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United Economy after Unification of Areas with Neoclassic Economies

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Summary:
We consider N areas ($N > 1$) the economies of which will mathematically be modelled by the Solow-Swan model with neoclassic production functions and which will be *equal* with respect to exogenous savings rates and depriciation rates. We show that the unification of these areas leads to a united economy which is also neoclassic. This suggests itself but will now be clarified with respect to suitable assumptions and will then be confirmed by a simple mathematical proof. Moreover, if the production functions are approximated by Cobb-Douglas functions, the parameters of the united economy can explicitly be computed from the parameters of the single economies involved, and this also holds for the steady state solution of the joint economy. In other situations, numerical methods concerning the computation of steady state solutions will be briefly discussed.

1 The Model

Assume $F_i(K_i, L_i)$ $(i = 1, \cdots, N)$ to be the production functions of the economies of N $(N > 1)$ areas depending on the physical capitals K_i in the i-th area and the labors L_i proportional to the number of inhabitants, with

$$0 < K_i < \infty \ , \ \ 0 < L_i < \infty \qquad (i = 1, \cdots, N) \ . \tag{1}$$

Let the *Capital Accumulation Equations* of the single economies be written as

$$\frac{\partial K_i}{\partial t} = s_i \ F_i(K_i, L_i) \ - \ \delta_i \, K_i \quad , \quad (i = 1, \cdots, N) \ , \ t : \text{time} \tag{2}$$

with the exogenous *savings rates* s_i $\quad (0 \leq s_i \leq 1) \quad$ of the particular regions and the exogenous *depriciation rates* δ_i $\quad (i = 1, \cdots, N)$ of the capitals. Hence, corresponding to the simplest form of the Solow-Swan Model [1], we assume the savings rates to be constant, particularly to be independent of the *production factors* capital and labor.

Assume that the depriciation rates do not differ very much from eachother in the areas involved so that

$$\delta_i = \delta \qquad (i = 1, \cdots, N) \tag{3}$$

[1] It does not lead to real difficulties if one tries to include also knowledges T_i or other more realistic models.

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holds approximately with a common rate δ .

Assume the physical capital K, the labor L and the output F of the united economy to be given by

$$K = \sum_{i=1}^{N} K_i \quad , \tag{4}$$

$$L = \sum_{i=1}^{N} L_i \quad , \tag{5}$$

$$F = \sum_{i=1}^{N} F_i \quad , \tag{6}$$

respectively.

In here, F is a function of the $2N$ variables K_i , L_i $(i = 1, \cdots, N)$, and we expect that for a certain time after the unification the structures of the outputs F_i do not extremely change.

(4), (3), (2) lead to

$$\frac{\partial K}{\partial t} = \sum_{i=1}^{N} s_i \, F_i(K_i, L_i) \; - \; \delta K \; . \tag{7}$$

We expect that all the economies of the different areas which are parts of the union are neoclassic in the sense of the Solow-Swan Model [1] , [2] [2].

Thus, particularly, the production functions exhibit constant returns to scale [3], i.e.

$$F_i(\nu K_i \, , \nu L_i) = \nu F_i(K_i \, , L_i) \quad , \quad \forall \, \nu > 0 \; , \; (i = 1, \cdots, N), \tag{8}$$

which leads to

$$F_i(K_i \, , L_i) = L_i \, F_i \left(\frac{K_i}{L_i} \, , 1 \right) = L_i \, f_i(k_i) \quad (i = 1, \cdots, N). \tag{9}$$

$k_i = \frac{K_i}{L_i}$ is the capital per capita in the i-th region and $f_i(k_i)$ the so-called *intensive form* of the production function F_i.

After the unification the per capita capital does not depend on the regions and we assume the process of adaptation of the economies to eachother to lead forthwith to identical ratios between capital and labor:

$$k_i = k = \frac{K}{L} \quad , \quad \forall \, i = 1, \cdots, N \; , \tag{10}$$

[2]This implies, that capital and labor are *essential*, so that $K_i = 0$ or $L_i = 0$ can not occur if $F_i > 0$ is expected.
[3]in mathematical language: the functions are homogeneous of degree 1 with respect to capital and labor.

hence

$$K_i = k\,L_i \quad (i = 1, \cdots, N) \tag{11}$$

and

$$\frac{\partial k}{\partial t} = \sum_{i=1}^{N} m_i\, s_i\, f_i(k) \;-\; (n+\delta)k\ . \tag{12}$$

In here,

$$n = \frac{1}{L}\frac{\partial L}{\partial t}$$

represents the growth rate of the population of the joint region and

$$m_i = \frac{L_i}{L}$$

measures the share of the population of i-th area compared with the united population.

Obviously (cf. (5)),

$$\sum_{i=1}^{N} m_i = 1 \tag{13}$$

and

$$n = \sum_{i=1}^{N} m_i n_i \tag{14}$$

with the population growth rate

$$n_i = \frac{1}{L_i}\frac{\partial L_i}{\partial t} \quad (i = 1, \cdots, N)$$

on the i-th area.

(12) represents the so-called *fundamental equation* of the united economies.

We had already expected that the single economies are *equal* with respect to the depreciation rates, and it makes sense to expect that this also holds for the savings rates after the unification of the areas.

