Hamburger Beiträge
zur Angewandten Mathematik

The Margulis Construction in the Nonuniformly Hyperbolic Case

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Nr. 2007-10
June 2007

THE MARGULIS MEASURE CONSTRUCTION IN THE NONUNIFORMLY HYPERBOLIC CASE

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Abstract. We generalize Margulis' construction of the measure of maximal entropy from to the case of nonpositively curved manifolds with geometric rank one.

Contents

2000 *Mathematics Subject Classification.* 37D40 (Dynamical systems of geometric origin and hyperbolicity (geodesic and horocycle flows, etc.)), 53D25 (Geodesic flows), 37B40 (Topological entropy), 37D25 (Nonuniformly hyperbolic systems (Lyapunov exponents, Pesin theory, etc.)).

Key words and phrases. Measure of maximal entropy; nonpositive curvature; geodesic flow; nonuniform hyperbolicity.

1. Introduction

The measure of maximal entropy for the geodesic flow is useful for many purposes, including precise asymptotics for the number of closed geodesics and of volume in the universal cover. (See [Gun2, section 1] and [Gun3] for a detailled description of these problems in nonpositive curvature, as well as [Kni1], [Kni2], [KHK]. For the negative curvature case, see [Mar3] and [Mar4]). One method to provide this measure for nonpositively curved manifolds of rank one (see [Bal] for the geometric background) is Knieper's construction, which in turn uses the Patterson-Sullivan construction. In this article we provide a different method: We consider the construction which Margulis used to obtain his famous asymptotics. He covered the case where the curvature is strictly negative on compact manifolds and hence the geodesic flow uniformly hyperbolic. In this article we show that Margulis' measure construction can actually be extended to nonpositively curved manifolds of rank one, where the geodesic flow is nonuniformly hyperbolic. Our construction works even if stable and unstable leaves become tangential.

See [Gun2, section 2] for a detailled description of the history of the study of the dynamics of the geodesic flow on nonpositively curved rank one manifolds, including Margulis' asymptotics, Knieper's multiplicative bounds, and Katok's entropy conjecture.

The *Liouville measure* on SM, denoted by λ, is a finite and smooth measure. Existence and uniqueness of a measure whose entropy equals the topological entropy h were established for nonpositive curvature by G. Knieper [Kni1] with the Patterson-Sullivan construction, building the measure as limit of measures supported on periodic orbits.

For the special case of strictly negative curvature, the measure of maximal entropy was constructed in a different way by G.A. Margulis ([Mar1], [Mar4]). His construction builds the measure as the product of limits of measures supported on pieces of stable and unstable leaves. (See also U. Hamenstädt's geometric description of this measure [Ham], and B. Hasselblatt's generalization to Anosov flows [Has].)

We will show how to carry out a Margulis-type construction to obtain a measure which is well adapted to dynamical properties of the flow. We show that it has maximal entropy. Hence in nonpositive curvature and rank one it agrees with Knieper's measure.

It is not a priori clear that the Knieper measure has all of the relevant properties which the Margulis measure has in the negatively curved case, such as holonomy invariance and uniform expansion. By generalizing Margulis' construction, we show that those properties are actually present. (These two properties can actually also be derived directly from Knieper's construction ([Gun2, Theorems 4.6 and 4.7]).)

Compared to Knieper's approach, the Margulis-type method in this article could be easier to use when applications require to generalize the measure to further spaces. Examples of such applications are geodesic flows on not necessarily nonpositively curved spaces (i.e. with some controlled amount of positive curvature), as well as studying non-geodesic flows satisfying suitable cone conditions (see [Kat] for those). These examples will be treated in a separate article.

2. Preliminaries

In the following, M will be a compact manifold with nonpositive sectional curvatures and (geometric) rank one. We summarize some known facts (which are explained in greater detail in [Gun2, section 3]).

Continuity of the stable and unstable foliations was proved by P. Eberlein [Ebe2] and J.-H. Eschenburg [Esch]. Due to compactness of M, the continuity is automatically uniform. It was demonstrated by Eberlein ([Ebe1]) that stable manifolds are dense. Similarly, unstable manifolds are dense. M. Babillot [Bab] has shown that the measure of maximal entropy for the geodesic flow on a compact rank one nonpositively curved manifold is mixing.

We call an open set $U \subset SM$ of size $\leq \delta$ regularly coordinated if for all $v, w \in U$ there are unique x, y such that

$$x \in W^u_\delta(v),\ y \in W^0_\delta(x),\ w \in W^s_\delta(y).$$

If v is regular then it has a regularly coordinated neighborhood ([Gun2, Proposition 3.14]).

Lemma 2.1. *The vector $v \in SM$ is regular if and only if $W^u(v)$, $W^s(v)$ and $W^0(v)$ intersect transversely at v.*

The proof is given in [Gun2, Lemma 3.12].

Let

$$\langle \xi, \eta \rangle := \langle d\pi\xi, d\pi\eta \rangle + \langle K\xi, K\eta \rangle$$

for $\xi, \eta \in T_vSM$ be the Sasaki metric on SM.

Let $b(., q, \xi)$ be the **Busemann function** centered at $\xi \in \tilde{M}(\infty)$ and based at $q \in \tilde{M}$:

$$b(p, q, \xi) := \lim_{p_n \to \xi} (d(q, p_n) - d(p, p_n)) = \lim_{t\to\infty} (d(c_{p,\xi}(t), q) - t)$$

(independent of the sequence $p_n \to \xi$) where $c_{p,\xi}$ is the geodesic with $c_{p,\xi}(0) = p$ and $c_{p,\xi}(t) \to \xi$ as $t \to \infty$.

For ξ, p fixed, we have

$$b(p, p_n, \xi) \to -\infty \quad \text{for} \quad p_n \to \xi$$

and

$$b(p, p_n, \xi) \to \infty \quad \text{for} \quad \lim_n p_n \in \tilde{M}(\infty) \setminus \{\xi\}.$$

Also: $b(p, q, \xi) = -b(q, p, \xi)$.

μ_p is a h-dimensional Busemann density (also called conformal density) if the following are true:

- For all $p \in \tilde{M}$, μ_p is a finite nonzero Borel measure on $\tilde{M}(\infty)$.
- μ_p is equivariant under deck transformations, i.e., for all $\gamma \in \pi_1(M)$ and $S \subset \tilde{M}(\infty)$ we have
$$\mu_{\gamma p}(\gamma S) = \mu_p(S).$$
- When changing the base point of μ_p, the density transforms as follows:
$$\frac{d\mu_p}{d\mu_q}(\xi) = e^{-hb(q,p,\xi)}.$$

Knieper has shown [Kni1] that μ_p is unique up to a multiplicative factor.

3. Existence of holonomies

This section recalls some facts from Margulis, although we have to use a somewhat different approach later since in our setup transversality is missing.

For $L_1, L_2 \subset W^{0u}$ (not necessarily in the same leaf) the holonomy map $\mathbf{H}$ from L_1 to L_2 can be defined by $\{\mathbf{H}(x)\} := W^s_D(x) \cap L_2$ for $x \in L_1$ where D is sufficiently large for the intersection to be nonempty and sufficiently small for it to be just one point. This intersection is well-defined if for all y in L_2 some local intersection $W^s_\gamma \cap W^u_\gamma$ consists of a single point, in particular if W^s and W^{0u} are transversal on L_2.

Lemma 3.1. *Assume that the vector v and the set L are regular, where L is open in a W^{0u}-leaf. Then for $\delta > 0$ sufficiently small and $D \in \mathbb{R}$ sufficiently large there is an open set $U \subset L$ such that $B^{0u}_\delta(v)$ is D-equivalent to U. Moreover, the holonomy between these sets is a homeomorphism.*

Proof. We use regularity and the same transversality argument as in the negatively curved case. Namely: Since $W^s(v)$ is dense, it approximates arbitrarily closely any point of L. By regularity, it is transversal to L, thus it passes through the interior of L. Hence, the holonomy map from a sufficiently small neighborhood of v maps this neighborhood to an open subset of L homeomorphically. □

Even though the preceding statement is what was used by Margulis to prove results in negative curvature, we find it more useful to look at holonomies with less restrictions to size of leaves but with more restrictions on closeness of them. This is done in the following.

4. Exponentially growing Lebesgue measure of projections: multiplicative bounds

4.1. Positive density of some projections.

Definition 4.1. We use the notation $L_t := g^t L$ for a set $L \subset W^{0u}$ (and for $v \in S\tilde{M}$). In particular, $L_\infty := g^\infty L = \lim_{t\to\infty} g^t L$.

