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# Finite element approximation of Dirichlet boundary control for elliptic PDEs on two- and three-dimensional curved domains

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**Abstract:** We consider the variational discretization of elliptic Dirichlet optimal control problems with constraints on the control. The underlying state equation, which is considered on smooth two- and three-dimensional domains, is discretized by linear finite elements taking into account domain approximation. The control variable is not discretized. We obtain optimal error bounds for the optimal control in two and three space dimensions and prove a superconvergence result in two dimensions provided that the underlying mesh satisfies some additional condition. We confirm our analytical findings by numerical experiments.

**Mathematics Subject Classification (2000):**

**Keywords:** Elliptic optimal control problem, boundary control, control constraints, error estimates

## 1 Introduction

Dirichlet boundary control plays an important role in many practical applications such as active boundary control of flows. If one is interested in control by blowing and suction on parts of the boundary only, boundary controls with low regularity should be admissible which even may develop jump discontinuities. In model based optimization with boundary controls the flow often is modeled with the help of the Navier-Stokes equations whose classical variational formulation does not allow for Dirichlet boundary data with jump discontinuities, see [6, 9], so that the concept of very weak solutions [11] has to be applied instead, see [2] for a more detailed discussion. Moreover, pointwise bounds on the control actions have to be considered in practice.

In the present work we consider as model problem Dirichlet boundary control of an elliptic equation with $L^2$-boundary controls subject to pointwise bounds on the controls. The state equation is posed on a bounded, smooth domain in $\mathbb{R}^d$, $d = 2, 3$. Our aim is to develop and analyze a finite element concept which is tailored to the numerical treatment of pointwise bounds, and at the same time is able to cope with the low regularity of the control and the state. To this purpose we propose an approximation of the state equation using piecewise linear, continuous finite elements taking into account domain approximation. The controls are not discretized explicitly, but implicitly (variationally) through the optimality conditions associated with the discrete optimal control problem. Our main result, see Theorem 4.1, is an

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$O(h\sqrt{|\log h|})$ bound for the $L^2$–error of optimal control and state. In two space dimensions and under additional conditions on the underlying mesh we are able derive the improved error bound $O(h^{\frac{3}{2}})$, which reflects a superconvergence effect.

There are only few contributions to Dirichlet boundary control reported in the literature. Casas and Raymond in [5] consider semilinear elliptic Dirichlet boundary control problems with pointwise bounds on two-dimensional convex polygonal domains $\Omega$. Denoting by $u$ the optimal control they are able to prove the optimal result

$$\|u - u_h\|_{0,\partial\Omega} \leq Ch^{1-1/p}.$$

Here, $u_h$ denotes the optimal discrete boundary control which they find in the space of piecewise linear, continuous finite elements on $\partial\Omega$, and $p \geq 2$ depends on the smallest angle of the boundary polygon. For control functions of the form

$$Bq := \sum_{i=1}^{n} q_i f_i$$

with given $f_i \in H^{5/2}(\Gamma)$ and box-constrained $q \in \mathbb{R}^n$, Vexler in [14] provides a finite element analysis for two-dimensional bounded polygonal domains and proves

$$\|q - q_h\| \leq Ch^2.$$

In a recent paper [12] May, Rannacher and Vexler consider Dirichlet boundary control without control constraints on two-dimensional convex polygonal domains, where they present optimal error estimates for the state and the adjoint state. Important ingredients are duality techniques and an optimal error estimate in $H^{-1/2}$ for the control.

Our paper is organized as follows. In the next section we present the mathematical setting and formulate the optimal control problem. In Section 3 we examine the finite element discretization of the state equation taking into account the approximation of the domain. In Section 4 we introduce the discrete control problem and prove an optimal error estimate for the discrete controls. Section 5 deals with superconvergence properties of boundary controls induced by finite element partitions with certain regularity properties. In Section 6 we finally present numerical results which confirm our analytical findings.

