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Abstract. In the present paper we complement the work in [2] with presenting the analytical framework for general optimal boundary control problems of the Boussinesq approximation. We prove existence of optimal controls, use results of [6] to prove existence and uniqueness of solutions to state and the adjoint system, and derive first order necessary as well as second order sufficient optimality conditions.

Keywords: Time–dependent Boussinesq approximation, optimal boundary control, optimality conditions.

1 Introduction

In the present work we analyze the following optimal boundary control problem considered in [2];

$$(P) \qquad \begin{cases} \min J(y,\theta,u) \\ \text{s.t.} u \in U_{ad} \text{ and} \end{cases} \tag{1}$$

$$y_t + (y\cdot\nabla)y - \nu\Delta y + \nabla p - Gr\,\theta\,\boldsymbol{g} = 0 \qquad \text{on } \Omega_T, \tag{2}$$

$$-div\; y = 0 \qquad \text{on } \Omega_T, \tag{3}$$

$$\theta_t + (y\cdot\nabla)\theta - \frac{1}{Pr}\Delta\theta - f = 0 \qquad \text{on } \Omega_T \quad . \tag{4}$$

$$\left.\begin{array}{r} y_1 = 0,\ y_2 = 0,\ y_3 = 0 \ \text{ on } \Gamma_T, \\ \frac{1}{Pr}\frac{\partial\theta}{\partial\mathbf{n}} + b\theta = Bu \quad \text{on} \quad \Gamma_T. \end{array}\right\} \tag{5}$$

Here, $\Omega \subset \mathbb{R}^2$ denotes the flow domain with the boundary Γ and $[0,T]$ the time horizon. Further, $y = (y_1, y_2, y_3)$ denotes the flow velocity vector field in the melt, and θ the temperature. The function Bu denotes the temperature flux on the boundary and u serves as abstract control variable.

Further $\Omega_T := \Omega \times [0,T]$, $\Gamma_T := \Gamma \times [0,T]$, and $b > 0$ denotes an appropriate coefficient. $B : U \to L^2(0,T;H^{-1/2}(\Gamma))$ denotes the bounded, linear control operator which maps abstract controls of the Hilbert space U to feasible temperature boundary controls, and $U_{ad} \subseteq U$ closed and convex the set of admissible controls.

The initial velocity is chosen as the neutral position of the crystal melt, i.e.

$$y = 0, \tag{6}$$

and as initial temperature field we take

$$\theta(0) = \theta_0 \text{ in } \Omega. \tag{7}$$

We consider cost functionals of separated type, i.e.

$$J(y, \theta, u) = J_1(y, \theta) + J_2(u) \ ,$$

where $J_1 : W_y \times W_\theta \to \mathbb{R}$, $J_2 : U \to \mathbb{R}$ are bounded from below, and are weakly lower semi–continuous, twice continuously Fréchet-differentiable with Lipschitz continuous second derivatives. Further, J_2 is assumed to be radially unbounded, i.e. $J_2(u) \to \infty$ for $\|u\|_U \to \infty$. The spaces W_y, W_θ are specified in Section 2.

Let us give some examples.

Example 1. Typical cost functionals are given by

$$J_1(y, \theta) = \frac{1}{2} \int_0^T \int_\Omega [|y - \bar{y}|^2 + |\theta - \bar{\theta}|^2] \, d\Omega dt \ , \quad J_2(u) = \frac{\alpha}{2} \|u\|_U^2 \ , \tag{8}$$

$$J_1(y, \theta) = \frac{1}{2} \int_0^T \int_\Omega [|curly|^2 + |\nabla \theta|^2] \, d\Omega dt \ , \quad J_2(u) = \frac{\alpha}{2} \|u\|_U^2 \ , \tag{9}$$

where $\bar{y}$ and $\bar{\theta}$ denote desired velocity and temperature fields respectively. The constant $\alpha > 0$ denotes a weighting factor for the control costs.

