Hamburger Beiträge
zur Angewandten Mathematik

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Nr. 2007-20
Dezember 2007

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December 7, 2007

Abstract For elliptic partial differential equations with periodically oscillating coefficients quadratic L^2-convergence of a corrected asymptotic expansion, which is motivated by the theory of homogenization, is proven in the one-dimensional case.

In the two-dimensional case the rate of convergence and its dependency on the symmetry of the diffusion coefficient is numerically analysed. The correction of the asymptotic expansion on a locally refined grid is then embedded inside a two-grid method and numerically compared with a classical PCG-method.

Keywords homogenization · asymptotic expansion · multigrid method · elliptic partial differential equation

Mathematics Subject Classification (2000) 35B27 · 74Q05 · 78M35 · 78M10

1 Introduction

Within this article the stationary diffusion equation

$$\begin{cases} -\operatorname{div}(A\nabla u) = f & \text{in } \Omega \subset \mathbb{R}^n, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

for a composite material with finely mixed constituents will be studied. If the material is additionally periodic, one can assume the existence of two characteristic

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scales, first a macroscopic scale, describing the global behaviour of the composite and second a microscopic scale, describing the heterogeneities on the unit cell $Y =]0, l_1[\times \cdots \times]0, l_n[$. To describe the dependency on the size ε of the unit cell, one writes (1) in the form

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

with

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) \tag{3}$$

for a Y-periodic A.

The theory of homogenization tries to characterize the "limit"

$$u^0 = \lim_{\varepsilon \to 0} u^\varepsilon.$$

Homogenization does not mean to determine u^0, but to establish a differential equation for u^0. In the case of the stationary diffusion equation this differential equation (called homogenized differential equation) is of the same type but with constant coefficient. The intention of homogenization is ultimately the calculation of the effective coefficient A^0, such that

$$\begin{cases} -\operatorname{div}(A^0 \nabla u^0) = f & \text{in } \Omega, \\ u^0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

So, replacing the original problem for u^ε by the homogenized problem means that one has to solve two problems instead of one. On the other hand the homogenized problem is a differential equation with constant coefficients and therefore (numerically) a lot easier to solve than the differential equation for u^ε.

The convergence of u^ε towards u^0 is only weak in $H_0^1(\Omega)$. The proof of the last statement using the two-scale expansion

$$u^\varepsilon(x) = u^0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \cdots$$

in [2], chapter 4, §1 shows, that the asymptotic expansion of first order $u_1^\varepsilon(x) = u^0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon})$ satisfies

$$u_1^\varepsilon(x) = u^0(x) - \varepsilon \sum_{i=1}^n \chi_i\left(\frac{x}{\varepsilon}\right) \frac{\partial u^0}{\partial x_i}(x),$$

where $\chi_i \in W_{\mathrm{per}}(Y) = \left\{ v \in H^1_{\mathrm{per}}(Y) \,|\, \langle v \rangle_Y = 0 \right\}$ for $i = 1, \ldots, n$ is the unique weak solution of the so called cell-problem

$$\begin{cases} -\operatorname{div}(A(y)\nabla \chi_i) = -\operatorname{div}(A(y)e_i) \text{ in } Y, \\ \chi_i \ Y\text{-periodic}, \\ \langle \chi_i \rangle_Y = 0 \end{cases} \tag{5}$$

on the unit cell Y. The various cell solutions are needed anyway, in order to determine the homogenized diffusion coefficient A^0 by means of the equation

$$A^0 e_i = \langle -A\nabla\chi_i + Ae_i\rangle_Y = \frac{1}{|Y|}\int_Y -A\nabla\chi_i + Ae_i\,dy$$

([5], Theorem 6.1, p. 112). Instead of settling for the homogenized solution u^0, one can use the asymptotic expansion of first order without significant additional expenses. This has the further advantage of approximating the solution of the original problem in the H^1-norm ([9], p. 28)

$$\|u^\varepsilon - u_1^\varepsilon\|_{H^1(\Omega)} = O(\sqrt{\varepsilon}).$$

Regarding the L^2-norm this gives no improvement of the approximative property ([9], S. 29, 30)

$$\|u^\varepsilon - u^0\|_{L^2(\Omega)} = O(\varepsilon),$$
$$\|u^\varepsilon - u_1^\varepsilon\|_{L^2(\Omega)} = O(\varepsilon).$$

On the other hand it is proved in [11], that by the right choice of the right hand sides of the equations for the finite element approximations u_h^ε, u_H^0 and $\chi_{i,\hat{h}}$ one obtains

$$\left\| u_h^\varepsilon(x) - I_h\left(u_H^0(x) - \varepsilon\sum_{i=1}^n \frac{\partial u_H^0}{\partial x_i}(x)\chi_{i,\hat{h}}\left(\frac{x}{\varepsilon}\right)\right)\right\|_{L^2(\Omega)} = O(\varepsilon^2),$$

if A is symmetric in the following sense

Definition 1 A Map $A : Y \to \mathbb{R}^{n\times n}$ is called *k-symmetric*, iff for $i,j = 1,\dots,n$

$$(-1)^{\delta_{ik}} a_{ij}^{\#}(y_1,\dots,y_{k-1},-y_k,y_{k+1},\dots,y_n)(-1)^{\delta_{kj}} = a_{ij}^{\#}(y),$$

where $a_{ij}^{\#}$ is the periodic continuation of a_{ij}. A is called *symmetric*, iff A is k-symmetric for all $k = 1,\dots,n$.

In the first part of this article this statement will be shown analytically for the one-dimensional case in terms of the L^∞-norm, provided that the source f is regular enough (Theorem 1). Furthermore in the non symmetric case an additional correction problem with constant coefficient will suffice to maintain the quadratic approximation property.

Afterwards this results will be numerically examined in the two-dimensional case using finite element approximations. The finite element spaces will be choosen so that the quadratic approximation property is translated into the finite element approximation (Theorems 8 and 9 as well as Corollaries 4 and 5). Thereby particularly the regularity of solutions of elliptic differential equations with periodic boundary conditions will be used. This follows from the inner regularity of solutions of elliptic equations ([7], Theorem 8.8, p. 183 and Theorem 8.10, p. 186) and the invariance under translation of Y-periodic functions ([5], Lemma 2.3, p. 27) so that it just depends on the regularity of the diffusion coefficient. Actual $a_{ij} \in C^{m-2,1}(\overline{Y}) \cap H^{m-1}(Y)$ implies $\chi_i \in H^m(Y)$.

Both the classical definition and the finite elements used can be found in [4]. Whereas finite elements for solving Dirichlet problems are described in almost every book about numerical methods for partial differential equations, there seems to exist no corresponding source for the case of periodic boundary conditions. The approach for the numeric calculations (probably already followed before) is based on "periodic" subdivisions (Definition 3), for which the interpolation of a periodic function itself is periodic. Since the periodicity cell is a rectangle, it is reasonable to decompose it into rectangles. To gain statements about the regularity of the global interpolation, one needs (just as for triangulations) another assumption on the geometry of the subdivision. Therefore rectangulations (Definition 2) are defined.

After summarizing the numeric results for isotropic materials a two-grid method, which is based on the components needed for the calculation of the corrected asymptotic approximation of first order, will be presented and compared with a preconditioned conjugate gradient method (PCG-method) for different sizes of the unit cell.

2 Asymptotic approximation

In this section it will be shown that the improved L^2-convergence of the first order asymptotic expansion applies analytically in the one-dimensional case for symmetric A.

An additional correction of u_1^ε by the solution of another elliptic boundary problem with constant coefficient preserves the improved L^2-convergence even in the asymmetric case. The proof will make use of an integral representation for the first order asymptotic expansion. It applies for $\Omega =]x_0,x_1[$, $Y =]0,l[$ and $f = f_0 + \frac{df_1}{dx}$ with $f_0, f_1 \in L^2(\Omega)$

$$
\begin{aligned}
u_1^\varepsilon(x) &= u^0(x) - \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\frac{du^0}{dx}(x) \\
&= \frac{1}{a^0}\left[-\int_{x_0}^{x}(F_0(s)+f_1(s))\,ds + \langle F_0+f_1\rangle_\Omega (x-x_0)\right] + \\
&\qquad \varepsilon\left[\left(A\left(\frac{x}{\varepsilon}\right) - \frac{1}{a^0}\frac{x}{\varepsilon}\right) - \left(\langle A\rangle_Y - \frac{l}{2a^0}\right)\right]\left[\langle F_0+f_1\rangle_\Omega - (F_0(x)+f_1(x))\right]
\end{aligned}
$$

with

$$
\begin{aligned}
a^0 &= l\left(\int_0^l \frac{ds}{a(s)}\right)^{-1}, \\
F_0(x) &= \int_{x_0}^{x} f_0\,ds, \\
A(y) &= \int_0^y \frac{ds}{a(s)}.
\end{aligned}
$$

2.1 Correction

Since u^0 satisfies the Dirichlet boundary conditions on $\partial\Omega$, the first order asymptotic expansion u_1^ε of u^ε is apparently no element of $H_0^1(\Omega)$. An immediate consequence is the lower order of convergence (compared to the symmetric case) of $u^\varepsilon - u_1^\varepsilon$ with respect to $\|\cdot\|_{H_0^1(\Omega)}$. The obvious correction of u_1^ε would be the solution of

$$\begin{cases} -\operatorname{div}(A^\varepsilon \nabla v^\varepsilon) = 0 & \text{in } \Omega, \\ v^\varepsilon = u_1^\varepsilon = -\varepsilon \sum_{k=1}^{n} \chi_k\left(\frac{x}{\varepsilon}\right) \frac{\partial u^0}{\partial x_k}(x) & \text{on } \partial\Omega. \end{cases}$$

But to calculate v^ε numerically would be as expansive as the original problem (2). Therefore the diffusion coefficient A^ε of the correction problem is replaced by the homogenized coefficient A^0. As a result the numerical calculation of the correction just needs a locally (near the boundary) refined grid.

$$\begin{cases} -\operatorname{div}(A^0 \nabla v_0^\varepsilon) = 0 & \text{in } \Omega, \\ v_0^\varepsilon = u_1^\varepsilon = -\varepsilon \sum_{k=1}^{n} \chi_k\left(\frac{x}{\varepsilon}\right) \frac{\partial u^0}{\partial x_k}(x) & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Now the improved L^2-convergence of $\tilde{u}_1^\varepsilon := u_1^\varepsilon - v_0^\varepsilon$ will be proven under appropriate assumptions on the source f.

Remark 1 If Ω is made up of unit cells and A is symmetric, then $v_0^\varepsilon \equiv 0$ ([11], Corollary 2.4, p. 5), so already the first order asymptotic expansion possess the improved L^2-convergence.