Thus,

$$s_i = s \quad (i = 1, \cdots, N) \quad , \tag{15}$$

and because of (6) and (11)

$$\sum_{i=1}^{N} s_i m_i f_i(k) = s \sum_{i=1}^{N} m_i f_i(k) = \frac{s}{L} \sum_{i=1}^{N} L_i f_i(k) = \frac{s}{L} \sum_{i=1}^{N} L_i \, F_i(k,1) = \frac{s}{L} \sum_{i=1}^{N} F_i(K_i, L_i) = \frac{s}{L} F \ . \quad (16)$$

We define

$$\sum_{i=1}^{N} m_i f_i(k) = f(k) \quad (17)$$

so that $\frac{F}{L}$ depends only on k, and with (12) we find the fundamental equation of the joint economy now in the usual form

$$\frac{\partial k}{\partial t} = s f(k) - (n + \delta) k \quad (18)$$

2 The Neoclassic Behaviour of the United Economy

Theorem: *The production function of the union of single neoclassic economies which are equal with respect to the exogenous savings- and depriciation rates, and which adapt to eachother with respect to the ratios between capital and labor, is also neoclassic.*

Proof:

Because of (10), (16) and (17), $\frac{F}{L} = f(k)$ depends only on k, hence $F = L f(k) = L f\left(\frac{K}{L}\right)$ depends only on K and L:

$$F = \tilde{F}(K, L)$$

with

$$\tilde{F}(\nu K, \nu L) = \nu L f\left(\frac{\nu K}{\nu L}\right) = \nu \tilde{F}(K, L)$$

so that F exhibits constant returns to scale.

And it is trivial to show that F exhibits positive and marginal products with respect to inputs, i.e. to show $f'(k) > 0 \quad , \quad f''(k) < 0$, and also the *Inada conditions* [3]

$$\lim_{k \to 0} f'(k) = \infty \quad , \quad \lim_{k \to \infty} f'(k) = 0 \quad (19)$$

are fulfilled, namely because all these properties hold for the single functions $f_i \quad (i = 1, \cdots, N)$, because the m_i are positive and because of (17).

q.e.d.

Therefore, the single economies as well as the united economy have unique steady state solutions $k_i^* \quad (i = 1, \cdots, N)$ and k^*, respectively.

3 Computation of the Steady State Solution of the United Economy

The steady state solution of (18) results from $\frac{\partial k}{\partial t} = 0$, i.e. from the nonlinear equation

$$k^* = \omega f(k^*) \quad \text{with} \quad \omega = \frac{s}{n+\delta} \quad , \tag{20}$$

and in the same way −because of (10)−

$$k_i^* = \omega_i f_i(k_i^*) \quad \text{with} \quad \omega_i = \frac{s}{n_i+\delta} \ \ (i = 1, \cdots, N). \tag{21}$$

Because the steady state solutions are the limits of $k(t)$ and $k_i(t)$, respectively, for $t \to \infty$, we find with (17) and (23) and by the assumption that the intensive production functions are continuous,

$$k^* = \omega \lim_{t\to\infty} f(k(t)) = \omega \lim_{t\to\infty} \sum_{i=1}^{N} m_i f_i(k_i(t)) = \omega \sum_{i=1}^{N} m_i f_i(k^*) \tag{22}$$

so that

$$k^* = \omega \sum_{i=1}^{N} \frac{m_i}{\omega_i} k_i^* \quad . \tag{23}$$

Normally, a nonlinear equation like (10) or (22) can not be solved explicitly, even if one knows −as in our situation− that there is a unique solution. But there are numerical procedures like the Newton method (see e.g. [4] , p. 216) to solve the problems approximately by iteration, in case of (20) by

$$k^{*[\nu+1]} = \omega f(k^{*[\nu]}) \frac{1 - \mathrm{Sh}(k^{*[\nu]})}{1 - \omega f'(k^{*[\nu]})} \quad , \quad (\nu = 0, 1, 2, \cdots) \quad , \tag{24}$$

where $k^{*[\nu]}$ means the approximation for k^* after the ν-th iteration step and where

$$\mathrm{Sh}(k) = \frac{k f'(k)}{f(k)} \tag{25}$$

represents the *capital share* belonging to the production function f. This capital share is obviously greater than 0 and −by means of the Euler Equation− smaller than 1:

$$0 < \mathrm{Sh}(k) < 1 \ . \tag{26}$$

The Euler Equation cited here follows from the return to scale of a neoclassic production function F, namely

$$F(K,L) = K\frac{\partial F}{\partial K} + L\frac{\partial F}{\partial L} \tag{27}$$

and in the same way for the functions $F_i \quad (i = 1, \cdots N)$.

The Newton method (24) converges quadratically to k^* if $f'(k^*) \neq \frac{1}{\omega}$ and if the value $k^{*[0]}$ at the beginning of the iteration procedure is already sufficiently close to the unknown value k^* [4].

One often tries to approximate the unknown steady state solution of a given fundamental equation by replacing this production function by a Cobb-Douglas production function [5] which reads in its intensive form as

$$f(k) = f(1)k^\alpha \quad (0 < \alpha = const < 1). \tag{28}$$

The Cobb-Douglas function is a first-order approximation to a neoclassic production function where α is a constant approximation to the capital share (25) of the approximated intensive production function f.

The steady state solutions which belong to our single production functions −provided that these functions are given as or approximated by Cobb-Douglas functions− read as

$$k_i^* = (f_i(1)\omega_i)^{\frac{1}{1-\alpha_i}} \tag{29}$$

so that the steady state solution of the united economy becomes or will be approximated by

$$k^* = \omega \sum_{i=1}^{N} \frac{m_i}{\omega_i} (f_i(1)\omega_i)^{\frac{1}{1-\alpha_i}} \quad . \tag{30}$$

Remark: *The fact that a differental equation (like the fundamental equation) with a certain initial value is a good approximation to another differential equation with the same initial value* **does not already guarantee** *that also the solution of the one is a good approximation to the solution of the other ! Stability arguments have to be fulfilled additionally.*

[4]e.g. by choosing $k^{*[0]}$ carefully as the value found by graphical construction over k of the particular value $k = k^*$ where the functions $\omega f(k)$ and k intersect.

References

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