For a domain or hypersurface $Q$ and $s \geq 0, 1 \leq p \leq \infty$ we denote by $W^{s,p}(Q)$ the usual Sobolev space and by $\|\cdot\|_{s,p,Q}$ its norm. If $p = 2$ we write $W^{s,2}(Q) = H^s(Q)$ with norm $\|\cdot\|_{s,Q}$.

## 2 Mathematical setting

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded domain with a smooth boundary $\Gamma := \partial\Omega$ and consider the differential operator

$$\mathcal{A}y := -\sum_{i,j=1}^{d} \partial_{x_j}(a_{ij}y_{x_i}) + a_0 y,$$

where for simplicity the coefficients $a_{ij}$ and $a_0$ are assumed to be smooth functions on $\bar{\Omega}$. In what follows we assume that $a_{ij} = a_{ji}$, $a_0 \geq 0$ in $\Omega$ and that there exists $c_0 > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^d \text{ and all } x \in \Omega.$$

Given $f \in L^2(\Omega), u \in L^2(\Gamma)$ we consider the boundary value problem

$$\mathcal{A}y = f \quad \text{in } \Omega, \quad y = u \text{ on } \Gamma. \tag{2.1}$$

It is well–known that (2.1) has a unique solution $y \in H^{\frac{1}{2}}(\Omega)$ which we denote by $y = \mathcal{G}(u)$. Note that $y$ solves the problem in the sense that

$$\int_\Omega y \mathcal{A}\phi = \int_\Omega f\phi - \int_\Gamma u \partial_{\nu_\mathcal{A}} \phi \qquad \forall \phi \in H^2(\Omega) \cap H^1_0(\Omega), \tag{2.2}$$

where $\partial_{\nu_\mathcal{A}} \phi = \sum_{i,j=1}^d a_{ij}\phi_{x_j}\nu_i$ and $\nu$ is the outer unit normal to $\Gamma$.
In order to define an approximation of (2.1) we also introduce the bilinear form $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ associated with the differential operator $\mathcal{A}$ as

$$a(y, z) := \sum_{i,j=1}^d \int_\Omega \big(a_{ij} y_{x_i} z_{x_j} + a_0 yz\big).$$

Next, let $\alpha > 0$ and $y_0 \in W^{1,\bar{r}}(\Omega)$, $\bar{r} > d$ be given. We then consider the Dirichlet boundary control problem

$$\begin{aligned} &\min_{u \in U_{\text{ad}}} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{\alpha}{2}\int_\Gamma |u|^2 \\ &\text{subject to } y = \mathcal{G}(u), \end{aligned} \tag{2.3}$$

where

$$U_{\text{ad}} = \{u \in L^2(\Gamma) \,|\, a \le u \le b \text{ a.e. on } \Gamma\}.$$

Existence of a unique solution $u \in U_{\text{ad}}$ of (2.3) follows by standard arguments. This solution is characterized by the variational inequality

$$\int_\Omega (y - y_0)(z - y) + \alpha \int_\Gamma u(v - u) \ge 0 \qquad \forall v \in U_{\text{ad}} \tag{2.4}$$

where $z = \mathcal{G}(v)$. Let us introduce the adjoint state $p \in H^2(\Omega) \cap H^1_0(\Omega)$ as the solution of the following boundary value problem:

$$\mathcal{A}p = y - y_0 \text{ in } \Omega, \quad p = 0 \text{ on } \Gamma. \tag{2.5}$$

It is not difficult to see that the optimal control $u$ is given by

$$u = P_{[a,b]}\big(\frac{1}{\alpha}\partial_{\nu_\mathcal{A}} p\big) \qquad \text{a.e. on } \Gamma \tag{2.6}$$

where $P_{[a,b]}$ denotes the pointwise projection onto the interval $[a, b]$.

