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Abstract: This paper contains a stability analysis of the plane-wave Riemann problem for the two-dimensional compressible Euler equation. It is proved that the two-dimensional contact discontinuity in the plane-wave Riemann problem is unstable under small perturbations. The implication for Godunov´s and Roe´s scheme are discussed and it is shown that numerical post shock noise can set off a two-dimensional contact instability resulting in a carbuncle phenomenon. A theoretical explanation is given why the HLLE scheme is not affected by this two-dimensional contact (carbuncle) instability.

Key words: Riemann solver, Godunov-type methods, hyperbolic conservation laws, gas dynamic, carbuncle instability

AMS(MOS) subject classifications: 65M05, 65M10, 76M05

1 Introduction

Consider the two dimensional compressible Euler equation in conservation form

$$\frac{\partial}{\partial t}\mathbf{u}(x,y,t) + \frac{\partial}{\partial x} f(\mathbf{u}(x,y,t) + \frac{\partial}{\partial y}\mathbf{g}(\mathbf{u}(\mathbf{x},\mathbf{y},\mathbf{t}) = 0 \tag{1}$$

where the conserved variable and flux functions are given by

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} \qquad f(\mathbf{u}) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(E+p) \end{pmatrix} \qquad g(\mathbf{u}) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(E+p) \end{pmatrix} \tag{2}$$

The first equation is the conservation of mass $\rho(x, y, t)$, the second and third are the conservation of x-momentum ρu and y-momentum ρv, respectively. The fourth equation expresses the conservation of energy. The velocity field will be denoted by $\mathbf{v} = (u, v)$.The pressure p is related to the conserved quantities through the equation of state. In case of an γ-law gas the equation of state is

$$p = (\gamma - 1)\rho e = (\gamma - 1)[E - \frac{1}{2}\rho(u^2 + v^2)] \tag{3}$$

where the total Energy E/ρ is the sum of kinetic $\frac{1}{2}(u^2 + v^2)$ and internal energy e.

The Euler equation are a system of hyperbolic conservation laws; i.e. for any value of $\mathbf{u_0} = (\rho_0, \rho_0 u_0, \rho_0 v_0, E_0)^T$ with positive density ρ_0 and positive internal energy e_0 the Jacobian matrix

$$\vec{n} D\vec{F}(\mathbf{u_0}) = n_x Df(\mathbf{u_0}) + n_y Dg(\mathbf{u_0}) \tag{4}$$

is diagonalizable with real eigenvalues for every unit vector $\vec{n} = (n_x, n_y)$. Therefore the structure of a plane-wave

$$\mathbf{u}(x, y, t) = \varphi(\vec{n} \cdot (x, y) - st) \tag{5}$$

propagating at speed s, is independent of the orientation in space; see for example [Lev;2002]. A special class of plane-waves is defined through an initial value problem for (1) with data

$$\mathbf{u}(\vec{x}) = \begin{cases} \mathbf{u}_l & \text{for } \vec{n}_0 \cdot (\vec{x} - \vec{x}_0) < 0 \\ \mathbf{u}_r & \text{for } 0 < \vec{n}_0 \cdot (\vec{x} - \vec{x}_0) \end{cases} \tag{6}$$

where $\vec{x}_0$ is a given point in (x,y)-plane and $\vec{n}_0$ a given unit-direction, $\mathbf{u}_l$ and $\mathbf{u}_r$ are initial states at time $t = t^n$

Finite-Volume Godunov-type methods are derived form the integral form of the conservation law for (1):

$$\begin{aligned} &\frac{\partial}{\partial t} \int_{D_{i,j}} \mathbf{u}(\xi, \eta, t)\, d\eta\, d\xi \\ &= - \int_{\partial D_{i,j}} \vec{n}(\xi) \cdot\ \vec{F}(\mathbf{u}^R(\vec{x}(\xi), t+; \vec{n}(\xi)))\, d\xi \end{aligned} \tag{7}$$

where $\vec{n}(\xi) = (n_x(\xi), n_y(\xi))$ is the outward pointing unit-normal vector on $\partial D_{i,j}$ at a point $\vec{x}(\xi) = (x(\xi), y(\xi))$ on $\partial D_{i,j}$, where ξ is the arclength parametrization of $\partial D_{i,j}$. We assume that $D_{i,j}$ is a close bounded set with a piecewise smooth boundary. $\mathbf{u}^R(\vec{x}(\xi), t+; \vec{n}(\xi))$ denotes the one sided limit in time of the solution to the plane-wave Riemann problem (6) at the cell boundary $\vec{x}(\xi)$ in the direction $\vec{n}(\xi)$; i.e.

$$u^R(\vec{x}(\xi), t+; \vec{n}(\xi)) := \lim_{\epsilon \to 0} u^R(\vec{x}(\xi), t+\varepsilon; \vec{n}(\xi)) \quad \text{on } \partial D_{i,j} \tag{8}$$

If the solution $\mathbf{u}$ is smooth and $D_{i,j}$ is a closed set with a piecewise smooth boundary, then we can apply the divergence theorem and obtain the differential form (1) from the integral form (7). Denote by

$$\bar{\mathbf{u}}_{i,j}(t) = \frac{1}{V(D_{i,j})} \int_{D_{i,j}} \mathbf{u}(\xi, \eta, t)\, d\eta\, d\xi \tag{9}$$

the cell-average, where $V(D_{i,j})$ is the Volume of $D_{i,j}$. The integral form (7) can be rewritten as

$$\frac{d}{dt}\bar{\mathbf{u}}_{i,j}(t) = -\frac{1}{V(D_{i,j})} \int_{\partial D_{i,j}} \vec{n}(\xi) \cdot\ \vec{F}(\mathbf{u}^R(\vec{x}(\xi), t+; \vec{n}(\xi)))\, d\xi \tag{10}$$

which says that we can evolve the cell averages in time, by solving one-dimensional Riemann problems at a the cell boundary at time $t = t^n$ and then solve the system of ordinary differential equation (10) to obtain the cell average at time $t = t^n + \tau$, $\tau > 0$. Taking for granted that the solution of the Riemann problem at the cell interface can be locally advanced in time; i.e. we require that

$$\frac{\partial}{\partial t}\mathbf{u}^R(\vec{x}(\xi), t; \vec{n}(\xi))|_{t=t^n+} \tag{11}$$

should exist where the boundary $\partial D_{i,j}$ is smooth.

