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Precise Volume Estimates in Nonpositive Curvature

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PRECISE VOLUME ESTIMATES IN NONPOSITIVE CURVATURE

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ABSTRACT. We show that on a compact manifold of nonpositive curvature and rank 1 the volume of spheres (hence also that of balls) has an exact asymptotic; it is purely exponential, and the growth rate equals the topological entropy.

The resulting formula is the sharpest one which is known. It generalizes results of G. A. Margulis to the nonuniformly hyperbolic case. It improves the multiplicative asymptotic bound by G. Knieper.

1. INTRODUCTION

Let M be a compact smooth Riemannian manifold whose sectional curvature is nonpositive. We assume the (geometric) *rank* of M to equal 1; that is, there exists a geodesic which has no parallel Jacobi field except multiples of its velocity vector. (For the geometric background, see [BuKa], [BuSp], [Esch], [Bal], [Ebe], [Gro], [BGS] and [KaHa].)

Let $b_r(x) := \operatorname{vol} B_r(x)$ be the Riemannian volume of the ball of radius r around x in $\tilde{M}$ (the universal cover of M). Let h be the topological entropy of the geodesic flow on the unit sphere bundle of M. We show that

$$b_r(x) \sim c(x)e^{hr}$$

for a continuous function $c : M \to \mathbb{R}$.

This result was obtained by G. A. Margulis in the special case that the curvature is strictly negative everywhere; in that case the geodesic flow is uniformly hyperbolic. His result was published in [Mar2]. The proofs were part of his doctoral dissertation [Mar1] and were finally published in [Mar4].

In our situation, the problem is more difficult since we are dealing with a *non-uniformly* hyperbolic system. In particular, in our setup one has to deal with the *singular set* where the product structure of

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stable and unstable manifolds breaks down. We show how to overcome this problem.

It is known ([Man]) that in nonpositive curvature the exponential growth rate of volume equals h. The best known result so far in this setting is the estimate in [Kni2] and [Kni3] provided by G. Knieper which says that there exists a constant C such that asymptotically $1/C < e^{-ht} b_t(x) < C$. However, the upper and lower bound in his estimates cannot be made to be asymptotically close with the methods he provides. Our methods presented in this article give upper and lower bounds which are asymptotically the same. We use Margulis' methods and refine them with techniques from the geometry from nonpositive curvature.

2. Construction of (un-)stable measures on fibers which are not in the (un-)stable foliation

Let SM be the unit sphere bundle of M and let $(g^t)_{t\in\mathbb{R}}$ be the geodesic flow. Recall that the regular set $\mathbf{Reg} := \{v \in SM : \text{rank}(v) = 1\}$ is open and dense in SM. Recall that the regular set has a local product structure with respect to the foliations W^s (stable manifolds), W^u (unstable manifolds), and flow lines of the geodesic flow. Denote by W^{0u} the weakly unstable leaves (integral manifolds of W^u and flow lines). The set $\mathbf{Reg}$ has full measure with respect to the measure m of maximal entropy, and m is supported on $\mathbf{Reg}$.

The articles [Gun1] and [Gun2] give independent proofs of the following: On $\mathbf{Reg}$, the measure of maximal entropy has conditional measures m^{0u}, m^s, supported on the weakly unstable and on the stable foliation, respectively. They have the property of being uniformly expanding and contracting, i.e. $m^{0u} \circ g^t = e^{ht} m^{0u}$ and $m^s \circ g^t = e^{-ht} m^s$. Moreover, they are holonomy invariant, i.e. two nearby sets in W^{0u} which are pointwise uniquely connected by short W^s-fibers have the same m^{0u}-measure. Those two articles also provide different constructions of the conditionals. Next we show how to extend the conditional measure m^s to a measure on sets which are not necessarily subsets of W^s; only transversality to W^{0u} is required. Similarly we show how to extend m^{0u} to arbitrary sets which are transversal to W^s.

Let $\mathcal{K} \subset SM$ be a compact submanifold of dimension $\dim M - 1$ which is transversal to W^{0u}. Let $\mathcal{L} \subset SM$ be a compact submanifold of dimension $\dim M$ which is transversal to W^s. We denote by $B_\varepsilon(p)$, $B^s_\varepsilon(p)$, $B^u_\varepsilon(p)$ etc. the open ball of size ε around p in M, in the stable leaf of p, in the unstable leaf of p, etc. If D is an open subset of the regular set which has diameter ε, which has such a product structure and which is topologically a ball, then we use the notation$B^s_D(p) := B^s_{2\varepsilon}(p) \cap D$, $B^{0u}_D(p) := B^{0u}_{2\varepsilon}(p) \cap D$.

Definition 2.1. For a point $p \in D$ define the projection $\pi^s_{D,p} : D \to B^s_D(p)$ by

$$\pi^s_{D,p}(x) := B^{0u}_\varepsilon(x) \cap B^s_D(p).$$

For a set $K \subset \mathcal{K} \cap D$ define the function

$$\mathrm{preimg}_{K,D,p}(x) := \#\{y \in K : \pi^s_{D,p}(y) = x\}.$$

This function is integer-valued and semicontinuous from below, hence integrable.

Definition 2.2. Define

$$m^s_{D,p}(K) := \int_{B^s_D(p)} \mathrm{preimg}_{K,D,p}(x) dm^s(x).$$

In the following, we will often deal with pairs of quantities whose proximity we want to quantify:

Definition 2.3. For $f_1,f_2 \in \mathbb{R}$ with $f_1/f_2 \in (1/2, 2)$ we define the **logarithmic distance** by

$$d_{\log}(f_1, f_2) := \left| \ln \frac{f_1}{f_2} \right|.$$

This quantifies the proximity of f_1 and f_2. Evidently for $f_1/f_2 \in [0.9, 1.1]$, the expressions $|f_1/f_2 - 1|$ and $d_{\log}(f_1, f_2)$ differ at most by a factor 2. However, error estimates are easier using $d_{\log}$ since it satisfies the triangle inequality $d_{\log}(f_1, f_3) \leq d_{\log}(f_1, f_2) + d_{\log}(f_2, f_3)$ and hence is a distance function.