**Lemma 2.1.** *Let $u \in U_{ad}$ be the solution of (2.3) with corresponding state $y$ and adjoint state $p$. Then*

$$u \in H^1(\Gamma), \quad y \in H^{\frac{3}{2}}(\Omega), \quad p \in W^{3,r}(\Omega) \textit{ for some } d < r \le \bar{r}.$$

*Proof.* Elliptic regularity implies that $p \in H^2(\Omega)$ and hence $\partial_{\nu_\mathcal{A}} p \in H^{\frac{1}{2}}(\Gamma)$. In view of (2.6) we then have $u \in H^{\frac{1}{2}}(\Gamma)$ (cf. [5], p. 1590) which in turn yields $y \in H^1(\Omega)$. Repeating the above argument we obtain $p \in H^3(\Omega)$ and then $\partial_{\nu_\mathcal{A}} p \in H^{\frac{3}{2}}(\Gamma)$. Therefore $u \in H^1(\Gamma)$ and $y \in H^{\frac{3}{2}}(\Omega)$. Using an embedding theorem, the above regularity of $\partial_{\nu_\mathcal{A}} p$ also implies that $u \in W^{1-\frac{1}{r},r}(\Gamma)$ for some $r > d$. Hence, $y \in W^{1,r}(\Omega)$ and since $y_0 \in W^{1,\bar{r}}(\Omega)$ we finally obtain $p \in W^{3,r}(\Omega)$ for some $d < r \le \bar{r}$ again by elliptic regularity. ∎

## 3 Finite element discretization

Let $\mathcal{T}_h$ be a triangulation of a polygonal domain $\Omega_h$ approximating $\Omega$. We assume that all vertices on $\partial\Omega_h =: \Gamma_h$ also lie on $\Gamma$ and that at most one face of a simplex $T \in \mathcal{T}_h$ belongs to $\Gamma_h$. Furthermore, we suppose that the triangulation is quasi-uniform in the sense that there exists a constant $\kappa > 0$ (independent of $h$) such that each $T \in \mathcal{T}_h$ is contained in a ball of radius $\kappa^{-1}h$ and contains a ball of radius $\kappa h$, where $h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$ is the maximum mesh size. For every $T \in \mathcal{T}_h$ there exists an invertible affine mapping

$$F_T : \mathbb{R}^d \to \mathbb{R}^d, \quad F_T(\hat{x}) = A_T\hat{x} + b_T,$$

which maps the standard $d$–simplex $\hat{T}$ onto $T$. Besides the triangulation $\mathcal{T}_h$ which will be used to define the discrete problem and to carry out the practical calculations we also introduce an exact triangulation $\tilde{\mathcal{T}}_h$ of $\Omega$. The existence of such a triangulation for sufficiently small $h$ is shown in [3]. In essence, for every $T \in \mathcal{T}_h$ there is a mapping $\Phi_T \in C^3(\hat{T}, \mathbb{R}^d)$ such that $\tilde{F}_T := F_T + \Phi_T$ maps $\hat{T}$ onto a curved $d$–simplex $\tilde{T} \subset \bar{\Omega}$ and $\bar{\Omega} = \bigcup_{\tilde{T} \in \tilde{\mathcal{T}}_h} \tilde{T}$. Furthermore, the mapping $G_h$ which is locally defined by $G_{h|T} := \tilde{F}_T \circ F_T^{-1}$ is a homeomorphism between $\Omega_h$ and $\Omega$. The construction in [3] also implies that $\Phi_T = 0$ if $T$ has at most one vertex on $\Gamma_h$ so that $G_h \equiv \mathrm{id}$ on all simplices which are disjoint from $\Gamma_h$. Furthermore, we have the estimates

$$\begin{aligned}
&\sup_{x \in T} \|(DG_{h\,|\,T} - I)(x)\| \le Ch, \quad \|G_h\|_{3,\infty,T} \le C, \quad T \in \mathcal{T}_h \\
&\sup_{\hat{x} \in \hat{T}} \|D\tilde{F}_T(\hat{x})\| \le C\|A_T\|, \quad \sup_{x \in \tilde{T}} \|D\tilde{F}_T^{-1}(x)\| \le C\|A_T^{-1}\|, \quad T \in \mathcal{T}_h \\
&c_1|\det A_T| \le |\det D\tilde{F}_T(\hat{x})| \le c_2|\det A_T|, \quad \hat{x} \in \hat{T}.
\end{aligned} \tag{3.1}$$