In this paper we consider the stability of the plane-wave Riemann problem and proof that the plane-wave Riemann problem is unstable under small perturbations. We show that this instability is related to the failing of Godunov´s and Roe´s method reported by [QUI;1994],[ROE;2007] and give a mathematical analysis that explains the machinery, which leads to the so called carbuncle instability.

The results of this paper can be extended to general systems of hyperbolic conservation laws, which have at least one genuinely nonlinear characteristic field and two linear degenerate fields with a single eigenvalue. For clarity we restrict the presentation to the Euler equation for a γ-law gas (1) which serves as a model equation for more general systems.
The outline of this paper is as follows. In section 2 we analyse the stability of the plane-wave Riemann problem and prove that the solution can be unstable in a bounded region under small perturbation. In section 3 we show that this instability is related to the known numerical instabilities in Godunov´s and Roe´s method reported in [QUI;1994]. In section 4 we proof that the HLLE method [EIN;1988] gets around this instability. The last section contains our conclusion.

2 An Instability in the plane-wave Riemann Problem

For first order finite volume methods the assumption is generally made, that the solution is constant inside a cell at a time-level $t = t^n$; e.g.

$$\bar{\mathbf{u}}(\vec{x}, t) := \mathbf{u}_{i,j} \text{ for } \vec{x} \,\epsilon\, D_{i,j} \tag{12}$$

and that discontinuities are moved to the cell boundary. We assume for simplicity that $D_{i,j} = I_{i,j} = [x_{i-1/2}, x_{i+1/2}] \times [y_{i-1/2}, y_{i+1/2}]$ is given by a rectangle defined through a constant cartesian grid $x_i = i\Delta x$ and $y_j = j\Delta y$ and that the plane-wave solution is moving in the x-direction; i.e $\vec{n} = (1, 0)$. If the solution of the plane-wave Riemann problem at a cell-boundary $\vec{x}_{i+1/2,j} = (x_{i+1/2}, y_j)$ is given by a single discontinuity, then the initial states $\mathbf{u}_l = \mathbf{u}_{i,j}$ and $\mathbf{u}_r = \mathbf{u}_{i+1,j}$ of the Riemann problem satisfy the Rankine-Hugoniot jump condition [CF;1948].

$$\dot{s}[\mathbf{u}_r - \mathbf{u}_l] = f(\mathbf{u}_r) - f(\mathbf{u}_l) \tag{13}$$

where $\dot{s} = \dot{s}_{i+1/2,j}(t = t^n)$ is the shock speed. The normal velocity of the discontinuity can be computed form (13). However, through nonlinear interactions of the components u^k of $\mathbf{u} = (u^1, \cdots, u^4)$ any perturbation of $\mathbf{u}_l$ or $\mathbf{u}_r$ results in a non unique solution for the shock speed in (13).

Let us consider a near stationary discontinuity for the two dimensional Euler Equations. We assume that the states $\mathbf{u}_l = \mathbf{u}_{i,j}$ and $\mathbf{u}_r = \mathbf{u}_{i+1,j}$ are connected by a 1-shock wave with normal shock speed $-\varepsilon^s < \dot{s}^1_{i+1/2,j}(t^n) = \dot{s}^1_{i+1/2,j} < 0$. Let $\tilde{\mathbf{u}}_r$ be a small perturbation of the right state. The perturbed solution $\tilde{\mathbf{u}}$ has the initial data

$$\tilde{\mathbf{u}}(x, y, t^n) = \begin{cases} \mathbf{u}_l & \text{for } x < x_{i+1/2} \\ \tilde{\mathbf{u}}_r & \text{for } x > x_{i+1/2} \end{cases} \tag{14}$$

and consists of a 1-shock, a contact discontinuity and a 3-wave, which is a weak rarefaction or weak shock wave. In the following we neglect, without loss of generality, the 3-wave such that $\tilde{\mathbf{u}}_{mr} = \tilde{\mathbf{u}}_r$; see Fig. (1) and assume for notational simplicity that $i = 0$. A smooth function F exists with

$$\tilde{\mathbf{u}}_r = \mathbf{u}_l + F(\varepsilon_1, \varepsilon_2, \varepsilon_3; \mathbf{u}_l) \tag{15}$$

where ε_2 represents a parameter for the strength of the contact discontinuity and ε_3 is a parameter for the strength of a 3-wave; see [SM;1983 Chapter 17]. ε_1 represents a parameter for the strength of a 1-shock. We choose ε_1 such that

