Hamburger Beiträge
zur Angewandten Mathematik

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constrained elliptic optimal control problems:
optimal error estimates and parameter adjustment

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August 13, 2007

Abstract: An adjustment scheme for the relaxation parameter of interior point approaches to the numerical solution of pointwise state constrained elliptic optimal control problems is introduced. The method is based on error estimates of an associated finite element discretization of the relaxed problems and optimally selects the relaxation parameter in dependence on the mesh size of discretization. The finite element analysis for the relaxed problems is carried out and a numerical example is presented which confirms our analytical findings.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain with a smooth boundary $\partial\Omega$. In this note, we are interested in the following control problem:

$$\min_{w \in U} J(w) = \tfrac{1}{2} \int_\Omega |\mathcal{G}(Bw) - y_0|^2 + \tfrac{\alpha}{2} \|w\|_U^2$$
$$\text{subject to } \mathcal{G}(Bw) \leq \overline{y} \text{ a.e. in } \Omega. \tag{1.1}$$

Here and throughout, "a.e." stands for "almost everywhere". We suppose that $\alpha > 0$, $y_0 \in H^1(\Omega)$ and $\overline{y} \in W^{2,\infty}(\Omega)$ are given. Further $(U, (\cdot, \cdot)_U)$ denotes a Hilbert space which we identify with its dual, and $B : U \to (H^1(\Omega))'$ a linear, continuous operator. For given $f \in (H^1(\Omega))'$ the function $\mathcal{G}(f)$ denotes the unique weak solution $y \in H^1(\Omega)$ to the elliptic boundary value problem

$$Ay = f \quad \text{in } \Omega,$$
$$\sum_{i,j=1}^d a_{ij} y_{x_i} \nu_j = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

Here, ν is the outward unit normal to $\partial\Omega$, and

$$Ay := -\sum_{i,j=1}^d \partial_{x_j}\big(a_{ij} y_{x_i}\big) + \sum_{i=1}^d b_i y_{x_i} + cy,$$

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where subsequently we assume that, for simplicity, the coefficients a_{ij}, b_i and c are smooth functions in $\bar{\Omega}$, and that there exists $c_0 > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \text{ and all } x \in \Omega.$$

We associate with A the bilinear form

$$a(y,z) := \int_{\Omega} \Big(\sum_{i,j=1}^{d} a_{ij}(x)y_{x_i}z_{x_j} + \sum_{i=1}^{d} b_i(x)y_{x_i}z + c(x)yz \Big) dx, \qquad y, z \in H^1(\Omega).$$

Furthermore we suppose that a is coercive on $H^1(\Omega)$, i.e., there exists $c_1 > 0$ such that

$$a(v,v) \geq c_1\|v\|_{H^1}^2 \qquad \text{for all } v \in H^1(\Omega). \tag{1.3}$$

Furthermore, if $f \in L^2(\Omega)$, then the solution y belongs to $H^2(\Omega)$ and satisfies

$$\|y\|_{H^2} \leq C\|f\|,$$

where have used $\|\cdot\|$ to denote the $L^2(\Omega)$-norm.
We note that the formal adjoint operator of A is given by

$$A^*y = -\sum_{i=1}^{d} \partial_{x_i} \Big(\sum_{j=1}^{d} a_{ij}y_{x_j} + b_i y \Big) + cy.$$

It is not hard to prove that problem (1.1) admits a unique solution $u \in U$. Moreover, from [5, Theorem 2] we deduce that there exist functions $\lambda \in \mathcal{M}(\bar{\Omega})$ and $p \in L^2(\Omega)$ satisfying, together with $y = \mathcal{G}(Bu)$, the dual system

$$\int_{\Omega} pAv = \int_{\Omega} (y - y_0)v + \int_{\bar{\Omega}} v d\lambda \qquad \forall v \in H^2(\Omega) \text{ with } \sum_{i,j=1}^{d} a_{ij}v_{x_i}\nu_j = 0 \text{ on } \partial\Omega, \tag{1.4}$$

$$B^*p + \alpha u = 0 \quad \text{in } U, \tag{1.5}$$

$$\lambda \geq 0, \ y \leq \bar{y} \text{ a.e. in } \Omega \text{ and } \int_{\bar{\Omega}} (\bar{y} - y)d\lambda = 0. \tag{1.6}$$

Here, $\mathcal{M}(\bar{\Omega})$ denotes the space of Radon measures, which is defined as the dual space of $C_0(\bar{\Omega})$. It is endowed with the norm

$$\|\lambda\|_{\mathcal{M}(\bar{\Omega})} = \sup_{f \in C_0(\bar{\Omega}), |f| \leq 1} \int_{\bar{\Omega}} f d\lambda.$$

Now suppose

Assumption 1.1.
$$\text{For } u \in U \text{ there holds } \mathcal{G}(Bu) \in C^0(\bar{\Omega}).$$

Example 1.2. There are several examples for the choice of B and U, for which Assumption 1.1 holds.