Let us next define the space of linear finite elements,

$$X_h := \{\phi_h \in C^0(\overline{\Omega_h}) \,|\, \phi_{h|T} \in P_1(T), T \in \mathcal{T}_h\}$$

as well as $X_{h0} := X_h \cap H_0^1(\Omega_h)$. Let $\gamma X_h$ be the restriction to $\Gamma_h$ of functions in $X_h$ and denote by $P_h : L^2(\Gamma_h) \to \gamma X_h$ the $L^2$–projection, i.e. for $v \in L^2(\Gamma_h)$ we have

$$\int_{\Gamma_h} v\chi_h = \int_{\Gamma_h} P_h v\, \chi_h \qquad \forall \chi_h \in \gamma X_h. \tag{3.2}$$

Let us introduce an approximation to the solution operator $\mathcal{G}$ as follows. For a given function $u_h \in L^2(\Gamma_h)$ we denote by $y_h = \mathcal{G}_h(u_h) \in X_h$ the unique solution of

$$a_h(y_h, \phi_h) = \int_{\Omega_h} f_h\phi_h, \quad \forall \phi_h \in X_{h0}, \quad y_h = P_h(u_h) \text{ on } \Gamma_h, \tag{3.3}$$

where

$$a_h(y_h, \phi_h) = \sum_{i,j=1}^{d} \int_{\Omega_h} \left(a_{h,ij}y_{h,x_i}\phi_{h,x_j} + a_{h,0}y_h\phi_h\right)$$

and $a_{h,ij} = a_{ij} \circ G_h$, $a_{h,0} = a_0 \circ G_h$ and $f_h = f \circ G_h$.

In order to deal with the problem that the solutions of (2.1) and (3.3) are defined on different domains we assign to each $\phi_h \in X_h$ a function $\tilde{\phi}_h : \bar{\Omega} \to \mathbb{R}$ by $\tilde{\phi}_h := \phi_h \circ G_h^{-1}$ and let

$$\tilde{X}_h := \{\tilde{\phi}_h \,|\, \phi_h \in X_h\} \quad \text{as well as } \gamma\tilde{X}_h = \{\tilde{\phi}_{h|\Gamma} \,|\, \tilde{\phi}_h \in \tilde{X}_h\}.$$

It is not difficult to verify with the help of (3.1) that for $y_h, \phi_h \in X_h$

$$|a(\tilde{y}_h, \tilde{\phi}_h) - a_h(y_h, \phi_h)| \le Ch\|\tilde{y}_h\|_{1,A_h}\|\tilde{\phi}_h\|_{1,A_h}, \tag{3.4}$$

where $A_h = \{x \in \Omega \,|\, \mathrm{dist}(x, \Gamma) < \gamma h\}$ and $\gamma$ is chosen so large that $\bigcup_{\tilde{T}\cap\Gamma\neq\emptyset} \tilde{T} \subset A_h$.

Next, by adapting the methods developed in [13] it is possible to show that there exists an interpolation operator $\tilde{\Pi}_h : L^1(\Omega) \to \tilde{X}_h$ such that for $\phi \in W^{l,p}(\Omega)$ ($1 \le l \le 2$ if $p = 1$, $\frac{1}{p} < l \le 2$ otherwise)

$$\|\phi - \tilde{\Pi}_h\phi\|_{m,p,\Omega} \le Ch^{l-m}\|\phi\|_{l,p,\Omega}, \quad 0 \le m \le \min(1, l). \tag{3.5}$$

In addition it is possible to construct $\tilde{\Pi}_h$ in such way that $\tilde{\Pi}_h\phi = 0$ on $\Gamma$ provided that $\phi = 0$ on $\Gamma$. If $\phi \in C^0(\bar{\Omega})$ then we can also define the usual Lagrange interpolation operator $\tilde{I}_h : C^0(\bar{\Omega}) \to \tilde{X}_h$ via $\tilde{I}_h\phi = I_h(\phi \circ G_h) \circ G_h^{-1}$ where $I_h$ is the Lagrange interpolation operator corresponding to $X_h$.