$$\mathbf{u}_r = \mathbf{u}_l + F(\varepsilon_1, 0, 0; \mathbf{u}_l) \tag{16}$$

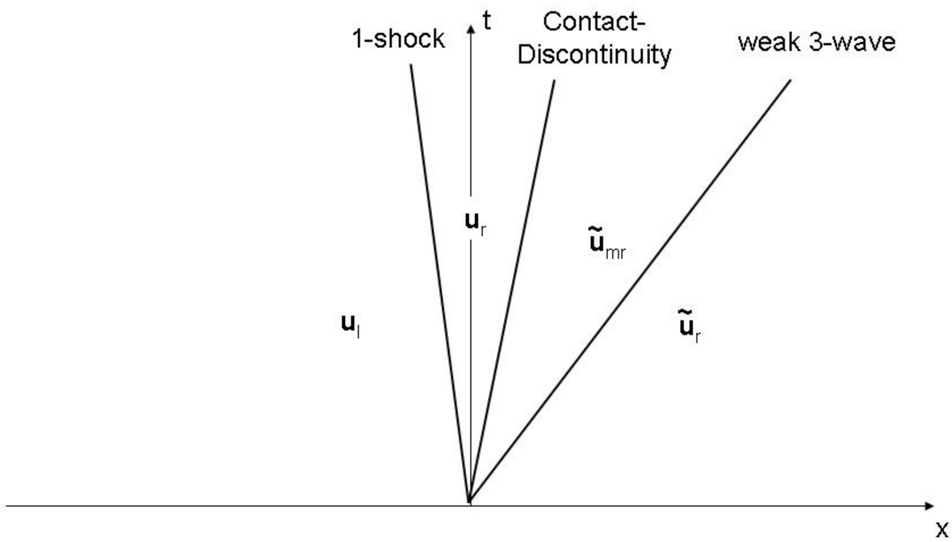

Figure 1: Riemann Problem for a perturbed shock wave

For a perturbation of ε_2 with $\varepsilon_3 = 0$ the pressure p and the x-component of the velocity u are constant behind the 1-shock. We obtain form the Lax shock conditions $\dot{s}^1_{1/2,j} < \lambda_2(\mathbf{u}_r)$ where $\lambda_2(\mathbf{u}_r)$ is the second eigenvalue of the Jacobian $A(\mathbf{u}_r)$. For a near stationary strong shock wave we can assume that $0 < \lambda_2(\mathbf{u}_r)$. Since the parameter ε_2 represents a contact discontinuity, the discontinuity speed $\dot{s}^2_{1/2,j}$ is equal to the characteristic speed $\lambda_2(\mathbf{u}_r)$, which is constant across a contact discontinuity; i.e. $\dot{s}^2_{1/2,j} = \lambda_2(\mathbf{u}_r) = \lambda_2(\tilde{\mathbf{u}}_r)$. Where $\tilde{\mathbf{u}}_r$ is given by

$$\tilde{\mathbf{u}}_r = \mathbf{u}_l + F(\varepsilon_1, \varepsilon_2, 0; \mathbf{u}_l) \tag{17}$$

The y-component of the velocity $v_l = v_{0,j}$ and $v_r = v_{1,j}$ enters the solution of the plane-wave Riemann problem essentially as a parameter. We can first solve the one dimensional Riemann problem ignoring the momentum equation for v and then introduce a jump in v at the contact discontinuity to obtain the full plane-wave solution. However, v_l and v_r enter the one dimensional Riemann problem through the pressure as a function of the total energy, density and velocity. We assume that $v_r = O(\varepsilon)$ and $v_l = O(\varepsilon)$. For a small perturbation $O(\varepsilon)$ the change in (3) is of order ε^2. Therefore we can assume that the wave structure for the perturbed plane-wave Riemann problem

$$\tilde{\mathbf{u}}(x, y, t^n) = \begin{cases} \mathbf{u}^\varepsilon_l(x_{1/2,j}-) & \text{for } x < x_{1/2,j} \\ \tilde{\mathbf{u}}^\varepsilon_r(x_{1/2,j}+) & \text{for } x_{1/2,j} < x \end{cases} \tag{18}$$

persists up to second order in ε, where $\mathbf{u}^\varepsilon_l(x) = (\rho_l, \rho_l u_l, \rho_l v^\varepsilon(x), E_l)^T$ and $\mathbf{u}^\varepsilon_r(x) = (\tilde{\rho}_r, \tilde{\rho}_r u_r, \tilde{\rho}_r v^\varepsilon(x), \tilde{E}_r)^T$.

Note: If $v^\varepsilon(x, t^n) = v^\varepsilon(x)$ is discontinuous at $x_{1/2,j}$ the perturbation introduces a jump in the y-component of the velocity at the contact discontinuity in the solutions. If $u_l = 0$ and $u_r = 0$ the perturbed plane-wave Riemann problem contains a tangential or shear instability. In [LL;§81] it is proved that such a tangential instability in an incompressible non viscous flow is absolutely unstable and may lead to a turbulent flow, and it is further mentioned that these instabilities exist also in compressible flows. However, we assume in the following that the perturbation $v^\varepsilon(x, t^n)$ is a smooth function.

We can consider the parameter ε_1, ε_2 in (17) as functions of y. We assume that the variation of ε_1 in y is small enough such that $\dot{s}^1_{1/2,j} < 0 < u_r$ still holds. This variation can be defined independently of the y-component of the velocity and we can assume that v is not affected through this perturbation.

The flow for this perturbed plane-wave Riemann problem is defined through the two-dimensional Euler equation (1). A change of the parameter ε_2 in (17) affects only the contact discontinuity. Since the pressure and the x-component of the velocity are constant across a contact discontinuity, we have behind the 1-shock $p(x,y,t) = p_r = \tilde{p}_r$ and $u(x,y,t) = u_r = \tilde{u}_r$ and we obtain with $v(x,y,t) = v(x,t)$ for the Euler equation :

$$\begin{aligned} &\frac{\partial}{\partial t}\rho + u_r \frac{\partial}{\partial x}\rho + v\frac{\partial}{\partial y}\rho = 0 \\ &\frac{\partial}{\partial t}v + u_r \frac{\partial}{\partial x}v = 0 \\ &\frac{\partial}{\partial t}E + u_r \frac{\partial}{\partial x}E + v\frac{\partial}{\partial y}E = 0 \end{aligned} \tag{19}$$

where the last equation follows from the equation of state (3) and from the first two. Therefore the perturbed solution satisfies behind the 1-shock the eqaution (19) with the initial data