Let $f \in L^1(\Omega) \subset H^{-1}(\Omega)$. Then $F : [x_0, x_1] \to \mathbb{R}$ defined by

$$F(x) := \int_{x_0}^{x} f \, ds$$

is an absolutely continuous function on $\Omega =]x_0, x_1[$ with $\frac{dF}{dx} = f$ (a.e. in Ω) and the first order asymptotic expansion satisfies

$$u_1^\varepsilon(x) = \frac{1}{a^0}\left[-\int_{x_0}^{x} F(s)\,ds + \langle F\rangle_\Omega (x - x_0)\right] + \\ \varepsilon\left[\left(A\left(\frac{x}{\varepsilon}\right) - \frac{1}{a^0}\frac{x}{\varepsilon}\right) - \left(\langle A\rangle_Y - \frac{l}{2a^0}\right)\right](\langle F\rangle_\Omega - F(x)).$$

Using the l-periodic function $h : Y \to \mathbb{R}$ defined by

$$h(y) = \left(A(y) - \frac{y}{a^0}\right) - \left(\langle A\rangle_Y - \frac{l}{2a^0}\right)$$

yields the following integral representation for the corrected first order asymptotic expansion

$$\tilde{u}_1^\varepsilon = u_1^\varepsilon - v_0^\varepsilon = \frac{1}{a^0}\left[-\int_{x_0}^{x} F(s)\,ds + \langle F\rangle_\Omega (x-x_0)\right] + \varepsilon\left[h\left(\frac{x}{\varepsilon}\right)(\langle F\rangle_\Omega - F(x)) - h\left(\frac{x_0}{\varepsilon}\right)\langle F\rangle_\Omega \frac{x_1-x}{x_1-x_0} - h\left(\frac{x_1}{\varepsilon}\right)(\langle F\rangle_\Omega - \langle f\rangle_\Omega (x_1-x_0))\frac{x-x_0}{x_1-x_0}\right]$$

Partial integration for absolutely continuous functions ([12], Theorem 14.8, p. 104) provides

$$u^\varepsilon(x) = \frac{1}{a^0}\left[-\int_{x_0}^{x} F(s)\,ds + \langle F\rangle_\Omega (x-x_0)\right] + \varepsilon\left[h\left(\frac{x}{\varepsilon}\right)(\langle F\rangle_\Omega - F(x)) - \frac{x-x_0}{x_1-x_0} h\left(\frac{x_1}{\varepsilon}\right)(\langle F\rangle_\Omega - \langle f\rangle_\Omega (x_1-x_0)) - \frac{x_1-x}{x_1-x_0} h\left(\frac{x_0}{\varepsilon}\right)\langle F\rangle_\Omega\right] + \varepsilon\left[\int_{x_0}^{x} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds - \frac{x-x_0}{x_1-x_0}\int_{x_0}^{x_1} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds\right] + O(\varepsilon^2).$$

In summary it can be said

$$u^\varepsilon(x) - \tilde{u}_1^\varepsilon(x) = \varepsilon\left[\int_{x_0}^{x} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds - \frac{x-x_0}{x_1-x_0}\int_{x_0}^{x_1} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds\right] + O(\varepsilon^2).$$

By definition $h_\varepsilon \rightharpoonup \langle h\rangle_Y = 0$ weak$*$ in $L^\infty(\mathbb{R})$ ([5], Theorem 2.6,p.33) implies

$$\int_{x_0}^{x} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds \to 0 \tag{7}$$

for all $x \in \overline{\Omega}$. The Hölder's inequality yields

$$\left|\int_{x_0}^{x} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds\right| \le \|f\|_{L^1(\Omega)}\|h\|_{L^\infty(\mathbb{R})}$$

so that for $1 \le p < \infty$ Lebesgue's theorem provides

$$\left\|\int_{x_0}^{x} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds\right\|_{L^p(\Omega)} \to 0.$$

Since

$$\begin{aligned}\left\|\frac{x-x_0}{x_1-x_0}\int_{x_0}^{x_1} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds\right\|_{L^p(\Omega)} &= \left|\int_{x_0}^{x_1} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds\right| \left\|\frac{x-x_0}{x_1-x_0}\right\|_{L^p(\Omega)} \\ &\le (x_1-x_0)\left|\int_{x_0}^{x_1} f(s) h\left(\frac{s}{\varepsilon}\right)\,ds\right| \to 0\end{aligned}$$

it follows for $1 \le p < \infty$

$$\|u^\varepsilon - \tilde{u}_1^\varepsilon\|_{L^p(\Omega)} = o(\varepsilon).$$

Remark 2 Statement (7) is just the lemma of Riemann-Lebesgue.

Let f be a piecewise H^1-function, i.e.

$$f = \sum_{k=1}^{N} f_k \chi_{]z_{k-1},z_k[},$$

with $f_k \in H^1(\Omega_k), \Omega_k :=]z_{k-1}, z_k[$ and $x_0 = z_0 < z_1 < \cdots < z_N = x_1$. Then after possibly adding the jump $x \in \Omega$, that means $z_i = x$ for some $i \in \{0, \ldots, N\}$, it follows with $H(y) = \int_0^y h(s)\, ds$

$$\begin{aligned}
&|u^\varepsilon(x) - \tilde{u}_1^\varepsilon(x)| \\
\leq &\varepsilon \left| \int_{x_0}^{x} f(s) h\left(\frac{s}{\varepsilon}\right) ds - \frac{x - x_0}{x_1 - x_0} \int_{x_0}^{x_1} f(s) h\left(\frac{s}{\varepsilon}\right) ds \right| + O(\varepsilon^2) \\
\leq &\varepsilon \left| \int_{x_0}^{x} f(s) h\left(\frac{s}{\varepsilon}\right) ds \right| + \varepsilon \left| \int_{x_0}^{x_1} f(s) h\left(\frac{s}{\varepsilon}\right) ds \right| + O(\varepsilon^2) \\
\leq &\varepsilon \sum_{k=1}^{i} \left| \int_{z_{k-1}}^{z_k} f_k(s) h\left(\frac{s}{\varepsilon}\right) ds \right| + \varepsilon \sum_{k=1}^{N} \left| \int_{z_{k-1}}^{z_k} f_k(s) h\left(\frac{s}{\varepsilon}\right) ds \right| + O(\varepsilon^2) \\
\leq &2\varepsilon^2 \sum_{k=1}^{N} \left| \left[f_k(s) H\left(\frac{s}{\varepsilon}\right) \right]_{z_{k-1}}^{z_k} - \int_{z_{k-1}}^{z_k} \frac{df_k}{dx}(s) H\left(\frac{s}{\varepsilon}\right) ds \right| + O(\varepsilon^2) \\
\leq &2\varepsilon^2 \|H\|_{L^\infty(\mathbb{R})} \left(\sum_{k=1}^{N} (|f_k(z_{k-1})| + |f_k(z_k)|) + \sum_{k=1}^{N} \left\| \frac{df_k}{dx} \right\|_{L^1(\Omega_k)} \right) + O(\varepsilon^2) \\
\leq &4N\varepsilon^2 \|H\|_{L^\infty(\mathbb{R})} \max_{i \in \{1,\ldots,N\}} \|f_i\|_{L^\infty(\Omega)} + O(\varepsilon^2),
\end{aligned}$$

where the term $O(\varepsilon^2)$ is independent of $x = z_i$. From this it follows

$$\|u^\varepsilon - \tilde{u}_1^\varepsilon\|_{L^\infty(\Omega)} = O(\varepsilon^2).$$

That proofs the following theorem.

Theorem 1 *Let $\Omega =]x_0, x_1[$, $Y =]0, l[$. If $f \in L^1(\Omega)$ the corrected first order asymptotic approximation $\tilde{u}_1^\varepsilon$ of u^ε fulfils*

$$u^\varepsilon(x) - \tilde{u}_1^\varepsilon(x) = \varepsilon \left[\int_{x_0}^{x} f(s) h\left(\frac{s}{\varepsilon}\right) ds - \frac{x - x_0}{x_1 - x_0} \int_{x_0}^{x_1} f(s) h\left(\frac{s}{\varepsilon}\right) ds \right] + O(\varepsilon^2).$$

with

$$\begin{aligned}
h(y) &= \left(A(y) - \frac{y}{a^0} \right) - \left(\langle A \rangle_Y - \frac{l}{2a^0} \right) \\
&= \left(\int_0^y \frac{ds}{a(s)} - \frac{y}{a^0} \right) - \left(\frac{1}{l} \int_0^l \int_0^y \frac{ds}{a(s)} dy - \frac{l}{2a^0} \right)
\end{aligned}$$

so that

$$\|u^\varepsilon - \tilde{u}_1^\varepsilon\|_{L^p(\Omega)} = o(\varepsilon)$$

for all $1 \leq p < \infty$.

For piecewise H^1-functions f (especially for step functions) even applies

$$\|u^\varepsilon - \tilde{u}_1^\varepsilon\|_{L^\infty(\Omega)} = O(\varepsilon^2).$$

2.2 Example

Now we exemplarily verify that the correction of the first order asymptotic expansion u_1^ε is necessary to obtain the quadratic convergence proved in theorem 1.

Example 1 Let $\Omega = Y =]0,1[\subset \mathbb{R}$, $f \equiv \lambda \in \mathbb{R}$ and

$$a(y) = \frac{1}{2+\sin(2\pi y)}.$$

Under the assumption $\varepsilon = \frac{1}{N}$ follows for the errors

$$\begin{aligned}
u^\varepsilon(x) - u^0(x) &= \varepsilon\lambda \sin\left(\frac{\pi x}{\varepsilon}\right) \frac{2\pi(1-2x) - \varepsilon\left(2\cos\left(\frac{\pi x}{\varepsilon}\right) + \sin\left(\frac{\pi x}{\varepsilon}\right)\right)}{4\pi^2}, \\
u^\varepsilon(x) - u_1^\varepsilon(x) &= \varepsilon\lambda \frac{2\pi(1-2x) - \varepsilon\left(1-\cos\left(\frac{2\pi x}{\varepsilon}\right) + 2\sin\left(\frac{2\pi x}{\varepsilon}\right)\right)}{8\pi^2}, \\
u^\varepsilon(x) - \tilde{u}_1^\varepsilon(x) &= -\varepsilon^2\lambda \frac{1+\cos\left(\frac{2\pi x}{\varepsilon}\right) - 2\sin\left(\frac{2\pi x}{\varepsilon}\right)}{8\pi^2}
\end{aligned}$$

so that

$$\begin{aligned}
\left\|u^\varepsilon(x) - u^0(x)\right\|_{L^\infty} &= O(\varepsilon), \\
\|u^\varepsilon(x) - u_1^\varepsilon(x)\|_{L^\infty} &= O(\varepsilon), \\
\|u^\varepsilon(x) - \tilde{u}_1^\varepsilon(x)\|_{L^\infty} &= O(\varepsilon^2).
\end{aligned}$$

Figure 1 clarifies that the order of magnitude of the error (with respect to the L^∞-norm) of the homogenized solution u^0 is not reduced by adding the term of first order. Furthermore the boundary condition is violated. Only the solution of the correction problem (6) lowers the order of magnitude.

Remark 3 The order of convergence with respect to the H^1-norm is (generally) not affected by the correction

3 Error estimations for elliptic problems

Definition 2 A *rectangulation* of a domain $\Omega \subset \mathbb{R}^n$ is a subdivision of Ω in n-rectangles for which any face of any n-rectangle K is either a subset of the boundary $\partial\Omega$, or a face of another n-rectangle K'.

Remark 4 For triangulations and rectangulation so called hanging nodes are excluded. For this reason local refinements of rectangulations (which are themselves rectangulations) are strongly limited.

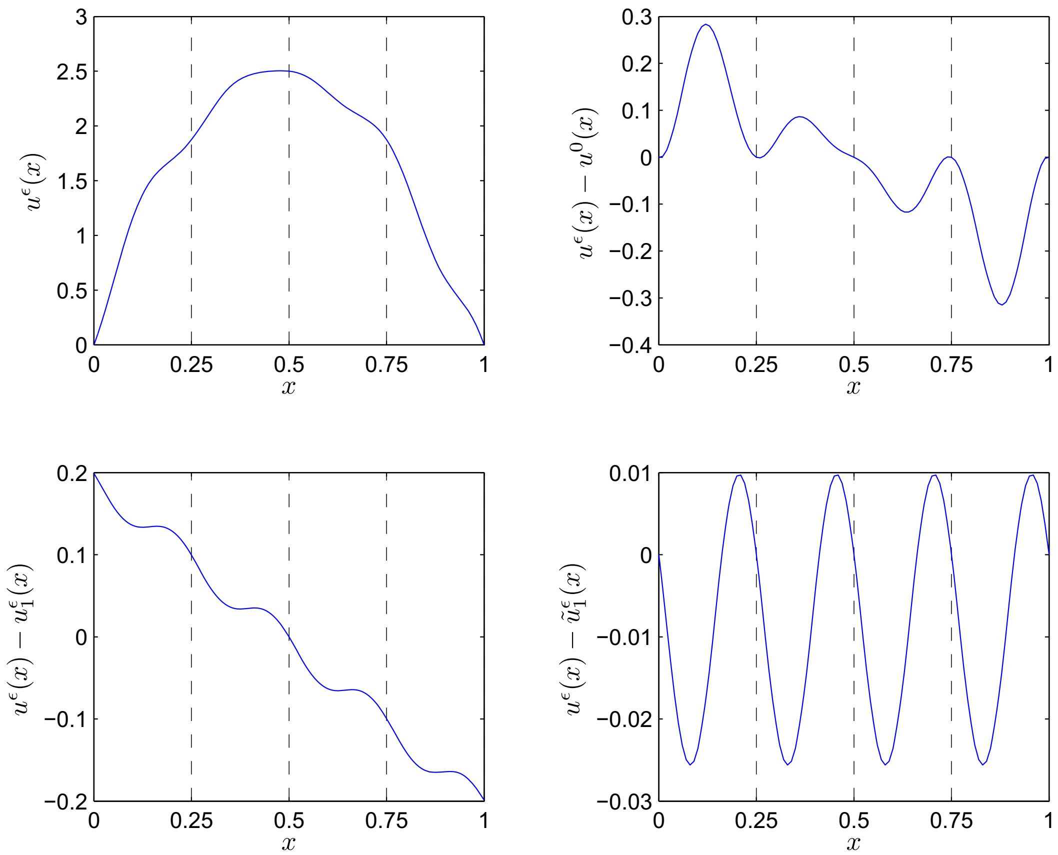


Fig. 1 Analytic solution and error of the approximations in the case of layered media for $\varepsilon = 1/4$.