In the following we consider functions $f_1 = f_1(t, x),\ f_2 = f_2(t, x) : [0, \infty) \times M \to \mathbb{R}$ be functions depending on a base point x and a parameter t (which will tend to ∞). We write $d_{\log}(f_1, f_2)$ as before; in that case this will depend on x, t. In particular, we will show that there is a continuous function $c : M \to \mathbb{R}$ so that for all $x \in M$ we have

$$\lim_{t\to\infty} d_{\log}(b_t(x), c(x)e^{ht}) = 0.$$

Lemma 2.4. *Let $\mathcal{K}$, ε and D be as above, let $D' \subset D$ be open and have a product structure. For $p \in D$, $p' \in D$ and $K \subset D \cap D'$ we have*

$$d_{\log}\left(m^s_{D,p}(K), m^s_{D',p'}(K)\right) < h\varepsilon.$$

Proof. Recall that the measure m^s contracts uniformly with exponent h in the time direction, i.e. $m^s \circ g^t = e^{-ht} m^s$, and is invariant under holonomy in u-direction. Using the product structures in D and D' there is a bijective $0u$-holonomy from $\pi^s_{D,p}(K)$ to $\pi^s_{D',p'}(K)$ that moves points by at most ε. □

Now we show how Margulis' construction of measures [Mar4] is done in nonpositive curvature:

Definition 2.5. Let M be a manifold of nonpositive curvature and rank 1 and let $\mathcal{K} \subset M$ be a compact submanifold. A **regular partition-cover** of $\mathcal{K}$ of size ε is a triple $(\mathbf{D}, \mathbf{K}, \mathbf{p})$ where $\mathbf{D} = (D_i)_{i\in\mathbb{N}}$ is an open cover of $\mathbf{Reg}$ so that all D_i have a product structure and are of diameter at most ε, where $\mathbf{p} = (p_i)_{i\in\mathbb{N}}$ with $p_i \in D_i$ for all i, and where $\mathbf{K} = (K_i)_{i\in\mathbb{N}}$ is a (disjoint) partition of $\mathcal{K} \cap \mathbf{Reg}$ such that $K_i \subset D_i$ for all i.

Note that a *regular partition-cover* does not cover the singular part of $\mathcal{K}$. To avoid multiindices we allow the K_i to be empty.

Definition 2.6. For a regular partition-cover $(\mathbf{D}, \mathbf{K}, \mathbf{p})$ we define a measure $m^s_{\mathbf{D},\mathbf{K},\mathbf{p}}$ on $\mathcal{K}$ by declaring that for $K \subset \mathcal{K} \cap \mathbf{Reg}$ the measure is

$$m^s_{\mathbf{D},\mathbf{K},\mathbf{p}}(K) := \sum_{i\in\mathbb{N}} m^s_{D_i,p_i}(K \cap K_i)$$

and declaring that it vanishes on the singular set (the complement of the regular set), i.e. $m^s_{\mathbf{D},\mathbf{K},\mathbf{p}}(\mathcal{K} \cap \mathbf{Sing}) := 0$. Hence $m^s_{\mathbf{D},\mathbf{K},\mathbf{p}}$ is defined on all of $\mathcal{K}$, including the singular part.

Lemma 2.7. *Let* $(\mathbf{D}, \mathbf{K}, \mathbf{p})$ *and* $(\mathbf{D}', \mathbf{K}', \mathbf{p}')$ *be regular partition-covers of* $\mathcal{K}$ *of size* ε*. Then*

$$d_{\log}\left(m^s_{\mathbf{D},\mathbf{K},\mathbf{p}}(K), m^s_{\mathbf{D}',\mathbf{K}',\mathbf{p}'}(K)\right) < 2h\varepsilon.$$

Proof. Use lemma 2.4 for a common refinement of $\mathbf{D}$ and $\mathbf{D}'$. Note that the common refinement contains only sets which are ε-small and ε-close to a set in each $\mathbf{D}$ and $\mathbf{D}'$. Thus each holonomy from $\pi^s_{D_i\cap D'_j,p}(K)$ to $\pi^s_{D_i\cap D'_j,p'}(K)$ moves points by distance at most 2ε. □

Definition 2.8. Choose a sequence $(\mathbf{D}_i, \mathbf{K}_i, \mathbf{p}_i)_{i\in\mathbb{N}}$ of regular partition-covers of $\mathcal{K}$ of size $1/i$. Let

$$m^s_{\mathcal{K}}(K) := \lim_{i\to\infty} m^s_{\mathbf{D}_i,\mathbf{K}_i,\mathbf{p}_i}(K).$$

By the previous lemma, this does not depend on the sequence $(\mathbf{D}_i, \mathbf{K}_i, \mathbf{p}_i)_{i\in\mathbb{N}}$ chosen. Note that for all $K \subset \mathcal{K}$ and for all $\varepsilon > 0$ there is $N(\varepsilon)$ so that for all $i > N(\varepsilon)$ we have

$$d_{\log}(m^s_{\mathcal{K}}(K), m^s_{\mathbf{D}_i,\mathbf{K}_i,\mathbf{p}_i}(K)) < h\varepsilon.$$

Hence $m^s_{\mathcal{K}}$ is an additive measure.

Similarly, for $\mathcal{L}$ compact and transversal to W^s we construct a measure $m^{0u}_{\mathcal{L}}$ by repeating the construction with s and $0u$ exchanged. Once again, we declare that $m^{0u}_{\mathcal{L}}(L)$ is zero for $L \subset$ **Sing**, which makes $m^{0u}_{\mathcal{L}}$ defined on all of $\mathcal{L}$, including the singular part.

It is interesting to note that for the construction of $m^{0u}_{\mathcal{L}}$, we do not even need the limit of $i \to \infty$ since the holonomy along s-fibers (as opposed to $0u$-fibers) leaves the measure in the $0u$-direction strictly invariant.

Similarly, for Λ compact and transversal to W^{0s} we get a measure m^u.

We normalize $dm = dm^u dm^s dt$ so that $m(SM) = 1$.