$$\tilde{\mathbf{u}}(x,y,t^n+) = \begin{cases} \mathbf{u}_r^\epsilon(x,y) & \text{for } x < x_{1/2} \\ \tilde{\mathbf{u}}_r^\epsilon(x,y) & \text{for } x_{1/2} < x \end{cases} \tag{20a}$$

with

$$\begin{aligned} &\mathbf{u}_r^\epsilon(x,y) = (\rho(x,y), \rho(x,y)u_r, \rho(x,y)v^\varepsilon(x), E(x,y))^T \\ &\rho(x,y) = \rho_r(1+\rho_0^\varepsilon(y)) \end{aligned} \tag{20b}$$

and

$$\begin{aligned} &\tilde{\mathbf{u}}_r^\epsilon(x,y) = (\rho(x,y), \rho(x,y)u_r, \rho(x,y)v^\varepsilon(x), E(x,y))^T \\ &\rho(x,y) = \tilde{\rho}_r(1+\rho_0^\varepsilon(y)) \end{aligned} \tag{20c}$$

and

$$E(x,y) = p_r/(\gamma-1) + \frac{1}{2}\rho(x,y)(u_r^2 + v^\varepsilon(x)^2) \tag{20d}$$

where $v_0^\varepsilon(x)$ and $\rho_0^\varepsilon(y)$ are arbitrary smooth initial perturbations.

Proposition: The solution of the initial vale problem (19), (20a) is given by:

$$\rho(x,y,t) = \rho^x(x,t)\rho^y(y,t) = \rho_0^x(x-u_rt)\rho_0^y(y-v^\varepsilon t) \tag{21a}$$

with

$$v^\varepsilon(x,t) = v_0^\varepsilon(x-x_{1/2}-u_rt)$$

and

$$\rho_0^x(x) = \begin{cases} \rho_r & \text{for } x < x_{1/2} \\ \tilde{\rho}_r & \text{for } x_{1/2} < x \end{cases} \tag{21b}$$

and

$$\rho_0^y(y) = 1 + \rho_0^\varepsilon(y) \tag{21c}$$

Proof: The solution of the plane-wave Riemann problem has behind the 1-shock, i.e. for $x > s_{1/2}^1 t + x_{1/2}$, a constant pressure $p_r = \tilde{p}_r$ and constant a x-component of the velocity $u_r = \tilde{u}_r$. Therefore the Euler equation reduce to (19) and the solution is completely defined through the density ρ and the y-component of the velocity v and the constant pressure and the x-component of the velocity. We have

$$\begin{aligned} &\frac{\partial}{\partial t}\rho + u_r\frac{\partial}{\partial x}\rho + v^\varepsilon\frac{\partial}{\partial y}\rho \\ &= \rho^y[\frac{\partial}{\partial t}\rho^x + u_r\frac{\partial}{\partial x}\rho^x + v^\varepsilon\frac{\partial}{\partial y}\rho^x] + \rho^x[\frac{\partial}{\partial t}\rho^y + u_r\frac{\partial}{\partial x}\rho^y + v^\varepsilon\frac{\partial}{\partial y}\rho^y] \end{aligned}$$

Furthermore

$$\begin{aligned} \frac{\partial}{\partial t}\rho^x &= -u_r\frac{d}{d\xi}\rho_0^x \\ \frac{\partial}{\partial x}\rho^x &= \frac{d}{d\xi}\rho_0^x \\ \frac{\partial}{\partial y}\rho^x &= 0 \end{aligned} \tag{22}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}\rho^y &= -(\frac{\partial}{\partial t}v^\varepsilon t + v^\varepsilon)\frac{d}{d\xi}\rho_0^y \\ \frac{\partial}{\partial x}\rho^y &= -\frac{\partial}{\partial x}v^\varepsilon t\frac{d}{d\xi}\rho_0^y \\ \frac{\partial}{\partial y}\rho^y &= \frac{d}{d\xi}\rho_0^y \end{aligned} \tag{23}$$

Thus we obtain

$$\frac{\partial}{\partial t}\rho^x + u_r\frac{\partial}{\partial x}\rho^x + v^\varepsilon\frac{\partial}{\partial y}\rho^x = -u_r\frac{d}{d\xi}\rho_0^x + u_r\frac{d}{d\xi}\rho_0^x = 0$$

and

$$\begin{aligned}
&\frac{\partial}{\partial t}\rho^y + u_r\frac{\partial}{\partial x}\rho^y + v^\varepsilon\frac{\partial}{\partial y}\rho^y \\
&= -(\frac{\partial}{\partial t}v^\varepsilon t + v^\varepsilon)\frac{d}{d\xi}\rho_0^y - u_r\frac{\partial}{\partial x}v^\varepsilon t\frac{d}{d\xi}\rho_0^y + v^\varepsilon\frac{d}{d\xi}\rho_0^y \\
&= -t\frac{d}{d\xi}\rho_0^y[\frac{\partial}{\partial t}v^\varepsilon + u_r\frac{\partial}{\partial x}v^\varepsilon] = 0
\end{aligned}$$

Therefore (21a) and the constant x-component $u = u_r = \tilde{u}_r$ and constant pressure $p = p_r = \tilde{p}_r$ define a solution of the Euler equation (1) in smooth parts of the solution. The Jump Condition for a plane-wave moving in the x-direction reduce for constant pressure and constant x-component of the velocity to

$$\begin{aligned}
&\dot{s}(t)[\rho(x+,y,t) - \rho(x-,y,t)] = u_r[\rho(x+,y,t) - \rho(x-,y,t)] \\
&\dot{s}(t)u_r[\rho(x+,y,t) - \rho(x-,y,t)] = (u_r)^2[\rho(x+,y,t) - \rho_r(x-,y,t)] \\
&\dot{s}(t)[\rho(x+,y,t)v^\varepsilon(x+,t) - \rho(x-,y,t)v^\varepsilon(x-,t)] = \\
&u_r[\rho(x+,y,t)v^\varepsilon(x+,t) - \rho(x-,y,t)v^\varepsilon(x-,t)] \\
&\dot{s}(t)[E(x+,y,t) - E(x-,y,t)] = u_r[E(x+,y,t) - E(x-,y,t)]
\end{aligned}$$