(i) Distributed control: $U = L^2(\Omega)$, $B = Id : L^2(\Omega) \to H^1(\Omega)'$.

(ii) Boundary control: $U = L^2(\Omega)$, $Bu(\cdot) = \int u\gamma_0(\cdot) \, dx : L^2(\Omega) \to H^1(\Omega)'$, where γ_0 is the trace operator in $H^1(\Omega)$. In this case Assumption 1.1 holds in the case $d = 2$.

(iii) Linear combinations of input fields: $U = \mathbb{R}^n$, $Bu = \sum_{i=1}^{n} u_i f_i$, $f_i \in H^{-s}(\Omega)$, $s < (d-1)/d$.

Supposing a Slater condition of the type

Assumption 1.3.
$$\text{There exist } \hat{u} \in U, \tau > 0 : \quad \mathcal{G}(B\hat{u}) \leq -\tau,$$

a finite element analysis of problem (1.1) was carried out in [7] (compare also [6]) for $\mathcal{I}mB \subseteq L^2(\Omega)$, yielding the following error bounds:

$$\|u - u_h\|_U, \ \|y - y_h\|_{H^1} = \begin{cases} O(h^{\frac{1}{2}}), & \text{if } d = 2, \\ O(h^{\frac{1}{4}}), & \text{if } d = 3, \end{cases} \tag{1.7}$$

where u_h and y_h are the discrete optimal control and state, respectively. If, in addition, $Bu \in W^{1,s}(\Omega)$ then

$$\|u - u_h\|_U, \ \|y - y_h\|_{H^1} \leq Ch^{\frac{3}{2} - \frac{d}{2s}} \sqrt{|\log h|}.$$

In the present paper, our aim is to investigate a finite element approximation of an interior point technique for the numerical solution of (1.1) and to provide optimal adjustment strategies for the relaxation parameter with respect to the finite element mesh size. From here onwards and without loss of generality it is convenient to set $\overline{y} \equiv 0$.

The regularized version of (1.1) considered in this paper reads

$$\min_{w \in U} J(w) = \frac{1}{2} \int_{\Omega} |\mathcal{G}(Bw) - y_0|^2 + \frac{\alpha}{2} \|w\|_U^2 - \mu \int_{\Omega} \log(-y) dx, \tag{1.8}$$

where $\mu > 0$ denotes the relaxation parameter. In order to make the functional well defined for all $w \in U$, we set $J(w) = +\infty$, if $y > 0$ on an non-zero set.

The rest of the paper is organized as follows: In Section 2 we collect basic results on (1.8). In Section 3 we present the finite element analysis of problem (1.8). Among other aspects we prove the error bounds

$$\|y^\mu - y_h^\mu\|_1 + \|u^\mu - u_h^\mu\|_U \leq Ch^{1 - \frac{d}{4}} \quad (\leq Ch^{2 - \frac{d}{2} - \epsilon} \text{ for } Bu \in W^{1,s}(\Omega)),$$

where y_h^μ, u_h^μ denote the finite element approximations to y^μ and u^μ, respectively. We note that the latter estimate is in the spirit of (1.7). In Section 4 we discuss the overall errors

$$\|y - y_h^\mu\|_1 \sim \|y - y^\mu\|_1 + \|y^\mu - y_h^\mu\|_1 \text{ and } \|u - u_h^\mu\|_U \sim \|u - u^\mu\|_U + \|u^\mu - u_h^\mu\|_U$$

and propose a-priori strategies for balancing μ and h.

In Section 5 we present numerical results which confirm our theoretical findings.

Let us comment on further approaches that tackle optimization problems for pdes with control and state constraints. In [14] Meyer considers a fully discrete strategy to approximate an elliptic control problem with pointwise state and control constraints. He obtains the approximation order $\mathcal{O}(h^{2-d/2-\epsilon})$ for the state in H^1 and for the control in L^2, where d denotes the spatial dimension and $\epsilon > 0$ can be chosen arbitrarily. His results confirm those obtained by Deckelnick and the first author in [6] for the purely state constrained case. A *Lavrentiev-type regularization* of problem (1.1) is investigated in [16]. In this approach the state constraint $y \leq b$ in (1.1) is replaced by the mixed constraint $\epsilon u + y \leq b$, with $\epsilon > 0$ denoting a regularization parameter. It turns out that the associated Lagrange multiplier μ_ϵ belongs to $L^2(\Omega)$. Numerical analysis for this approach with emphasis on the coupling of gridsize and regularization parameter ϵ is presented by the first author and Meyer in [13]. The resulting optimization problems are solved either by interior-point methods or primal-dual active set strategies, compare [15]. The development of numerical approaches to tackle (1.1) is ongoing. An excellent overview is given by Hintermüller and Kunisch in in [9, 10]. An introductory text is provided by Tröltzsch with [24].