3. Fiberwise ergodic theorems

Definition 3.1. A family $\mathbf{F}$ of functions $SM \to \mathbb{R}$ is called **uniformly equicontinuous in the $0u$-direction** iff $\forall\varepsilon > 0\, \exists\delta > 0\, \forall f \in \mathbf{F}, \forall p, q \in SM$ the condition $d^{0u}(p,q) < \delta$ implies $|f(p) - f(q)| < \varepsilon$.

We apply this later to nonnegative functions in $C^0(SM, [0,\infty))$.

Lemma 3.2. *For any f which is continuous in the $0u$-direction, the family $\mathbf{F} = \{f \circ g^t : t \leq 0\}$ is uniformly equicontinuous in the $0u$-direction.*

Proof. Note that by compactness of SM, the function f is automatically uniformly continuous in $0u$-direction, i.e. $\forall\varepsilon > 0\, \exists\delta > 0\, \forall p, q \in SM : d^{0u}(p,q) < \delta$ implies $|f(p) - f(q)| < \varepsilon$. By nonpositivity of the curvature, d^{0u} is nondecreasing with the flow (in positive time direction), i.e. $d^{0u}(p,q) < \delta$ implies $d^{0u}(g^t p, g^t q) < \delta$ for all $t \leq 0$, thus $|f(g^t p) - f(g^t q)| < \varepsilon$. □

Note that $d^{0u}(p,q) < \delta$ does not imply that $d^{0u}(g^t p, g^t q) \to 0$ as $t \to -\infty$, unlike in the hyperbolic case.

Definition 3.3. If the set D has a product structure, then for all p in D, the measure $m^{0u}(B^{0u}_D(p))$ is the same (because of strict holonomy invariance along W^s-fibers). We call this number $S(D)$.

Lemma 3.4. *Assume $K \subset W^s$, $D \subset SM$ open and with product structure, for all $i \in \mathbb{N}$ let D_i be open in D, with $B_D(p) \subset D_i$ for $p \in D_i$, with*

$K = \cap_{i\in\mathbb{N}} D_i$ and $D_i \subset B^{0u}_{1/i}(K)$. Let $\mathbf{F}$ be uniformly equicontinuous in $0u$-direction. Then, for all $f \in \mathbf{F}$ we have

$$\lim_{i\to\infty} \frac{1}{S(D_i)} \int_{D_i} f\, dm = \int_K f\, dm^s.$$

Proof. By uniform equicontinuity, for all $\gamma > 0$ there exists $i_0 \in \mathbb{N}$ such that for all $x, y \in M$ with $d^{0u}(x, y) < 2/i$ and for all $f \in \mathbf{F}$ we have $|f(x) - f(y)| < \gamma$. Now we use holonomy invariance and uniform contraction of the conditional measure on stable leaves and deduce that if $H : D_i \to K$ is the holonomy map which moves along $0u$-leaves in D_i, i.e. $H(q) = K \cap B^{0u}_{D_i}(q)$, then we have

$$d_{\log}\left(dm^s, dm^s \circ H\right) < h/i$$

i.e. if $v \in D_i$ and $w \in K \cap B^{0u}_{D_i}(v)$, then for all u we have

$$d_{\log}\left(dm^s(v), dm^s(w)\right) < h/i.$$

Therefore for all $f \in \mathbf{F}$ we have

$$\begin{aligned}&\left|\frac{1}{S(D_i)} \int_{D_i} f\, dm - \int_K f\, dm^s\right| \\ &\qquad \le \int_K \left(e^{h/i} \max_{x\in B^{0u}_D(x)\cap K} f(x) - e^{-h/i} \min_{x\in B^{0u}_D(x)\cap K} f(x)\right) \le e^{2h/i} 2\gamma.\end{aligned}$$

□

Proposition 3.5. *Assume $K \subset W^s$, $D \subset SM$ open and with product structure, for all $i \in \mathbb{N}$ let D_i be open in D, with $B_D(p) \subset D_i$ for $p \in D_i$, with $K = \cap_{i\in\mathbb{N}} D_i$ and $D_i \subset B^{0u}_{1/i}(K)$. Let $f : D \to [0,\infty)$ be uniformly continuous in the $0u$-direction. Then*

$$\int_{g^t K} f\, dm^s \sim e^{-ht} m^s(K) \int_{SM} f\, dm$$

for $t \to \infty$. Indeed,

$$d_{\log}\left(\int_{g^t K} f\, dm^s, e^{-ht} m^s(K) \int_{SM} f\, dm\right) < c_1(\mathrm{diam}\,(D_i)) + c_2(t).$$

Proof. Using the previous two lemmata, for all $\varepsilon > 0$, for i large enough and for all $t \le 0$,

$$\begin{aligned}&d_{\log}\left(\frac{1}{S(D_i)} \int_{D_i} f \circ g^t\, dm, \int_K f \circ g^t\, dm^s\right) \\ &= d_{\log}\left(\frac{1}{S(D_i)} \int_{D_i} f\, dm, \int_K f\, dm^s\right) < \gamma.\end{aligned}$$

Using the mixing property (see [Bab]),

$$\int_{D_i} f \circ g^t \, dm = \int_{SM} \chi_{D_i} \cdot (f \circ g^t) dm \to m(D_i) \int_{SM} f \, dm$$

for $t \to -\infty$. (Here χ_{D_i} is the characteristic function of D_i.) In other words, there exists a function $c_2 : [0, \infty) \to [0, \infty)$, $c_2 = c_2(i, t)$, so that for all i we have $\lim_{t \to \infty} c_2(i, t) = 0$.

$$d_{\log} \left(\int_{D_i} f \circ g^t \, dm, m(D_i) \int_{SM} f \, dm \right) < c_2(i, t).$$

Hence

$$d_{\log} \left(\int_K f \circ g^t dm^s, \frac{m(D_i)}{S(D_i)} \int_{SM} f \, dm \right) < \gamma + c_2(i, t).$$