Based on the interpolation theory for finite element spaces ([3], Corollary 4.4.24, p. 110) one can estimate the error of finite element approximations of elliptic boundary problems. To simplify matters the interpolation error bounds are quoted first ($\mathscr{P}_{m-1}$ denotes the set of polynomials of degree $\leq m-1$).

Theorem 2 *Let $(\mathscr{K}_h)_{0<h\leq 1}$ be a non-degenerate family of subdivisions of Ω, i.e. there exists $\rho > 0$ satisfying for all $\tilde{K} \in \mathscr{K}_h$ and all $0 < h \leq 1$*

$$\operatorname{diam} B_{\tilde{K}} \geq \rho \operatorname{diam} \tilde{K},$$

where $B_{\tilde{K}}$ is the largest ball contained in $\tilde{K}$ such that $\tilde{K}$ is star-shaped with respect to $B_{\tilde{K}}$. Let the finite element $(K, \mathscr{P}, \mathscr{N})$ and $1 \leq p \leq \infty$ satisfy

1. *K is star-shaped with respect to some ball,*
2. *$\mathscr{P}_{m-1} \subset \mathscr{P} \subset W^{m,\infty}(K)$,*
3. *$\mathscr{N} \subset (C^l(\overline{K}))'$*
4. *(a) $m - l - n \geq 0$, if $p = 1$,*
   (b) $m - l - n/p > 0$, if $p > 1$,

for adequate m and l. Suppose all finite elements $(\tilde{K}, \mathscr{P}_{\tilde{K}}, \mathscr{N}_{\tilde{K}})$, $\tilde{K} \in \mathscr{K}_h$, $0 < h \leq 1$ are affine interpolation-equivalent to $(K, \mathscr{P}, \mathscr{N})$, then for $0 \leq s \leq m$

$$\|u - I_h u\|_{s,p,h} \leq C_{l,m,n,p,\rho} h^{m-s} |u|_{W^{m,p}(\Omega)}, \quad \forall u \in W^{m,p}(\Omega).$$

Additionally for $0 \leq s \leq l$

$$\|u - I_h u\|_{s,\infty,h} \leq C_{l,m,n,p,\rho} h^{m-s-n/p} |u|_{W^{m,p}(\Omega)}, \quad \forall u \in W^{m,p}(\Omega).$$

3.1 Dirichlet problems

Theorem 3 *Let* $(\mathscr{K}_h)_{0<h\leq 1}$ *be a non-degenerate family of triangulations or rectangulations of* $\Omega \subset \mathbb{R}^n$ *with* C^0*-elements, which satisfy the assumptions of theorem 2 for* $p = 2$*. If* u *is the solution of*

$$a(u,v) = (f,v)_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega)$$

for given $f \in H^{-1}(\Omega)$ *and* u_h *is the finite element approximation in*

$$V_h = \left\{ v = (v_K)_{K\in\mathscr{K}_h} \in \prod_{K\in\mathscr{K}_h} \mathscr{P}_K \,\middle|\, \exists u \in H_0^1(\Omega), u|_K \in C^l(K), v_K = I_K u, \forall K \in \mathscr{K}_h \right\},$$

i.e.

$$a(u_h, v_h) = (f, v_h)_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v_h \in V_h,$$

then $u \in H^m(\Omega)$ *implies*

$$\|u - u_h\|_{H_0^1(\Omega)} \leq C_{\alpha,\beta,l,m,n,\rho} h^{m-1} |u|_{H^m(\Omega)}.$$

Proof The definition assures

$$I_h(H_0^1(\Omega) \cap C^l(\Omega)) \subset V_h.$$

Since C^0-elements are used

$$V_h \subset H_0^1(\Omega).$$

Therefore a combination of Theorem 2 and Céa's theorem yields the statement. □

Corollary 1 *Let* $\Omega \subset \mathbb{R}^2$ *or* $\Omega \subset \mathbb{R}^3$ *be a convex polygonal domain,* $L = \mathrm{div}(A\nabla\cdot)$ *be uniformly elliptic with uniformly Lipschitz continuous coefficient* A *and* $f \in L^2(\Omega)$*. If additionally* $m = 2$ *as well as* $l = 0$ *in the assumptions of Theorem 2 then*

$$\|u - u_h\|_{H_0^1(\Omega)} \leq C_{\alpha,\beta,\rho} h |u|_{H^2(\Omega)} \leq C_{\alpha,\beta,\rho,A,\Omega} h \|f\|_{L^2(\Omega)}.$$

Proof The regularity of u is proved in [10] or rather [8], Theorem 3.2.1.2 and Theorem 3.2.1.3 □

Corollary 2 *Under the assumptions of Theorem 3*

$$\|u - u_h\|_{L^2(\Omega)} \leq \frac{\beta}{\alpha} h \|u - u_h\|_{H_0^1(\Omega)}.$$

If additionally $f \in L^2(\Omega)$ *so that* $u \in H^2(\Omega)$*, then*

$$\|u - u_h\|_{L^2(\Omega)} \leq C_{\alpha,\beta,\rho,A,\Omega} h^2 \|f\|_{L^2(\Omega)}.$$

Proof Aubin-Nitsche's theorem. □

3.2 Periodic boundary conditions

Now the statement of Corollary 2 on periodic boundary conditions will be carried over. In order to use Céa's theorem, the finite element space V_h must satisfy the condition $V_h \subset V = \left\{u \in H^1_{\text{per}}(Y) | \langle u \rangle_Y = 0\right\}$. This leads to the following definition.

Definition 3 A subdivision $\mathscr{K}$ of $Y \subset \mathbb{R}^n$ is called *periodic*, iff for all sufficiently regular Y-periodic functions u the global interpolation $I_{\mathscr{K}} u$ is also Y-periodic.

Remark 5 The easiest way to obtain a periodic subdivision consists in using "symmetric" rectangle elements: If $\mathscr{K}$ is a rectangulation of Y with except for translation identic C^0 rectangle elements $K_i = y_i +]a_1, b_1[\times \cdots \times]a_n, b_n[$ and if the nodal variables are just evaluations of the function and its derivatives, which are symmetrically distributed with respect to the bisector $\left\{x | x_j = (y_i)_j + (b_j - a_j)/2\right\}$, then the rectangulation is periodic.

The Bogner-Fox-Schmitt rectangle for instance satisfies this condition. The use of n-simplices would also be possible, but not very reasonable, because the domain Y is a n-rectangle.

Theorem 4 *Let $u \in V = \left\{u \in H^1_{\text{per}}(Y) | \langle u \rangle_Y = 0\right\}$ be a weak solution of*

$$\begin{cases} -\operatorname{div}(A \nabla u) = f \text{ in } Y, \\ \qquad\quad u \; Y\text{−periodisch}, \\ \quad \langle u \rangle_Y = 0. \end{cases}$$

Let $A \in M(\alpha, \beta, Y)$ be Y-periodic, $a_{ij} \in C^{m-2,1}(\overline{Y})$, $f \in H^{m-2}(Y)$ (with $\langle f \rangle_Y = 0$) and let $(\mathscr{K}_h)_{0<h\leq 1}$ be a non-degenerate family of periodic triangulation or periodic rectangulations of $Y \subset \mathbb{R}^n$, which satisfies the assumptions of Theorem 2 for $p = 2$ and $m \geq 2$. Then the finite element approximation u_h in

$$V_h = \left\{ v = (v_K)_{K \in \mathscr{K}_h} \in \prod_{K \in \mathscr{K}_h} \mathscr{P}_K \,\middle|\, \langle v \rangle_Y = \sum_{K \in \mathscr{K}_h} \langle v_K \rangle_K = 0, \right.$$
$$\left. \exists u \in H^1_{\text{per}}(Y), u|_K \in C^l(K), v_K = I_K u, \forall K \in \mathscr{K}_h \right\}$$

satisfies the error estimate

$$\|u - u_h\|_{H^1(Y)} \leq C_{\alpha,\beta,l,m,n,\rho,Y} h^{m-1} |u|_{H^m(Y)} \leq C_{\alpha,\beta,l,m,n,\rho,M,Y} h^{m-1} \|f\|_{H^{m-2}(Y)},$$

where

$$M = \max\left\{ \|a_{ij}\|_{C^{m-2,1}(\overline{Y})} \right\}.$$

Proof The regularity of the diffusion coefficient and the source yield $u \in H^m(Y)$ with

$$\|u\|_{H^m(Y)} \leq C_{n,\alpha,M,Y} \|f\|_{H^{m-2}(Y)}.$$

Theorem 2 implies for the global interpolation $I_h u$

$$\begin{aligned} \|u - I_h u\|_{H^1(Y)} &\le C_{l,m,n,\rho} h^{m-1} |u|_{H^m(Y)}, \\ \|u - I_h u\|_{L^2(Y)} &\le C_{l,m,n,\rho} h^m |u|_{H^m(Y)}. \end{aligned}$$

Since $\langle u \rangle_Y = 0$ and $\|\cdot\|_{L^1(Y)} \le C_{|Y|} \|\cdot\|_{L^2(Y)}$ ([1], Theorem 2.14, p. 28) with $\tilde{I}_h u = I_h u - \langle I_h u \rangle_Y$, it follows

$$\begin{aligned} \left\|u - \tilde{I}_h u\right\|_{H^1(Y)} &= \|u - I_h u + \langle I_h u \rangle_Y\|_{H^1(Y)} \\ &\le \|u - I_h u\|_{H^1(Y)} + \|\langle I_h u \rangle_Y\|_{H^1(Y)} \\ &= \|u - I_h u\|_{H^1(Y)} + |Y|\,|\langle u - I_h u \rangle_Y| \\ &\le \|u - I_h u\|_{H^1(Y)} + |Y|\,\|u - I_h u\|_{L^1(Y)} \\ &\le \|u - I_h u\|_{H^1(Y)} + C_{|Y|}\,\|u - I_h u\|_{L^2(Y)} \\ &\le C_{l,m,n,\rho,|Y|} h^{m-1} |u|_{H^m(Y)}. \end{aligned}$$

As a complete subspace of $H^1(Y)$ $(V, \|\cdot\|_{H^1(Y)})$ is a Hilbert space. Due to $V_h \subset V$ Céa's theorem yields

$$\|u - u_h\|_{H^1(Y)} \le \frac{\beta}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|_{H^1(Y)},$$

thus the statement follows from $\tilde{I}_h u \in V_h$. □

Theorem 5 *Under the assumptions of Theorem 4 for $m = 2$ without $f \in L^2(Y)$ follows*

$$\|u - u_h\|_{L^2(Y)} \le C_{\alpha,\beta,l,m,n,\rho,M,Y}\, h\, \|u - u_h\|_{H^1(Y)}.$$

Proof Let

$$H = \left\{ u \in L^2(Y) | \langle u \rangle_Y = 0 \right\}.$$

Since $L^2(Y) = \overline{C_0^\infty(Y)}$ with respect to $\|\cdot\|_{L^2(Y)}$ ([1], Corollary 2.30, p. 38), so particularly $L^2(Y) = \overline{H^1_{\text{per}}(Y)}$, follows $H = \overline{V}$: Let $u_i \in H^1_{\text{per}}(Y)$ be a sequence with $u_i \to u \in H$ in $L^2(Y)$. Since $|\langle u_i - u \rangle_Y| \le \|u_i - u\|_{L^1(Y)} \le C_{|Y|}\,\|u_i - u\|_{L^2(Y)} \to 0$ ([1], Theorem 2.14, p. 28), it follows $\langle u_i \rangle_Y \to \langle u \rangle_Y = 0$. Defining $v_i := u_i - \langle u_i \rangle_Y \in V$ yields $\|v_i - u\|_{L^2(Y)} \le \|u_i - u\|_{L^2(Y)} + |Y|\,|\langle u_i \rangle_Y| \to 0$.