2 The regularized problem

Problems of the type (1.8) have been analyzed in [21] for the case $B = Id$. However, by straightforward modifications this analysis can be extended to the more general case considered here, provided Assumption 1.1 holds.

First, for each $\mu \geq 0$ problem (1.8) admits a unique solution $u^\mu \in U$ with associated state $y^\mu = \mathcal{G}(Bu^\mu)$. Furthermore there exists a function $p^\mu \in W^{1,s}(\Omega)$ and a regular Borel measure $\lambda^\mu \in M(\overline{\Omega})$ which satisfy the adjoint system

$$a(v, p^\mu) = \int_\Omega (y^\mu - y_0)v + \int_\Omega v\, d\lambda^\mu \ \forall v \in H^2(\Omega) \text{ with } \sum_{i,j=1}^d a_{ij} v_{x_i} \nu_j = 0 \text{ on } \partial\Omega, \qquad (2.9)$$

$$B^* p^\mu + \alpha u^\mu = 0 \text{ in } U. \qquad (2.10)$$

Moreover, y^μ is strictly feasible a.e. in Ω, the measure $\lambda^\mu \in M(\overline{\Omega})$ is non-negative, and splits into two parts:

$$\int_{\overline{\Omega}} v\, d\lambda^\mu = -\int_\Omega \frac{\mu}{y^\mu} v + \int_{y^\mu = 0} v\, d\lambda. \qquad (2.11)$$

The measure λ is non-negative and vanishes, if y^μ is strictly feasible. In this case λ^μ and p^μ are uniquely defined. As a consequence the following complementarity condition

$$\int_{\overline{\Omega}} y^\mu\, d\lambda = 0 \qquad (2.12)$$

holds. Furthermore, from [21, Proposition 4.5] we deduce

$$\|\lambda^\mu\|_{M(\overline{\Omega})}, \ \|\frac{\mu}{y^\mu}\|_{L^1(\Omega)}, \|\lambda\|_{M(\overline{\Omega})}, \text{ and } \|p^\mu\|_{W^{1,s}(\Omega)} \leq C \qquad (2.13)$$

with some positive constant C that is independent of μ.

Remark 2.1. The potential occurrence of λ in (2.11) motivates to consider rational barrier functionals $\int_\Omega b(y; \mu)$ of the form

$$b(y; \mu) = \frac{\mu^q}{(q-1)y^{q-1}}.$$

Their (formal) gradients read

$$b'(y, \mu) = -\frac{\mu^q}{y^q}.$$

In [21] it is shown that for sufficiently high *order* q, the non-regular part λ vanishes for all $\mu > 0$. The analysis of this paper also applies to this class of functionals. An appropriate order q depends on the dimension of the problem and on the regularity of y^μ, and can be chosen a-priori for certain classes of problems.

The convergence analysis of the regularization path is also covered by the results of [21]. In Lemma 5.1 and Theorem 5.3 there it is proven that

$$J(u^\mu) - J(u) \leq C\mu$$

and

$$\|u^\mu - u\| \leq C\sqrt{\mu}$$

hold.

From

$$c_1\|y - y^\mu\|_{H^1}^2 \leq a(y - y^\mu, y - y^\mu) = \langle B(u - u^\mu), y - y^\mu \rangle \leq C\|u - u^\mu\|_U \|y^\mu - y\|_{H^1}$$

we immediately infer

$$\|y^\mu - y\|_{H^1} + \|u^\mu - u\|_U \leq C\|u^\mu - u\|_U. \qquad (2.14)$$

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3 Finite element discretization and error analysis for (1.8)

Let $\mathcal{T}_h$ be a triangulation of Ω with maximum mesh size $h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$ and vertices $x_1, \ldots, x_m$. We suppose that $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$ so that element edges lying on the boundary are curved. In addition, we assume that the triangulation is quasi-uniform in the sense that there exists a constant $\kappa > 0$ (independent of h) such that each $T \in \mathcal{T}_h$ is contained in a ball of radius $\kappa^{-1}h$ and contains a ball of radius κh. Let us define the space of linear finite elements,

$$X_h := \{v_h \in C^0(\bar{\Omega}) \,|\, v_h \text{ is a linear polynomial on each } T \in \mathcal{T}_h\}$$

with the appropriate modification for boundary elements.