Definition 4.2. Let $L \subset W^{0u}$ be a set in the weakly unstable foliation. We call L a **tall** set if it is nonempty, regular and open in W^{0u}.

In particular, any regular vector has a tall neighborhood in its $0u$-leaf. Technically, the empty set is singular, and hence the requirement of nonemptyness in this definition is already implied by regularity.

Definition 4.3. For a set $U \subset \tilde{M}$ we define the **projection** $\text{proj}_p U$ from a point $p \in \bar{M}$ into the boundary at infinity to be the set of endpoints of all geodesic rays emanating from p and passing through the set U.

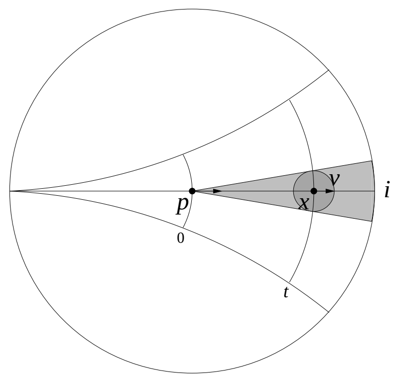

Figure 4.1. Projecting into the boundary at infinity.

From Knieper's results we derive:

Proposition 4.4. *There is $\rho = \rho(M)$ such that if $L \subset W^{0u}$ contains a weakly unstable ball B_r^{0u} of radius $r > \rho$ then $\mu_p L_\infty > 0$ for some (hence all) $p \in M$.*

Proof. Knieper showed in [Kni1] that there are constants $\rho, A > 0$ such that balls of radius at least ρ project to sets of $\mu_p > A$, i.e. for any $x \in \tilde{M}$ and $q \in \bar{M}$ it is true that

$$\mu_x \text{proj}_q B_\rho(x) > A.$$

Let $C_0 = C_0(M) \geq 1$ be such that $\pi B^{0u}_{C_0 r}(v) \supset B_r(\pi v)$ for all $v \in X = S\tilde{M}$. C_0 can be chosen to be finite by compactness of M. For $r \leq r_0$, C_0 can also be assumed independent of r.

Let $v \in X$, $r \geq C_0\rho$ be such that $B^{0u}_r(v) \subset L$. Let $x := \pi v$.

For the unique q with $\{q\} = L_{-\infty}$ we get

$$g^{\infty} B_r^{0u}(v) = \text{proj}_q \pi B_r^{0u}(v) \supset \text{proj}_q B_{r/C_0}(x) \supset \text{proj}_q B_\rho(x).$$

Hence

$$\mu_p L_\infty \geq \mu_p \text{proj}_q \pi B_r^{0u}(v) \geq \mu_p \text{proj}_q B_\rho(x) > A > 0.$$

□

Proposition 4.5. *Let $L \subset W^{0u}$ be regular and open in W^{0u} (i.e. tall). Then it satisfies $\mu_p L_\infty > 0$ for some (hence all) $p \in M$.*

Proof. If there exist two vectors $v, w \in L$ for which the iterations $g^t v$, $g^t w$ stay at a bounded distance for all positive t (hence for all t) then the geodesics through v and w bound a flat strip, hence lie on the same geodesic by regularity of L.

Now pick any $v \in L$. Pick $R > 0$ small enough so that $g^{[-R,R]} \bar{B}_R^u v \subset L$. We know that for all $w \in L \cap B^u(v) \setminus \{v\}$:

$$d(\pi g^t v, \pi g^t w) \to \infty \qquad \text{as} \quad t \to \infty.$$

Thus

$$\min_{w \in \partial B_R^u(v)} d(\pi g^t v, \pi g^t w) \to \infty \qquad \text{as} \quad t \to \infty$$

by the compactness of $\partial B_R^u(v)$. In particular, $g^t B_R^u(v)$ contains arbitrarily large u-balls for t sufficiently large. By Proposition 4.4 it follows that $\mu_p L_\infty > 0$. □

4.2. **Projecting distant balls to infinity.** The following theorem uses the results of G. Knieper [Kni1]:

Theorem 4.6 (Projection of distant balls)**.** *Given some arbitrary $p \in \tilde{M}$, for $r > \rho$ there exists some number $a = a(r)$ such that for all $x \in \tilde{M}$:*

$$\frac{1}{a} \leq \frac{\mu_p proj_p B_r(x)}{e^{-ht}} \leq a, \tag{4.1}$$

where $t := d(p, x)$.

Proof. As we pointed out before, Knieper showed that there are constants $\rho = \rho(M), A = A(M) > 0$ such that for any $x \in \tilde{M}$ and $q \in \bar{M}$ and $r > \rho$ it is true that

$$\mu_x \text{proj}_q B_r(x) > A.$$

(Actually, Knieper's formulation assumes that moreover $d(x, p) > r$. Note that the claim of the Theorem is true even without this stipulation since for $d(x, p) < r$ we get $\mu_x \text{proj}_p B_r(x) = \mu_x \tilde{M}(\infty) > A$.)

Recall that μ_x is a finite measure for all x. Thus for all $r > \rho$ there exists $A > 0$ such that for any $q \in \tilde{M}$:

$$A < \mu_x \text{proj}_q B_r(x) \leq \mu_x \tilde{M}(\infty) < \infty.$$

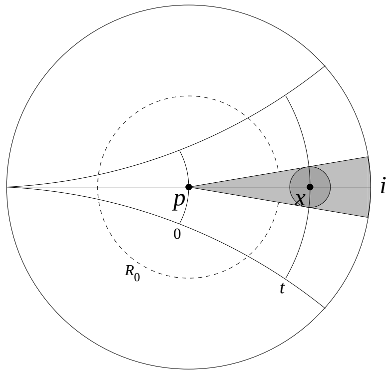

FIGURE 4.2. Projection of a distant ball.

Next we change the base point of μ :

$$\begin{aligned}\mu_p \mathrm{proj}_p B_r(x) &= \int_{\mathrm{proj}_p B_r(x)} 1 d\mu_p(\xi) \\ &= \int_{\mathrm{proj}_p B_r(x)} \frac{d\mu_p}{d\mu_x}(\xi) d\mu_x(\xi) \\ &= \int_{\mathrm{proj}_p B_r(x)} e^{-hb(x,p,\xi)} d\mu_x(\xi).\end{aligned}$$

Let $\phi(\xi) := b(x, p, \xi)$. For $\xi_0 := \mathrm{proj}_p x$ we see that $\phi(\xi_0) = d(x, p)$ and $\phi(\xi_0) \geq \phi(\xi)$ for all other $\xi \in \tilde{M}(\infty)$.

We are interested in distant x and p, so assume now that $d(x, p) > R_0$ for some large R_0. Let

$$y := c_{p,\xi} \cap b(x, ., \xi)^{-1}(0),$$

i.e., y is the point on the intersection of the geodesic from p to ξ with the horosphere centered at ξ based at x. In yet different terms, y is given by

$$y = c_{p,\xi}(b(x, p, \xi)).$$

Note that for R_0 sufficiently large, for all $\xi \in \mathrm{proj}_p B_r(x)$ the triangle inequality shows that x and y are at a bounded distance, i.e., we have $d(x, y) < 2r$.

Hence for all $\xi \in \mathrm{proj}_p B_r(x)$ we have

$$0 \leq \phi(\xi_0) - \phi(\xi) \leq C$$

independent of p, x (as long as p, x are more than R_0 apart). Therefore

$$1 \geq \frac{e^{-hb(x,p,\xi)}}{e^{-hd(x,p)}} \geq e^{-C}$$

and thus

$$\frac{1}{a} \leq \frac{\mu_p \mathrm{proj}_p B_r(x)}{e^{-ht}} \leq a.$$

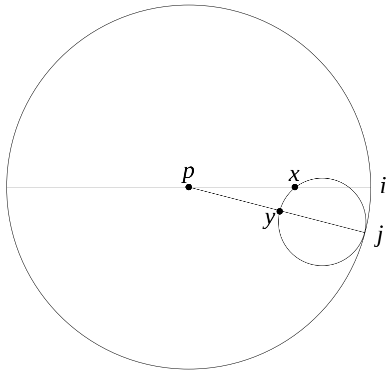

FIGURE 4.3. Construction of y.