Note that

$$\frac{m(D_i)}{S(D_i)} \to m^s(K) \quad \text{for} \quad i \to \infty.$$

Thus, using the uniform expansion property

$$\int_{g^t K} f \, dm^s = e^{-ht} \int_K f \circ g^t \, dm^s,$$

we get the claim. □

Lemma 3.6. *For $D \subset SM$ open and $K \subset W^s \cap \mathbf{Reg}$:*

$$m^s(D \cap g^t K) \sim e^{ht} m^s(K) m(D).$$

Proof. First assume that D has a product structure. Choose a decreasing nested sequence $(D_i)_{i \in \mathbb{N}}$ of open sets with $D = \bigcap_{i \in \mathbb{N}} D_i$. Choose a pointwise nonincreasing sequence of continuous functions f_i which are 1 on D and 0 outside D_i. Then the previous proposition states that

$$\int_{g^t K} f_i \, dm^s \sim e^{-ht} m^s(K) \int_{SM} f_i \, dm,$$

and letting $i \to \infty$ shows the claim (for this D). Next, note that both sides of the claimed equation are additive in D. Any open subset of $\mathbf{Reg}$ is the union of product cubes, thus the claim is proven for regular D. □

4. Holonomy continuity and regular neighborhoods

The counting argument in section 7 requires a certain function to be continuous. That property is easily established in the uniformly hyperbolic case; however, for the nonuniform case that we are dealing with in this article, it is quite nontrivial. This section is devoted entirely to that point. We use several fairly new results about the measure of maximal entropy for the geodesic flow, in particular existence of conditional measures, holonomy invariance and uniform expansion for those. The holonomy continuity discussed here differs from the holonomy invariance proved in [Gun1] and [Gun2]: Instead of taking a set and its holonomic counterpart and showing that the conditional measure is preserved, we show that nearby sets of given geometry have similar conditional measure.

Definition 4.1. For $x, y \in SM$ let $d^s(x, y)$ be the distance of x and y along stable leaves; if $x \notin W^s(y)$ then $d^s(x, y) = \infty$. For $r \in [0, \infty)$ define $\vartheta_r : [0, \infty] \to [0, \infty)$ by

$$\vartheta_r(t) := \max(0, r - t).$$

For $r \in [0, \infty)$ and $v, w \in SM$ define

$$\sigma_r(v, w) := \vartheta_r(d^s(v, w)).$$

Finally, define

$$\psi_r(v) := \int_{z \in B_r^s(v)} \sigma_r(v, z)\, dm^s(z).$$

Evidently $\sigma_r(.,.)$ is symmetric. Note that it is also Lipschitz with Lipschitz constant 1 along W^s-leaves, i.e. for all z contained in at least one of the leaves $W^s(x)$ and $W^s(y)$ we have

$$|\sigma_r(v, z) - \sigma_r(w, z)| \le d^s(v, w).$$

This is so because ϑ is 1-Lipschitz.

Theorem 4.2. *$\psi_r(v)$ is continuous in r. It is continuous in v along any W^s-leaf. If $\overline{B_r^s}(v) \subset$ **Reg** then $\psi_r(v)$ is continuous in all variables at v.*

Proof. Continuity of $\psi_r(v)$ in r easily follows from the fact that for all v, w the map $r \mapsto \sigma_r(v, w)$ is continuous.

To show continuity of ψ_r in v in the s-direction, let v, w be such that $d^s(v, w) < \delta$. Write $A_1 := B_r^s(v) \cap B_r^s(w)$ and $A_2 := B_r^s(v) \triangle B_r^s(w)$;

then

$$|\psi_r(v) - \psi_r(w)| \le m^s(A_1) \sup_{z\in A_1} |\sigma_r(v,z) - \sigma_r(w,z)|$$
$$+ m^s(A_2) \cdot \left(\sup_{z\in A_2} |\sigma_r(w,z)| + \sup_{z\in A_2} |\sigma_r(v,z)| \right).$$

Note that if $z \in A_2$ then $d^s(v,w) \in [0,\delta)$, thus $\sigma_r(v,z) < \delta$ and $\sigma_r(w,z) < \delta$. Thus the first summand on the right hand side is at most $\delta m^s(A_1)$ and the second at most $2\delta m^s(A_2)$. Thus $\psi_r(v)$ and $\psi_r(w)$ are arbitrarily close for v, w sufficiently close. Thus ψ_r is continuous along any W^s-leaf.

Now let $\overline{B^s_r}(v) \subset \mathbf{Reg}$; we want to show continuity in v. Continuity in the s-direction is shown above. Next we deal with the flow direction. Let $w = g^\delta v$. Then

$$\begin{aligned} \psi_r(w) &= \int_{z\in B^s_r(g^\delta v)} \sigma_r(g^\delta v, z)\, dm^s(z) \\ &= \int_{z'\in g^{-\delta} B^s_r(g^\delta v)} \sigma_r(g^\delta v, g^\delta z')\, dm^s(g^\delta z'). \qquad (1) \end{aligned}$$

Recall that on a manifold of nonpositive curvature, any stable Jacobi field is nonincreasing in length; hence the map $F : t \mapsto d^s(g^t v, g^t w)$ is nonincreasing, thus for $t \ge 0$ its values are bounded by $d^s(v,w)$. Thus

$$g^\delta B^s_r(v) \subset B^s_r(g^\delta v).$$

On the other hand, the decrease of F is bounded by the derivative of the unstable Jacobi field, which is bounded due to compactness of M. Hence for all $\varepsilon > 0$ there is $\delta_0 > 0$ so that for all $\delta < \delta_0$ we have $B^s_\varepsilon(g^\delta B^s_r(v)) \supset B^s_r(g^\delta v)$.

First note that $\delta \mapsto \sigma_r(g^\delta v, g^\delta z')$ is continuous in δ because the foliation W^s is continuous. Next note that $dm^s(g^\delta z')$ is continuous in δ because it is uniformly expanding in δ and $e^{h\delta}$ is arbitrarily close to 1 for δ sufficiently small. Finally note that $\forall v \in SM\, \forall \varepsilon > 0\, \exists \delta_0 > 0\, \forall \delta < \delta_0$ we have

$$B^s_r(g^\delta v) \supset g^\delta B^s_r(v) \supset B^s_{r-\varepsilon}(g^\delta v),$$

hence the value of $\sigma_r$is at most ε on the set $B^s_r(g^\delta v) \setminus g^\delta B^s_r(v)$. This shows that $\psi_r(w)$ and $\psi_r(v)$ are close because in the last line of equation (1) all terms are continuous with respect to δ.