In what follows it is convenient to introduce a discrete approximation of the operator $\mathcal{G}$. In fact, for a given function $v \in H^1(\Omega)'$ we denote by $z_h = \mathcal{G}_h(v) \in X_h$ the solution of the discrete Neumann problem

$$a(z_h, v_h) = \langle v, v_h \rangle \text{ for all } v_h \in X_h.$$

It is well-known [22] that for all $v \in L^2(\Omega)$

$$\|\mathcal{G}(v) - \mathcal{G}_h(v)\| \leq Ch^2\|v\|, \tag{3.15}$$

$$\|\mathcal{G}(v) - \mathcal{G}_h(v)\|_{L^\infty} \leq Ch^{2-\frac{d}{2}}\|v\|. \tag{3.16}$$

The estimate (3.16) can be improved provided one strengthens the assumption on v, compare [7, Lemma 2.1]. To simplify the exposition we from here onwards assume $\mathcal{I}mB \subseteq L^2(\Omega)$.

3.1 Estimate for $\|y^\mu - y_h^\mu\|_{H_1} + \|u^\mu - u_h^\mu\|_U$

Problem (1.8) is now approximated by the following sequence of control problems depending on the mesh parameter h:

$$\min_{u \in U} J_h(u) := \frac{1}{2}\int_\Omega |\mathcal{G}_h(Bu) - y_0|^2 + \frac{\alpha}{2}\|u\|_U^2 - \mu \int_\Omega \log\left(-\mathcal{G}_h(Bu)\right). \tag{3.17}$$

Problem (3.17) represents a convex infinite-dimensional optimization problem of a similar structure as problem (1.8). It admits a unique solution $u_h^\mu \in U$ with corresponding state $y_h^\mu \in X_h$. Furthermore, in accordance with problem (1.8), there exist a unique function $p_h^\mu \in X_h$ and a regular, non–negative Borel measure λ_h^μ satisfying

$$a(v_h, p_h^\mu) = \int_\Omega (y_h^\mu - y_0)v_h + \int_{\bar{\Omega}} v_h d\lambda_h^\mu \text{ for all } v_h \in X_h, \text{ and} \tag{3.18}$$

$$\alpha u_h^\mu + B^* p_h^\mu = 0 \text{ in } U. \tag{3.19}$$

The function y_h^μ is strictly feasible a.e. in Ω. The measure can be represented in the form

$$\int_{\bar{\Omega}} f d\lambda_h^\mu = -\int_\Omega \frac{\mu}{y_h^\mu} f dx + \sum_{i=1}^m \mu_i \delta_{x_i} f \text{ for all } f \in C^0(\bar{\Omega}), \tag{3.20}$$

where $\mu_1, \ldots, \mu_m \geq 0$ and δ_{x_i} denotes the Dirac measure concentrated at the finite element node x_i $(i = 1, \ldots, m)$. Furthermore, the measure vanishes if y_h^μ is strictly feasible. As a consequence we also have strict complementarity

$$\int_{\bar{\Omega}} y_h^\mu d(\sum_{i=1}^m \mu_i \delta_{x_i}) = 0. \tag{3.21}$$

We note that the control is not discretized in (3.17), compare [6, 7, 12, 20] for a more detailed discussion of this discretization approach.

Next we prove an error estimate in h which is independent of μ. For this purpose we first prove an uniform bounds w.r.t. μ for $\|\frac{\mu}{y_h^\mu}\|_{L^1(\Omega)}$, $\|\lambda_h^\mu\|_{M(\bar\Omega)}$, and $\sum_{i=1}^m \mu_i$.

Lemma 3.1. Let Assumption 1.3 be satisfied, let u_h^μ denote the unique solutions to (3.17), and let y_h^μ denote the corresponding state. Then

$$\|\lambda_h^\mu\|_{M(\bar\Omega)}, \ \|\frac{\mu}{y_h^\mu}\|_{L^1(\Omega)} \text{ and } \sum_{i=1}^m \mu_i \leq C \tag{3.22}$$

with some positive constant C independent of μ and of h.