Typical control operators and control spaces are given by

1. $U = L^2(\Gamma_T)$, $B := \mathcal{I}$, where $\mathcal{I} : U \to L^2(0, T; H^{-1/2}(\Gamma))$ denotes the injection. $U_{ad} := \{u \in L^2(\Gamma_T); a \leq u \leq b \text{ a.e. on } \Gamma_T\}$, where a, $b \in L^\infty(\Gamma_T)$.
2. $U = L^2(0, T)^m$, $Bu(t) := \sum_{i=1}^m u_i(t) f_i$, where $f_i \in H^{-1/2}(\Gamma)$, $i = 1, \ldots, m$. $U_{ad} := \{u \in L^2(0, T); a \leq u \leq b \text{ a.e. on } (0, T)\}$, where a, $b \in L^\infty(0, T)$.
3. $U = \mathbb{R}^m$, $Bu(t) := \sum_{i=1}^m u_i g_i(t)$, where $g_i \in L^2(0, T; H^{-1/2}(\Gamma))$, $i = 1, \ldots, m$. $U_{ad} := \{u \in \mathbb{R}^m; a \leq u \leq b \text{ a.e. on } (0, T)\}$, where $a \leq b$ componentwise in $\mathbb{R}^m$.
4. $U = u_0 + \{u \in H^1(0, T; L^2(\Gamma)); u(0) = 0\} \equiv u_0 + H_0$, where $u_0 \in L^2(\Gamma)$, and $B := \mathcal{I}$, where $\mathcal{I} : U \to L^2(0, T; H^{-1/2}(\Gamma))$ denotes the injection. $U_{ad} = U$.

A discussion of related literature is given in [2]. For the convenience of the reader we provide the list of related references [1,4,5,7–14]. Let us emphasize that Belmiloudi in [5] discusses Robin boundary control in three spatial dimensions for a model related to the Boussinesq approximation, where he imposes certain smallness assumptions on the data and on the controls. The scope of the present paper differs from that of Belmiloudi [5] in that it focusses on a functional analytic framework which is tailored to the algorithmic approach presented in [2]. In particular more general boundary controls are allowed and first and second order necessary, and second order suffient optimality conditions are presented. Moreover a different proof technique is used which is based on Gajewski's work [6].

Above, and from now onwards derivatives w.r.t. time are denoted by the subscript t, i.e. $v_t := \frac{\partial v}{\partial t}$.

2 Mathematical model

In order to formulate problem (P) mathematically we introduce the solenoidal spaces

$$H = \{v \in L^2(\Omega),\, div\, v = 0\}\ , \quad V = \{v \in H_0^1(\Omega),\, div\, v = 0\}\ ,$$

and set

$$W_y = \{v_t \in L^2(V^*),\ v \in L^2(V)\}\ , \quad W_\theta = \{\theta_t \in L^2(H^1(\Omega)^*),\ \theta \in L^2(H^1(\Omega))\}\ .$$

Here we abbreviate $L^p(Z) = L^p(0,T;Z)$ for Z denoting a Banach space.

Next we introduce the operator

$$e : W_y \times W_\theta \times U \to L^2(V^*) \times H \times L^2(H^1(\Omega)^*) \times L^2(\Omega) =: Z^*$$

by

$$\begin{aligned} e(y,\theta,u) = (&y_t + (y\cdot\nabla)y - \nu\Delta\, y - Gr\,\theta g, y(0),\\ &\theta_t + (y\cdot\nabla)\theta - \frac{1}{Pr}\Delta\,\theta + E(\frac{1}{Pr}\frac{\partial\theta}{\partial\mathbf{n}} + b\theta - Bu), \theta(0)) \end{aligned}$$

where

$$E : L^2(H^{-1/2}(\Gamma)) \to L^2(H^1(\Omega)^*)\ ,$$

defines a linear bounded operator whose action is defined by

$$\langle E\, z, v\rangle_{L^2(H^1(\Omega)^*),L^2(H^1(\Omega))} = \int_0^T \langle z, \gamma v\rangle_{H^{-1/2}(\Gamma)),H^{1/2}(\Gamma)}\, dt,$$

where $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ denotes the trace operator. The action of e applied to an element
$z = (z_1, z_2, z_3, z_4) \in Z = L^2(V) \times H \times L^2(H^1(\Omega)) \times L^2(\Omega)$ is given by