So far we have assumed that $d(x,p) > R_0$. However, since the ball of radius R_0 is compact, by increasing a we can ensure that the claim is true also if $d(x,p) \le R_0$, hence holds for all x, p. $\square$

From this we immediately deduce the following, which is the same formula but where the variables involved have a slightly different meaning:

Corollary 4.7 (Projection of balls in distant leaves)**.** *Let $r > \rho$. Let L be tall and bounded. Let $p \in \tilde{M}$ be arbitrary. Then there exists some number $b = b(p, L)$ such that for all $t \ge 0$ and for all $x \in \pi L_t$ the estimate*

$$\frac{1}{b} \le \frac{\mu_p proj_p B_r(x)}{e^{-ht}} \le b$$

holds.

The same statement is true with $B_r(x)$ replaced by $B_r^{0u}(v)$:

Theorem 4.8 (Projection of pieces of distant leaves)**.** *Let $r > \rho$. Let L be tall and bounded. Let $p \in \tilde{M}$ be arbitrary. Then there exists some number $c = c(p, L)$ such that for all $t \ge 0$ and for all $v \in L_t$:*

$$\frac{1}{c} \le \frac{\mu_p proj_p \pi B_r^{0u}(v)}{e^{-ht}} \le c. \tag{4.2}$$

4.3. **A multiplicative bound for the growth of Lebesgue measure.** Our efforts are now rewarded by the following Theorem:

Theorem 4.9 (Tall sets grow exponenetially)**.** *If $L \subset W^{0u}$ is bounded and tall then there is a constant $C = C(L)$ such that for all $t \ge 0$:*

$$\frac{1}{C} \le \frac{\lambda^{0u}(g^t L)}{e^{ht}} \le C.$$

Proof. Write

$$l_t := \lambda^{0u}(L_t).$$

Fix $p \in \tilde{M}$ and $r > \rho$. From Theorem 4.8 we see that each $0u$-ball of radius r casts a projection from p of μ_p-measure at most ce^{-ht}. Thus it takes at least $\frac{1}{cC_0}e^{ht} \cdot \mu_p \pi L_\infty$ such $0u$-balls of radius r to cover πL_t, with c as in Theorem 4.8, even if their projections to L_∞ do not overlap. Hence we get the lower bound

$$l_t = \lambda^{0u}(g^t L) \geq C_1 e^{ht}$$

with

$$C_1 := \frac{A}{C_0} \cdot \min_{x \in \tilde{M}} \lambda^{0u}(B_r^{0u}(x))$$

where A is as in the proof of Proposition 4.4.

The minimum exists and is positive because the minimum over $\tilde{M}$ is the minimum over M, which is compact. Note that C_1 depends only on L and M, not on t.

Now take any maximal set Z of points in πL_t which are at least distance $2r$ apart from each other and at least at distance r from the boundary of πL_t. (It is easy to see that without the restriction of staying away from the boundary, this set would contain at most C_0' times as many points, where C_0' just depends on the manifold and is independent of L and t.) This set contains at most $ce^{ht} \cdot \mu_p \pi L_\infty$ such points since the balls $(B_r^{0u}(x_i))_{i \in Z}$ are disjoint and project to disjoint subsets of πL_∞, each of which has μ_p-measure at least $\frac{1}{c}e^{-ht}$. Thus it takes at most $ce^{ht} \cdot \mu_p \pi L_\infty$ such $0u$-balls of radius $2r$ to cover πL_t. This gives the upper bound

$$\lambda^{0u}(g^t L) \leq C_2 e^{ht}$$

with

$$C_2 := cC_0 \mu_p(\tilde{M}(\infty)) \cdot \max_{x \in \tilde{M}} \lambda^{0u}(B_{2r}^{0u}(x)).$$

The maximum exists and is finite because the maximum over $\tilde{M}$ is the maximum over M, which is compact. The number C_2 again depends only on L and M, not on t. Hence taking $C := \max(\frac{1}{C_1}, C_2)$ we have proved the claim:

$$\frac{1}{C} \leq \frac{\lambda^{0u}(g^t L)}{e^{ht}} \leq C. \tag{4.3}$$

□

Remark 4.10. The right hand inequality of 4.3 still holds if L is not tall, even if L does not contain any regular vectors. Also, it is true even if L is not open, since any set sits inside a bigger open set.

Remark 4.11. The left hand inequality of 4.3 is not necessarily true if L does not contain a regular vector.

5. Construction of conditionals of the measure of maximal entropy

5.1. Integration and relative independence of the function.

Lemma 5.1. *For any compact L which is contained in a W^{0u}-leaf and which is the closure of a tall set and for any $L' \subset W^{0u}$ compact (not necessarily containing regular vectors or the closure of an open set) there exists a constant $C(L, L')$ such that for all $t \geq 0$:*

$$l'_t \leq C(L, L')l_t$$

where $l'_t := \lambda^{0u}(L'_t)$.

Proof. Immediate from Theorem 4.9:

$$l'_t \leq C^{(1)}(L')e^{ht}, \quad l_t \geq \frac{1}{C^{(2)}(L)}e^{ht}$$

where the right inequality is true since L is a tall set. If L' is not a tall set, then the left inequality is still true by Remark 4.10. □

Definition 5.2. Whenever a function f has support in a W^{0u}-leaf, we simply write $\int f$ for $\int_{\mathrm{supp}(f)} f \mathrm{d}\lambda^{0u}$.

Corollary 5.3. *For all nonnegative $f_1 \in C(W^{0u})$ with support in a compact set L which is the closure of a tall set and all $L' \subset W^{0u}$ there is a constant $C(L', f_1)$ such that for any bounded measurable function f_2 which is supported on L' and for any $t > 0$:*

$$\int f_2 \circ g^{-t} < C(L', f_1)||f_2||_\infty \int f_1 \circ g^{-t}.$$

Proof. Choose $\varepsilon \in (0, \max f_1)$ such that $A := \{x : f_1(x) > \varepsilon\}$ still is a tall set (in particular nonempty). Then

$$l_t \leq C(A, L) \cdot \lambda^{0u}(A_t)$$

by Corollary 5.1. Thus

$$\begin{aligned}
\int f_2 \circ g^{-t} &\leq ||f_2||_\infty l'_t \\
&\leq C(L, L')||f_2||_\infty l_t \\
&\leq C(A, L)C(L, L')||f_2||_\infty \lambda^{0u}(A_t) \\
&\leq \varepsilon^{-1}C(A, L)C(L, L')||f_2||_\infty \int f_1 \circ g^{-t}.
\end{aligned}$$

Choosing $C(L, f_1) := \varepsilon^{-1}C(A, L)C(L, L')$ gives the result. □

In particular, this proof shows that for any tall set L and for any nonzero bounded function f (not necessarily supported on a tall set) the following holds:

$$||f||_\infty^{-1} \cdot \int f \circ g^{-t} \leq \lambda^{0u}(g^t \mathrm{supp}(f)) \leq C(L, \mathrm{supp}(f)) \cdot l_t. \tag{5.1}$$

5.2. **Linear functionals on leaves.** Next we can define objects F on $C(W^{0u})$ which are linear functionals in the sense that they satisfy

$$F(a \cdot f_1 + b \cdot f_2) = a \cdot F(f_1) + b \cdot F(f_2)$$

whenever $f_1, f_2 \in C(W^{0u})$ and the sum $f_1 + f_2$ of those functions still is supported on one single leaf:

Definition 5.4. Let $t \geq 0$. Define

$$F_t(f) := \int_{g^t L} f \circ g^{-t} d\lambda^{0u}$$

for a function f with support in $L \subset W^{0u}$. Then F_t is a linear functional, depending on a parameter t.

Remark 5.5. Note that this functional is **positive**, i.e. $F_t(f) \geq 0$ whenever $f \geq 0$.

Remark 5.6. The set C^* of all functionals on $C(W^{0u})$ naturally is equipped with a topology by embedding it into a product of real lines.

Definition 5.7. Given $t \geq 0$, define

$$F'_t := e^{-ht} F_t.$$

Moreover, given numbers $t_i \geq 0$, define

$$\begin{aligned} C_0^* &:= \left\{ \sum_i c_i F'_{t_i} : 0 \leq c_i \leq 1, \sum c_i = 1 \right\} \\ &= \left\{ \sum_i c_i e^{-ht_i} F_{t_i} : 0 \leq c_i \leq 1, \sum c_i = 1 \right\}. \end{aligned}$$

Finally, let

$$C^{\#} := \bar{C}_0^*.$$

($C^{\#}$ is the closure of C_0^*.)

5.3. **Note on Margulis' construction.** Unlike in the hyperbolic case, we have to work without uniform transversality of the stable and unstable foliations and without uniform exponential convergence along stable manifolds. A very detailed description of the construction in the particular case that zero curvature is absent (i.e. on compact manifolds whose curvature is strictly negative) can be found in [KaHa].