Aubin-Nitsche's theorem and theorem 4 then imply the statement. □

Corollary 3 *If in addition to the assumptions of Theorem 5 the source satisfies $f \in L^2(Y)$ (with $\langle f \rangle_Y = 0$), then even*

$$\|u - u_h\|_{L^2(Y)} \le C_{\alpha,\beta,l,m,n,\rho,M,Y}\, h^2\, \|f\|_{L^2(Y)}.$$

Proof Reapplication of Theorem 4. □

Remark 6 In the case of $m > 2$ the quadratic convergence (with respect to the L^2-norm) follows directly from Theorem 4.

Remark 7 If the coefficient satisfies $a_{ij} \in C^{m-2,1}(\overline{Y}) \cap H^{m-1}(Y)$ Theorem 4 implies for an adequate finite element approximation $\chi_{i,h}$ of the solution χ_i of the cell-problems (5)

$$\left\|\chi_i - \chi_{i,h}\right\|_{H^1(Y)} \leq C_{\alpha,\beta,l,m,n,\rho,M,Y} h^{m-1} \left\|\mathrm{div} A^i\right\|_{H^{m-2}(Y)}, \tag{8}$$

where $A^i = (a_{1i}, \ldots, a_{ni})$ is the i-th column of A. Corollary 3 additionally implies for $m = 2$

$$\left\|\chi_i - \chi_{i,h}\right\|_{L^2(Y)} \leq C_{\alpha,\beta,l,m,n,\rho,M,Y} h^2 \left\|\mathrm{div} A^i\right\|_{L^2(Y)}. \tag{9}$$

4 Finite element approximation

In this section the finite element spaces will be choosen so that a potential analytic quadratic L^2-convergence of the corrected first order asymptotic expansion carries over to the finite element approximation.

Theorem 1 states the quadratic L^2-convergence in the one-dimensional case. The following numeric analysis will be an indication under which assumption on the coefficient and source quadratic L^2-convergence of the corrected first order asymptotic expansion can be expected in the two-dimensional case.

Because of the C^∞-regularity of the boundary and the compact embedding $H^1(\Omega) \subset C^0(\overline{\Omega})$ the one-dimensional case is considerably easier to treat. Therefore the following analysis will be restricted to the two-dimensional case and a comparison of the corresponding results for the one-dimensional case.

For the calculation of the finite element approximations of the first order asymptotic expansion, the correction and the solution of the original problem the finite element spaces of Table 1 will be used. At the moment it is not clear, why one should use these finite element spaces. Despite of that it is useful to take a look at the notation.

4.1 First order asymptotic approximation

At first the difference between the first order asymptotic expansion u_1^ε and its finite element approximation will be estimated. Therefore it is mandatory to analyse the influence of using the finite element approximations $\chi_{i,\hat{h}}$ on the homogenized diffusion coefficient A^0.

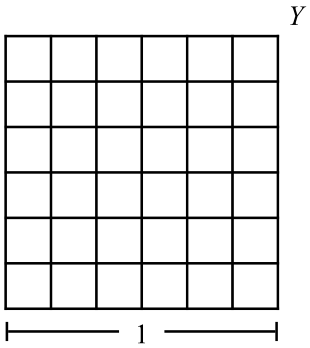
The first error arises at interpolating $A = (a_{ij}) \in M(\alpha, \beta, Y)$. Suppose $(K_{\tilde{h}})$ is a non-degenerate family of subdivisions of Y, then Theorem 2 implies for the interpolation $A_{\tilde{h}}$ of $A \in H^4(Y)$ using cubic Lagrange elements

$$\left\|A - A_{\tilde{h}}\right\|_{L^\infty} \leq C_\rho \tilde{h}^3 \left|A\right|_{H^4(Y)}.$$

Hence for sufficiently small $\tilde{h} > 0$

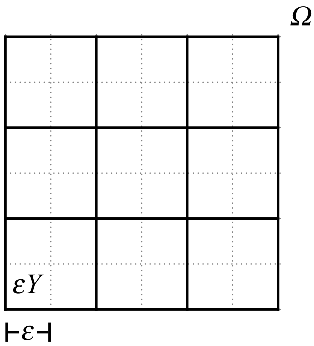
$$(A_{\tilde{h}}(y)\lambda, \lambda) \geq \frac{\alpha}{2} \|\lambda\|_2^2. \tag{10}$$

Table 1 Description of the used finite element spaces.



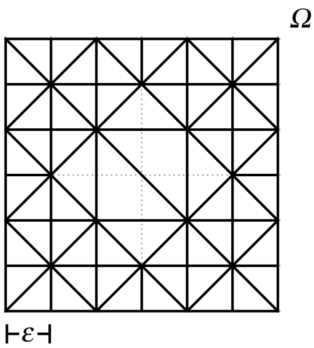
Finite element spaces on the unit cell

- $(\mathscr{K}_{\tilde{h}})_{0\leq\tilde{h}\leq 1}$: non-degenerate family of periodic rectangulations with bi-cubic rectangular elements for interpolating the diffusion coefficient $A \in M(\alpha,\beta,Y)$,
- $(\mathscr{K}_{\hat{h}})_{0\leq\hat{h}\leq 1}$: quasi-uniform family of periodic rectangulations with Bogner-Fox-Schmitt rectangles or Hermite elements (in the one-dimensional case) for approximating the cell-solutions χ_i.

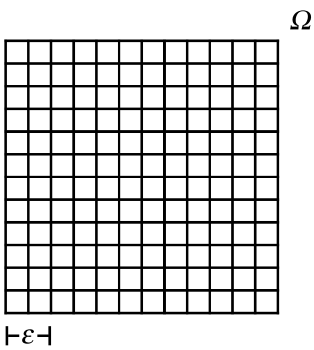


Finite element spaces on Ω

- $(\mathscr{K}_{\tilde{H}})_{0\leq\tilde{H}\leq 1}$: non-degenerate family of bi-cubic rectangulations for interpolating the source $f \in H^{-1}(\Omega)$,
- $(\mathscr{K}_{H})_{0\leq H\leq 1}$: quasi-uniform family of rectangulations with Bogner-Fox-Schmitt rectangles or Hermite elements (in the one-dimensional case) for approximating the homogenized solution u^0,



- $(\mathscr{T}_{\hat{H}})_{0\leq\hat{H}\leq 1}$: non-degenerate family of triangulations with quadratic triangular elements for approximating the correction v_0^{ε},



- $(\mathscr{K}_{h})_{0\leq h\leq 1}$: quasi-uniform family of rectangulations with bi-quadratic rectangular elements for approximating the solution u^{ε} of the original problem.

This makes it possible to estimate the error of the finite element approximation $\chi_{i,\hat{h}}$ in consequence of interpolating the diffusion coefficient. Let $\tilde{\chi}_{i,\hat{h}}$ be the finite element approximation of the differential equation with interpolated diffusion coefficient. Since

$$-\operatorname{div}(A_{\tilde{h}}\nabla(\chi_{i,\hat{h}}-\tilde{\chi}_{i,\hat{h}})) = -\operatorname{div}((A_{\tilde{h}}-A)\nabla\chi_{i,\hat{h}}+(A-A_{\tilde{h}})e_i)$$

it follows using (10)

$$\begin{aligned}\left\|\chi_{i,\hat{h}}-\tilde{\chi}_{i,\hat{h}}\right\|_{H^1(Y)} &\leq \frac{2}{\alpha}\left\|\operatorname{div}((A_{\tilde{h}}-A)\nabla\chi_{i,\hat{h}}+(A-A_{\tilde{h}})e_i)\right\|_{(\mathscr{W}_{\mathrm{per}}(Y))'}\\ &\leq \frac{2}{\alpha}\left\|(A_{\tilde{h}}-A)\nabla\chi_{i,\hat{h}}+(A-A_{\tilde{h}})e_i\right\|_{L^2(Y)}\\ &\leq \frac{2}{\alpha}\left(\left\|A_{\tilde{h}}-A\right\|_{L^\infty(Y)}\left\|\chi_{i,\hat{h}}\right\|_{H^1(Y)}+\left\|A-A_{\tilde{h}}\right\|_{L^2(Y)}\right)\\ &\leq C_{\alpha,\rho,Y}\tilde{h}^3\,|A|_{H^4(Y)}\,.\end{aligned}$$

For measuring the error of $\tilde{\chi}_{i,\hat{h}}$ concerning the analytic solution χ_i only an estimation for the error of $\chi_{i,\hat{h}}$ lacks.

For this the use of an inverse estimate will be necessary, which in turn requires an affine family of finite element spaces. Since also the C^1-differentiability of the finite element approximation will be needed, one automatically arrives at using Bogner-Fox-Schmitt elements ([4], Theorem 2.2.15, p. 77 and p. 85), which are normally used for fourth-order elliptic problems on rectangular domains. Therefore the cell-solutions χ_i have to satisfy higher regularity conditions:

$$\chi_i \in H^4(\Omega).$$

This is true for $A \in C^{2,1}(\overline{Y}) \cap H^3(Y)$.

For a quasi-uniform family $(\mathscr{K}_{\hat{h}})_{0\leq\hat{h}\leq 1}$ of periodic rectangulations with Bogner-Fox-Schmitt elements the Theorems 2 and 4 yield

$$\begin{aligned}\|\chi-I_{\hat{h}}\chi\|_{H^1(Y)} &\leq C_\rho\hat{h}^3\,|\chi|_{H^4(Y)}\,,\\ \|\chi-\chi_{\hat{h}}\|_{H^1(Y)} &\leq C_{\alpha,\beta,\rho,Y}\hat{h}^3\,|\chi|_{H^4(Y)}\,.\end{aligned}$$

Since the subdivision is especially quasi-uniform, the inverse estimate stated in [3], Theorem 4.5.11, p. 112, even implies

$$\|\chi-\tilde{\chi}_{\hat{h}}\|_{W^{1,\infty}(Y)} \leq C_{\alpha,\beta,\rho,Y}\hat{h}^2\,|\chi|_{H^4(Y)}+C_{\alpha,\rho,Y}\frac{\tilde{h}^3}{\hat{h}}\,|A|_{H^4(Y)}\,. \tag{11}$$

The next step consists in determing how this error influences the calculation of the homogenized diffusion coefficient. Let

$$\tilde{a}^0_{ik} = \left\langle a_{ij,\tilde{h}}\delta_{jk}+a_{ij,\tilde{h}}\frac{\partial\tilde{\chi}_{k,\hat{h}}}{\partial y_j}\right\rangle_Y$$

be the components of the homogenized diffusion coefficient, calculated using the finite element approximation $\tilde{\chi}_{k,\hat{h}}$ as well as the interpolation $a_{ij,\tilde{h}}$ of the diffusion coefficient a_{ij} with cubic Lagrange elements. Then

$$a^0_{ik} - \tilde{a}^0_{ik} = \left\langle a_{ik} - a_{ik,\tilde{h}} \right\rangle_Y + \left\langle a_{ij,\tilde{h}} \frac{\partial(\chi_k - \tilde{\chi}_{k,\hat{h}})}{\partial y_j} \right\rangle_Y + \left\langle (a_{ij} - a_{ij,\tilde{h}}) \frac{\partial \chi_k}{\partial y_j} \right\rangle_Y ,$$

so

$$\begin{aligned}
|Y| \cdot |a^0_{ik} - \tilde{a}^0_{ik}| \le& \left\| a_{ik} - a_{ik,\tilde{h}} \right\|_{L^1(Y)} + \left\| A_{i,\tilde{h}} \right\|_{L^2(Y)} \left\| \chi_k - \tilde{\chi}_{k,\hat{h}} \right\|_{H^1(Y)} + \\
& \qquad \left\| A_i - A_{i,\tilde{h}} \right\|_{L^2(Y)} \|\chi_k\|_{H^1(Y)} \\
\le& \left\| a_{ik} - a_{ik,\tilde{h}} \right\|_{L^1(Y)} + \left\| A_{i,\tilde{h}} - A_i \right\|_{L^2(Y)} \left\| \chi_k - \tilde{\chi}_{k,\hat{h}} \right\|_{H^1(Y)} + \\
& \|A_i\|_{L^2(Y)} \left\| \chi_k - \tilde{\chi}_{k,\hat{h}} \right\|_{H^1(Y)} + \left\| A_i - A_{i,\tilde{h}} \right\|_{L^2(Y)} \|\chi_k\|_{H^1(Y)} \\
\le& C_{\rho,Y} \tilde{h}^3 \, |A|_{H^4(Y)} \, (1 + \|\chi\|_{H^1(Y)}) + \\
& C_{\alpha,\beta,\rho,Y} \hat{h}^3 \, |\chi|_{H^4(Y)} \, (C_{\rho,Y} \tilde{h}^3 \, |A|_{H^4(Y)} + \|A\|_{L^2(Y)}) \\
\le& C_{\rho,Y,\chi} \tilde{h}^3 \, |A|_{H^4(Y)} + C_{\alpha,\beta,\rho,Y,A} \hat{h}^3 \, |\chi|_{H^4(Y)} ,
\end{aligned}$$

where $A_i = (a_{i1}, \dots, a_{in})$. Using the norm $\|A\|_G = n \cdot \max_{\{i,j=1,\dots,n\}} |a_{ij}|$ (which is consistent with the Euclidean norm) follows immediately

$$\|A^0 - \tilde{A}^0\|_G \le C_{\rho,Y,\chi} \tilde{h}^3 \, |A|_{H^4(Y)} + C_{\alpha,\beta,\rho,Y,A} \hat{h}^3 \, |\chi|_{H^4(Y)} . \tag{12}$$