$$\begin{aligned} \langle e(y,\theta,u), z\rangle_{Z^*Z} = &\int_0^T \langle y_t, z_1\rangle_{V^*V}\, dt + \int_0^T\int_\Omega \nu\nabla y\nabla z_1 + (y\cdot\nabla)yz_1\, dxdt+\\ &+ (y(0), z_2)_H + \int_0^T \langle\theta_t, z_3\rangle_{H^{1*}H^1}\, dt + \int_0^T\int_\Omega \frac{1}{Pr}\nabla\theta\nabla z_3 + (y\cdot\nabla)\theta z_3\, dxdt+\\ &+ \int_0^T\int_\Gamma (b\theta - Bu)z_3\, ds + (\theta(0), z_4)_{L^2(\Omega)} =: \langle(e^1,e^2,e^3,e^4)(y,\theta,u), z\rangle_{Z^*Z}\ . \end{aligned}$$

Using e the system of state equations can be written in the form

$$e(y, \theta, u) = 0 \text{ in } Z^* . \tag{10}$$

Considering [6, Bemerkung 2.1] together with [6, Satz 3.1,Folgerung 5.1] gives

Theorem 1. *Let $f \in L^2(V^*)$, $q \in L^2(H^1(\Omega)^*)$, $y_0 \in H$, $\theta_0 \in L^2(\Omega)$ and $u \in U$. Then*

$$e(y, \theta, u) = (f, y_0, q, \theta_0) \textit{ in } Z^*$$

admits a unique solution $(y, \theta) \in W_y \times W_\theta$.

For the differentiability of the operator e we prove

Theorem 2. *The operator e is infinitely often Fréchet-differentiable with Lipschitz continuous second derivatives and vanishing derivatives of third and higher order. For the first and second derivatives there holds*

$$\begin{aligned}
e_y(y,\theta,u)\, v &= (v_t + (v\cdot\nabla)y + (y\cdot\nabla)v - \nu\Delta\, v, v(0), (v\cdot\nabla)\theta, 0)\\
e_\theta(y,\theta,u)\, s &= (-Gr\, sg, 0, s_t + (y\cdot\nabla)s - \tfrac{1}{Pr}\Delta\, s + E(\tfrac{1}{Pr}\tfrac{\partial s}{\partial \mathbf{n}} + Bs), s(0))\\
e_u(y,\theta,u)\, \tilde{u} &= (0, 0, E(-B\tilde{u}), 0)\\
e_{y\theta}[v,s] &= (0, 0, (v\cdot\nabla)s, 0)\\
e_{\theta y}[s,v] &= (0, 0, (v\cdot\nabla)s, 0)\\
e_{yy}[v_1,v_2] &= ((v_2\cdot\nabla)v_1 + (v_1\cdot\nabla)v_2, 0, 0, 0)\ ,
\end{aligned}$$

all other derivatives vanish.

Proof. Since e^2, e^4 represent linear operators it is sufficient to consider e^1 and e^3. Let

$$b_\theta(y, \theta, \chi) := \langle (y\cdot\nabla)\theta, \chi\rangle_{H^{1*}H^1}$$

and

$$b_y(y, v, \phi) := \langle (y\cdot\nabla)v, \phi\rangle_{V^*V}\ .$$

Since we only consider two-dimensional spatial domains it follows from [15, p. 293] that

$$|b_\theta(y,\theta,\chi)|^2 \le 2\|y\|_H\|y\|_V\|\theta\|_{L^2}\|\theta\|_{H^1}\|\chi\|^2_{H^1} \tag{11}$$