Finally, we show that ψ_r is continuous in the u-direction. Assume that $w \in B^u_\delta(v)$. Recall that the measure m^{0s} is invariant under holonomy along W^u-fibers. Hence if H is a holonomy map along W^{0u} from some set $A \subset W^s$ to some set $B \subset W^s$ so that for all $v_1 \in A$ the points v_1 and Hv_1 are ε_1-close, then $d_{\log}(m^s(A), m^s(B)) < h\varepsilon_1$. We are interested in the case $A = B^s_r(v),\ w \in B$.

Note that due to the condition that $\overline{B^s_r}(v) \subset \mathbf{Reg}$ there is some open neighborhood of $\overline{B^s_r}(v)$ which lies inside $\mathbf{Reg}$ and on which W^s, W^{0u} are uniformly transversal.

Note that for H as above we have $|\sigma_r(Hv_1, Hv_2) - \sigma_r(v_1, v_2)| \le 2\varepsilon_1$ by 1-Lipschitzness of σ_r. Thus

$$\begin{aligned}\psi_r(w) &= \int_{z\in B^s_r(w)} \sigma_r(z,w)\,dm^s(z)\\ &\le \int_{z'\in H^{-1}B^s_r(w)} (\sigma_r(z',v) + 2\varepsilon_1)\ dm^s(Hz')\\ &\le \int_{z'\in B^s_{r+\varepsilon_2}(v)} (\sigma_r(z',v) + 2\varepsilon_1)\ dm^s(z')\\ &\le \psi_r(v) + 3\varepsilon_1 m^s(B^s_{r+\varepsilon_1}(v)).\end{aligned}$$

Letting $\varepsilon_1 \to 0$ shows that $\psi_r(w)$ and $\psi_r(v)$ are arbitrarily close for v, w close enough. □

5. Measuring Riemannian volume by counting intersections

Let $x, y \in M$. Define

$$a_r(x,y) := \#(B_r(x) \cap (\pi_1(M)\cdot y))$$

to be the number of copies of y under Deck transformations that are inside the ball of radius r around x in the universal cover of M.

Lemma 5.1. *For all $x \in M$, $r \in \mathbb{R}$ we have*

$$b_r(x) = \int_{y\in M} a_r(x,y)d\operatorname{vol}(y),$$

where vol *is the Riemannian volume on M.*

Proof. ([Mar4]) Let F be a fundamental domain of M. Denote the characteristic function of B by χ_B. Then

$$\begin{aligned}b_r(x) &= \sum_{\gamma\in\pi_1(M)} \operatorname{vol}(\gamma F \cap B_r(x)) = \sum_{\gamma\in\pi_1(M)} \int_{y\in\gamma F} \chi_{B_r(x)} d\operatorname{vol}(y)\\ &= \int_{y\in F} \sum_{\gamma\in\pi_1(M)} \chi_{B_r(x)} d\operatorname{vol}(\gamma^{-1}y) = \int_{y\in M} a_r(x,y)d\operatorname{vol}(y).\end{aligned}$$

□

For $x, y \in M$ assume $\mathcal{K} := S_yM$ to be transversal to W^{0u} and $\Lambda_0 := S_xM$ to be transversal to W^s. Let $\mathcal{L} := g^{[0,a]}\Lambda_0$. Let $\mathcal{K}, \Lambda_0$ be the disjoint unions $\mathcal{K} = \bigcup_j K_j$, $\Lambda_0 = \bigcup_i L_i$. Then

$$\begin{aligned} a_t(x,y) &= \#((\pi_1(M)\cdot y) \cap B_t(x)) \\ &= \#(S_yM \cap g^{[0,t]}S_xM) \\ &= \#(\mathcal{K} \cap g^{[0,t]}\Lambda_0) \\ &= \sum_{i,j} \#(K_j \cap g^{[0,t]}L_i). \end{aligned}$$

Therefore it suffices to be able to count these intersections in order to find b_r. Note that these intersections are always finite, even though the components of intersection of stable and unstable manifolds can be uncountable.

For uniform hyperbolicity, it would now suffice to treat the single choice of submanifolds $\mathcal{K} = S_yM,\ \mathcal{L} = S_xM$ and calculate the number $a_t(x,y)$ in one single counting step. The counting step requires a product structure in the neighborhood of the submanifolds $\mathcal{K},\ \mathcal{L}$. In our setup of nonpositive curvature, no such (global) product neighborhood exists. Instead we proceed as follows: We decompose our submanifolds into the regular and singular part. Concerning the latter, we show that it asypmtotically contributes only a zero proportion and hence can be ignored. The former is decomposed into countably many pieces, each of which has a product neighborhood. For each piece we calculate a multiplicative asymptotic, and then we show that the asymptotics can be combined in a multiplicative asymptotic for the whole set.

Definition 5.2. Let $x, y \in M$, $A \subset S_xM$. Define

$$a_t(A,y) := \#(S_yM \cap g^{[0,t]}A)$$

to be the number of copies of y under Deck transformations which can be reached from x by a geodesic of length at most t and with initial velocity in A.

Lemma 5.3. *For all $x, y \in M$ we have*

$$\lim_{t\to\infty} \frac{a_t(S_xM \cap \mathbf{Sing}, y)}{a_t(S_xM \cap \mathbf{Reg}, y)} = 0.$$

In fact, there exists $\alpha > 0$ and $t_0 < \infty$ so that for all $t > t_0$ we have

$$\frac{a_t(S_xM \cap \mathbf{Sing}, y)}{a_t(S_xM \cap \mathbf{Reg}, y)} \le e^{-\alpha t}.$$

Obviously the same holds with the denominator replaced by $a_t(x, y)$.