Remark 5.8. We could have normalized, similar to the way Margulis does, as follows: Let $\mathbf{K} \subset W^{0u}$ be a fixed set which is open in W^{0u}, has compact closure and is a tall set. E.g. a sufficiently large W^{0u}-ball will do. Let $\theta \in C(W^{0u})$ be a fixed continuous and integrable function with $\theta > 1$ on $\mathbf{K}$. Then instead of F'_t use

$$\hat{F}_t(f) := \frac{F_t(f)}{F_t(\theta)}.$$

Our formalism appears slightly simpler, although they are of course similar. We are able to simplify the notation because we already know that the measure we are about to construct has entropy h, hence is the unique maximal measure, a fact not available to Margulis when he pioneered his construction.

5.4. **Uniform bounds for the functionals.**

Proposition 5.9 (Uniform upper bound)**.** *For any $f \in C(W^{0u})$, there exist $C > 0$ such that for all $F \in C^{\#}$ the bound*

$$|F(f)| \leq C$$

holds.

Proof. Define

$$S := \{F'_t : t \in \mathbb{R}, t \geq 0\}.$$

Then the convex hull of S equals C_0^* and $C^{\#}$ is the closure of that.

First of all, fix some tall set L. Then inequality 5.1 shows that

$$\frac{F_t(f)}{||f||_\infty} \leq C(L, K)\lambda^{0u}(L_t)$$

where K is the support of f. By Theorem 4.9 and Remark 4.10, there is a constant C_1 such that

$$\lambda^{0u}(L_t) \leq C_1 e^{ht}.$$

Replacing F_t by F'_t means introducing another factor e^{-ht} and hence the claim

$$|F(f)| \leq C = C(f)$$

is true for all $\phi \in S$.

Second, any $\phi \in C_0^*$ can be written as $\phi = \sum a_i \phi_i$ with $\phi_i \in S$ where $a_i \in [0, 1]$ and $\sum a_i = 1$. Hence

$$|\phi(f)| \leq \sum a_i |\phi_i(f)| \leq C(f).$$

So the claim is true for $\phi \in C_0^*$.

Finally, if $\phi \in C^{\#}$ then $\phi = \lim_i \phi_i$ with $\phi_i \in C_0^*$, thus

$$|\phi(f)| = \lim_i |\phi_i| \leq C(f).$$

Therefore the claim is true for $\phi \in C^{\#}$ as well. □

Proposition 5.10 (Uniform lower bound)**.** *For any $f \in C(W^{0u})$, $f \geq 0$, such that for some $\varepsilon > 0$ the set $\{x : f(x) > \varepsilon\}$ is a tall set, there exist $C > 0$ such that for all $F \in C^{\#}$ the bound*

$$1/C \leq |F(f)|$$

holds.

Proof. Let $A := \{x : f(x) > \varepsilon\}$ which we assume to be tall. Then we proceed with a comparison similar to that in the proof of Lemma 5.3:

$$\begin{aligned} \varepsilon C_2 e^{ht} &\leq \varepsilon \lambda^{0u}(L_t) \\ &\leq C(L,A) \int_{A_t} f \circ g^{-t} d\lambda^{0u} \\ &\leq C(L,A) F_t(f). \end{aligned}$$

Hence $F_t'(f) \geq C_2 \varepsilon^{-1} C(L,A)^{-1}$ as claimed. Thus the claim is true for S.

Again, any $\phi \in C_0^*$ can be written as $\phi = \sum a_i \phi_i$ with $\phi_i \in S$ where $a_i \in [0,1]$ and $\sum a_i = 1$, hence $|\phi(f)| = |\sum a_i \phi_i(f)| \geq C(f)^{-1}$ for some $C(f)$ and therefore the claim is true for $\phi \in C_0^*$ as well.

And as in the previous proof, if $\phi \in C^\#$ then $\phi = \lim_i \phi_i$ with $\phi_i \in C_0^*$, thus $|\phi(f)| = \lim_i |\phi_i| \geq C(f)^{-1}$. Therefore the claim is true for $\phi \in C^\#$ as well. □

5.5. **The conditional measure as a fixed point.**

Definition 5.11. Let $(G^t)_{t\in\mathbb{R}}$ be the flow on $C^\#$ defined by

$$G^t F := F \circ g^t.$$

Let $(\hat{G}^t)_{t\in\mathbb{R}}$ be the flow on $C^\#$ defined by

$$\hat{G}^t F := e^{-ht} F \circ g^t.$$

Proposition 5.12. *There is a measure m' which is a fixed point for $\hat{G}^t$.*

Proof. For each t, the map g^t is a smooth diffeomorphism. Hence $\hat{G}^t$ is a continuous map from $C^\#$ to itself. $C^\#$ is a convex compact subset of a locally convex topological vector space. Thus the Tychonoff fixed point theorem [KaHa] applies and gives a fixed point for $\hat{G}^t$. □

Note that for any $t_1, t_2 \in \mathbb{R} \setminus \{0\}$ the fixed point for $\hat{G}^{t_1}$ is the fixed point for $\hat{G}^{t_2}$. This is clear for $t_1/t_2 \in \mathbb{N}$, hence for $t_1/t_2 \in \mathbb{Q}$ and thus for all t_1, t_2 since $g^t \to \mathrm{Id}$ as $t \to 0$.

Definition 5.13. Later we will denote m' by m^{0u}.

Remark 5.14 (Comment on Margulis' method). Note that m^{0u} (the family of conditional measures on the weakly unstable leaves) is already normalized due to the fact that m^{0u} lies in $C^\#$.

5.6. **Uniform expansion and contraction of the conditionals.** The preceding fixed point statement immediately shows the following:

Theorem 5.15 (Uniform expansion on weakly unstable leaves)**.** *The measure m^{0u} satisfies*

$$m^{0u} \circ g^t = e^{ht} \cdot m^{0u}.$$

This is what we actually need for the calculation of the asymptotics of periodic orbits. Before we do so, we verify that these conditionals are indeed those of the maximal measure.

6. Holonomy invariance of the measure on leaves

Definition 6.1. If the leaves $L, L' \subset W^{0u}$ are related by a holonomy map $\mathbf{H}$, i.e. $L' = \mathbf{H}(L)$ and all connecting pieces of W^s between $v \in L$ and $\mathbf{H}(v) \in L'$ are of length at most γ then L and L' are called γ-**holonomic**. If $f_1, f_2 \in C(W^{0u})$ and the support of f_2 is obtained from the support of f_1 via a holonomy $\mathbf{H}$ of length at most γ and $f_2 = f_1 \circ \mathbf{H}$ then f_1 and f_2 are called γ-**equivalent** or also γ-**holonomic**.

Theorem 6.2. *Let M be a manifold of nonpositive curvature with geometric rank one. Then the measure m^{0u} is locally holonomy invariant inside regular neighborhoods.*

In other words, if $L, L' \subset W^{0u}$ are tall and ε-equivalent for ε sufficiently small then

$$m^{0u}(L) = m^{0u}(L').$$

6.1. Strategy for showing holonomy invariance.

The strategy is as follows: To show that the Jacobian between two holonomic leaves in W^u is 1, we first show in Lemma 6.19 that it is between $1-\varepsilon$ and $1+\varepsilon$ whenever the leaves are δ-close and lie in the same chart. That by itself would not necessarily be sufficient (e.g. this would also be true for Lebesgue measure); we also show that the leaves which initially are δ_0-close (where δ_0 is small but fixed) become δ-close for any $\delta > 0$ as $t \to \infty$. We show this in Lemma 6.20. Then the claim follows by noting that the Jacobian converges to 1 uniformly and that any two holonomic leaves can be broken into pieces of holonomic pairs which lie within one chart. Note that δ_0 is assumed sufficiently small for the arguments to work. However, if we chose it small enough then we can fix it and will then get $\delta \to 0$ as $t \to \infty$.

Remark 6.3. Unlike in the case of strictly negative curvature, we do not have uniform transversality of the stable and unstable manifolds at our disposal. In fact, as we have seen before, the angle between these is zero on the singular set. This restricts our arguments to the regular set. Therefore, when in our setup pieces of holonomic leaves approach each other as $t \to \infty$, it is not a priori clear that their sizes also become very close. Instead, it is necessary to estimate of the size of the "overhanging" part, i.e. the part where the two plaques do not "overlap" in suitable coordinates. This estimate is also achieved in this subsection.

6.2. Geometric properties of stable and unstable leaves.

Recall that $\pi : S\tilde{M} \to \tilde{M}$ is the canonical projection $\pi v := x$ with $v \in S_x\tilde{M}$.