According to [5], Proposition 6.12, p. 118, for $A \in M(\alpha, \beta, Y)$ exists an $\alpha_0 > 0$ with $(A^0 \lambda, \lambda) \ge \alpha_0 \|\lambda\|_2^2$. Hence for sufficiently small $\tilde{h}, \hat{h} > 0$

$$(\tilde{A}^0 \lambda, \lambda) \ge \frac{\alpha_0}{2} \|\lambda\|_2^2 .$$

The last two results make it possible to determine the difference of the analytic solution u^0 of the homogenized equation and the finite element approximation of the erroneous homogenized equation. Let $\tilde{u}^0$ be the solution of the homogenized equation with erroneous diffusion coefficient $\tilde{A}^0$ and $\tilde{u}^0_H$ be the finite element approximation of $\tilde{u}^0$, i.e.

$$\int_\Omega \tilde{A}^0 \nabla \tilde{u}^0_H \cdot \nabla v \, dx = \int_\Omega f_{\tilde{H}} v \, dx$$

for all $v \in V_H$, where $f_{\tilde{H}}$ is the interpolation of f with bi-quadratic rectangular elements of a non-degenerate family $(\mathscr{K}_{\tilde{H}})_{0 \le \tilde{H} \le 1}$ of rectangulations.

From

$$\mathrm{div}(\tilde{A}^0 \nabla (u^0 - \tilde{u}^0)) = \mathrm{div}(\tilde{A}^0 \nabla u^0) + f = \mathrm{div}((\tilde{A}^0 - A^0) \nabla u^0)$$

it follows according to [5], theorem 4.16, p. 72 and equation (12)

$$
\begin{aligned}
\left\|u^0-\tilde{u}^0\right\|_{H_0^1(\Omega)} &\le \frac{2}{\alpha_0}\left\|\operatorname{div}((\tilde{A}^0-A^0)\nabla u^0)\right\|_{H^{-1}(\Omega)} \le \frac{2}{\alpha_0}\left\|(\tilde{A}^0-A^0)\nabla u^0\right\|_{L^2(\Omega)} \\
&\le \frac{2}{\alpha_0}\|A^0-\tilde{A}^0\|_G\|u^0\|_{H_0^1(\Omega)} \\
&\le \left(C_{\alpha_0,\rho,Y,\chi}\tilde{h}^3\,|A|_{H^4(Y)}+C_{\alpha_0,\alpha,\beta,\rho,Y,A}\hat{h}^3\,|\chi|_{H^4(Y)}\right)\|u^0\|_{H_0^1(\Omega)}.
\end{aligned}
$$

The (analytic) solution of

$$
\begin{cases}
-\operatorname{div}(\tilde{A}^0\nabla\hat{u}^0_{\tilde{H}})=f_{\tilde{H}} & \text{in } \Omega, \\
\hat{u}^0_{\tilde{H}}=0 & \text{on } \partial\Omega
\end{cases}
$$

satisfies

$$
-\operatorname{div}(\tilde{A}^0\nabla(\tilde{u}^0-\hat{u}^0_{\tilde{H}}))=f-f_{\tilde{H}}.
$$

Therefore Theorem 2 yields for $f\in H^2(\Omega)$

$$
\left\|\tilde{u}^0-\hat{u}^0_{\tilde{H}}\right\|_{H_0^1(\Omega)} \le \frac{2}{\alpha_0}\|f-f_{\tilde{H}}\|_{H^{-1}(\Omega)} \le \frac{2}{\alpha_0}\|f-f_{\tilde{H}}\|_{L^2(\Omega)} \le C_{\alpha_0,\rho}\tilde{H}^2\,|f|_{H^2(\Omega)}.
$$

Finally Corollary 1 yields

$$
\left\|\hat{u}^0_{\tilde{H}}-\tilde{u}^0_H\right\|_{H_0^1(\Omega)} \le C_{\alpha,\beta,\rho,A,\Omega}H\,\|f_{\tilde{H}}\|_{L^2(\Omega)}.
$$

From this we have

Theorem 6 *Let $\Omega\subset\mathbb{R}^2$ be a convex domain, which can be subdivided into rectangles, $f\in H^2(\Omega)$ and $A\in C^{2,1}(\overline{Y})\cap H^4_{\text{per}}(Y)$. Then using the finite element spaces of Table 1 yields*

$$
\begin{aligned}
\left\|u^0-\tilde{u}^0_H\right\|_{H_0^1(\Omega)} \le{}& C_{\alpha_0,\rho,Y,\chi}\tilde{h}^3\,|A|_{H^4(Y)}\,\|u^0\|_{H_0^1(\Omega)}+ \\
& C_{\alpha_0,\alpha,\beta,\rho,Y,A}\hat{h}^3\,|\chi|_{H^4(Y)}\,\|u^0\|_{H_0^1(\Omega)}+ \\
& C_{\alpha_0,\alpha,\beta,\rho,A,\Omega}\tilde{H}^2\,|f|_{H^2(\Omega)}+C_{\alpha,\beta,\rho,A,\Omega}H\,\|f\|_{L^2(\Omega)}.
\end{aligned}
$$

Using the last theorem one gets

Theorem 7 *Under the assumptions of Theorem 6*

$$
\begin{aligned}
&\left\|\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x)-\tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla\tilde{u}^0_H(x)\right\|_{L^2(\Omega)} \\
&\qquad\le C_{\alpha_0,\alpha,\beta,\rho,Y,A,f,\chi,u^0}\left(\hat{h}^2+\frac{\tilde{h}^3}{\hat{h}}+\tilde{H}^2+H\right).
\end{aligned}
$$

Proof Using the L^∞-estimate (see (11))

$$\|\chi - \tilde{\chi}_{\hat{h}}\|_{L^\infty(Y)} \leq C_{\alpha,\beta,\rho,Y} \hat{h}^2 |\chi|_{H^4(Y)} + C_{\alpha,\rho,Y} \frac{\tilde{h}^3}{\hat{h}} |A|_{H^4(Y)}$$

yields

$$\begin{aligned}
&\left\|\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) - \tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_H^0(x)\right\|_{L^2(\Omega)} \\
=&\left\|\chi\left(\frac{x}{\varepsilon}\right)\left(\nabla u^0(x) - \nabla \tilde{u}_H^0(x)\right) + \left(\chi\left(\frac{x}{\varepsilon}\right) - \tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\right)\nabla \tilde{u}_H^0(x)\right\|_{L^2(\Omega)} \\
=&\left\|\chi\left(\frac{x}{\varepsilon}\right)\left(\nabla u^0(x) - \nabla \tilde{u}_H^0(x)\right) + \left(\chi\left(\frac{x}{\varepsilon}\right) - \tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\right)\left(\nabla \tilde{u}_H^0(x) - \nabla u^0\right) + \right. \\
&\qquad\qquad\qquad\qquad\left.\left(\chi\left(\frac{x}{\varepsilon}\right) - \tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\right)\nabla u^0\right\|_{L^2(\Omega)} \\
\leq& \|\chi\|_{L^\infty(Y)} \left\|\nabla u^0 - \nabla \tilde{u}_H^0\right\|_{L^2(\Omega)} + \|\chi - \tilde{\chi}_{\hat{h}}\|_{L^\infty(Y)} \left\|\nabla \tilde{u}_H^0 - \nabla u^0\right\|_{L^2(Y)} + \\
&\qquad\qquad\qquad\qquad \|\chi - \tilde{\chi}_{\hat{h}}\|_{L^\infty(Y)} \left\|\nabla u^0\right\|_{L^2(\Omega)} \\
\leq& C_{\alpha_0,\rho,Y,\chi} \tilde{h}^3 |A|_{H^4(Y)} \|u^0\|_{H_0^1(\Omega)} + C_{\alpha_0,\alpha,\beta,\rho,Y,A,\chi} \hat{h}^3 |\chi|_{H^4(Y)} \|u^0\|_{H_0^1(\Omega)} + \\
&\qquad C_{\alpha_0,\alpha,\beta,\rho,A,\chi,\Omega} \tilde{H}^2 |f|_{H^2(\Omega)} + C_{\alpha,\beta,\rho,A,\chi,\Omega} H \|f\|_{L^2(\Omega)} + \\
&\left(C_{\alpha,\beta,\rho,Y} \hat{h}^2 |\chi|_{H^4(Y)} + C_{\alpha,\rho,Y} \frac{\tilde{h}^3}{\hat{h}} |A|_{H^4(Y)}\right)\left(C_{\alpha_0,\rho,Y,\chi} \tilde{h}^3 |A|_{H^4(Y)} \|u^0\|_{H_0^1(\Omega)} + \right. \\
&\qquad\qquad C_{\alpha_0,\alpha,\beta,\rho,Y,A} \hat{h}^3 |\chi|_{H^4(Y)} \|u^0\|_{H_0^1(\Omega)} + \\
&\qquad\left. C_{\alpha_0,\alpha,\beta,\rho,A,\Omega} \tilde{H}^2 |f|_{H^2(\Omega)} + C_{\alpha,\beta,\rho,A,\Omega} H \|f\|_{L^2(\Omega)} + \left|u^0\right|_{H^1(\Omega)}\right) \\
\leq& C_{\alpha_0,\alpha,\beta,\rho,Y,A,f,\chi,u^0} \left(\hat{h}^2 + \frac{\tilde{h}^3}{\hat{h}} + \tilde{H}^2 + H\right).
\end{aligned}$$

□

4.2 Original problem

It follows from the above and Corollary 2.

Theorem 8 *Under the assumptions of Theorem 6*

$$\left\|u^\varepsilon - u^0 + \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x)\right\|_{L^2(\Omega)} = O(\varepsilon^2),$$

iff

$$\left\|u^\varepsilon - \tilde{u}_H^0 + \varepsilon\tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_H^0(x)\right\|_{L^2(\Omega)} = O(\varepsilon^2)$$

for $\tilde{h} = \hat{h} = H = \tilde{H} = \varepsilon$.

In the one-dimensional case the following result can be proven.

Theorem 9 *Suppose $\Omega \subset \mathbb{R}$ be an interval, $a \in C^{2,1}(\overline{Y}) \cap H^4_{\mathrm{per}}(Y) \subset C^3_{\mathrm{per}}(Y)$ and $f \in H^2(\Omega) \subset C^1(\Omega)$, then using the finite element spaces of Table 1 yields*

$$\left\| u^\varepsilon - u^0 + \varepsilon \chi \left(\frac{x}{\varepsilon}\right) \frac{du^0}{dx} \right\|_{L^2(\Omega)} = O(\varepsilon^2),$$

iff

$$\left\| u^\varepsilon - \tilde{u}^0_H + \varepsilon \tilde{\chi}_{\hat{h}} \left(\frac{x}{\varepsilon}\right) \frac{d\tilde{u}^0_H}{dx} \right\|_{L^2(\Omega)} = O(\varepsilon^2)$$

for $\tilde{h} = \hat{h} = H = \tilde{H} = \varepsilon$.