Proof. For any ε smaller than half the injectivity radius of M, any two geodesics from x to y of length at most T are (T, ε)-separated. Therefore $a_T(\mathcal{L}_{\mathbf{Sing}}, y) \leq N(g|_{\mathbf{Sing}}, T, \text{inj}(M)/2)$ where $N(g|_{\mathbf{Sing}}, T, r)$is the maximum cardinality of an (T, r)-separated set with respect to the flow $g|_{\mathbf{Sing}}$ (the geodesic flow restricted to the singular set) and with respect to the distance function on SM induced by the Sasaki metric. Taking growth rates, we get

$$\lim_{t\to\infty} \frac{1}{t} a_t(\mathcal{L}_{\mathbf{Sing}}, y) \leq h_{\text{top}}(g|_{\mathbf{Sing}}) < h_{\text{top}}(g) = h$$

where the last inequality was shown in [Kni1]. On the other hand, for all x, y there exists $c(x, y) > 0$ and $t_0 < \infty$ such that for all $t > t_0$ we have $a_t(x, y) \geq c(x, y)e^{ht}$. This is shown in more detail later, but for this lower estimate it suffices to notice that the growth rate of volume equals h ([Man], [Kni3]) and hence $a_t(x, y)$ has growth rate h. □

However, in our case we achieve this:

Lemma 5.4. *Fix $x, y \in M$. If $\mathcal{L}_{\mathbf{Reg}} = S_xM \cap \mathbf{Reg} = \bigcup_{i\in\mathbb{N}} A_i$ is a decomposition of $\mathcal{L}_{\mathbf{Reg}}$ into countably many disjoint open regular sets A_i and for each i there exists $C(i)$ such that $a_t(A_i, y) \sim C(i)e^{ht}$ then $a_t(\mathcal{L}_{\mathbf{Reg}}, y) \sim Ce^{ht}$ with $C = \sum_{i\in\mathbb{N}} C(i)$.*

Proof. Let $\hat{A}_i := \bigcup_{k<i} A_i$ and $R_i := \mathcal{L}_{\mathbf{Reg}} \setminus \hat{A}_i$. It is easy to see that the counting functions a_t and hence their asymptotics are finitely additive in A_i, i.e., we have $a_t(\hat{A}_i, y) = \sum_{k<i} a_t(A_k, y)$ and thus we have $a_t(\hat{A}_i, y) \sim \left(\sum_{k<i} C(i)\right) e^{ht}$.

We know that $a_t(R_i, y) \leq C(i)e^{ht}$ for some $C(i)$. We will show that $C(i) \to 0$ as $i \to \infty$.

Assume the contrary. Then there exists $C > 0$ such that for all $i \in \mathbb{N}$ and for all $t > t_0(i)$ we have $a_t(R_i, y) \geq Ce^{ht}$. By closedness and nestedness of the R_i and continuity of the flow and the uniform projection property of the conformal density (Busemann density) it follows that also $a_t(\bigcap_{i\in\mathbb{N}} R_i, y) \geq Ce^{ht}$. But $\bigcap_{i\in\mathbb{N}} R_i$ is empty and hence its a_t-value zero. □

Summarizing, we have shown:

Theorem 5.5. *Let $\mathcal{L}_{sing} \cup \bigcup_{i\in\mathbb{N}} A_i$ be a decomposition of $\mathcal{L} = S_xM$ (or $\mathcal{L} \subset S_xM$) into the singular part and countably many regular open sets A_i*

which are pairwise disjoint. Assume that for each i there is a constant $C(i)$ such that $a_t(A_i, y) \sim C(i)e^{ht}$. Then $a_t(x, y) \sim Ce^{ht}$ with $C = \sum_{i\in\mathbb{N}} C(i)$.

6. PRODUCT NEIGHBORHOODS

Let D be open with $\overline{D} \subset \mathbf{Reg}$. Hence it has a local product structure and transversality is uniform on D. Let $L \subset D$ be transversal to W^s. Note that for each $v \in L$ we have

$$\lim_{r\to 0} \psi_r(v) = 0.$$

Note that this is not true without assuming $L \subset D$ and $\bar{D} \subset \mathbf{Reg}$.

By compactness of $\overline{D}$ and continuity of $\psi_r(v)$ in r, the convergence $\psi_r(v) \to 0$ as $r \to 0$ is uniform with respect to v. Hence there exists $c_0 > 0$ so that for all $c < c_0$ and all $v \in \overline{L}$ there exists $r_c = r_c(v)$ so that $\psi_{r_c(v)}(v) = c$. From the product structure of D follows that for c sufficiently small, for each $v \in L$, the intersection of $B^s_{r_c(v)}(v)$ contains exactly one point of L.

Definition 6.1. Write

$$Z = Z(L, c) := B^s_{r_c}(L) := \bigcup_{v\in L} B^s_{r_c(v)}(v).$$

Let $\pi_L : Z \to L$ be the projection defined by $\pi_L(z) = l$ for $z \in B^s_{r_c(l)}(l)$. Define the function $f_{L,c}$ supported on $\overline{Z}$ by

$$f_{L,c}(z) := \sigma_{r_c}(z, \pi_L(z))$$

for $z \in Z$ and $f_{L,c}(z) := 0$ otherwise.

Then

$$\int_Z f_{L,c} dm = cm^{0u}_{\mathcal{L}}(L)$$

since $dm = dm^s dm^{0u}$ on D and since the stable measure of each s-fiber equals c.

For two subspaces E_1, E_2 of a vector space let

$$\mathbf{d}(E_1, E_2) := d_H(E_1 \cap S^1, E_2 \cap S^1)$$

be the distance in the Grassmannian bundle induced by the Hausdorff distance on unit spheres.

Lemma 6.2. *Let $\mathcal{K}$ be compact and transversal to W^{0u}. Then for all $\varepsilon > 0$ there exists t_0 so that for all $t > t_0$ we have*

$$\mathbf{d}(Tg^{-t}\mathcal{K}, TW^s) < \varepsilon$$

uniformly on $\mathcal{K}$.

