are taken to keep the flow inside the tube one-dimensional.

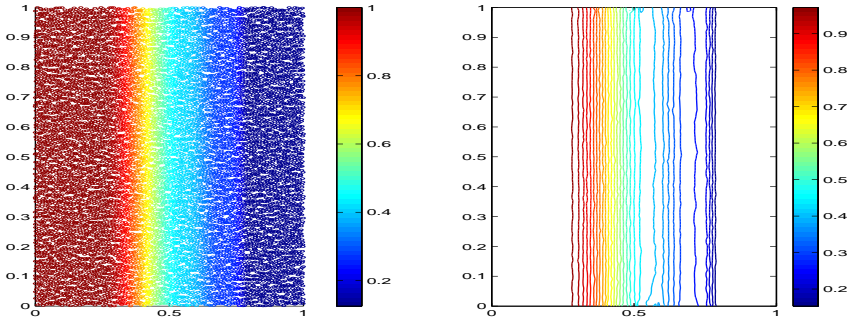


FIG. 3.5.  $N = 100 \times 100$  quasi-random particles and their corresponding density (left), with  $h = 1.5\Delta x$ , and isolines of the density reconstructed on a uniform grid (right), for the shock tube problem

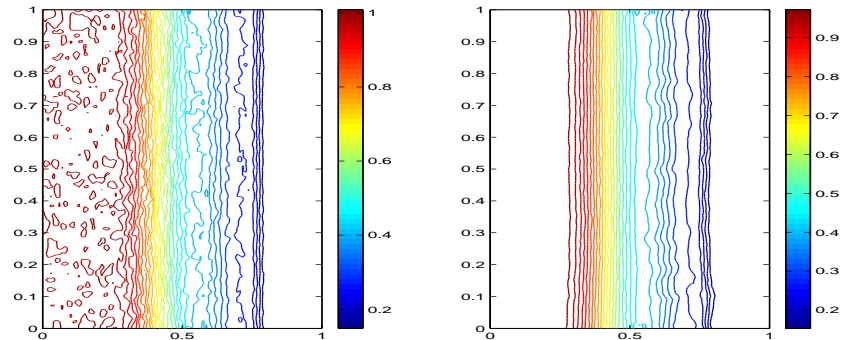


FIG. 3.6. Isolines of the density reconstructed on a uniform grid, without correcting the geometrical coefficients (left), and with Keck's correction procedure (right)

In Fig. 3.5(left) are shown the positions of  $100 \times 100$  quasi-random distributed particles together with their corresponding density. Then the solution is reconstructed on a  $100 \times 100$  uniform grid, and isolines of the density are shown in Fig. 3.5(right).

As mentioned in §1.2, if the coefficients  $\beta_{ij}$  are not corrected such that condition (1.20) is satisfied, the constant states are not preserved, as can be seen in Fig. 3.6(left) and 3.7. Furthermore, we have also implemented the correction procedure proposed by Keck in [10], and isolines of the density are shown in Fig. 3.6(right). After a rough comparison between the density obtained using our and Keck's correction, see Figure 3.7, we conclude that the results are similar and therefore we will restrict in the following to our correction procedure.

In Fig. 3.8, we compare the density obtained with exact and computed coefficients  $\beta_{ij}$ , for a uniform distribution of particles with  $h = \Delta x$ . For a better visualization, we used only  $N = 50 \times 50$  particles. As expected, the results are almost the same.



The influence of the smoothing length  $h$  on the solution is shown in Fig. 3.9, where cuts through the density profile obtained for different  $h$  are shown. Again only  $N = 50 \times 50$ , but quasi-random distributed, particles were used. As pointed out in [6] for Burgers' equation, the FVPM is quite robust when changing the smoothing length  $h$ . Decreasing the smoothing length yields a better resolution of the waves, without producing oscillations.