Remark 8 Of course, it would be interesting to estimate the error $\|u^\varepsilon - u^\varepsilon_h\|_{L^2(\Omega)}$ in order to be able to adjust the parameter h appropriately in the numeric analysis. Though according to Corollary 2

$$\|u^\varepsilon - u^\varepsilon_h\|_{L^2(\Omega)} \le C_{\alpha,\beta,\rho,A^\varepsilon,\Omega} h^2 \|f\|_{L^2(\Omega)},$$

this is not possible since the dependency of the constant $C_{\alpha,\beta,\rho,A^\varepsilon,\Omega}$ on ε is unknown. At least in the one-dimensional case it can be proven that $C_{\alpha,\beta,\rho,A^\varepsilon,\Omega} = O(\varepsilon^{-1})$.

4.3 Correction

Now only the last term, i.e. the correction, has to be analysed. Therefore it is useful to estimate $\left\|\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) - \tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}^0_H(x)\right\|_{H^1(\Omega)}$. For this to be defined one has to suppose $\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x), \tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}^0_H(x) \in H^1(\Omega)$. A sufficient condition would be the following properties of the homogenized solution u^0, the vector χ of the cell-solutions as well as the corresponding finite element approximations $\tilde{u}^0_H$ and $\tilde{\chi}_{\hat{h}}$

$$\begin{aligned} u^0, \tilde{u}^0_H &\in H^2(\Omega), \\ \chi, \tilde{\chi}_{\hat{h}} &\in W^{1,\infty}. \end{aligned}$$

While this is true for χ and $\tilde{\chi}_{\hat{h}}$, u^0 satisfies this assumption according to [10] respectively [8], theorem 3.2.1.2 and theorem 3.2.1.3. The assumption on $\tilde{u}^0_H$ requires using C^1 finite elements. Since again an inverse estimate will be used, the family of finite elements spaces has to be affine. All these conditions are satisfied by the Bogner-Fox-Schmitt element. For this to be defined one has to impose the condition $u^0 \in H^4(\Omega)$.

Since u^0 satisfies this condition normally only if $\partial\Omega$ is a C^4-boundary ([6], Theorem 5, p. 323), the usage of Bogner-Fox-Schmitt elements has to be critically considered, because domains, which are made up of rectangles do not possess a C^4-boundary. Therefore the following should only be considered as motivation for the choice of the finite element spaces.

Theorem 2 yields for a non-degenerate family (K_H) of rectangulations with Bogner-Fox-Schmitt elements,

$$\begin{aligned}\left\|u^0 - I_H u^0\right\|_{H^2(Y)} &\leq C_\rho H^2 \left|u^0\right|_{H^4(Y)},\\ \left\|u^0 - I_H u^0\right\|_{H^1(Y)} &\leq C_\rho H^3 \left|u^0\right|_{H^4(Y)}.\end{aligned}$$

From this with Ceá's theorem it follows that

$$\left\|u^0 - u_H^0\right\|_{H^1(Y)} \leq C_{\alpha,\beta,\rho} H^3 \left|u^0\right|_{H^4(Y)}.$$

The inverse estimate ([3], theorem 4.5.11, p. 112) implies for a quasi-uniform subdivision

$$\left\|u_H^0 - I_H u^0\right\|_{H^2(Y)} \leq \frac{C_\rho}{H} \left\|u_H^0 - I_H u^0\right\|_{H^1(Y)}.$$

Therefore

$$\left\|u^0 - u_H^0\right\|_{H^2(Y)} \leq C_{\alpha,\beta,\rho} H^2 \left|u^0\right|_{H^4(Y)}.$$

Theorem 10 *Let $\Omega \subset \mathbb{R}^2$ be a rectangle, $A \in C^{2,1}(\overline{Y}) \cap H^4_{\text{per}}(Y)$ and $f \in H^3(\Omega)$, then using the finite element spaces of Table 1 yields*

$$\begin{aligned}\left\|u_H^0 - \tilde{u}_H^0\right\|_{H_0^1(\Omega)} \leq\; & C_{\alpha,\beta,\rho,Y,\chi,u^0} \tilde{h}^3 |A|_{H^4(Y)} +\\ & C_{\alpha,\beta,\rho,Y,A,u^0} \hat{h}^3 |\chi|_{H^4(Y)} + C_{\alpha_0,\rho} H^3 |f|_{H^3(\Omega)},\\ \left\|u_H^0 - \tilde{u}_H^0\right\|_{H^2(\Omega)} \leq\; & C_{\alpha,\beta,\rho,Y,\chi,u^0} \frac{\tilde{h}^3}{H} |A|_{H^4(Y)} +\\ & C_{\alpha,\beta,\rho,Y,A,u^0} \frac{\hat{h}^3}{H} |\chi|_{H^4(Y)} + C_{\alpha_0,\rho} H^2 |f|_{H^3(\Omega)}.\end{aligned}$$

Proof Since

$$-\operatorname{div}(\tilde{A}^0 \nabla(u_H^0 - \tilde{u}_H^0)) = -\operatorname{div}((\tilde{A}^0 - A^0)\nabla u_H^0) + f - f_H$$

the first part of the statement follows from estimate (12). The second part is just an application of the inverse estimate ([3], theorem 4.5.11, p. 112). □

From this it follows

Theorem 11 *Let $\Omega \subset \mathbb{R}^2$ be a rectangle, $A \in C^{2,1}(\overline{Y}) \cap H^4_{\text{per}}(Y)$ and $f \in H^3(\Omega)$, then using the finite element spaces of Table 1 with $\tilde{h} = \hat{h} = H = \tilde{H} = \varepsilon$ yields for $u^0 \in H^4(\Omega)$*

$$\left\|\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) - \tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_H^0(x)\right\|_{H^1(\Omega)} = O(\varepsilon).$$

Proof

$$\begin{aligned}&\left\|\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) - \tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_H^0(x)\right\|_{H^1(\Omega)} = \Big\|\chi\left(\frac{x}{\varepsilon}\right)\left(\nabla u^0(x) - \nabla u_H^0(x)\right) +\\ &\quad (\chi - \tilde{\chi}_{\hat{h}})\left(\frac{x}{\varepsilon}\right)\nabla u_H^0(x) + \tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\left(\nabla u_H^0(x) - \nabla \tilde{u}_H^0(x)\right)\Big\|_{H^1(\Omega)} = O(\varepsilon).\end{aligned}$$

□

Theorem 12 *The solution v_0^ε of the correction problem*

$$\begin{cases} -\operatorname{div}(A^0\nabla v_0^\varepsilon) = 0 & \text{in } \Omega, \\ v_0^\varepsilon = -\varepsilon\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) & \text{on } \partial\Omega \end{cases}$$

and the finite element approximation $\tilde{v}_{0,\hat{H}}^\varepsilon$ of the solution $\tilde{v}_0^\varepsilon$ of the problem

$$\begin{cases} -\operatorname{div}(\tilde{A}^0\nabla \tilde{v}_0^\varepsilon) = 0 & \text{in } \Omega, \\ \tilde{v}_0^\varepsilon = -\varepsilon\tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_H^0(x) & \text{on } \partial\Omega \end{cases}$$

satisfy using the finite element spaces of table 1 with $\tilde{h} = \hat{h} = H = \tilde{H} = \varepsilon$

$$\left\|v_0^\varepsilon - \tilde{v}_{0,\hat{H}}^\varepsilon\right\|_{L^2(\Omega)} = O\left(\varepsilon^2 + \left\|v_0^\varepsilon - v_{0,\hat{H}}^\varepsilon\right\|_{L^2(\Omega)} + \varepsilon^3\left\|v_0^\varepsilon - v_{0,\hat{H}}^\varepsilon\right\|_{H^1(\Omega)}\right).$$

Thereby $\mathscr{T}_{\hat{H}}$ must be choosen such that the boundary conditions $\varepsilon\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x)$ and $\varepsilon\tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_H^0(x)$ can be described in $\mathscr{T}_{\hat{H}}$.

Proof Let $\hat{v}_0^\varepsilon$ be the solution of

$$\begin{cases} -\operatorname{div}(\tilde{A}^0\nabla \hat{v}_0^\varepsilon) = 0 & \text{in } \Omega, \\ \hat{v}_0^\varepsilon = -\varepsilon\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) & \text{on } \partial\Omega. \end{cases}$$

From

$$\operatorname{div}(\tilde{A}^0\nabla(v_{0,\hat{H}}^\varepsilon - \hat{v}_{0,\hat{H}}^\varepsilon)) = \operatorname{div}((\tilde{A}^0 - A^0)\nabla v_{0,\hat{H}}^\varepsilon)$$

follows with estimate (12)

$$\begin{aligned} &\left\|v_{0,\hat{H}}^\varepsilon - \hat{v}_{0,\hat{H}}^\varepsilon\right\|_{H_0^1(\Omega)} \\ \le &C_{\alpha_0,\beta_0}\|A^0 - \tilde{A}^0\|_G\left(\left\|\varepsilon\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x)\right\|_{H^1(\Omega)} + \left\|v_0^\varepsilon - v_{0,\hat{H}}^\varepsilon\right\|_{H^1(\Omega)}\right) \\ = &O\left(\varepsilon^3 + \varepsilon^3\left\|v_0^\varepsilon - v_{0,\hat{H}}^\varepsilon\right\|_{H^1(\Omega)}\right). \end{aligned}$$

For $u^0 \in H^4(\Omega)$ and $\tilde{h} = \hat{h} = H = \tilde{H} = \varepsilon$

$$\begin{aligned} \left\|\tilde{v}_{0,\hat{H}}^\varepsilon - \hat{v}_{0,\hat{H}}^\varepsilon\right\|_{H^1(\Omega)} &\le C_{\alpha_0,\beta_0}\left\|\left(\varepsilon\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) - \varepsilon\tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla\tilde{u}_H^0(x)\right)\right\|_{H^1(\Omega)} \\ &= O(\varepsilon^2). \end{aligned}$$

A simple application or the triangle inequality proves the statement. □

Remark 9 The last theorem suggests the usage of a locally refined grid for the correction problem.

Corollary 4 *Let $\Omega \subset \mathbb{R}^2$ be a rectangle, $A \in C^{2,1}(\overline{Y}) \cap H^4_{\mathrm{per}}(Y)$ and $f \in H^3(\Omega)$. Suppose $u^0 \in H^4(\Omega)$, then using the finite element spaces of Table 1 with $\tilde{h} = \hat{h} = H = \tilde{H} = \varepsilon$ and appropriate $\mathcal{T}_{\hat{H}}$ yields*

$$\left\| u^\varepsilon - u^0 + \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\nabla u^0(x) + v_0^\varepsilon(x)\right\|_{L^2(\Omega)} = O(\varepsilon^2),$$

iff

$$\left\| u^\varepsilon - \tilde{u}_H^0 + \varepsilon\tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\nabla \tilde{u}_H^0(x) + \tilde{v}_{0,\hat{H}}^\varepsilon(x)\right\|_{L^2(\Omega)} = O(\varepsilon^2).$$

Proof Application of the last theorems. □

In the one-dimensional case the H^4-regularity of u^0 is satisfied without further assumption. Therefore

Corollary 5 *Let $\Omega \subset \mathbb{R}$ be an interval, $a \in C^{2,1}(\overline{Y}) \cap H^4_{\mathrm{per}}(Y)$ and $f \in H^3(\Omega)$, then using the finite element spaces of Table 1 with $\tilde{h} = \hat{h} = H = \tilde{H} = \varepsilon$ yield*

$$\left\| u^\varepsilon - u^0 + \varepsilon\chi\left(\frac{x}{\varepsilon}\right)\frac{du^0}{dx}(x) + v_0^\varepsilon(x)\right\|_{L^2(\Omega)} = O(\varepsilon^2),$$

iff

$$\left\| u^\varepsilon - \tilde{u}_H^0 + \varepsilon\tilde{\chi}_{\hat{h}}\left(\frac{x}{\varepsilon}\right)\frac{d\tilde{u}_H^0}{dx}(x) + \tilde{v}_{0,\hat{H}}^\varepsilon(x)\right\|_{L^2(\Omega)} = O(\varepsilon^2).$$

5 Numeric results

For the numeric results the finite element spaces of Table 1 have been implemented using the C++-library getfem++[1], extended by the Bogner-Fox-Schmitt element and periodic boundary conditions. The calculations have been made on an AMD Opteron[2].