Proof. Each $\xi \in Tg^{-t}\mathcal{K}$ can be written as $\xi = \xi^{\|} + \xi^{\perp}$ with $\xi^{\|} \in TW^s$, $\xi^{\perp} \perp TW^s$ (perpendicular with respect to the Sasaki metric on SM). Recall that for $t \to -\infty$, any unstable Jacobi field is bounded and any stable Jacobi field is unbounded. Since $(dg^t\xi)^{\|}$ is unbounded for $t \to -\infty$ and $(dg^t\xi)^{\perp}$ is bounded, it follows that $\angle((dg^t\xi), TW^s) \to 0$ for $t \to -\infty$. Compactness gives uniformity. □

7. Intersection estimate

Definition 7.1. For $\mathcal{K}$ compact and transversal to W^{0u} and $\mathcal{L}$ compact and transversal to W^s, $K \subset \mathcal{K}$, $L \subset \mathcal{L}$ define

$$Q(K, L, t) := \#(L \cap g^{-t}K),$$

define $\Phi(K, L, c, t)$ to be the set of connected components φ of $Z \cap g^{-t}\mathcal{K}$ such that if $p \in \varphi$ and $p \in B^s_{r_c}(l)$ for $l \in L$ then $B^s_{r_c}(l) = \varphi$, and finally define

$$N(K, L, c, t) := \#\Phi(K, L, c, t).$$

For a set K as above with the property that $K \subset D$ where D is open, has diameter $< \varepsilon$, and $\overline{D}$ carries a product structure, we abbreviate the notation $\pi_{D,p}(K)$ by K'. This means that the choice of p is suppressed in the notation and all of the following estimates are true for any choice of p.

Note that K' may be disconnected even if K is connected.

For a set $K' \subset W^s$ with $B_\alpha K' \subset D$ for $\alpha > 0$ we write

$$\hat{B}^s_\alpha K' := \pi_{D,p} B_\alpha K',$$

i.e. the set K' is (arbitrarily) extended by distance α in the s-direction. Similarly, for $\alpha < 0$ we write

$$\hat{B}^s_\alpha K' := (K \setminus B_{-\alpha}(\partial K))'$$

for the opposite, namely shrinking K' by distance α from the boundary of K. (We use the boundary of K, not that of K', because the transition from K to K' introduces new boundary points of K' which do not correspond to the boundary of K.

Note that by the previous lemma, for each $\mathcal{K}, \mathcal{L}$ there exists t_0 such that for $t > t_0$ the number $Q(K, L, t)$ is finite.

Clearly for K, L as before and for all $c > 0, t \geq 0$:

$$N(K, L, c, t) \leq Q(K, L, t).$$

On the other hand, unboundedness of stable Jacobi fields for $t \to -\infty$ and hence unboundedness of the diameter of the image of the annulus $(\hat{B}^s_\alpha K') \setminus K'$ under g^{-t} gives $\forall K, L,\ \forall c > 0\, \forall \alpha > 0\, \exists T = T(\alpha)\, \forall t > T$:

$$Q(K', L, t) \leq N(\hat{B}^s_\alpha K', L, t).$$

Moreover, $\forall \varphi \in \Phi(K', L, c, t)$:

$$\int_\varphi f_{L,c} dm^s = c.$$

Hence $\forall K, L,\ \forall c > 0,\ \forall t$:

$$N(K', L, c, t) \leq \frac{1}{c} \int_{SM} f_{L,c} \cdot (\chi_{K'} \circ g^t) dm^s.$$

On the other hand, $\forall \alpha > 0\, \exists T = T(\alpha)\, \forall t > T$:

$$N(\hat{B}^s_\alpha K', L, c, t) \geq \frac{1}{c} \int_{SM} f_{L,c} \cdot (\chi_{K'} \circ g^t) dm^s.$$

Note that $m^s_{\mathcal{K}}(\partial K') = 0 = m^{0u}_{\mathcal{L}}(\partial L)$.

In the following, the only type of L we need to consider is $L = g^{[0,t_0]}\Lambda$ for some Λ contained in some D_i with a product structure (which makes Λ transversal to W^{0s}). This L is the disjoint union $L = \bigcup_{i=0}^{n-1} L_i$ with $L_i \subset g^{[it_0/n,(i+1)t_0/n)}\Lambda$, and we can further assume without loss of generality that $L_i \subset D_i$ for $i < n$ (by renumbering the D_i appropriately).

Lemma 7.2. *For all $\alpha > 0$ there exists T such that for all $t > T$:*

$$\sum_{i=2}^{k-2} Q(K', \hat{B}^u_{-\alpha} L_i, t) \leq \sum_{i=1}^{k-1} Q(K, L_i, t) \leq \sum_{i=0}^{k} Q(K', \hat{B}^u_\alpha L_i, t).$$

Proof. Note that once more the nonpositivity of the curvature gives unboundedness of the boundary annulus as $t \to -\infty$. Moreover $g^{-t}K'$ is arbitrarily close to W^s as t becomes large. Note also that each point $p \in K$ which lies on L_i gets moved by at most ε in the flow direction under the map $K \mapsto K'$ and hence gets mapped back to L_i or gets mapped to $L_j$with $|i-j|\, a/k < \varepsilon$. Note that, for $i \neq j$, increasing the term $Q(K', \hat{B}^u_\alpha L_j, t)$ by 1 and simultaneously decreasing the term $Q(K', \hat{B}^u_\alpha L_i, t)$ by 1 does not change the sum in the statement of the lemma. $\square$

Theorem 7.3. *The following two estimates hold:*

$$\text{(1):}\ \limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k-1} Q(K,L_i,t) \ \leq\ e^{2h\varepsilon} m^s_{\mathcal{K}}(K)\sum_{i=0}^{k} m^{0u}_{\mathcal{L}}(L_i),$$

$$\text{(2):}\ \liminf_{t\to\infty} e^{-ht}\sum_{i=0}^{k} Q(K,L_i,t) \ \geq\ e^{-2h\varepsilon} m^s_{\mathcal{K}}(K)\sum_{i=1}^{k-1} m^{0u}_{\mathcal{L}}(L_i).$$

Proof. For the first inequality (1), we see that for $\alpha > 0$

$$\limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k} Q(K,L_i,t) \leq \limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k} Q(K',\hat{B}^u_\alpha L_i,t).$$

Hence for the limit as $\alpha \to 0$:

$$\begin{aligned}
\limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k} Q(K,L_i,t) &\leq \lim_{\alpha\to 0}\limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k} Q(K',\hat{B}^u_\alpha L_i,t) \\
&\leq \lim_{\beta\to 0}\lim_{\alpha\to 0}\limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k} N(\hat{B}^s_\beta K',\hat{B}^u_\alpha L_i,t) \\
&\leq \frac{1}{c}\lim_{\beta\to 0}\lim_{\alpha\to 0}\limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k}\int_{SM} f_{\hat{B}^u_\alpha L_i,c}\cdot(\chi_{\hat{B}^s_\beta K'}\circ g^t)\,dm^s.
\end{aligned}$$