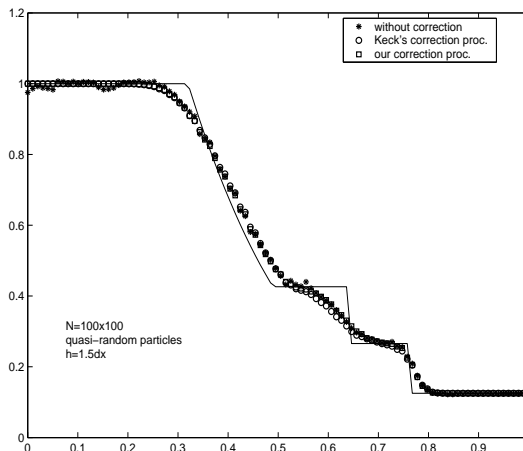


FIG. 3.7. Cut in  $x$ -direction at the middle of the domain, through the density profiles obtained in Fig. 3.5(right), 3.6(left) and (right)

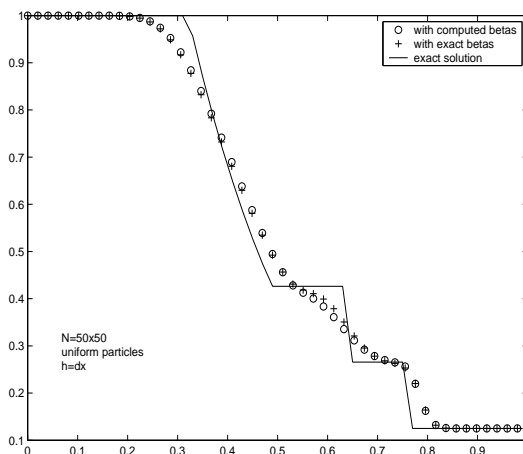


FIG. 3.8. The density obtained with exact and with computed coefficients  $\beta_{ij}$

Finally, we compute numerical convergence rates in the discrete  $L_1$ - and  $L_2$ -norm for this problem. The error between the exact solution and the approximate solution is computed for a sequence of  $N = 20 \times 20, 40 \times 40, \dots, 320 \times 320$  quasi-random distributed particles, with different smoothing lengths  $h$ . In Table 3.1 the errors for the density, together with the experimental order of convergence (EOC) computed from two meshes of size  $N_1 = n_1 \times n_1$  and  $N_2 = n_2 \times n_2$  as:

$$EOC_k = \log \frac{|\rho_{N_1} - \tilde{\rho}_{N_1}|_k}{|\rho_{N_2} - \tilde{\rho}_{N_2}|_k} / \log \left( \frac{n_2}{n_1} \right), \quad k = 1, 2 \quad (3.5)$$

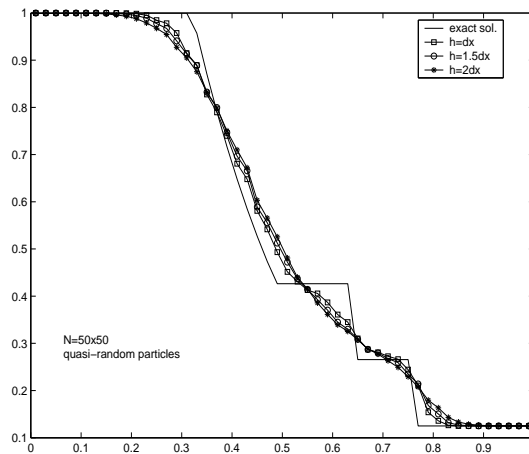
FIG. 3.9. Influence of the smoothing length  $h$  on the density

TABLE 3.1

Discrete  $L_1$ - and  $L_2$ -errors of the density for the shock tube problem, for different  $h$ 

$N$	$h = \Delta x$	$ \rho - \tilde{\rho} _1$	$EOC_1$	$ \rho - \tilde{\rho} _2$	$EOC_2$
$20 \times 20$	5.000000e-02	3.553001e-02		5.088527e-02	
$40 \times 40$	2.500000e-02	2.603677e-02	0.4485	4.052730e-02	0.3284
$80 \times 80$	1.250000e-02	1.723002e-02	0.5956	2.910075e-02	0.4778
$160 \times 160$	6.250000e-03	1.120914e-02	0.6202	2.104956e-02	0.4673
$320 \times 320$	3.125000e-03	7.298787e-03	0.6189	1.571074e-02	0.4220
$N$	$h = 1.5\Delta x$	$ \rho - \tilde{\rho} _1$	$EOC_1$	$ \rho - \tilde{\rho} _2$	$EOC_2$
$20 \times 20$	7.500000e-02	4.427848e-02		5.966602e-02	
$40 \times 40$	3.750000e-02	3.029744e-02	0.5474	4.500231e-02	0.4069
$80 \times 80$	1.875000e-02	2.056287e-02	0.5592	3.276041e-02	0.4580
$160 \times 160$	9.375000e-03	1.368476e-02	0.5875	2.404254e-02	0.4464
$320 \times 320$	4.687500e-03	9.053464e-03	0.5960	1.813783e-02	0.4066
$N$	$h = 2\Delta x$	$ \rho - \tilde{\rho} _1$	$EOC_1$	$ \rho - \tilde{\rho} _2$	$EOC_2$
$20 \times 20$	1.000000e-01	4.924631e-02		6.400495e-02	
$40 \times 40$	5.000000e-02	3.534180e-02	0.4786	5.035410e-02	0.3461
$80 \times 80$	2.500000e-02	2.460084e-02	0.5227	3.740869e-02	0.4287
$160 \times 160$	1.250000e-02	1.666850e-02	0.5616	2.770045e-02	0.4335
$320 \times 320$	6.250000e-03	1.110999e-02	0.5853	2.084552e-02	0.4102