We start with some elementary considerations:

Lemma 6.4. *$\pi W^u(p) \perp \pi g^{\mathbb{R}} p$ at p. Similarly, $\pi W^s(p) \perp \pi g^{\mathbb{R}} p$ at p.*

Proof. Let $b := b(\pi p, ., g^\infty p)$ be the Busemann function based at πp and centered at $g^\infty p$. Then $\pi W^u(p) = b(\pi p, ., g^\infty p)^{-1}(0)$ and $p = \operatorname{grad} b(\pi p, ., g^\infty p)|_{\pi p}$. The gradient of a function is perpendicular to its level sets. □

Corollary 6.5. *$\pi W^u(p)$ is tangent to $\pi W^s(p)$ at p.*

Proof.

$$T_p \pi W^u(p) = \left(\frac{d}{dt} \pi g^t p|_{t=0} \right)^\perp = T_p \pi W^s(p).$$

□

Definition 6.6. Let J_v^s be the **stable Jacobi field** along the geodesic c with $c'(0) = v$, defined by $J_v^s(0)$ and the condition that it is bounded for $t \geq 0$. Similarly, let J_v^u be the **unstable Jacobi field** along the geodesic c with $c'(0) = v$, defined by its initial value at 0 and the condition that it is bounded for $t < 0$.

Unless we specify otherwise, we assume that $|J_v^i(0)| = 1$ (for $i = s, u$), and we reserve the right not to explicitly assign a value to $J_v^i(0)$.

6.3. **Contact structure of the geodesic flow.** The following is well-known (see [Pat]): The geodesic flow has a **contact structure**, i.e. there exists a one-form α on SM, called the **contact form**, such that $\alpha \wedge (d\alpha)^{n-1}$ is a volume form. There exists a vector field $V : SM \mapsto TSM$ with $\alpha(V) \equiv 1$, $d\alpha(V, .) \equiv 0$. We can express α as $\alpha_v(\xi) = \langle v, d\pi\xi \rangle$ for $v \in SM$, $\xi \in T_v SM$.

If we disregard the V-direction (i.e. restrict to $\ker(\alpha) = TSM/\mathbb{R}V$), then $d\alpha =: \omega$ is a nondegenerate closed 2-form. We would call it a *symplectic form* if the s- and u-distributions were jointly integrable, but of course they are not.

Note that the whole tangent bundle TM, which is even-dimensional, does admit a **symplectic form** Ω. The restriction of Ω to the odd-dimensional sphere bundle SM acquires a direction of degeneracy, namely the flow direction V. In other words, $\Omega(V, X) = 0$ for all $X \in TSM$.

Analogously to the Sasaki metric, we can write an explicit formula for the form:

$$\omega(\xi, \eta) = \langle d\pi\xi, K\eta \rangle - \langle K\xi, d\pi\eta \rangle.$$

(This is in fact exactly the expression for the **almost complex structure** with respect to the Sasaki metric, but we do not use this structure in the sequel.)

Lemma 6.7. *The spaces $E^u := T_v W^u$ and $E^s := T_v W^s$ are Lagrangian subspaces.*

Proof. We use the formula $\omega(\xi,\eta) = \langle d\pi\xi, K\eta\rangle - \langle K\xi, d\pi\eta\rangle$. Consider $\xi,\eta \in E^u_v$. Find curves x, y representing them, i.e.

$$\xi = \frac{d}{dt}\Big|_{t=0} x(t),\ \eta = \frac{d}{dt}\Big|_{t=0} y(t),\ x(0) = v = y(0).$$

Then

$$(d\pi, K)(\xi) = \left(\frac{d}{dt}\Big|_{t=0} \pi x(t), \nabla_{\frac{d}{dt}|_{t=0}\pi x} x\right)$$

and

$$(d\pi, K)(\eta) = \left(\frac{d}{dt}\Big|_{t=0} \pi y(t), \nabla_{\frac{d}{dt}|_{t=0}\pi y} y\right).$$

Extend $\frac{d}{dt}\big|_{t=0} \pi x(t)$ to a vector field X in a neighborhood of x such that $X \in T\pi W^u(v)$ on $\pi W^u(v)$ and similarly extend $\frac{d}{dt}\big|_{t=0} \pi y(t)$ to a vector field Y. Note that curves in $W^u(v)$ consist of unit vectors and therefore their derivative is perpendicular to them, i.e. tangential to $\pi W^u(v)$. Thus in particular we have $y \perp X$ at $\pi y(t)$ and $x \perp Y$ at $\pi x(t)$. Thus

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \langle Y, x\rangle \\ &= \frac{d}{dt}\Big|_{t=0} \langle Y(x(t)), x(t)\rangle \\ &= \langle \nabla_X Y, x\rangle + \langle Y, \nabla_{\frac{d}{dt}|_{t=0}\pi x} x\rangle \end{aligned}$$

and

$$0 = \langle \nabla_Y X, y\rangle + \langle X, \nabla_{\frac{d}{dt}|_{t=0}\pi y} y\rangle.$$

Hence

$$\begin{aligned} \omega(\xi,\eta) &= \left\langle \frac{d}{dt}\Big|_{t=0} \pi x(t), \nabla_{\frac{d}{dt}|_{t=0}\pi y} y\right\rangle \\ &\quad - \left\langle \frac{d}{dt}\Big|_{t=0} \pi y(t), \nabla_{\frac{d}{dt}|_{t=0}\pi x} x\right\rangle \\ &= \langle X, \nabla_{\frac{d}{dt}|_{t=0}\pi y} y\rangle - \langle Y, \nabla_{\frac{d}{dt}|_{t=0}\pi x} x\rangle \\ &= -\langle \nabla_Y X, v\rangle + \langle \nabla_X Y, v\rangle \\ &= \langle \nabla_Y X - \nabla_X Y, v\rangle. \end{aligned}$$

Note that since the Riemannian connection is torsion free,

$$\omega(\xi,\eta) = \langle [X,Y], v\rangle.$$

This term is zero since by Frobenius' theorem, $[X,Y] \in T\pi W^u$ and $v \perp T\pi W^u$. (A similar argument for general normal bundles can be found in [Pat].)

Hence we have shown that E^u_v is an isotropic subspace. The exact same argument with u replaced by s shows that E^s_v is also isotropic. If

v is regular, they are disjoint. Their dimension is equal. Hence they are maximal isotropic subspaces, i.e. Lagrangian. □

6.4. **Dynamical proprerties of Jacobi fields.**

Proposition 6.8 (Stable Jacobi fields become short or parallel)**.** *Let v be regular. Then $J_v^u(t)$ (the unstable Jacobi field in the direction v) is unbounded as $t \to \infty$.*

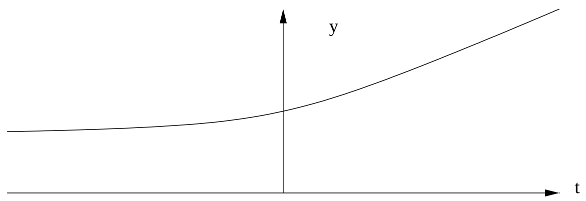

FIGURE 6.1. The length of an unstable Jacobi field is convex.

Proof. Let
$$y(t) := |J_v^s(t)|.$$
Let $c = c(t, s)$ be a variation of geodesics in $\tilde{M}$ associated with the geodesic c_v, i.e. $c(t, 0) = c_v(t)$ and $\frac{\partial}{\partial s}c(t, s) = J_v^s(t)$ and such that $c(s, .) = (t \mapsto c(s, t))$ is a geodesic for all s close to 0. Then $y(t) = |\frac{\partial}{\partial s}c(t, s)| = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} d(c(t, s), c(t, s + \varepsilon))$. This expression is a convex function of t since distance along geodesics is convex, i.e. the function $t \mapsto d(c_1(t), c_2(t))$ is convex and in particular $t \mapsto d(c(s, t), c(s, t + \varepsilon))$ is convex.

Hence y is convex (and in particular never zero since it corresponds to the stable Jacobi field). Since y is bounded for $t < 0$, y is either constant or unbounded for $t > 0$. Since v is regular, J_v^s cannot be parallel and perpendicular, hence y cannot be constant. Hence y is unbounded. □

We will write η_t for $dg^t\eta$ and similarly ξ_t.

Theorem 6.9. *Let v be regular. Then*
$$J_v^s(t) \to 0 \quad \text{or} \quad \angle(\eta_t, TW^u(g^t v)) \to 0 \qquad \text{as} \qquad t \to \infty$$
where $J_v^s = J_\eta$ is the stable Jacobi field along c_v with $d\pi\eta = J_\eta(0), K\eta = J'_\eta(0)$.