and

$$|b_y(y,v,\phi)|^2 \le 2\|y\|_H\|y\|_V\|v\|_H\|v\|_V\|\phi\|^2_V\ . \tag{12}$$

hold. In order to argue Lipschitz continuity of e^1 we estimate the difference

$$
\begin{aligned}
&\langle e^1(y,\theta,u) - e^1(\tilde{y},\tilde{\theta},\tilde{u}), \phi\rangle_{L^2(V^*)L^2(V)} = \\
&\qquad \langle (y-\tilde{y})_t + [(y-\tilde{y})\cdot\nabla]\tilde{y} + (y\cdot\nabla)(y-\tilde{y}), \phi\rangle_{L^2(V^*)L^2(V)} \\
&\qquad + \nu\int_0^T (\nabla(y-\tilde{y}), \nabla\phi)_{(L^2)^2} dt - \int_0^T ((\theta-\tilde{\theta})g, \phi)_{L^2} dt \\
&\leq \sqrt{2}\int_0^T \|y-\tilde{y}\|_H^{1/2}\|y-\tilde{y}\|_V^{1/2}(\|\tilde{y}\|_H^{1/2}\|\tilde{y}\|_V^{1/2} + \|y\|_H^{1/2}\|y\|_V^{1/2})\|\phi\|_V\, dt \\
&\qquad + C\{\|\theta-\tilde{\theta}\|_{W_\theta} + \|y-\tilde{y}\|_{W_y}\}\|\phi\|_{L^2(V)} \\
&\leq C\left\{ \|y-\tilde{y}\|_{C(H)}^{1/2}\left(\|\tilde{y}\|_{C(H)}^{1/2} + \|y\|_{C(H)}^{1/2}\right)\left(\int_0^T \|y-\tilde{y}\|_V(\|\tilde{y}\|_V + \|y\|_V)\, dt\right)^{1/2}\right. \\
&\qquad \left. + \|\theta-\tilde{\theta}\|_{W_\theta} + \|y-\tilde{y}\|_{W_y}\right\}\|\phi\|_{L^2(V)} \\
&\qquad \leq C(\|y\|_{W_y} + \|\tilde{y}\|_{W_y})\|y-\tilde{y}\|_{W_y}\|\theta-\tilde{\theta}\|_{W_\theta}\|\phi\|_{L^2(V)} \ .
\end{aligned}
$$

Here we have used the continuous embeddings

$$W_y, W_\theta \hookrightarrow C([0,T];H), C([0,T];L^2(\Omega)) \ .$$

For e^3 a similar estimate holds. In order to argue Fréchet differentiability it is sufficient to consider component e^3, since component e^1 admits a similar structure. We obtain

$$
\begin{aligned}
e^3(y,\theta,u) - e^3(\tilde{y},\tilde{\theta},\tilde{u}) &= (y\cdot\nabla)\theta - \tilde{y}\cdot\nabla)\tilde{\theta} - (y-\tilde{y})\nabla\theta - (y\cdot\nabla)(\theta-\tilde{\theta}) \\
&= [(y-\tilde{y})\cdot\nabla](\theta-\tilde{\theta}) \ ,
\end{aligned}
$$

so that estimation similar as above yields

$$
\begin{aligned}
&\|e^3(y,\theta,u) - e^3(\tilde{y},\tilde{\theta},\tilde{u}) - e^3_{(y,\theta,u)}(y,\theta,u)(y-\tilde{y},\theta-\tilde{\theta},u-\tilde{u})\|_{L^2(H^1)} \\
&\qquad = \sup_{\|\chi\|_{L^2(H^1)}=1}\int_0^T |b_\theta(y-\tilde{y},\theta-\tilde{\theta},\chi)|\, dt \\
&\qquad\qquad \leq C\|y-\tilde{y}\|_{W_y}\|\theta-\tilde{\theta}\|_{W_\theta} \ .
\end{aligned}
$$

The expression for the second derivative can be verified by an estimate analogous to the one for the first derivative. The second derivative is independent of the point at which is taken, and thus it is necessarily Lipschitz continuous.

¿From here onwards it is convenient to set $x = (y,\theta)$, $W = W_y \times W_\theta$, and to denote derivatives with respect to (y,θ) accordingly.