where $\dot{s}$ is the speed of the discontinuity in the x-direction. Therefore we see that the jump conditions are satisfied along the curve $x = s_{1/2}^2 t + x_{1/2} = u_r t + x_{1/2}$, and (21a) is a weak solution of (1). For $x = u_r t + x_{1/2}$ we have

$$\begin{aligned}
&v^\varepsilon(x+,t) = v_0^\varepsilon(0) \\
&\rho(x+,y,t) = \rho_0^x(x_{1/2}+)\rho_0^y(y - v_0^\varepsilon(0)t) = \tilde{\rho}_r\rho_0^y(y - v_0^\varepsilon(0)t) \\
&\text{and} \\
&v^\varepsilon(x-,t)t = v_0^\varepsilon(0) \\
&\rho(x-,y,t) = \rho_0^x(x_{1/2}-)\rho_0^y(y - v_0^\varepsilon(0)t) = \rho_r\rho_0^y(y - v_0^\varepsilon(0)t)
\end{aligned} \tag{24}$$

and we see that the initial conditions are satisfied. This completes the proof. □

For a small positive constant $\delta < \varepsilon$ we define

$$v_0^\varepsilon(x) = \delta\sin(x\frac{\pi}{\delta^2}) \tag{25}$$

Then we obtain for a point with $x - u_r t = x_{1/2}$ on the contact-shear discontinuity

$$\begin{aligned}
\lim_{\epsilon\to 0+} \rho(x_{1/2} + u_r t + \epsilon, y, t) = \\
\lim_{\epsilon\to 0+} \rho_0^x(x_{1/2} + \epsilon)\rho_0^y(y - \delta t \sin(\varepsilon \frac{\pi}{\delta^2})) = \\
\tilde{\rho}_r \lim_{\epsilon\to 0} \rho_0^y(y - \delta t \sin(\varepsilon \frac{\pi}{\delta^2})) = \\
\tilde{\rho}_r \lim_{\epsilon\to 0} \rho_0^y(y - \delta(x - x_{1/2})/u_r \sin(\varepsilon \frac{\pi}{\delta^2})) = \tilde{\rho}_r(1 + \rho_0^\varepsilon(y))
\end{aligned} \tag{26}$$

as we would expect. However, for the time derivative of ρ we get

$$\frac{\partial}{\partial t}\rho = \frac{\partial}{\partial t}[\rho_0^x(x - u_r t)\rho_0^y(y - v^\varepsilon t)] \tag{27}$$

$$= \rho^x \frac{\partial}{\partial t}\rho^y + \rho^y \frac{\partial}{\partial t}\rho^x$$

$$= -\rho^x(v^\varepsilon + t\frac{\partial}{\partial t}v^\varepsilon)\frac{d}{d\xi}\rho_0^y - u_r\rho^y \frac{d}{d\xi}\rho_0^x \tag{28}$$

and we obtain with

$$\begin{aligned}
\frac{\partial}{\partial t}v^\varepsilon &= \frac{\partial}{\partial t}v_0^\varepsilon(x - x_{1/2} - u_r t) \\
&= \frac{\partial}{\partial t}[\delta \sin((x - x_{1/2} - u_r t)\frac{\pi}{\delta^2})] \\
&= -u_r\frac{\pi}{\delta^2}[\delta \cos((x - x_{1/2} - u_r t)\frac{\pi}{\delta^2})] \\
&= -u_r\frac{\pi}{\delta}\cos((x - x_{1/2} - u_r t)\frac{\pi}{\delta^2})
\end{aligned} \tag{29}$$

for a point with $x - u_r t = x_{1/2}$ on the contact discontinuity

$$\begin{aligned}
&\lim_{\epsilon\to 0+} \frac{\partial}{\partial t}\rho(x_{1/2} + u_r t + \varepsilon, y, t) \\
&= \lim_{\epsilon\to 0+}[-\rho_0^x(x_{1/2} + \varepsilon)v_0^\varepsilon(\varepsilon) + \rho_0^x(x_{1/2} + \varepsilon)tu_r\frac{\pi}{\delta}\cos(\varepsilon\frac{\pi}{\delta^2})]\frac{d}{d\xi}\rho_0^y - \lim_{\epsilon\to 0+}[u_r\rho^y\frac{d}{d\xi}\rho_0^x] \\
&= \lim_{\epsilon\to 0+} \tilde{\rho}_r t u_r \frac{\pi}{\delta}\frac{d}{d\xi}\rho_0^y \\
&= \lim_{\epsilon\to 0+} \tilde{\rho}_r t u_r \frac{\pi}{\delta}\frac{d}{d\xi}\rho_0^\varepsilon \\
&= \tilde{\rho}_r t u_r \frac{\pi}{\delta}\frac{d}{d\xi}\rho_0^\varepsilon \mid_{\xi=y}
\end{aligned} \tag{30}$$

Let $t > 0$ and (x, y) a point on the discontinuity with $\frac{d}{dy}\rho_0^\varepsilon(y) \neq 0$, then the last term grows to infinity if $\lim \delta \to 0+$ in (25). Therefore the one sided limit for

$\frac{\partial}{\partial t}\rho(x_{1/2} + u_r t + \varepsilon, y, t)$ can increase for arbitrary small perturbation of the form (25). Since we obtain the same result for any δ^n in (25), where $n > 2$ is an arbitrary integer (30) grows exponential. The two-dimensional contact discontinuity is unstable for $t > 0$ and $\frac{d}{dy}\rho_0^\varepsilon(y_0) \neq 0$. We obtained the following

Theorem: If the solution of the plane-wave Riemann problem (14) contains a contact discontinuity, then the solution is not stable under small perturbations and (11) can grow exponentially, if the instability coincides with the cell boundary for t > 0.