Proof. The symplectic form ω on SM is invariant. Note that since ω is a 2-form, it satisfies
$$\omega(\xi, \eta) = |\xi| \cdot |\eta| \cdot \angle(\xi, \eta) \cdot \phi(\sigma(\xi, \eta))$$
where $\sigma(\xi, \eta)$ is the 2-plane spanned by (ξ, η) and ϕ is a (continuous) function on the Grassmannian 2-plane-bundle with $|\phi| \leq C$ where C

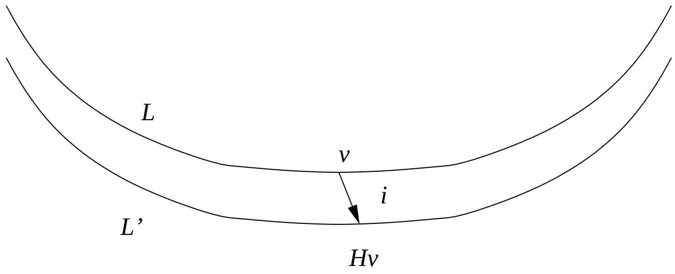

FIGURE 6.2. First possibility: J^s_v is short.

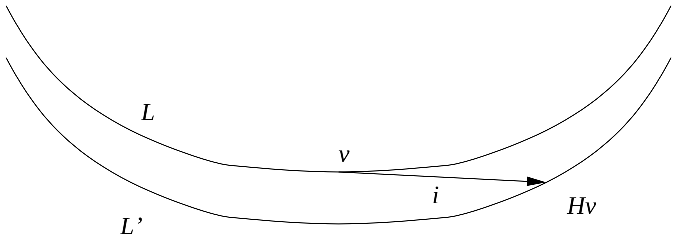

FIGURE 6.3. Second possibility: J^s_v has small angle with the tangent of L.

depends only on M. Note that for the minimal choice of C (i.e. $C := \max_{|\xi|=|\eta|=1} \omega(\xi, \eta)$), by Darboux' Theorem for any η there exists ξ such that $\phi(\sigma(\xi, \eta)) \geq C/(n-1)$ because we can find symplectic coordinates $P_1, \ldots, P_{n-1}, Q_1, \ldots, Q_{n-1}$ such that $P_1 := \eta$ and then choose $\xi := Q_1$. Since $E^s := TW^s(v)$ and $E^u := TW^u(v)$ are Lagrangian subspaces, ξ has nonzero projection to E^u since it obviously cannot lie entirely in E^s. Hence we may assume ξ to lie in E^u.

Now fix some $\eta \in E^s$. In the sequel, ξ will be an arbitrary element of E^u. Note that by invariance of ω either $\phi(\sigma(\xi, \eta))$ is zero or for all t the term $\phi(\sigma(\xi_t, \eta_t))$ is nonzero. Note that due to the previous Lemma (Lemma 6.8), $|\xi_t| \to \infty$ as $t \to \infty$ for all $\xi \in E^u$. This convergence is uniform on the compact set $S := \{\xi \in E^u : |\xi| = 1\}$. We see that

$$|\eta_t| \cdot \angle(\eta_t, \xi_t) \cdot \phi(\eta_t, \xi_t) = \frac{\omega(\eta_t, \xi_t)}{\xi_t} = \text{const} \cdot |\xi_t|^{-1}.$$

Therefore for $\gamma > 0$ there is $T > 0$ such that for all $t > T$ and for all $\xi \in S$ we have $|\eta_t| \cdot \angle(\xi_t, \eta_t) \cdot \max_\xi \phi(\sigma(\xi_t, \eta_t)) < \gamma$. Hence $|\eta| \cdot \angle(E^u, \eta) < \gamma n/C$. Thus the product $|\eta_t| \cdot \angle(\eta_t, TW^u(g^t v))$ converges to zero. $\square$

Remark 6.10. For $\xi \in W^u$ and $\eta \in W^s$ note that $\angle(\xi, \eta)$ is independent of whether we measure the angle in $\ker \alpha$ or in SM.

Remark 6.11. In the special case of a surface, i.e. $\dim(M) = 2$, it is actually possible to show that of the two possibilities "$J^s_v(t) \to 0$" or "$\angle(\eta_t, TW^u(g^t v)) \to 0$" mentioned in the previous Proposition, the former one always applies, i.e.,

$$J_v^s(t) \to 0 \quad \text{as} \quad t \to \infty.$$

However, we do not need this fact in what follows.

6.5. **Geometric comparisons.**

Proposition 6.12. *Let $\mathbf{F}, \mathbf{F}^*$ be foliations of M with C^2-leaves satisfying $|c''| < a$ for any L -geodesic and any L^* -geodesic c parameterized by arc length for any leaf L of $\mathbf{F}$ and for any leaf L^* of $\mathbf{F}^*$. Let c^* be a geodesic in $\mathbf{F}^*$ with $(c^*)'(0) = v \in SM$. Then*

$$d(c^*(\varepsilon), B_{2\varepsilon}^{\mathbf{F}}(\pi v)) < 4(a\varepsilon^2 + \varepsilon\angle_{\pi v}(F, F^*).$$

Proof. First note that if N is a C^2-submanifold of M and N satisfies $|c''| < a$ for any N-geodesic c parameterized by arc length (the second derivative is taken in M), then for any $v \in SN$:

$$d(\exp(\varepsilon v), N) < 2a\varepsilon^2.$$

This is true because of the tangency.

Next observe that if M, N are as before and if $\angle(v, N) < \gamma$ then

$$d(\exp(\varepsilon v), N) < 2(a\varepsilon^2 + \varepsilon\gamma).$$

This follows from adding the previous estimate and a linear term, which of course has slope γ.

Now let $\mathbf{F}$ be a foliation of M with C^2-leaves satisfying $|c''| < a$ for any $\mathbf{F}$ -geodesic c parameterized by arc length. Then

$$d(\exp(\varepsilon v), B_{2\varepsilon}^{\mathbf{F}}(\pi v)) < 2(a\varepsilon^2 + \varepsilon\angle(v, \mathbf{F})).$$

This is true because is the previous statement with the leaf through v being the submanifold N.

Now the claim of the Proposition is evident from applying the previous estimate to both foliations and comparing with the corresponding geodesic in M. □

Now we show that we can apply Proposition 6.12 to any piece L with the some uniform constant a:

Proposition 6.13 (Bending of (un)stable leaves is bounded)**.** *Let M be nonpositively curved and compact. Then there exists $a > 0$ such that $|c''| < a$ for any geodesic parameterized by arclength in πL for $L \subset W^s$ or $L \subset W^u$.*

Proof. By compactness, the curvature is bounded below. Note that for the stable Jacobi field J_v^s along $g^t v$ with initial length 1, the initial derivative $|(J_v^s)'(0)|$ is determined up to an error of order $1/R$ by the sectional curvature K along $g^t v$ for $t \in [0, 2R]$. Since K is smooth on M, the sectional curvature K along $g^t v$ for $t \in [0, 2R]$ depends smoothly on v (and of course on t). Hence $|(J_v^s)'(0)|$ is continuous in v up to an error

$1/R$. Since R can be chosen arbitrarily large, $|(J_v^s)'(0)|$ is continuous in v. Since SM is compact, this continuity is uniform.

The second derivative $|c''|$ of the horosphere is given (up to a constant c_i depending on the chart) by $|(J_v^s)'(0)|$ where c parameterizes $\pi W_\delta^s(v)$. Since it is continuous, by compactness and since it suffices to consider finitely many charts, $|c''|$ is globally bounded above.

This proves the claim for W^s. Exchanging J_v^s with J_v^u (the unstable Jacobi field) gives the claim for W^u. □

Definition 6.14. For a smooth curve c on $\tilde{M}$ from p to q let

$$\mathrm{Par}_c : T_p\tilde{M} \to T_q\tilde{M}$$

denote the parallel transport along c. We define a distance function $\mathbf{d}$ on $S\tilde{M}$ as follows: For $v, w \in S\tilde{M}$ (not necessarily in the tangent space of the same point) let

$$\mathbf{d}(v, w) := d(\pi v, \pi w) + |w - \mathrm{Par}_{c_{\pi v, \pi w}} v|,$$

where $c_{\pi v,\pi w}$ is the geodesic from πv to πw. Note that $c_{\pi v,\pi w}$ is unique in $\tilde{M}$.

Lemma 6.15. *The following are equivalent:*

(1) $\mathbf{d}(g^t v, g^t w) \to 0 \quad$ *as* $\quad t \to \infty$,
(2) $d(\pi g^t v, \pi g^t w) \to 0 \quad$ *as* $\quad t \to \infty$.