The locally (near the boundary) refined grid for the correction has been computed by iterative division of the outermost triangles until the size of them has been reduced by the factor ε.

Definition 4 The *Estimated Order of Convergence (EOC)* of a sequence of approximative solutions is the sequence defined by

$$\mathrm{EOC}\left(\frac{\varepsilon}{2}\right) = \log_2\left(\frac{\left\|u^\varepsilon - u_{\mathrm{app}}^\varepsilon\right\|}{\left\|u^{\varepsilon/2} - u_{\mathrm{app}}^{\varepsilon/2}\right\|}\right).$$

Remark 10 All iterative solvers use the stopping criterion

$$\frac{\|r^k\|_2}{\|r^0\|_2} < 10^{-9},$$

where r^k is the residual of the k-th approximative solution.

[1] http://home.gna.org/getfem/

[2] CPU-frequence 2.3 GHz, 16 GB RAM

The numeric analysis is restricted to isotropic materials, i.e.

$$A(y) = a(y)I,$$

because the influence of the regularity as well as the symmetry of the diffusion coefficient is the primary focus. Nevertheless also anisotropic problems have to be solved, because the homogenized diffusion coefficient of isotropic materials is generally anisotropic. Furthermore $\Omega = Y =]0,1[^2$ and $f \equiv 10$ in all examples. For the examples with discontinuous diffusion coefficient the finite element spaces of Table 1 have to be adjusted appropriately.

5.1 Symmetric diffusion coefficients

For symmetric diffusion coefficients the solution v_0^ε of the correction problem (6) equals zero ([2], §6.3 and [11], Corollary 2.4, p. 5).

As first example let

$$a(y) = \frac{1}{2+\cos(2\pi y_1)\cos(2\pi y_2)}.$$

For the homogenized diffusion coefficient one gets numerically

$$A^0 = \begin{pmatrix} 0.52 & 0 \\ 0 & 0.52 \end{pmatrix}.$$

Tabular 2 shows that already for the first order approximation u_1^ε the order of convergence equals 2 with respect to the L^2-norm and equals 1 with respect to the H^1-norm. The latter is theoretically assured ([9], S. 29, 30).

The comparison of the runtimes for solving the approximations and the original problem is done in Tabular 3. Thereby one has to bear in mind that the subdivisions of the unit cell Y and the domain Ω are refined for ε becoming smaller, so that especially the calculation of the homogenized solution, which is independent of ε, becomes more costly for smaller ε.

Loosening the assumption on the regularity of the coefficient the does not influence the order of convergence qualitatively (Tables 4, 5 and 6, 7) as the following discontinuous examples shows. Therefore the coefficients are approximated by step functions. Furthermore the Bogner-Fox-Schmitt elements are replaced by Lagrange elements, because the cell-solutions can not be smooth for discontinuous coefficients.

Let

$$a(y) = \begin{cases} 1, & \|y-(0.5,0.5)\|_\infty > 0.25, \\ 2, & \|y-(0.5,0.5)\|_\infty < 0.25. \end{cases}$$

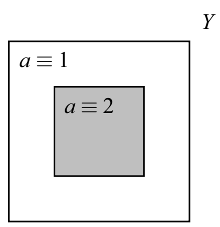


Table 2 EOC for smooth symmetric diffusion coefficient (2d).

ε	$\lVert u^\varepsilon - u^0 \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u^0 \rVert_{H^1(\Omega)}$	EOC
1/2	0.0453033143	–	0.7396763548	–
1/4	0.0198023429	1.1939454650	0.6924529308	0.0951781502
1/8	0.0096168501	1.0420347991	0.6826984688	0.0204674917
1/16	0.0047723836	1.0108544194	0.6803285078	0.0050169725
1/32	0.0023817780	1.0026710405	0.6797724979	0.0011795488
ε	$\lVert u^\varepsilon - u_1^\varepsilon \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u_1^\varepsilon \rVert_{H^1(\Omega)}$	EOC
1/2	0.0150663057	–	0.2189079145	–
1/4	0.0035537233	2.0839224641	0.1194965457	0.8733552000
1/8	0.0008830151	2.0088212755	0.0617623306	0.9521698162
1/16	0.0002193059	2.0094937688	0.0310311422	0.9930103922
1/32	0.0000546916	2.0035525323	0.0155313801	0.9985307736

Table 3 Runtimes (s) for smooth symmetric diffusion coefficient (2d).

ε	u^0	u_1^ε	u^ε
1/2	0.02	0.03	0.03
1/4	0.02	0.05	0.14
1/8	0.07	0.57	3.30
1/16	0.31	9.42	116.37
1/32	1.52	170.09	12828.12

Table 4 EOC for discontinous symmetric diffusion coefficient (2d).

ε	$\lVert u^\varepsilon - u^0 \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u^0 \rVert_{H^1(\Omega)}$	EOC
1/2	0.0239518289	–	0.3416978653	–
1/4	0.0117225822	1.0308454275	0.3406784255	0.0043106475
1/8	0.0057796767	1.0202296871	0.3450890791	−0.0185582308
1/16	0.0028760931	1.0068784421	0.3465445275	−0.0060719180
1/32	0.0014361379	1.0019160630	0.3469459848	−0.0016703345
ε	$\lVert u^\varepsilon - u_1^\varepsilon \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u_1^\varepsilon \rVert_{H^1(\Omega)}$	EOC
1/2	0.0074396050	–	0.0951826775	–
1/4	0.0027864411	1.4168023867	0.0489383481	0.9597336338
1/8	0.0007968061	1.8061230500	0.0256343033	0.9328897161
1/16	0.0002080745	1.9371286759	0.0130790994	0.9708124860
1/32	0.0000528195	1.9779580265	0.0065997939	0.9867703186

The (numerically calculated) homogenized diffusion coefficient then fulfils

$$A^0 = \begin{pmatrix} 1.2 & 0.001 \\ 0.001 & 1.2 \end{pmatrix}.$$

The last symmetric example is the symmetric checkerboard media defined by

Table 5 Runtimes (s) for discontinuous symmetric diffusion coefficient (2d).

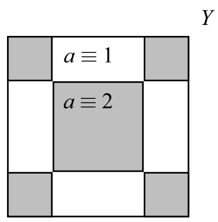
ε	u^0	u_1^ε	u^ε
1/2	0.01	0.02	0.01
1/4	0.01	0.04	0.04
1/8	0.04	0.53	1.78
1/16	0.18	9.28	84.16
1/32	1.02	169.24	6453.78

Table 6 EOC for symmetric checkerboard media (2d).

ε	$\left\|u^\varepsilon - u^0\right\|_{L^2(\Omega)}$	EOC	$\left\|u^\varepsilon - u^0\right\|_{H^1(\Omega)}$	EOC
1/2	0.0163724770	–	0.3095758450	–
1/4	0.0076974322	1.0888234487	0.3139580561	−0.0202789239
1/8	0.0037765208	1.0273195215	0.3219953159	−0.0364678695
1/16	0.0018822135	1.0046274532	0.3248096859	−0.0125549533
1/32	0.0009406925	1.0006351948	0.3258576955	−0.0046474118

ε	$\left\|u^\varepsilon - u_1^\varepsilon\right\|_{L^2(\Omega)}$	EOC	$\left\|u^\varepsilon - u_1^\varepsilon\right\|_{H^1(\Omega)}$	EOC
1/2	0.0059345065	–	0.1032529360	–
1/4	0.0013423883	2.1443260663	0.0480595898	1.1032865655
1/8	0.0003378550	1.9903259254	0.0245391079	0.9697415346
1/16	0.0000843014	2.0027753245	0.0123250493	0.9934893838
1/32	0.0000210614	2.0009553368	0.0061709050	0.9980394212

$$a(y) = \begin{cases} 1, & 0.25 < y_1 < 0.75 \text{ und } y_2 < 0.25, \\ 1, & 0.25 < y_1 < 0.75 \text{ und } y_2 > 0.75, \\ 1, & 0.25 < y_2 < 0.75 \text{ und } y_1 < 0.25, \\ 1, & 0.25 < y_2 < 0.75 \text{ und } y_1 > 0.75, \\ 2, & \text{otherwise.} \end{cases}$$



For the homogenized diffusion coefficient one gets (also compare with [9], p. 37)

$$A^0 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

Table 7 Runtimes (s) for symmetric checkerboard media (2d).

ε	u^0	u_1^ε	u^ε
1/2	0.01	0.02	0.02
1/4	0.03	0.06	0.05
1/8	0.05	0.53	1.77
1/16	0.18	9.35	84.43
1/32	1.01	169.45	6470.10

Table 8 EOC for smooth layered media (2d).

ε	$\lVert u^\varepsilon - u^0 \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u^0 \rVert_{H^1(\Omega)}$	EOC
1/2	0.0753795658	–	0.7568077129	–
1/4	0.0524915563	0.5220881141	0.8229203871	−0.1208260732
1/8	0.0290220394	0.8549364701	0.8596651469	−0.0630219519
1/16	0.0148975437	0.9620744180	0.8691848233	−0.0158881673
1/32	0.0074988942	0.9903247135	0.8716373795	−0.0040650825
ε	$\lVert u^\varepsilon - u_1^\varepsilon \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u_1^\varepsilon \rVert_{H^1(\Omega)}$	EOC
1/2	0.0811675407	–	0.6017144340	–
1/4	0.0452225062	0.8438619550	0.3111532438	0.9514536780
1/8	0.0239634065	0.9162079399	0.1571932788	0.9850857540
1/16	0.0121963564	0.9743827882	0.0789971925	0.9926662461
1/32	0.0061268676	0.9932286336	0.0396156716	0.9957301213
ε	$\lVert u^\varepsilon - \tilde{u}_1^\varepsilon \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - \tilde{u}_1^\varepsilon \rVert_{H^1(\Omega)}$	EOC
1/2	0.0304643433	–	0.3000975354	–
1/4	0.0083997149	1.8587093738	0.1552774971	0.9505827016
1/8	0.0021197667	1.9864348440	0.0799556498	0.9575768840
1/16	0.0005338653	1.9893577599	0.0406563081	0.9757207669
1/32	0.0001343803	1.9901539728	0.0205461175	0.9846134151

5.2 Asymmetric diffusion coefficient

Again the first example has a smooth diffusion coefficient. More precisely the first example generalises the one-dimensional Example 1

$$a(y) = \frac{1}{2 + \sin(2\pi y_1)}.$$

From [5], Theorem 5.10, p. 99 follows for the homogenized diffusion coefficient

$$A^0 = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/\sqrt{3} \end{pmatrix}.$$

Unlike the case of symmetric diffusion coefficients the correction does not vanish any more. Tabular 8 shows that the correction enhances the order of convergence with respect to the L^2-norm by one. The additional effort for determing the correction is contained, as Tabular 9 shows.

Table 9 Runtimes (s) for smooth layered media (2d).

ε	u^0	u_1	$\tilde{u}_1^\varepsilon$	u^ε
1/2	0.02	0.03	0.04	0.04
1/4	0.02	0.05	0.09	0.14
1/8	0.08	0.59	0.98	3.35
1/16	0.30	9.66	13.19	117.86
1/32	1.48	170.51	211.23	12469.08

Table 10 EOC for layered media (2d).