Lemma 1. *Let $(x,u) \in W \times U$. Then $e_x(x,u) : W \to Z^*$ is a homeomorphism, and thus also $e_x^* : Z \to W^*$.*

Proof. Let $(f, v_0, h, s_0) \in Z^*$. It suffices to prove that the system

$$v_t - \nu \Delta v + (y \cdot \nabla)v + (v \cdot \nabla)y + \nabla p_v - Gr\, sg = f \tag{13}$$
$$v(0) = v_0 \tag{14}$$
$$s_t - \frac{1}{Pr}\Delta s + (y \cdot \nabla)s + (v \cdot \nabla)\theta = h \tag{15}$$
$$s(0) = s_0 \tag{16}$$

has a unique solution. With $a = (a^1, a^2, a^3)$, $b = (b^1, b^2, b^3)$ we set

$$B(a,b) = \begin{pmatrix} (a^{1,2} \cdot \nabla)b^{1,2} \\ (a^{1,2} \cdot \nabla)b^3 \end{pmatrix} , \quad C(b) = \begin{pmatrix} -Gr\, b_3 g \\ 0 \end{pmatrix}$$

and for $u = (v_1, v_2, s)^T$ we define

$$Au = \begin{pmatrix} \nu \Delta v_1 \\ \nu \Delta v_2 \\ \frac{1}{Pr}\Delta s \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} f \\ h \end{pmatrix} .$$

Then system (13) may be written as initial value problem in the form

$$u' + Au + +B(x,u) + B(u,x) + C(u) = F \ , \quad u \in W \ , \quad u(0) = (v_0, s_0) \ ,$$

and admits exactly the form of the system [6, 2.15] if one replaces there $B_r(u,u)$ by $B(x,u)+B(u,x)+C(u)$. This completes the proof since the analysis presented in [6] also applies to this slightly modified situation. □

The action of the adjoint operator e_x applied to $z \in Z$ as an element of W^* is defined as

$$\langle e_x(x,u)^* z, \tilde{x} \rangle_{W^* W} = \langle e_x(x,u)\tilde{x}, z \rangle_{Z^* Z} \ .$$

¿From lemma 1 we have

$$\|e_x(x,u)^*\|_{\mathcal{L}(Z,W^*)} = \|e_x(x,u)\|_{\mathcal{L}(W,Z^*)} \ ,$$

and for $g \in W^*$ the unique solution $w \in Z$ of $e_x(x,u)w = g$ in W^* satisfies

$$\|w\|_Z \leq C\|g\|_{W^*} \ . \tag{17}$$

The constant C depends on u through $x(u)$. Due to theorem 1 it is meaningful to define the reduced functional

$$\hat{J}(u) = J(x(u), u) \ ,$$

where for given $u \in U$ the function $x(u)$ denotes the unique solution of $e(x,u) = 0$. Our optimization problem (1) can be rewritten in the form

$$(\hat{P}) \quad \min_{u \in U_{ad}} \hat{J}(u) \ .$$

Theorem 3. *Problem $(\hat{P})$ admits a solution.*

Proof. Since $\hat{J}$ is bounded from below we have $d := \inf_{u \in U_{ad}} \hat{J} \geq -\infty$. Let (u^n) denote a minimizing sequence, i.e. $\hat{J}(u^n) \to d$ for $n \to \infty$. Since J_2 is radially unbounded we infer $\|u^n\|_U \leq M$ uniformly in n, so that $u^n \rightharpoonup u$ for a subsequence. Since U_{ad} is convex and closed it is weakly closed so that $u \in U_{ad}$ holds. For all u^n exists a unique x^n satisfying $e(x^n, u^n) = 0$ and $\|x^n\|_W \leq M$ for all n. Since W is a Hilbert space we have $x^n \rightharpoonup x$ for a further subsequence. Because of the compact embedding $W \hookrightarrow L^2([0,T]; H) \times L^2(Q)$ we have $x^n \to x \in W$ in $L^2(H) \times L^2(Q)$ for a further subsequence. Since B is bounded and linear it is weakly continuous so that we finally can pass to the limit in the equation $e(x^n, u^n) = 0$, and thus $e(x, u) = 0$ so that $x = x(u)$. Finally, u is a solution to $(\hat{P})$ since the cost functional is weakly lower semi-continuous, so that $\hat{J}(u) \leq \liminf_{n \to \infty} \hat{J}(u^n) = d$.