Proof. Since $\mathbf{d}(v, w) \geq d(\pi v, \pi w)$, the implication from (1) to (2) is trivial. For the converse, note that d is convex. This means that for v, w fixed, the function $\phi : t \mapsto d(\pi g^t v, \pi g^t w)$ is convex. Hence the function ϕ is either monotonous or it satisfies $\phi(t) \to \infty$ for $t \to -\infty$ as well as $\phi(t) \to \infty$ for $t \to \infty$. In particular, if $d(\pi g^t v, \pi g^t w) \to 0$ and $|g^t w - \mathrm{Par}_{c_{\pi g^t v, \pi g^t w}} g^t v| \not\to 0$, then v, w are not asymptotic, which contradicts $d(\pi g^t v, \pi g^t w) \to 0$. □

Definition 6.16. Define the distance between two vector spaces $V, V' \subset T_pM$ as

$$\mathbf{d}_H(V, V') := d_H(V \cap S_pM, V' \cap S_pM)$$

where d_H denotes the Hausdorff distance of compact sets.

Lemma 6.17. *Let* $v \in \mathbf{Reg}, w \in \mathbf{Reg} \cap W^s(v)$, $\mathbf{d}(v, w) < \delta/2$. *Let* $L := W_\delta^u(v)$, $L' := W_\delta^u(w)$. *Then*

$$d(g^t v, L'_t) \to 0$$

and

$$d(g^t w, L_t) \to 0$$

as $t \to \infty$.

Proof. The claim is evidently true if $\mathbf{d}(g^t v, g^t w)$ converges to zero.

Now assume that $\mathbf{d}(g^t v, g^t w)$ does not converge to zero. Then Theorem 6.9 shows that if δ is suitably small (but fixed) and if t is sufficiently large then the angle between $\eta \in TW^s(v)$ and $TW^u(v)$ becomes smaller than any $\gamma > 0$. Hence by Proposition 6.12 the distance under consideration becomes smaller than e.g. $\text{const} \cdot \delta^2$ (in particular smaller than $\delta/2$). Reapplying the argument to $g^t v$ and L'_t shows that the distance becomes arbitrarily small as $t \to \infty$. □

6.6. **Dynamical properties of stable and unstable leaves.**

Definition 6.18. Let $L, L' \subset W^{0u}$ be bounded, tall, δ-holonomic pieces of leaves. We can assume that there exist a tubular neighborhood of L in SM which contains $B^{0u}_\delta L$ and $B^{0u}_\delta L'$. Since we are interested in the case where L and L' are close and holomic, we can assume without loss of generality that for each $v \in L$, the shortest geodesic segment from v to $v' \in L'$ is contained in the tubular neighborhood. Define the **Riemannian projection** $\mathcal{P} : L \to B^{0u}_\delta L'$ to be the vector $\mathcal{P}v := w$ so that $d(w, v)$ is minimal with respect to the Sasaki distance on SM. Let $\operatorname{Jac}\mathcal{P}$ be the **Jacobian** of the Riemannian projection $\mathcal{P}$, given by

$$(\operatorname{Jac}\mathcal{P})(v) := \lim_{\delta_1 \to 0} \frac{\lambda^{0u}(\mathcal{P}(L \cap B_{\delta_1}(v)))}{\lambda^{0u}(L \cap B_{\delta_1}(v))}.$$

Let $\mathcal{P}_t$ be the Riemannian projection from L_t to $B^u_\delta L'_t$. Let $\operatorname{Jac}\mathcal{P}_t$ be the Jacobian of $\mathcal{P}_t$.

Similarly, let $\mathcal{P}'_t$ be the Riemannian projection from L'_t to $B^u_\delta L_t$ and let $\operatorname{Jac}\mathcal{P}'_t$ be the Jacobian of $\mathcal{P}'_t$.

Now we are going to show that this projection is arbitrarily close to preserving Liouville measure if the leaves sufficiently close:

Lemma 6.19. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $L, L' \subset W^u$ are such that they are tall, bounded, holonomic, δ-close and such that their tangents are δ-close with respect to* $\mathbf{d}$, *then*

$$|\operatorname{Jac}\mathcal{P} - 1| < \varepsilon.$$

Proof. Let N be the perpendicular distribution to the distribution TW^{0u}. This distribution is uniformly continuous since TW^{0u} is.

Let γ be any N-geodesic of length s from L to L' (hence $\gamma'(0) \in L^\perp$, $\gamma'(s) \in (L')^\perp$).

Let c be a geodesic in SM with $c'(0) = \gamma'(0)$. Then c crosses L' at some parameter value σ with $|\sigma - s| < \sigma^2$ and $|\angle(c'(\sigma), L') - \pi/2| < \sigma^2$ by uniform continuity of the two distributions. Hence $d(c(\sigma), \mathcal{P}c(0)) < \sigma^2$ whenever the distance δ between L and L' was small enough. Thus for r sufficiently small, $\lambda^{0u}\mathcal{P}B_r/\lambda^{0u}B_r$ is arbitrarily close to 1. □

From this we deduce that if we start with holonomic leaves at a fixed distance and wait sufficiently large, $\mathcal{P}_t$ will again be arbitrarily close to preserving Liouville measure:

Corollary 6.20. *Let $L, L' \subset W^u$ be such that they are tall with compact closure, holonomic, δ_0-close and such that their tangents are δ_0-close w.r.t.* **d**. *For all $\varepsilon > 0$ there exists $T > 0$ such that whenever $t > T$ then*

$$|\operatorname{Jac} \mathcal{P}_t - 1| < \varepsilon.$$

Proof. Simply observe that by Lemma 6.17, for any $v \in \bar{L}$, after flowing for sufficiently large time t, v_t is δ-close to L'_t for δ arbitrarily small. Observe that by compactness of $\bar{L}$, this can be guaranteed uniformly for all v in L. Hence for all $\delta > 0$ there exists $T > 0$ such that $t > T$ implies that for all $v \in L$ the estimate $d(v_t, L'_t) < \delta$ holds. In particular, L_t and L'_t are δ-holonomic. Increasing T further we can assure that also the tangents of L_t and L'_t are δ-close. Hence L_t and L'_t satisfy the assumptions of Lemma 6.19, which shows that the Riemannian Jacobian after flowing time t is smaller than ε. □

We have already seen that the leaves L, L' get arbitrarily close everywhere except possibly on the δ-neighborhood of their boundaries. Next we will show that this neighborhood does not contribute a substantial part.

Lemma 6.21 (No boundary effects)**.** *Let L, L' be bounded, tall and δ_0-holonomic. Define*

$$O(t) := L_t \setminus \mathcal{P}'_t(L'_t), \quad O'(t) := L'_t \setminus \mathcal{P}_t(L_t).$$

Then

$$\frac{\lambda^{0u} O(t)}{\lambda^{0u} L_t} \to 0, \quad \frac{\lambda^{0u} O(t)}{\lambda^{0u} L_t} \to 0$$

as $t \to \infty$.

Proof. First note that it suffices to show the claim for *strictly* stable holonomic leaves $L, L' \subset W^u$. It suffices to consider the case that $L = B^u_\gamma(v)$ for some arbitrarily small γ. It is obvious that for each $\varepsilon_2, \varepsilon_3 > 0$ the variable γ can be chosen such that the stable measure of the annulus

$$A := \bar{B}^u_{\gamma+\varepsilon_2}(v) \setminus B^u_{\gamma-\varepsilon_2}(v)$$

satisfies $m^u(A)/m^u(B^u_\gamma) < \varepsilon_3$; this is because the ball $B^u_{\gamma+\varepsilon_2}(v)$ of finite measure can be cut into arbitrarily many nested annuli, so some of the annuli must have small measure. Note that the convexity of the distance along geodesics implies that if $w \neq w'$ are in A then $d(w_t, w'_t) \to \infty$, and in particular this distance $d(w_t, w'_t)$ will become larger than δ_0 (hence larger than the distance between L and L'). Since the set

$$K := \{(w, w') : w, w' \in B^{0u}_{2\gamma}(v), d(v, w) = \gamma + \varepsilon_2/2,\ d(v, w') = \gamma - \varepsilon_2/2\}$$

is compact and disjoint from the diagonal, we deduce that $d(w_t, w'_t) \to \infty$ uniformly for $(w, w') \in K$. Thus we can choose T such that for all $t > T$ and all $(w, w') \in K$ the lower bound $d(w_t, w'_t) > \delta_0$ holds.