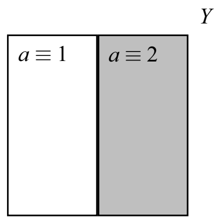
ε	$\lVert u^\varepsilon - u^0 \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u^0 \rVert_{H^1(\Omega)}$	EOC
1/2	0.0301482923	–	0.2790714535	–
1/4	0.0200507515	0.5884199752	0.2977509503	−0.0934715550
1/8	0.0109363794	0.8745211066	0.3083434409	−0.0504320450
1/16	0.0055957736	0.9667256927	0.3114974056	−0.0146819900
1/32	0.0028144330	0.9914932777	0.3123237718	−0.0038222334

ε	$\lVert u^\varepsilon - u_1^\varepsilon \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u_1^\varepsilon \rVert_{H^1(\Omega)}$	EOC
1/2	0.0333610942	–	0.2312360181	–
1/4	0.0182724957	0.8684929129	0.1192249442	0.9556800265
1/8	0.0095836605	0.9310249823	0.0601919623	0.9860433521
1/16	0.0048601412	0.9795785886	0.0302273055	0.9937184678
1/32	0.0024391302	0.9946314686	0.0151488619	0.9966429737

ε	$\lVert u^\varepsilon - \tilde{u}_1^\varepsilon \rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - \tilde{u}_1^\varepsilon \rVert_{H^1(\Omega)}$	EOC
1/2	0.0102038362	–	0.1044334018	–
1/4	0.0028560276	1.8370298092	0.0532558581	0.9715710809
1/8	0.0007309319	1.9662010154	0.0272786145	0.9651698590
1/16	0.0001846220	1.9851624713	0.0138590398	0.9769430634
1/32	0.0000464546	1.9906809929	0.0069990899	0.9855880614

The next example is the problem of (discontinuous) layered media.

$$a(y) = \begin{cases} 1, & y_1 < 0.5, \\ 2, & y_1 > 0.5. \end{cases}$$



As proven in [5], Theorem 5.10, p. 99, one gets for the homogenized diffusion coefficient

$$A^0 = \begin{pmatrix} 4/3 & 0 \\ 0 & 3/2 \end{pmatrix}.$$

According to Tabular 10 the first order approximation u_1^ε has only the order of convergence 1 with respect to the L^2-norm, whereas the corrected approximation has the order of convergence 2 with respect to the L^2-norm. The order of convergence with respect to the H^1-norm is not considerably influenced by the correction.

As in the case of smooth coefficients, the additional expanses for the correction problem are justifiable (Tabular 11).

If the problem is asymmetric for both directions, there exists examples for which the correction does not enhance the order of convergence. A possible example is the checkerboard, i.e.

Table 11 Runtimes (s) for layered media (2d).

ε	u^0	u_1^ε	$\tilde{u}_1^\varepsilon$	u^ε
1/2	0.02	0.02	0.03	0.01
1/4	0.01	0.04	0.08	0.04
1/8	0.05	0.55	0.93	1.79
1/16	0.19	9.31	12.77	88.50
1/32	1.00	169.62	209.55	6907.70

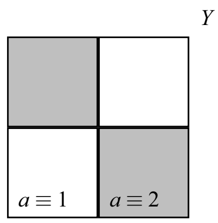
Table 12 EOC for checkerboard media (2d).

ε	$\lVert u^\varepsilon - u^0\rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u^0\rVert_{H^1(\Omega)}$	EOC
1/2	0.0161774046	–	0.2268342841	–
1/4	0.0085648440	0.9174812869	0.2563242817	−0.1763314439
1/8	0.0046063517	0.8948024023	0.2891273304	−0.1737348370
1/16	0.0024018905	0.9394541991	0.3075452366	−0.0890936420
1/32	0.0012272049	0.9687942027	0.3170389096	−0.0438612794

ε	$\lVert u^\varepsilon - u_1^\varepsilon\rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - u_1^\varepsilon\rVert_{H^1(\Omega)}$	EOC
1/2	0.0200220247	–	0.3052071004	–
1/4	0.0080652016	1.3118053713	0.2196768661	0.4744055763
1/8	0.0036548677	1.1418914178	0.1594285293	0.4624731308
1/16	0.0017417208	1.0693057915	0.1135195716	0.4899687691
1/32	0.0008500261	1.0349343740	0.0804082929	0.4975248426

ε	$\lVert u^\varepsilon - \tilde{u}_1^\varepsilon\rVert_{L^2(\Omega)}$	EOC	$\lVert u^\varepsilon - \tilde{u}_1^\varepsilon\rVert_{H^1(\Omega)}$	EOC
1/2	0.0105382132	–	0.1795852569	–
1/4	0.0053155046	0.9873517113	0.0626434086	1.5194342947
1/8	0.0029842713	0.8328279685	0.0461270740	0.4415489371
1/16	0.0015751167	0.9219199254	0.0325400238	0.5033984726
1/32	0.0008084339	0.9622569956	0.0228024854	0.5130242220

$$a(y) = \begin{cases} 1, & y_1, y_2 < 0.5, \\ 1, & y_1, y_2 > 0.5, \\ 2, & \text{otherwise.} \end{cases}$$



The homogenized diffusion coefficient suffices ([9], p. 37)

$$A^0 = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix}.$$

Tabular 12 shows that the correction does not enhance the order of convergence for this problem.

Besides the short runtimes another advantage of approximating the solution by means of homogenization is the considerably smaller memory requirement,

Table 13 Runtimes (s) for checkerboard media (2d).

ε	u^0	u_1^ε	$\tilde{u}_1^\varepsilon$	u^ε
1/2	0.01	0.02	0.03	0.01
1/4	0.02	0.05	0.08	0.05
1/8	0.05	0.54	0.91	1.76
1/16	0.18	9.30	12.67	84.52
1/32	1.03	169.62	208.99	6461.73

because the original problem does not have to be assembled (not until the two-grid method of the next section).

Ω is made up of $(1/\varepsilon)^n$ $(n = 1,2)$ unit cells. Constructing the grid for Ω by copying the grid of the unit cell increases the number of degrees of freedom by the factor $(1/\varepsilon)^n$ $(n = 1,2)$ (same choice of shape functions). Therefore assembling the linear system of equations for the original problem produces a matrix which has compared to the matrix for the cell-problems a by the factor $(1/\varepsilon)^n$ $(n = 1,2)$ increased amount of non-zero entries.

On the other hand using the corrected first order asymptotic expansion means to solve $n+2$ small linear systems of equations (homogenized problem + n cell-problems + correction problem) and thereby only the memory capacity for three small matrices has to be allocated. Therewith also problems can be (approximately) solved even if the original problem can not be assembled due to lack of memory.

6 Two-grid method

The numeric results of the last section show, that the correction does not always enhance the order of convergence. Hence it is all the more important to construct an effective algorithm for solving the original problem. The following numeric analysis verifies, that this is possible using the finite elements used in order to determine the corrected asymptotic expansion.

The error functions in Figure 2 show the the corrected first order asymptotic expansion is well suited for a coarse-grid correction on a locally near the boundary refined grid. Exactly such a grid was already used for solving the correction problem. Therefore all components for the tow-grid method are at hand.

1. The already calculated interpolation from $\mathscr{T}_{\hat{H}}$ to $\mathscr{K}_h$ is used as prolongation for the coarse-grid correction.
2. The transposed interpolation matrix is used as restriction (Galerkin choice).
3. The discretised homogenized problem is used to determine the coarse grid correction.
4. The PCG-method also used for solving the original problem directly is applied as smoother.
5. The method starts with the corrected first order asymptotic expansion.

The most interesting feature of this two-grid method is the fact, that it amends compared to the pure PCG-method for ε becoming smaller. This has the following reason: For large ε the mainpart of the numerical expenses consists in calculating the restriction as well as the prolongation. On the other hand for small ε the

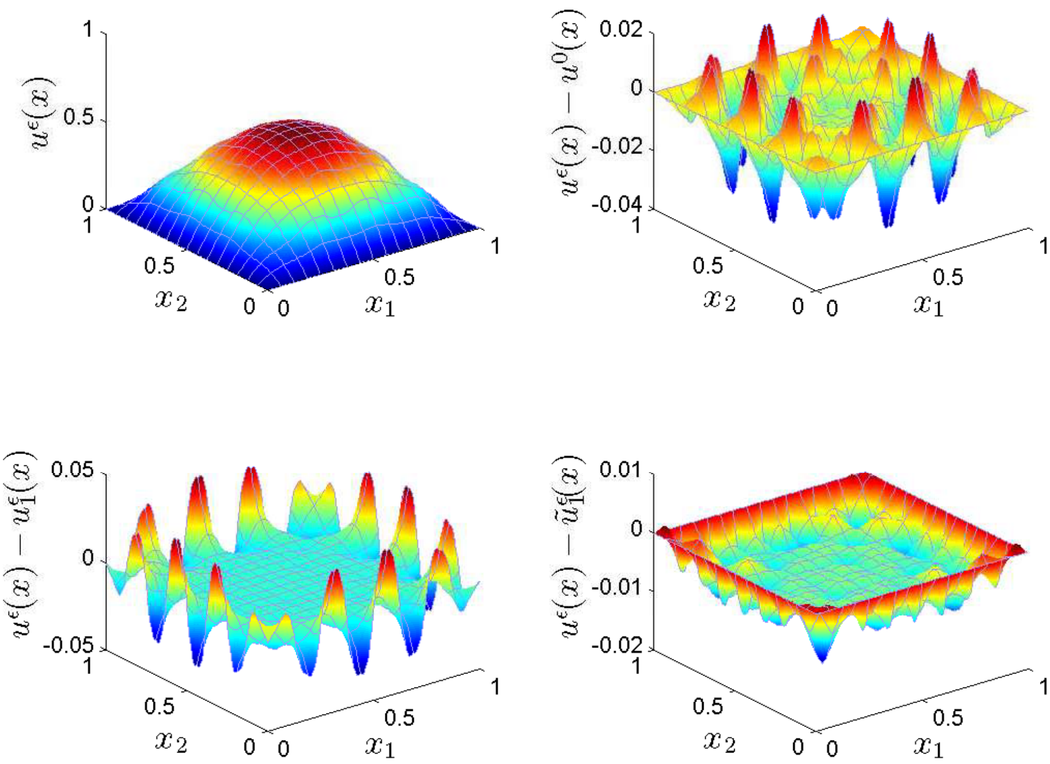


Fig. 2 Numeric solution and approximation error for checkerboard media ($\varepsilon = 1/4$).

smoothing steps are rather expensive. But exactly the number of smoothing steps is substantially reduced by the coarse-grid correction.

The exact runtimes and rate of convergence of the two-grid method for the the checkerboard media is listed in Tabular 14. Here the *rate of convergence* q is defined as the quotient of the Euclidean norms of the last two residuals before the abort

$$q^m = \frac{||r^m||_2}{||r^{m-1}||_2}.$$

The corresponding runtimes of the pure PCG-method can be found in Tabular 13.

Table 14 Runtimes (s) and rates of convergence for checkerboard media (2d).

ε	Runtime(s)	Iterations	Rate of convergence
1/2	0.04	6	0.043
1/4	0.12	6	0.018
1/8	1.81	8	0.057
1/16	33.11	14	0.263
1/32	1003.48	36	0.621

The runtimes suggest to use the two-grid-method for $\varepsilon \leq 1/16$. For $\varepsilon = 1/32$ the the two-grid method reduces the runtime by the factor 6.4, so already for $\varepsilon = 1/64$ one can expect that the runtime is reduced by one order.

References

1. Adams, R.A., Fournier, J.J.F.: Sobolev Spaces, *Pure and Applied Mathematics*, vol. 65, 2 edn. Academic Press (2003)
2. Bakhvalov, N.S., Panasenko, G.: Homogenisation: Averaging Process in Periodic Media, *Mathematics and its Applications: Soviet Series*, vol. 36. Kluwer Academic Publishers (1989)
3. Brenner, S.C., Scott, L.R.: The Mathematical Theory of Finite Element Methods, *Texts in Applied Mathematics*, vol. 15, 2 edn. Springer (2002)
4. Ciarlet, P.G.: The Finite Element Method for Elliptic Problems, *Studies in Mathematics and its Applications*, vol. 4. North-Holland (1978)
5. Cioranescu, D., Donato, P.: An Introduction to Homogenization, *Oxford Lecture Series in Mathematics and its Applications*, vol. 17. Oxford University Press (1999)
6. Evans, L.C.: Partial Differential Equations, *Graduate Studies in Mathematics*, vol. 19. AMS (1998)
7. Gilbarg, D., Trudinger, N.: Elliptic Partial Differential Equations of Second Order. Classics in Mathematics. Springer (2001)
8. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. Pitman (1985)
9. Jikov, V.V., Kozlov, S.M., Oleinik, O.A.: Homogenization of Differential Operators and Integral Functionals. Springer (1994)
10. Kadlec, J.: O reguljarnosti rešenija sadači puassona na oblasti s granicej, lokol'no podobnoj granice vypukloj oblasti. Czechoslovak Math. J. **14**, 386–393 (1964)
11. Neuss, N., Jäger, W., Wittum, G.: Homogenization and multigrid. Computing **66**, 1–26 (2001)
12. Saks, S.: Theory of the Integral, *Monografie Matematyczne*, vol. 7, 2 edn. New York, G. E. Stechert (1937)