are given. In Fig. 3.10 we show the errors versus the number of points in one direction, in order to synthesize the results from the Table 3.1 and for a better comparison. As expected, the numerical convergence rate does not depend on  $h$ . Furthermore, we remark that the error slightly increases with  $h$ . This is in agreement with the results obtained in Fig. 3.9. But, in this case, the convergence rate does not approach the theoretical order one of convergence. A similar convergence study, obtaining a similar convergence rate, is done in [6], also for the shock tube problem considered here, but the computations are 1D.

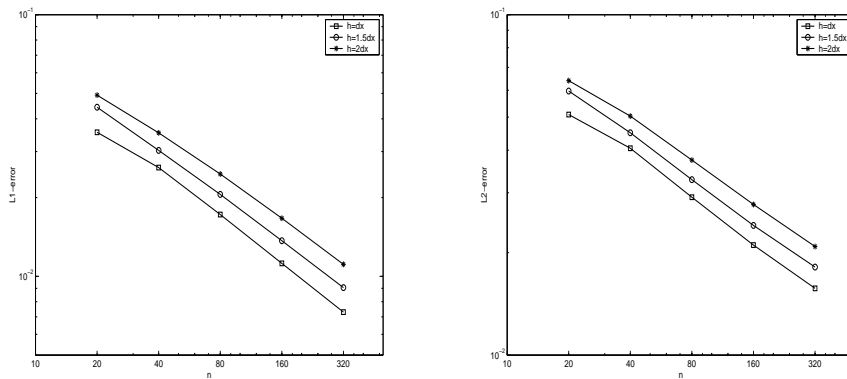


FIG. 3.10. Discrete  $L_1$ -error (left) and  $L_2$ -error (right) of the density for the shock tube problem, at time  $T = 0.15$ , using  $N = n^2$  quasi-random particles, for different  $h$

**3.3. A linear advection equation.** Let us consider a simple linear equation in order to study further the numerical convergence of the FVPM, namely the following linear advection equation

$$\partial_t u + \partial_x u = 0, \quad \text{on } [0, 1] \times [0, 1] \quad (3.6)$$

with a discontinuous initial condition

$$u_0(x, y) = \begin{cases} 1, & x \leq 0.5 \\ 0, & x > 0.5 \end{cases}, \quad (3.7)$$

or with a smooth initial condition

$$u_0(x, y) = \exp \{-100((x - 0.25)^2 + (y - 0.25)^2)\}. \quad (3.8)$$

The exact solution of the equation (3.6) is simply given by  $u(x, y, t) = u_0(x - t, y)$ .

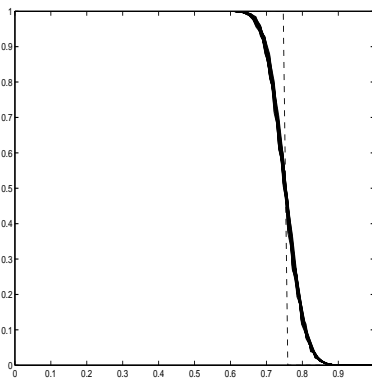


FIG. 3.11. Solution of the linear advection equation with the discontinuous initial condition (3.7), projected onto the  $u$ - $x$ -plane

The approximate solution is computed at  $T = 0.25$ , using  $N = 80 \times 80$  quasi-random distributed and non-moving particles with  $h = 1.5\Delta x$ . Outflow boundary conditions are used on all boundaries. In Fig. 3.11, the solution of the linear advection