As a consequence of the previous theorems and the implicit function theorem the functional $\hat{J}$ is infinitely often Fréchet differentiable. In order to formulate necessary and sufficient optimality conditions we next specify the first and the second derivative of $\hat{J}$. For the first derivative we obtain

$$\langle \hat{J}'(u), \delta u \rangle_{U^*U} = \langle J_x(x,u), x'(u)\delta u \rangle_{W^*W} + \langle J_u(x,u), \delta u \rangle_{U^*U} \ .$$

Differentiation of the state equation $e(x, u) = 0$ yields

$$e_x(x,u)x'(u) + e_u(x,u) = 0 \quad \text{in} \quad Z^*$$

and thus

$$x'(u)\delta u = -e_x(x,u)^{-1} e_u(x,u)\delta u \ .$$

Introducing the adjoint variable $\lambda = (\mu, \mu_0, \kappa, \kappa_0) \in Z$ by

$$\lambda = e_x(x,u)^{-*} J_x(x,u)$$

we obtain

$$\hat{J}'(u) = J_u(x,u) - e_u(x,u)^* \lambda \ .$$

Note that in our setting $e_u(x, u^*) = (0, 0, -B^*, 0)$ holds.

Analogously we obtain the second derivative of $\hat{J}$ by differentiating $e_x(x, u) = 0$ one more time to obtain

$$x''(u)(\delta u, \delta v) = -e_x^{-1} e_{xx}(x,u)(x'(u)\delta u, x'(u)\delta v) \ .$$

Using this we find

$$\langle \hat{J}''(u)\delta u, \delta v \rangle_{U^*U} = \langle J_{xx}(x,u)x'(u)\delta u, x'(u)\delta v \rangle_{W^*W} - \\ \langle \lambda, e_{xx}(x,u)(x'(u)\delta u, x'(u)\delta v) \rangle_{Z,Z^*} + \langle J_{uu}(x,u)\delta u, \delta v \rangle_{U^*,U} \ .$$

For example 1 we have

- $B^* : L^2(0,T;H^{1/2}(\Gamma)) \to L^2(\Gamma_T)^* \equiv L^2(\Gamma_T)$, so that B^* denotes the injection.
- $B^* : L^2(0,T;H^{1/2}(\Gamma)) \to L^2(0,T)^m,\ v \mapsto (B^*v)_i(t) = \langle f_i, v\rangle_{H^{-1/2}(\Gamma)H^{1/2}(\Gamma)}$, for $i = 1,\dots,m$.
- $B^* : L^2(0,T;H^{1/2}(\Gamma)) \to \mathbb{R}^m\ v \mapsto (B^*v)_i(t) = \int\limits_0^T \langle g_i(t), v(t)\rangle_{H^{-1/2}(\Gamma)H^{1/2}(\Gamma)} dt$, for $i = 1,\dots,m$.
- $B^* : L^2(0,T;H^{1/2}(\Gamma)) \to \{u \in H^1(0,T;L^2(\Gamma))\}^*$ denotes the injection and we have

$$\langle Bu, f\rangle_{L^2(0,T;H^{-1/2}(\Gamma))L^2(0,T;H^{1/2}(\Gamma))} = \langle u, B^*f\rangle_{H_0H_0^*} = \langle u, f\rangle_{H_0H_0^*} = (u, Rf)_{H_0} = \int_0^T \int_\Gamma [uRf + u_t(Rf)_t]dt\ ,$$

where $R : H_0^* \to H_0$ denotes the Riesz operator, whose action in the present situation is defined through

$$w = Rf \iff \int_0^T \int_\Gamma [vw + v_t w_t] d\Gamma dt = \langle v, f\rangle_{H_0H_0^*}\ \forall\, v \in H_0\ .$$