Note: To derive the Rankine-Hugoniot Jump Condition (13) form the integral form (7) of the conservation law, we must assume that the integral

$$\int_{x_{i-1/2}}^{x_{i+1/2}} u_t(\xi, y, t^n)\, d\xi \tag{31}$$

approaches zero for $\lim \Delta \to 0$. This assumption fails for the perturbed two dimensional plane-wave Riemann problem.

As Courant and Friedrichs [CF;1948] mentioned: "It is obvious that in reality such a contact surface cannot be maintained for an appreciable length of time; heat conduction between the permanently adjacent particles on either side of the discontinuity would soon make our idealized assumption unrealistic. While gas particles crossing a shock front are exposed to heat conduction for only a very short time, those that remain adjacent on either side of a contact surface are exposed to heat conduction all the time. Hence it is clear that a contact layer will gradually fade out."

In a Finite Volume method discontinuities are moved to the cell boundary. This small displacement of a shock is accompanied by infinitesimal disturbances of the dependent variable (pressure, velocity, etc.). These disturbances originate near the shock wave and spread with sound speed relative to the gas. This does not hold for the entropy disturbances which are carried with fluid away form the shock; see [LL;1959]. We have proved in this section that infinitesimal density changes (which are caused by the entropy disturbance behind a shock), can lead to an exponential growth of the time-derivative of the density at the two-dimensional contact discontinuity. If this 2d-contact instability enters the numerical method through the solution of the Riemann problem at the cell-boundary, then the absence of sufficient numerical viscosity can lead to a failing of the numerical method. This will be shown for Godunov´s and Roe´s method in the next secton.

3 Instability of Godunov´s method

We have seen in the last section that the solution of the plane-wave Riemann problem is unstable under perturbations at the contact discontinuity. If this instability enters a numerical scheme through the use of the plane-wave Riemann problem in the numerical flux function at the cell boundary, the numerical method itself can fail.
In Godunov´s method a plane-wave Riemann problem is solved at the cell boundary for example at the cell-boundary $(x_{i+1/2}, y_j)$ at time $t = t^n$ in the x-direction. Denote by $u^R(x, y, t; (1, 0))$ the solution of the plane-wave Riemann problen in the x-direction at the cell boundary.
Then $u^n_{i+1/2,j} := \lim_{t \to t^n+} u^R(x_{i+1/2}, y_j, t; (1, 0))$ is computed and used to evaluate the physical flux function. The following conditions must be met if the instability at the contact discontinuity can affect Godunov´s method:

(i)Perturbation must be present in the approximate solution.

(ii)There is not enough numerical dissipation to suppress perturbations.

(iii)The physical signal speeds satisfy $\dot{b}^l_{i+1/2,j} < 0 < \dot{b}^r_{i+1/2,j}$

The smallest numerical signal speed $\dot{b}^l_{i+1/2,j}$ in the solution of the plane-wave Riemann problem at the cell boundary $(x_{i+1/2}, y_j)$ is either the 1-shock speed $s^1_{i+1/2,j}$ or the characteristic speed $u_{i,j} - c_{i,j}$, where $u_{i,j}$ is the x-component of the velocity and $c_{i,j}$ the sound speed of the state $\mathbf{u}_{i,j}$. The largest numerical signal speed $\dot{b}^r_{i+1/2,j}$ is either the 3-shock speed $s^3_{i+1/2,j}$ or the characteristic speed $u_{i+1,j} + c_{i+1,j}$. If the solution satisfies

$$u^n_{i+1/2,j} = \begin{cases} u_{i,j} & \text{for } 0 < \dot{b}^l_{i+1/2,j} \\ u_{i+1,j} & \text{for } \dot{b}^r_{i+1/2,j} < 0 \end{cases} \tag{32}$$

then Godunov´s scheme reduces to an upwind scheme and the internal wave structure of the Riemann problem is ignored. The instability at the contact discontinuity does not enter the numerical solution.

If the first and third nonlinear waves are weak waves, no significant noise is generated in Godunov´s method. For a strong rarefaction wave there is a significant amount of dissipation in Godunov´s method. Therefore, if we are looking for numerical solutions which contain perturbations, a strong 1-shock or 3-shock should be present in the numerical solution. It is well known that numerical noise is generated in a moving shock wave. The source of small perturbation is the displacement of the shock curve at the cell-boundary and the nonlinear interaction of the dependent variables in the numerical shock layer. A non stationary or stationary displaced shock wave is approximated through a smeared profile with at least one intermediate cell. Whenever the smeared shock profile changes, small (acoustic and entropy) perturbations are generated from the characteristic fields . If the shock curve is not exactly a plane-wave in the x-direction the shock-curvature will introduce an additional perturbation which depends on y. This is the situation discussed in the previous section for a plane-wave 1-shock. Small perturbations from the smeared shock profile generate a contact discontinuity downstream of the shock. The curvature introduces a y-dependence of the density shock-perturbations, while the pressure and the x-component of the velocity at the contact discontinuity remain constant. If condition (iii) holds for the signal speeds, then a two-dimensional contact instability can enter Godunov´s scheme through the evaluation of the flux-function at the cell-boundary. This instability at the two-dimensional contact discontinuity can lead to large disturbances of the density, which are transported into the solution; see the Theorem in the previous section.

Based on numerical results, Abouziarov et. al. [AAT; 2001] found that the carbuncle phenomenon is related to an amplification of an error in the conserved variable ρv, in regions where the density has a discontinuity or very strong gradient.