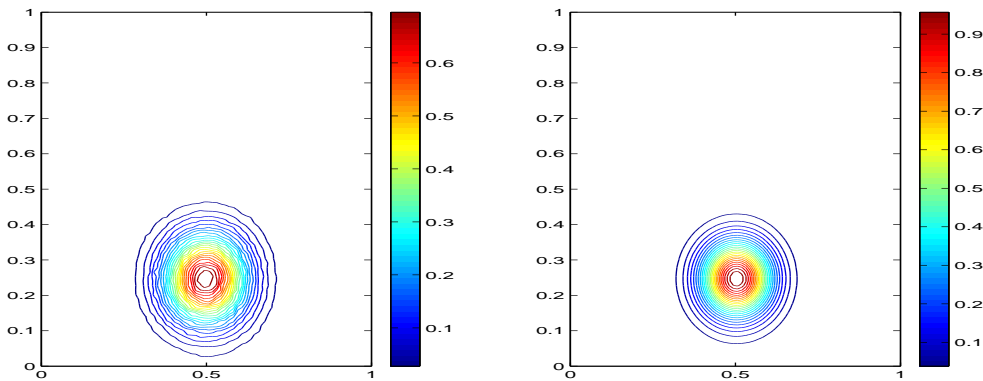


FIG. 3.12. *Isolines of the approximate solution (left) and of the exact solution (right) at  $T = 0.25$ , for the linear advection equation (3.6) with the smooth initial condition (3.8)*

equation (3.6) with the discontinuous initial condition (3.7), projected onto the  $x$ -plane, is shown. Then, in Fig. 3.12, isolines of the exact solution and of the approximate solution of (3.6) with the smooth initial condition (3.8) can be found.

Finally, we compute numerical convergence rates in the discrete  $L_1$ - and  $L_2$ -norm for this linear problem, to see if we obtain better rates than in the nonlinear case. The error between the exact solution and the approximate solution is computed for a sequence of  $N = 20 \times 20, 40 \times 40, \dots, 320 \times 320$  quasi-random distributed particles, with different smoothing lengths  $h$ . In Tables 3.2 and 3.3 the errors in the case of the jump solution and of the smooth solution, together with the experimental order of convergence (EOC) computed from (3.5), are given. In Fig. 3.13 we show the errors versus the number of points in one direction, in order to synthesize the results from the Tables 3.2 and 3.3, and for a better comparison. As expected, the numerical convergence rate does not depend on  $h$ . Also we obtain almost the expected convergence order of one in the case of the smooth solution.

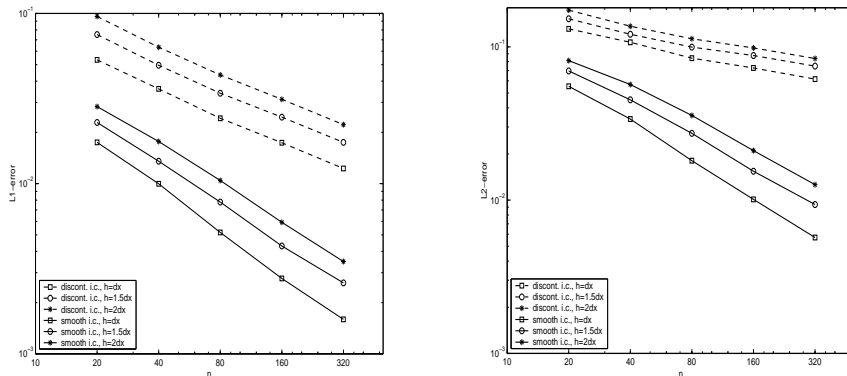


FIG. 3.13. *Discrete  $L_1$ -error (left) and  $L_2$ -error (right) for the linear advection equation with smooth and discontinuous initial condition at time  $T = 0.25$ , using  $N = n^2$  quasi-random particles, for different  $h$*

TABLE 3.2

Discrete  $L_1$ - and  $L_2$ -errors for the linear advection equation with a discontinuous initial condition, for different  $h$