On the other hand, since distances along weakly stable fibers are nonincreasing, the distance $d(v_t, v'_t)$ is bounded by δ_0 for all positive time t. We have shown before that for t sufficiently large, the graphs are arbitrarily close. Thus the "overhang" $O(t)$ is contained in a δ_0-neighborhood of ∂L_t. Hence for t large enough, $O(t)$ is contained in A_t, since A_t expands without bounds. Therefore the uniform expansion property implies that

$$\lim_{t\to\infty} \frac{m^u(O(t))}{m^u(B^u_\gamma(v))} \leq \lim_{t\to\infty} \frac{m^u(A_t)}{m^u(g^t B^u_\gamma(v))} = \frac{m^u(A)}{m^u(B^u_\gamma(v))} < \varepsilon_3.$$

Since ε_3 is arbitrary, this shows that

$$\lim_{t\to\infty} m^u(O(t))/m^u(B^u_\gamma(v)) = 0$$

as claimed. Analogously we see for $L' = B^u_\gamma(v)$ that

$$\lim_{t\to\infty} m^u(O'(t))/m^u(B^u_\gamma(v')) = 0.$$

□

Now we have all the components to formulate the immediate precursor to holonomy invariance:

Theorem 6.22 (Asymptotic equality of Liouville measure for holonomic pieces)**.** *Let L, L' be bounded, tall and holonomic. Then*

$$\lim_{t\to\infty} \frac{l_t}{l'_t} = 1.$$

Proof. Corollary 6.20 shows that this is true for large t up to an error term at the boundary. Lemma 6.21 shows that this error term is asymptotically zero. □

Lemma 6.23. *Let L, L' be tall, holonomic and bounded. Then*

$$\frac{m^{0u}(L)}{m^{0u}(L')} = \lim_{t\to\infty} \frac{\lambda^{0u}(L_t)}{\lambda^{0u}(L'_t)}.$$

Proof. The characteristic function on L, L' can be approximated by continuous functions. Thus we can write

$$\frac{F_t\chi_L}{F_t\chi_{L'}} = \frac{\int \chi_L d\lambda^{0u}}{\int \chi_{L'} d\lambda^{0u}} = \frac{l_t}{l'_t} \to 1$$

as $t \to \infty$. Hence clearly also

$$\frac{F'_t\chi_L}{F'_t\chi_{L'}} = \frac{e^{-ht}F_t\chi_L}{e^{-ht}F_t\chi_{L'}} \to 1$$

as $t \to \infty$. Thus the term

$$\frac{\sum_i c_i F'_{t_i}\chi_L}{\sum_i c_i F'_{t_i}\chi'_L}$$

is arbitrarily close to 1 if $t_i > T_0$ for all i and T_0 is sufficiently large. Now let F^j be a sequence converging to m^{0u} with each F^j of the form

$\sum_i c_i F'_{t_i}$. Then we can assume that $t_i > T_0$ for all i and for sufficiently large j, where T_0 is sufficiently large. Hence $\frac{F^j \chi_L}{F^j \chi_{L'}}$ is arbitrarily close to 1 for j large. □

Now we can finish the proof of Theorem 6.2 (holonomy invariance of m^{0u}):

End of the proof of holonomy invariance. The previous statement (Theorem 6.22) shows that the quotient of Liouville measures, $\lambda^{0u}(L)/\lambda^{0u}(L')$, converges to 1. Lemma 6.23 shows that this quotient converges to

$$m^{0u}(L)/m^{0u}(L').$$

Hence $m^{0u}(L)/m^{0u}(L') = 1$. □

7. Assembling the measure of maximal entropy from measures on leaves

Now we are able to show the following:

Theorem 7.1. *There exists a measure m^{0u} on tall sets with compact closure in W^{0u}, which extends to a measure m^{0u} on* **Reg** *satisfying:*

- $m^{0u}(g^t(U)) = e^{ht} m^{0u}(U)$,
- $0 < m^{0u}(U) < \infty$ *for any $U \subset W^{0u}$ tall (and in particular nonempty),*
- *γ-equivalent tall sets U_1, U_2 in W^{0u} have the same m^{0u}-measure.*

Proof. Define

$$T(p) := \{\text{tall sets with compact closure in } W^{0u}(p)\},$$

$$T := \bigcup_{p \in M} T(p).$$

Let

$$C_U(T) := \{f \in C(W^{0u}) : \operatorname{supp}(f) \subset \bar{U}\}$$

for $U \in T$.

We have seen that m' (hereafter called m^{0u}) is a positive linear functional on $C_U(W^{0u})$. A suitable version of the Hahn-Banach Theorem shows that m' extends to $C(\bar{U})$. This extension is still positive, thus an appropriate version of the Riesz Representation Theorem [Fol] shows that there is a measure m_U on $\bar{U}$ such that $m'(f) = \int f dm_U$ for all $f \in C_U(W^{0u})$. □

Remark 7.2. Nonpositivity of the curvature suffices to see that there exist $t_0 = t_0(M) > 0$, $r_0 = r_0(M) > 0$ such that for all $U \in W^u$ with diameter at most r_0 and for all $0 \leq t < t' \leq t_0$ the iterates $g^t U$ and $g^{t'} U$ are disjoint. Hence we can define:

Definition 7.3.

$$m^u(U) := m^{0u}(L_U)$$

where

$$L_U := g^{[0,t_0]}U.$$

Remark 7.4. This gives a measure on W^u. This definition is valid whether U is regular or not, although we will be interested in m^u (and m^s, see below) on the regular set.

Proposition 7.5 (Uniform expansion on W^u)**.**

$$m^u \circ g^t = e^{ht} \cdot m^u.$$

Proof. Immediate from the analogous property of m^{0u} :

$$\begin{aligned} (m^u \circ g^t)(U) &= m^{0u}(g^t L_U) \\ &= e^{ht} m^{0u}(L_U) \\ &= e^{ht} m^u(U). \end{aligned}$$

□

Remark 7.6. If we apply time reversal (i.e. considering the flow $\bar{g}$ with $\bar{g}^t := g^{-t}$), then W^u of $\bar{g}$ is W^s of g and analog for W^{0u} etc. Thus the preceding construction gives another measure m^s (and a measure m^{0s}) with the same properties as time is reversed. In particular:

Proposition 7.7 (Uniform contraction on W^s)**.**

$$m^s \circ g^t = e^{-ht} \cdot m^s.$$

Lemma 7.8. *For all $\varepsilon > 0$ there is $\gamma > 0$ such that for γ-equivalent sets A_1, A_2 we have*

$$\left| \frac{m^s(A_1)}{m^s(A_2)} - 1 \right| < \varepsilon.$$

Thus for all $\gamma > 0$ there is C such that for γ-equivalent sets A_1, A_2 we have

$$1/C < m^s(A_1)/m^s(A_2) < C.$$

The same statement is true with m^s replaced by m^u.

Proof. As in [KaHa], we consider $g^{[0,t_0]}A_1$ and $g^{[0,t_0]}A_2$ and use holonomy invariance for those, after cutting off the non-overlapping part. □

We are now able to define a product measure on the regular set as follows. The regular set is open. As we have shown in Lemma 2.1, a local product structure exists in the neighborhood of any regular point. Thus if v is regular, there is a regular neighborhood U of v of the form $U = U^s \times U^{0u}$.

Definition 7.9. Let $O \subset U$ be regular and open. Define a function α_O by

$$\alpha_O(q) := m^s((\{q\} \times U^s(p)) \cap O).$$

This function can easily be verified to be semicontinuous and thus integrable on U. Therefore we can define:

Definition 7.10.
$$m_q(L) := m^{0u}(L \times \{q\}),$$
$$m(O) := \int \alpha_O dm_q.$$

This gives a well-defined measure on the regular set, since holonomy invariance shows that m_q does in fact not depend on q.

In the sequel we consider m to be normalized to 1.

Lemma 7.11. *The measure m has maximal entropy, i.e.*
$$h(m) = h_{\text{top}}.$$

Proof. This follows immediately from the property
$$m^{0u}(g^t(B)) = e^{ht} \cdot m^{0u}(B).$$
□

Theorem 7.12. *The measure m which we have constructed equals the Knieper measure whenever M is compact, nonpositively curved and of rank one.*

Proof. Since the measure of maximal entropy is unique [Kni1] and both our measure and the Knieper measure have maximal entropy, these measures are the same. □

Remark 7.13. The uniqueness of the maximal measure shows that m has a unique extension to the entire SM given by $m(U) := 0$ for all $U \subset \mathbf{Sing}$.

This completes the construction.

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