We have for the perturbation discussed in the last section (with δ^2 replaced by δ^n, with $n > 2$) for a point $x - u_r t = x_{1/2}$ on the contact-shear discontinuity

$$\begin{aligned}
&\lim_{\epsilon \to 0+} \frac{\partial}{\partial t}[\rho(x_{1/2} + u_r t + \epsilon, y, t) v(x_{1/2} + u_r t + \epsilon, t)] \qquad (33)\\
&= \lim_{\epsilon \to 0+} v(x_{1/2} + u_r t + \epsilon, t) \lim_{\epsilon \to 0+} \frac{\partial}{\partial t}\rho(x_{1/2} + u_r t + \epsilon, y, t)\\
&+ \lim_{\epsilon \to 0+} \rho(x_{1/2} + u_r t + \epsilon, y, t) \lim_{\epsilon \to 0+} \frac{\partial}{\partial t} v(x_{1/2} + u_r t + \epsilon, t)\\
&= \lim_{\epsilon \to 0+} v_0^{\varepsilon}(\varepsilon)\tilde{\rho}_r t u_r \frac{\pi}{\delta^{n-1}} \frac{d}{d\xi}\rho_0^{\varepsilon} \,|_{\xi=y} - \tilde{\rho}_r(1 + \rho_0^{\varepsilon}(y)) u_r \frac{\pi}{\delta^{n-1}}\\
&= -\tilde{\rho}_r(1 + \rho_0^{\varepsilon}(y)) u_r \frac{\pi}{\delta^{n-1}}\\
&\simeq -\tilde{\rho}_r u_r \frac{\pi}{\delta^{n-1}}
\end{aligned}$$

Therefore a small density perturbation from the 1-shock can generate an exponential growth of the conserved variable ρv at the contact discontinuity, in agreement with the findings in [AAT; 2001].

Based on the numerical observation, that shock capturing methods which try to capture contact discontinuities exactly, generally suffer from failings, a link between the carbuncle phenomenon and the resolution of the contact discontinuities was suggested by [GRE; 1998].

In [ROE; 2007], one-dimensional oscillations at a shock are denoted as a carbuncle instability. This 1D numerical "instability" has a different character then the carbuncle phenomenon reported by [Quirk; 1994] and will not be denoted as a carbuncle instability in this paper; see also [EIN3;2008] for a discussion of 1D post-shock oscillations.
Numerical examples for a plane shock wave aligned with the grid, which is moving down a duct form left to right is also given in [QUI 1994; Figure 5]. At the grid center line, a small perturbation is introduced in the computation. Downstream of the shock an unstable density profile develops, which over time leads to an unstable numerical shock front. If we associate the center of the duct with the x-axis, then the perturbation introduced in the computation depends at the shock front on y. This numerical example reflects the situation discussed analytically in the previous section; the perturbation from the shock leads to an unstable growth of the density at the 2d contact discontinuity. In [Roe; 2007]results for a similar problem, denoted as 1 1/2 dimensional, are reported. It is claimed that 1d "stability" is important for 1 1/2d stability. In agreement with the results in this paper, since if a numerical scheme generates noise at the shock, the instability at the 2d-contact discontinuity is activated, whereas if very little or no noise is generated at the shock the dissipation of the method may be sufficient to suppress the unstable growth of the density at the 2d-contact discontinuity.

Roe´s method [Roe; 1980] is more vulnerable to the 2d-contact instability then Godunov´s method, due to the fact that a rarefaction wave is replaced by a rarefaction shock. Therefore noise form the rarefaction shock, can also lead to an unstable growth of the density at the contact discontinuity, in addition to the noise from the shock.

An example from an aerodynamic simulation, which results in incorrect numerical results is given in [Peery; 1988], for a bow shock over a blunt body placed in a high Mach number flow. Along the stagnation line the bow shock is approximately aligned with grid used for the calculation. A perturbation normal to shock is given through the curvature of the shock. At the stagnation point we have approximately a plane-wave near stationary shock wave, with a disturbance normal to the shock. This again is the situation discussed in the previous section.

Since small disturbance propagate along characteristics, the instability in the plane-wave Riemann problem at the two-dimensional contact discontinuity can spread into the solution not faster then the smallest and largest characteristic speed; i.e. we can assume that the unstable flow region is restricted to an area $\lambda^1(\mathbf{u}_r)t = (u_r - c_r)t < x$ and $x < \lambda^3(\tilde{\mathbf{u}}_r)t = (\tilde{u}_r + \tilde{c}_r)t$ where c_r and $\tilde{c}_r$ are the sound speeds to left and right of the contact discontinuity. From the Lax shock conditions we have $u_r - c_r < s^1(\mathbf{u}_r, \mathbf{u}_l)$ and the unstable flow can overtake the 1-shock. Then the assumption of a constant 1-shock speed in the Riemann problem is no longer valid - this explains the implications of the contact-shear instability on the shock itself, which are visible in the numerical results [Roe 2007; Fig.5] and [Quirk 1994; Fig 5].

4 HLLE Methods

The instability of the two-dimensional contact discontinuity can lead in a very short time to an extreme irregular and disorderly change of the density. If values from this unstable region enter the numerical method, nonphysical numerical artifacts can occur in the solution and, if the flow structure is not known a priori, the numerical solution may be useless. In this section we review a method which circumvents this problem.

We can assume, that for the plane Riemann problem in the x-direction curves $b^l_{l,r}(y,t)$ and $b^r_{l,r}(y,t)$ exists such that

$$\mathbf{u}(x,y,t) = \begin{cases} \mathbf{u}_l & \text{for } x < b^l_{l,r}(y,t) \\ \mathbf{u}_r & \text{for } b^r_{l,r}(y,t) < x \end{cases} \tag{34}$$

where we assumed that $b^l_{l,r}(y,0) = x_{i+1/2.j} = b^l_{l,r}(y,0)$. We denote by $b^l_{l,r}(y,0)$ and $b^r_{l,r}(y,0)$ the smallest and largest signal curves at the cell interface for a plane-wave Riemann problem in the x-direction and assume for notational simplicity that $t^n = 0$.