$N$	$h = \Delta x$	$ u - \tilde{u} _1$	$EOC_1$	$ u - \tilde{u} _2$	$EOC_2$
20×20	5.000000e-02	5.333467e-02		1.310182e-01	
40×40	2.500000e-02	3.603141e-02	0.5658	1.070453e-01	0.2915
80×80	1.250000e-02	2.419657e-02	0.5745	8.434356e-02	0.3439
160×160	6.250000e-03	1.738028e-02	0.4774	7.278524e-02	0.2126
320×320	3.125000e-03	1.227533e-02	0.5017	6.159035e-02	0.2409
$N$	$h = 1.5\Delta x$	$ u - \tilde{u} _1$	$EOC_1$	$ u - \tilde{u} _2$	$EOC_2$
20×20	7.500000e-02	7.515702e-02		1.524758e-01	
40×40	3.750000e-02	4.959395e-02	0.5997	1.208166e-01	0.3358
80×80	1.875000e-02	3.397842e-02	0.5455	9.962925e-02	0.2782
160×160	9.375000e-03	2.457986e-02	0.4671	8.771474e-02	0.1838
320×320	4.687500e-03	1.746742e-02	0.4928	7.491543e-02	0.2276
$N$	$h = 2\Delta x$	$ u - \tilde{u} _1$	$EOC_1$	$ u - \tilde{u} _2$	$EOC_2$
20×20	1.000000e-01	9.600310e-02		1.735468e-01	
40×40	5.000000e-02	6.332254e-02	0.6004	1.362808e-01	0.3487
80×80	2.500000e-02	4.349752e-02	0.5418	1.129455e-01	0.2710
160×160	1.250000e-02	3.128702e-02	0.4754	9.843819e-02	0.1983
320×320	6.250000e-03	2.221442e-02	0.4941	8.400084e-02	0.2288

TABLE 3.3

Discrete  $L_1$ - and  $L_2$ -errors for the linear advection equation with a smooth initial condition, for different  $h$

$N$	$h = \Delta x$	$ u - \tilde{u} _1$	$EOC_1$	$ u - \tilde{u} _2$	$EOC_2$
20×20	5.000000e-02	1.747204e-02		5.529881e-02	
40×40	2.500000e-02	9.979376e-03	0.8080	3.382673e-02	0.7091
80×80	1.250000e-02	5.163706e-03	0.9505	1.808657e-02	0.9032
160×160	6.250000e-03	2.772294e-03	0.8973	1.011878e-02	0.8379
320×320	3.125000e-03	1.594091e-03	0.7983	5.705053e-03	0.8267
$N$	$h = 1.5\Delta x$	$ u - \tilde{u} _1$	$EOC_1$	$ u - \tilde{u} _2$	$EOC_2$
20×20	7.500000e-02	2.287496e-02		6.967677e-02	
40×40	3.750000e-02	1.353946e-02	0.7566	4.515581e-02	0.6258
80×80	1.875000e-02	7.793727e-03	0.7968	2.727371e-02	0.7274
160×160	9.375000e-03	4.302940e-03	0.8570	1.544056e-02	0.8208
320×320	4.687500e-03	2.610491e-03	0.7210	9.359223e-03	0.7223
$N$	$h = 2\Delta x$	$ u - \tilde{u} _1$	$EOC_1$	$ u - \tilde{u} _2$	$EOC_2$
20×20	1.000000e-01	2.834719e-02		8.127777e-02	
40×40	5.000000e-02	1.769121e-02	0.6802	5.676342e-02	0.5179
80×80	2.500000e-02	1.043911e-02	0.7610	3.569118e-02	0.6694
160×160	1.250000e-02	5.926685e-03	0.8167	2.104068e-02	0.7624
320×320	6.250000e-03	3.485152e-03	0.7660	1.262132e-02	0.7373

**4. Conclusions.** In this paper we have further investigated the FVPM from a numerical and theoretical point of view. We derived the FVPM for a system of conservation laws in a bounded and time-dependent domain  $\Omega(t) \subset \mathbb{R}^d$ . Then we

proved an approximation property of the reconstruction formula used in the FVPM, namely the fact that the reconstruction formula is only of order one. Also a new correction procedure, although only approximately conservative, was proposed.

Furthermore, we presented here an application of the FVPM to a spatial two-dimensional problem posed on a moving domain, where the meshless character of the method is exploited. The particles are irregularly distributed in the domain and they are moving in a non-Lagrangian way such that they smoothly follow the time-dependent computational domain. Numerical results indicate that the method is well-suited for such problems. Also the discretization of the boundary conditions works very satisfactory. Thus, a first step to applying the FVPM to real fluid-structure interaction problems, which in general limit the use of grid-based methods, is done.

Finally, a convergence study for a shock tube problem and a simple linear advection equation was done.

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