Note that $f_{\hat{B}^u_\alpha L_i,c}$ is continuous by Theorem 4.2. Hence by Proposition 3.5 we have

$$\begin{aligned}
\limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k}\int_{SM} f_{\hat{B}^u_\alpha L_i,c}\cdot(\chi_{\hat{B}^s_\beta K'}\circ g^{t+\varepsilon})\,dm^s& \\
\leq e^{2h\varepsilon} m^s_{\mathcal{K}}\left(\hat{B}^s_\beta K\right)\sum_{i=1}^{k} m^{0u}_{\mathcal{L}}\left(\hat{B}^u_\alpha L_i,c\right).&
\end{aligned}$$

Taking $\lim_{\beta\to 0}\lim_{\alpha\to 0}$ we get

$$\begin{aligned}
\limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k}\int_{SM} f_{\hat{B}^u_\alpha L_i,c}\cdot(\chi_{\hat{B}^s_\beta K'}\circ g^{t})\,dm^s& \\
\leq e^{2h\varepsilon} m^s_{\mathcal{K}}(K)\sum_{i=1}^{k} m^{0u}_{\mathcal{L}}(L_i).&
\end{aligned}$$

Hence

$$\limsup_{t\to\infty} e^{-ht}\sum_{i=1}^{k} Q(K,L_i,t) \le e^{2h\varepsilon} m^s_{\mathcal{K}}(K)\sum_{i=1}^{k} m^{0u}_{\mathcal{L}}(L_i).$$

By mixing, the summands for $i = 0$ and $i = k$ are not bigger than the others, hence can be ignored in the limit.

The second inequality (2) is proven the same way, using the opposite estimates for exchanging Q with N and N with the integral. □

Using the fact that $m^s_{\mathcal{K}}(\partial K') = 0 = m^{0u}_{\mathcal{L}}(\partial L)$, we reach the following conclusion:

Corollary 7.4. *For L regular and K arbitrary (or for K regular and L arbitrary), we get*

$$\lim_{t\to\infty} e^{-ht}\sum_{i=1}^{k} Q(K,L_i,t) = m^s_{\mathcal{K}}(K)\sum_{i=1}^{k} m^{0u}_{\mathcal{L}}(L_i).$$

Note that regularity is invariant under the flow. Hence if L is regular, any intersection of L with $g^{-t}K$ can only occur at regular points. It is therefore sufficient if just one of the two sets L, K is regular. Hence the case we have treated before suffices.

If neither K nor L is regular, then we split it as $K = \bigcup_j K_j$ and $L = \bigcup_i \tilde{L}_i$ where each K_j and L_i is a subset of $\mathbf{Reg}$ or $\mathbf{Sing}$. We have dealt with the former and are going to show that the latter does not distort the count. Without loss of generality $\tilde{L}_i = \bigcup_{k=0}^{n-1} g^{kt/n} L_i$ and $L_i = g^{[0,t_0]}\Lambda_i$.

Proposition 7.5. *There exists $\gamma > 0$ such that for $K_j \subset \mathbf{Sing}$ or $L_i \subset \mathbf{Sing}$ there exists $T \in \mathbb{R}$ such that for $t > T$ we have*

$$0 \le \#\left(K_j \cap g^t L_i\right) \le e^{(h-\gamma)t}.$$

Proof. Let $v, w \in K \subset S_yM$ be such that $g^{-t}v, g^{-t+a_1}w \in \Lambda_i \subset S_xM$ for $t \ge 0$ and $0 \le a_1 \le t_0$. Then the geodesic segments $g^{[-t,0]}v$ and $g^{[-t+a_1,0]}w$ either form a geodesic biangle (i.e. 2-gon, bounding a topological 2-disc) or form a topologically nontrivial loop.

Note that in a space of nonpositive curvature, any geodesic biangle is degenerate, i.e. subset of a single geodesic. This is so because any biangle is in particular a triangle with one side of zero length, and by triangle comparison with flat 2-space the nonpositivity of the curvature shows that both interior angles of the biangle are 0.

Hence either $v = w$ or the orbits of v, w are (t, l_0)-separated, where $2l_0 > 0$ is the length of the shortest closed geodesic in M.

Since the growth rate of any separated set in **Sing** is less than or equal to the topological entropy of **Sing** and since $h_{\mathbf{Sing}} < h$ ([Kni1]), the claim follows. ☐

From this we deduce:

Theorem 7.6. *For each $x \in M$ there exists $c(x)$ so that*

$$\operatorname{vol} B_t(x) \sim c(x)e^{ht}.$$

The function $c : M \to \mathbb{R}$ is continuous. It satisfies

$$c(x) = \frac{1}{h} \int_{y \in M} a(x,y) d \operatorname{vol}(y)$$

where

$$a(x,y) = m^s_{\mathcal{K}}(S_x M) m^u_{\Lambda_0}(S_y M).$$

Proof. For x as above, the preceding arguments show that

$$a_t(x,y) \sim e^{ht} m^s_{\mathcal{K}}(S_x M) m^u_{\Lambda_0}(S_y M).$$

In particular, $a_t(x,y) \sim e^{ht} a(x,y)$ where

$$a(x,y) = m^s_{\mathcal{K}}(S_x M) m^u_{\Lambda_0}(S_y M).$$

The function $a(x,y)$ is evidently independent of t.

For continuity, simply note that if y is another point of M with $d(x,y) \le \varepsilon$ then $b_t(x) = \operatorname{vol} B_t(x)$ satisfies

$$b_{t-\varepsilon}(x) \le b_t(y) \le b_{t+\varepsilon}(x).$$

Thus

$$e^{-h\varepsilon} \le \frac{b_t(y)}{c(x)e^{ht}} \le e^{h\varepsilon}.$$

Thus $b_t(y)$ and $b_t(x)$ are arbitrarily close for x and y sufficiently close. ☐

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