Proof. Using (3.16) we obtain for some small enough $0 < h_0$

$$\mathcal{G}_h(B\hat u) \leq -\frac{\tau}{2} \text{ for all } 0 < h \leq h_0. \tag{3.23}$$

Therefore,

$$\frac{\tau}{2}\int_{\bar\Omega} d\lambda_h^\mu \leq \int_{\bar\Omega} -\mathcal{G}_h(B\hat u)d\lambda_h^\mu =$$

$$= \int_\Omega (y_h^\mu - y_0)\mathcal{G}_h(B\hat u) - a(\mathcal{G}_h(B\hat u), p_h^\mu) = \int_\Omega (y_h^\mu - y_0)\mathcal{G}_h(B\hat u) - \int_\Omega B\hat u p_h^\mu =$$

$$= \int_\Omega (y_h^\mu - y_0)\mathcal{G}_h(B\hat u) + \alpha(\hat u, u_h^\mu) \leq C.$$

Since

$$\sup_{\|f\|_\infty \leq 1} \int_{\bar\Omega} f d\lambda_h^\mu = -\int_\Omega \frac{\mu}{y_h^\mu}f dx + \sum_{i=1}^m \mu_i \delta_{x_i} f \leq \|\frac{\mu}{y_h^\mu}\|_{L^1(\Omega)} + \sum_{i=1}^m \mu_i = \int_{\bar\Omega} d\lambda_h^\mu \leq \frac{2C}{\tau},$$

the claim follows for $\|\lambda_h^\mu\|_{M(\bar\Omega)}$. Since $\sum_{i=1}^m \mu_i \geq 0$ the last estimate also gives the claim for $\|\frac{\mu}{y_h^\mu}\|_{L^1(\Omega)}$ and $\sum_{i=1}^m \mu_i$.

We are now prepared to prove

Theorem 3.2. Let u^μ denote the solution of (1.8) with $y^\mu = \mathcal{G}(Bu^\mu)$, and u_h^μ the solution to (3.17) with $y_h^\mu = \mathcal{G}_h(Bu_h^\mu)$. Then there exists some $h_0, 1 \geq h_0 > 0$, and a constant independent of μ and h such that

$$\|u^\mu - u_h^\mu\|_U + \|y^\mu - y_h^\mu\|_{H^1} \leq Ch^{1-\frac{d}{4}} \text{ for all } 0 < h \leq h_0. \tag{3.24}$$

Proof. Let $y^h, p^h \in X_h$ denote the finite element functions defined by

$$a(y^h, v_h) = \int_\Omega Bu^\mu v_h, \text{ and } a(v_h, p^h) = \int_\Omega (y^\mu - y_0)v_h + \int_{\bar\Omega} v_h d\lambda^\mu \text{ for all } v_h \in X_h$$

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and set $\lambda_h := \sum_{i=1}^{m} \mu_i \delta_{x_i}$. Next we test the difference of (2.10) and (3.19) with $u^\mu - u_h^\mu$. This gives

$$\alpha \|u^\mu - u_h^\mu\|_U^2 = \int_\Omega (p^\mu - p_h^\mu) B(u_h^\mu - u^\mu) dx =$$

$$= \int_\Omega (p^\mu - p_h) B(u_h^\mu - u^\mu) dx + \int_\Omega (p^h - p_h^\mu) B(u_h^\mu - u^\mu) dx =: (1) + (2).$$

We proceed with estimating

$$(1) \leq \frac{\alpha}{2} \|u_h^\mu - u^\mu\|_U^2 + \frac{C}{\alpha} \|p^\mu - p^h\|^2 \leq \frac{\alpha}{2} \|u_h^\mu - u^\mu\|_U^2 + \frac{C}{\alpha} h^{4-d} \left(\|y^\mu - y_0\|^2 + \|\lambda^\mu\|_{M(\bar\Omega)}^2 \right),$$

where we have used a result of Casas [4] to estimate the finite element error $\|p^\mu - p^h\|$. Further, using the definition of the auxiliary functions y^h, p^h and the optimality conditions we get

$$(2) = a(y_h^\mu - y^h, p^h - p_h^\mu) = \int_\Omega (y^\mu - y_h^\mu)(y_h^\mu - y^h) dx + \int_{\bar\Omega} y_h^\mu - y^h d(\lambda^\mu - \lambda_h^\mu) =$$

$$= -\|y^\mu - y_h^\mu\|^2 + \int_\Omega (y^\mu - y_h^\mu)(y^\mu - y^h) dx + \int_{\bar\Omega} y_h^\mu - y^h d(\lambda^\mu - \lambda_h^\mu) \leq$$

$$\leq -\frac{1}{2} \|y^\mu - y_h^\mu\|^2 + \frac{1}{2} \|y^\mu - y^h\|^2 + \int_{\bar\Omega} y_h^\mu - y^h d(\lambda^\mu - \lambda_h^\mu) =$$

$$= -\frac{1}{2} \|y^\mu - y_h^\mu\|^2 + \frac{1}{2} \|y^\mu - y^h\|^2 + \underbrace{\int_\Omega (\frac{\mu}{y_h^\mu} - \frac{\mu}{y^\mu})(y_h^\mu - y^\mu) dx}_{\leq 0 \text{ by monotonicity}} +$$