Given the largest and smallest signal curves we can define the average

$$\bar{\mathbf{u}}_{l,r}(y,t) := \frac{1}{b^r_{l,r}(y,t) - b^l_{l,r}(y,t)} \int_{b^l_{l,r}(y,t)}^{b^r_{l,r}(y,t)} u(\xi,y,t)\ d\xi \tag{35}$$

From the integral form of the conservation law (7) we obtain with (34) for $b^l_{l,r}(y,t) < 0 < b^r_{l,r}(y,t)$:

$$\begin{aligned} \bar{\mathbf{u}}_{l,r}(y,t) = & \frac{b^r_{l,r}(y,t) - x_{i+1/2,j}}{b^r_{l,r}(y,t) - b^l_{l,r}(y,t)}\mathbf{u}_r - \frac{b^l_{l,r}(y,t) - x_{i+1/2,j}}{b^r_{l,r}(y,t) - b^r_{l,r}(y,t)}\mathbf{u}_l \\ & - \frac{t}{b^r_{l,r}(y,t) - b^l_{l,r}(y,t)}[f(\mathbf{u}_r) - f(\mathbf{u}_l)] \end{aligned} \tag{36}$$

Assuming that $b^l_{l,r}(y,t)$ and $b^r_{l,r}(y,t)$ has onesided derivatives for $t = 0$; i.e.

$$\begin{aligned} \frac{\partial}{\partial t} b^l_{l,r}(y,t)|_{t=0+} &:= \dot{b}^l_{l,r}(y) \\ \frac{\partial}{\partial t} b^r_{l,r}(y,t)|_{t=0+} &:= \dot{b}^r_{l,r}(y) \end{aligned} \tag{37}$$

we obtain

$$\begin{aligned} b^l_{l,r}(t,y) &= b^l_{l,r}(0,y) + \dot{b}^l_{l,r}(y)t + O_2(\Delta t) \\ &= x_{i+1/2} + \dot{b}^l_{l,r}(y)t + O_2(\Delta t) \\ b^r_{l,r}(t,y) &= b^r_{l,r}(0,y) + \dot{b}^r_{l,r}(y)t + O_2(\Delta t) \\ &= x_{i+1/2} + \dot{b}^r_{l,r}(y)t + O_2(\Delta t) \end{aligned} \tag{38}$$

and

$$\begin{aligned} \bar{\mathbf{u}}_{l,r}(y,t) = &\frac{\dot{b}^r_{l,r}(y)}{\dot{b}^r_{l,r}(y) - \dot{b}^l_{l,r}(y)}\mathbf{u}_r - \frac{\dot{b}^l_{l,r}(y)}{\dot{b}^r_{l,r}(y) - \dot{b}^r_{l,r}(y)}\mathbf{u}_l \\ &- \frac{1}{\dot{b}^r_{l,r}(y) - \dot{b}^l_{l,r}(y)}[f(\mathbf{u}_r) - f(\mathbf{u}_l)] + O_2(\Delta t) \end{aligned} \tag{39}$$

follows.
Therefore the average (39) is well defined for small times, although the solution between the smallest and largest signal curves may be unstable at the two-dimensional contact discontinuity. This result is also reasonable from a physical point of view, then an unstable solution should fluctuate around a mean value.
An approximate solution to the Riemann problem which is only based on the largest and smallest signal speeds in the plane-wave Riemann problem and the corresponding average value (39), results in a HLLE method [EIN 1988]. This explains why the HLLE methods are not affected by the carbuncle instability.

5 Conclusion

The instability of the two-dimensional contact discontinuity in the plane wave Riemann problem under perturbations requires a rethinking concerning the use of the internal wave structure from approximate or exact solutions of plane wave Riemann problems in Finite Volume methods.
Immanent numerical noise and the instability of the two-dimensional contact discontinuity are fundamental aspects of shock capturing methods for the Euler equations, which must be considered in a numerical stability and convergence analysis. The class of HLLE schemes can address these aspects.

6 References

[LEV;2002] R. J. LeVeque, Finite Volume Methods for Hyperbolic Problems, Cambridge University Press, Cambridge, 2002.
[QUI;1994] J.J. Quirk, A Contribution to the Great Riemann Solver Debate, International Journal For Numerical Methods in Fluids, Vol. 18, 555-574 (1994)
[EIN;1988] B. Einfeldt, On Godunov-type methods for gas dynamics, SIAM J. Numer. Anal., 25, 294-318 (1988).
[CF;1948] R. Courant and K. 0. Friedrichs, Supersonic Flow and Shock Waves, Interscience, New York, 1948
[LL;1959] L. D. Landau and E. M Lifchitz, E. M. 1959 Fluid Mechanics Addison-Wesley Publishing Co..
[AAT 2001], M. Abouziarov, H. Aiso and T. Takahashi, Analysis for Numerical Inconvenience which happens in Numerical Computation for Conservation Laws, ICRACM 2001.
[ROE;2007] P. L. Roe, "An Evaluation of Euler Fluxes for Hypersonic Flow Computations", in AIAA Computational Fluid Dynamics Conference, Miami, FL, Jun. 25-28, 2007
[SM;1983] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer, 1983.
[HLL;1983] A. Harten P.D. Lax and B. van Leer, On upstream differencing and Godunov-type schemes for hyperbolic conservation laws, SIAM Rev. 25 (1983), 35-61.
[EIN;1991] B. Einfeldt, C. D. Munz, P. L. Roe and B. Sjogreen, "On Godunov-type methods near low densities", J. Comput. Phys., 92, 273-295 (1991)]
[GRE;1998]J. Gressier and J.-M. Moschetta "Robustness versus Accuracy in Shock-Wave Computations", International Journal of Numerical Methods in Fluids, July 1999
[EIN3;2008] B. Einfeldt et. al. "On Dissipation and Anti-Diffusion in the HLLE schemes"; to be published

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