Thus

$$B^*f = -w_{tt} + w\ .$$

For the cost functionals of example 1 we find

$$\langle J_x(x,u), \tilde{x}\rangle_{W^*W} = \begin{cases} \int_Q [(y-\bar{y})\tilde{y} + (\theta-\bar{\theta})\tilde{\theta} dxdt \\ \int_Q [-curl\, y\ curl \tilde{y} + \nabla\theta\, \nabla\tilde{\theta}] dxdt, \end{cases}$$

and $\langle J_u(x,u), v\rangle_{U^*,U} = \alpha\langle u, v\rangle_{U^*,U}$, so that in fact $J_x(x,u)$ is a element of $L^2(Q)^2 \times L^2(Q)$, or $L^2(V^*) \times L^2(0,T;H^1(\Omega)^*)$, respectively. In this case the adjoint variable has the form $\lambda = (\mu, \mu(0), \kappa, \kappa(0))$ with $\mu \in L^2(V)$, $\mu_t \in L^{4/3}(V^*) \cap W_y^*$, $\kappa \in L^2(H^1(\Omega))$, $\kappa_t \in L^{4/3}(H^1(\Omega))^* \cap W_\theta^*$ (compare [11]), and satisfies the system

$$\left.\begin{array}{rll} -\mu_t - \nu\Delta\mu + (\nabla y)^t\mu - (y\cdot\nabla)\mu + \nabla\xi = -\kappa\nabla\theta + \begin{cases} (y-\bar{y}) \\ -\text{curl curl}\, y \end{cases} & \text{in } \Omega_T, \\ -\text{div}\ \mu = 0 & \text{in } \Omega_T, \\ \mu = 0 & \text{on } \Gamma_T, \\ \mu(T) = 0 & \text{in } \Omega, \\ & \\ -\kappa_t - \frac{1}{Pr}\Delta\kappa - y\cdot\nabla\kappa = Gr\, g\cdot\mu + \begin{cases} (\theta-\bar{\theta}) \\ -\Delta\theta \end{cases} & \text{in } \Omega_T, \\ \frac{1}{Pr}\frac{\partial\kappa}{\partial n} + b\kappa = 0 & \text{on } \Gamma, \\ \kappa(T) = 0 & \text{in } \Omega. \end{array}\right\} \tag{18}$$

We are now in the position to specify the first order necessary optimality condition for problem $\hat{P}$. Since $\hat{J}$ is Fréchet differentiable it reads

$$\langle \hat{J}'(u), v - u\rangle_{U^*U} \geq 0 \quad \text{for all } v \in U_{ad} \, .$$

Let us finally specify a second order sufficient condition for a solution u of our control problem.

Theorem 4. *Let u denote a solution of $(\hat{P})$, such that $J_x(x,u)$ is sufficiently small, where x denotes the state associated to u. Further let us assume that $J_{uu}(x,u)$ is positive definite, i.e.*

$$\langle J_{uu}(x,u)v, v\rangle_{U^*U} \geq C\|v\|_U^2 \, ,$$

holds with some positive constant C, and $J_{xx}(x,u)$ is positive semi definite. Then $\hat{J}''(u)$ is positive definite.

Proof. We have

$$\begin{aligned}
\langle \hat{J}''(u)v, v\rangle_{U^*U} &= \langle J_{xx}(x,u)x'(u)v, x'(u)v\rangle_{W^*W} - \\
&\qquad \langle \lambda, e_{xx}(x,u)(x'(u)v, x'(u)v)\rangle_{Z,Z^*} + \langle J_{uu}(x,u)v, v\rangle_{U^*,U} \\
&\geq C\|v\|_U^2 - \langle \lambda, e_{xx}(x,u)(x'(u)v, x'(u)v)\rangle_{Z,Z^*} \\
&\geq C\|v\|_U^2 - c(u)\|v\|_U^2\|\lambda\|_Z \geq C\|v\|_U^2 - c(u)\|v\|_U^2\|J_x(x,u)\|_{W^*} \geq \frac{C}{2}\|v\|_U^2 \, ,
\end{aligned}$$

if $c(u)\|J_x(x,u)\|_{W^*} \leq \frac{C}{2}$.

3 Conclusion

In the present work we provide an analytical framework for optimal boundary control of instationary Boussinesq systems in two spatial dimensions. Among other things we use the results of [6] to prove existence and uniqueness of solutions to the adjoint of the Boussinesq Approximation. Further we derive a first order necessary optimality condition and prove a second order sufficient optimality condition under some smallness assumptions on the derivatives of the cost functional.

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