$$+ \int_\Omega \frac{\mu}{y_h^\mu}(y^\mu - y^h) dx + \int_\Omega \frac{\mu}{y^\mu}(y^h - y^\mu) dx + \int_{\bar\Omega} y_h^\mu - y^h d(\lambda - \lambda_h) \leq$$

$$\leq -\frac{1}{2} \|y^\mu - y_h^\mu\|^2 + \frac{1}{2} \|y^\mu - y^h\|^2 + \{\|\lambda\|_{M(\bar\Omega)}\|, \|\lambda_h\|_{M(\bar\Omega)}, \|\frac{\mu}{y^\mu}\|_{L^1}, \|\frac{\mu}{y_h^\mu}\|_{L^1}\} \|y^\mu - y^h\|_\infty,$$

where we have used the complementarity conditions (2.12) for λ and (3.21) for λ_h. Combining (1) and (2) we obtain with the help of (3.15), (3.16) and the bounds (2.13), (3.22)

$$\alpha \|u^\mu - u_h^\mu\|_U^2 + \frac{1}{2} \|y^\mu - y_h^\mu\|^2 \leq C \left(\frac{1}{\alpha} h^{4-d} \left(\|y^\mu - y_0\|^2 + \|\lambda_h^\mu\|_{M(\bar\Omega)}^2 \right) + h^2 \|u^\mu\|_U^2 \right)$$

$$+ C \left(h^{2-\frac{d}{2}} \{\|\lambda\|_{M(\bar\Omega)}\|, \|\lambda_h\|_{M(\bar\Omega)}, \|\frac{\mu}{y^\mu}\|_{L^1}, \|\frac{\mu}{y_h^\mu}\|_{L^1}\} \right).$$

Using this estimate and

$$\|y^\mu - y_h^\mu\|_{H^1} \leq \|y^\mu - y^h\|_{H^1} + \|y^h - y_h^\mu\|_{H^1} \leq C \{h\|u_\mu\|_U + \|u^\mu - u_h^\mu\|_U\},$$

we finally get the desired result, since $h_0 \leq 1$.

Let us recall that by [7, Lemma 2.1] for $v \in W^{1,s}(\Omega)$ $(1 < s < \frac{d}{d-1})$ we have

$$\|\mathcal{G}(v) - \mathcal{G}_h(v)\|_\infty \leq C h^{3-\frac{d}{s}} \|v\|_{W^{1,s}}. \tag{3.25}$$

Now let us assume that Bu^μ is uniformly bounded in $W^{1,s}(\Omega)$ for some $1 < s < \frac{d}{d-1}$. Then we deduce from the proof of the previous Theorem

Corollary 3.3.

$$\|u^\mu - u_h^\mu\|_U + \|y^\mu - y_h^\mu\|_{H^1} \le Ch^{\frac{3}{2} - \frac{d}{2s}} \sqrt{|\log h|} \text{ for all } 0 < h \le h_0. \tag{3.26}$$

Remark 3.4. Let us provide the following comments.

(i) For (i) of Example 1.2 we have $Bu = u \in W^{1,s}(\Omega)$ by (2.13) combined with (2.10), so that (3.26) holds in this case.

(ii) We note that the analysis carried out in this section with obvious modifications also applies to Neumann boundary control. In this case the control operator is defined as $B : L^2(\Gamma) \to H^1(\Omega)'$ and maps boundary controls $u \in L^2(\Gamma)$ to the functional $Bu(v) := \int_\Gamma u\gamma_0 v d\Gamma$, where $v \in H^1(\Omega)$. Thus, $B = \gamma_0^*$, where $\gamma_0 : H^1(\Omega) \to H^{1/2}(\Gamma)$ denotes the trace operator, see (ii) of Example 1.2. In order to carry out the error analysis, only finite element error bounds corresponding to (3.16) have to be provided.

(iii) The proof of Theorem 3.2 carries over to rational barrier functionals as considered in Remark 2.1 by dropping the terms containing λ, λ_h and by substituting the barrier integrals accordingly.

3.2 The use of quadrature rules

So far we have assumed that the integrals $\langle \mu/y^h, v_h \rangle$ are evaluated exactly. Now let us replace this term in (3.18) by a quadrature rule of the form

$$\left\langle \frac{\mu}{y_h^\mu}, v_h \right\rangle_Q := \sum_i \omega_i \frac{\mu}{y_h^\mu(x_i)} v_h(x_i),$$

so that the discrete optimality system (3.18)–(3.19) is replaced by

$$a(v_h, p_h^\mu) = \int_\Omega (y_h^\mu - y_0) v_h + \left\langle \frac{\mu}{y_h^\mu}, v_h \right\rangle_Q \quad \text{for all } v_h \in X_h, \text{ and} \tag{3.27}$$

$$\alpha u_h^\mu + B^* p_h^\mu = 0 \text{ in } U. \tag{3.28}$$

We will now study discretization errors for this case under the following minimal assumptions;

- the quadrature rule yields positive values for positive functions

- it integrates constant functions exactly

- it yields a solution y_h^μ which is feasible

Note that assumptions on the error introduced by the quadrature rule are not needed.

It is easy to see that y_h^μ is strictly feasible at x_i. Otherwise the discrete barrier functional would be ∞. These assumptions are valid a-priori for linear finite elements, if the bound $\overline{y} \in X_h$, the quadrature points are taken at the nodes of the discretization, and the weights are chosen appropriately (from the trapezoidal rule, say).

We apply the quadrature rule only for the evaluation of the barrier term, while all other integrals are assumed to be evaluated exactly.

Theorem 3.5. Let u^μ denote the solution of (1.8) with $y^\mu = \mathcal{G}(Bu^\mu)$, and u_h^μ the solution to (3.17) with $y_h^\mu = \mathcal{G}_h(Bu_h^\mu)$. Then there exists some $h_0, 1 \ge h_0 > 0$, and a constant independent of μ and h such that

$$\|u^\mu - u_h^\mu\|_U + \|y^\mu - y_h^\mu\|_{H^1} \le C\sqrt{h^{2 - \frac{d}{2}} + 2\mu} \text{ for all } 0 < h \le h_0. \tag{3.29}$$

Proof. The proof runs along the lines of the proof of Theorem 3.2 with the exception that the difference of the barrier terms

$$\int_\Omega (y_h^\mu - y^h) d(\lambda^\mu - \lambda_h^\mu)$$

has to be replaced by

$$\int_\Omega (y_h^\mu - y^h) d\lambda^\mu + \left\langle \frac{\mu}{y_h^\mu}, (y_h^\mu - y^h) \right\rangle_Q$$

$$= \int_\Omega (y_h^\mu - y^\mu + y^\mu - y^h) d\lambda^\mu + \left\langle \frac{\mu}{y_h^\mu}, (y_h^\mu - y^\mu + y^\mu - y^h) \right\rangle_Q$$

$$= \underbrace{\int_\Omega y_h^\mu d\lambda^\mu}_{\leq 0} + \mu|\Omega| + \int_\Omega (y^\mu - y^h) d\lambda^\mu + \mu|\Omega| - \underbrace{\left\langle \frac{\mu}{y_h^\mu}, y^\mu \right\rangle_Q}_{\geq 0} + \left\langle \frac{\mu}{y_h^\mu}, (y^\mu - y^h) \right\rangle_Q$$

$$\leq 2\mu|\Omega| + \int_\Omega (y^\mu - y^h) d\lambda^\mu + \left\langle \frac{\mu}{y_h^\mu}, (y^\mu - y^h) \right\rangle_Q$$

$$\leq 2\mu|\Omega| + \left(\|\lambda^\mu\|_{M(\Omega)} + \left\| \frac{\mu}{y_h^\mu} \right\|_{L_1(Q)} \right) \|y^\mu - y^h\|_{L_\infty}.$$

With the same argumentation as in Lemma 3.1 we show that $\left\| \frac{\mu}{y_h^\mu} \right\|_{L_1(Q)} < C$ independently of μ. Proceeding as in the proof of Theorem 3.2 yields the stated result. ∎

Hence, using this type of quadrature rules introduces an additional error, which is of the order of the remaining length of the central path.
Again, if Bu^μ is uniformly bounded in $W^{1,s}(\Omega)$ for some $1 < s < \frac{d}{d-1}$. Then, as before

Corollary 3.6.

$$\|u^\mu - u_h^\mu\|_U + \|y^\mu - y_h^\mu\|_{H^1} \leq C\sqrt{h^{3-\frac{d}{s}}|\log h| + 2\mu} \text{ for all } 0 < h \leq h_0. \tag{3.30}$$

Using the trapezoidal rule as a quadrature rule and a fixed grid we may interpret the resulting numerical scheme as an interior point method for solving the discrete optimization problem obtained by discretization in the spirit of [7].

4 Parameter adjustment

The analysis in the preceding two sections allows us to optimally select μ for given h, or vice versa. In fact, its was shown that there exists a constant C independent of μ such that

$$\|u - u_h^\mu\|_U \leq \|u - u^\mu\|_U + \|u^\mu - u_h^\mu\|_U \leq C\left(\sqrt{\mu} + h^{1-\frac{d}{4}} \right). \tag{4.31}$$

If we have that Bu^μ is uniformly bounded in $W^{1,s}(\Omega)$ for some $1 < s < \frac{d}{d-1}$ we would obtain

$$\|u - u_h^\mu\|_U \leq \|u - u^\mu\|_U + \|u^\mu - u_h^\mu\|_U \leq C\left(\sqrt{\mu} + h^{\frac{3}{2}-\frac{d}{2s}} \sqrt{|\log h|} \right). \tag{4.32}$$

For given $\mu > 0$ the mesh size h on the right hand side can be adjusted on the basis of (4.31) as

$$h(\mu) \sim \mu^{\frac{2}{4-d}}. \tag{4.33}$$

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The optimal error bound in (4.31) then is

$$\|u - u_h^\mu\|_U \leq Ch^{1-\frac{d}{4}} \tag{4.34}$$

if μ is chosen proportional to $h^{\frac{4-d}{2}}$. If we have the regularity of Corollary 3.3 the optimal adjustment is given by

$$h(\mu) \sim \begin{cases} \sqrt{\mu} & \text{if } d = 2, \\ \mu & \text{if } d = 3. \end{cases}$$

This provides a qualitative guideline when to stop the interior point method for a fixed discretization, or how to refine the discretization for a given μ.

5 Numerical examples

Finally we illustrate our theoretical findings by a numerical example. We choose $\Omega = [0; 1] \times [0; 1]$, $A = -\Delta + I$, $B = Id$, $\overline{y} = 0.5$, $y_0 = 2 \cdot x_1 \cdot x_2$, $\alpha = 10^{-3}$. The discretization of y and p is performed by a linear finite element method, based on the DUNE library [1]. For the evaluation of the barrier integrals we use the trapezoidal rule, as analyzed in Section 3.2. For the numerical solution we use an interior point Newton path-following method in the variables y and p similar to the one analyzed in [20]. The resulting linear systems of equations are solved by the direct sparse solver $PARDISO$ [18].

We compute y_h^μ and p_h^μ and estimate their overall errors w.r.t. y_*, p_* by comparing them with a discrete solution that is computed on a very fine grid for a very small μ. The choices of h were $h = 2^{-k}$ with $k = 2 \ldots 8$. The plots in Figure 1 show $\|y_h^\mu - y_*\|_{H^1} + \sqrt{\alpha} \|u_h^\mu - u_*\|$.

As one can see in the left plot, the error introduced by the regularization dominates, until a break even point is reached. Even on the finest level of discretization, this is for relatively large $\mu \approx 10^{-4} - 10^{-5}$. In this range would be the most efficient point to stop the algorithm. It is interesting to note that for this particular problem the convergence of the path is slightly faster than $O(\sqrt{\mu})$. Hence, an a-posteriori estimate for the remaining length of the central path as described, for example, in [19, Section 8.2] may be appropriate. In the right plot we observe that the discretization error for a small μ behaves like $O(h)$ as predicted by our theory.
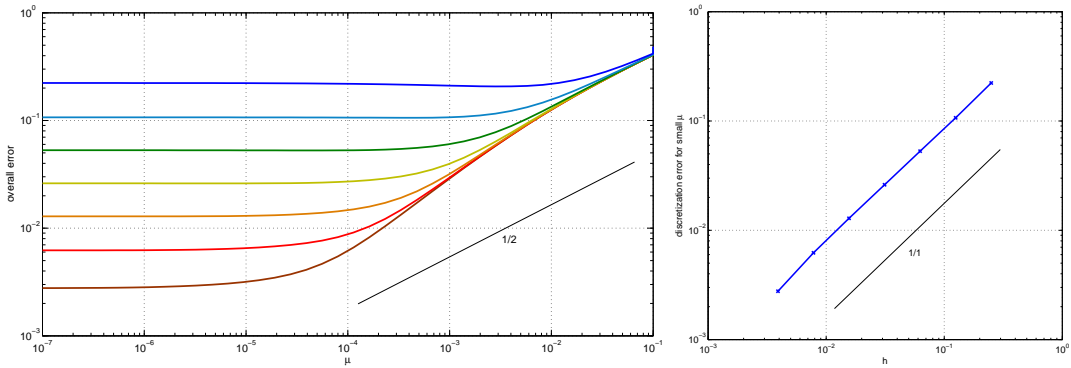

Figure 1: Left: Overall errors plotted against μ for $h = 2^{-k}$, $k = 2 \ldots 8$. Right: Discretization errors plotted against h for small μ.

Acknowledgements

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