Abbreviating $g_h := G_{h|\Gamma_h}$ we define for $v \in L^2(\Gamma)$ the projection $\tilde{P}_h v := [P_h(v \circ g_h)] \circ g_h^{-1} \in \gamma\tilde{X}_h$. In view of Lemma 3.1 in [10] we have

$$\int_{\Gamma_h} v = \int_\Gamma v \circ g_h^{-1}\, d_h \quad \text{ where } d_h = \det DG_h^{-1}|(DG_h)^T \circ G_h^{-1}\nu|. \tag{3.6}$$

Applying (3.6) to (3.2) we see that $\tilde{P}_h$ is characterized by the relation

$$\int_\Gamma v\tilde{\chi}_h\, d_h = \int_\Gamma \tilde{P}_h v\; \tilde{\chi}_h\, d_h \qquad \forall \tilde{\chi}_h \in \gamma\tilde{X}_h. \tag{3.7}$$

Furthermore one can show that

$$\|v - \tilde{P}_h v\|_{0,\Gamma} \le Ch^s\|v\|_{s,\Gamma}, \quad v \in H^s(\Gamma),\ 0 \le s \le 2. \tag{3.8}$$

Next we prove an $L^2$ error estimate for $\tilde{y}_h$, compare also [2].

**Lemma 3.1.** *Suppose that $f \in L^2(\Omega), u \in H^s(\Gamma)$ $(0 \le s \le 1)$ and that $y \in H^{s+\frac{1}{2}}(\Omega)$, $y_h \in X_h$ are the solutions of (2.1) and (3.3) with $u_h = u \circ g_h$ respectively. Then there exists $h_0 > 0$ such that for $0 < h \le h_0$*

$$\|y - \tilde{y}_h\|_{0,\Omega} \le Ch^{s+\frac{1}{2}}\big(\|u\|_{s,\Gamma} + \|f\|_{0,\Omega}\big). \tag{3.9}$$

*Proof.* In view of the linearity of $\mathcal{A}$ it is sufficient to consider the problems where either $f \equiv 0$ or $u \equiv 0$.

Let us first assume that $f \equiv 0$ and take $s = 1$. We denote by $y^h \in H^{\frac{3}{2}}(\Omega)$ the solution of

$$a(y^h, \phi) = 0 \quad \forall\phi \in H_0^1(\Omega), \quad y^h = \tilde{P}_h u \quad \text{ on } \Gamma. \tag{3.10}$$

Clearly,

$$\|y^h\|_{s+\frac{1}{2},\Omega} \le C\|\tilde{P}_h u\|_{s,\Gamma}, \quad 0 \le s \le 1. \tag{3.11}$$

Let us choose $\tilde{\phi}_h = \tilde{\Pi}_h[y^h - \tilde{y}_h] = \tilde{\Pi}_h y^h - \tilde{y}_h$. Note that $\phi_h \in X_{h0}$ since $y^h = \tilde{y}_h$ on $\Gamma$. The ellipticity of $\mathcal{A}$ and the fact that $a_0 \ge 0$ imply together with (3.10) and (3.3)

$$\begin{aligned} c_0 \int_\Omega |\nabla(y^h - \tilde{y}_h)|^2 &\le a(y^h - \tilde{y}_h, y^h - \tilde{y}_h) \\ &= a(y^h - \tilde{y}_h, y^h - \tilde{\Pi}_h y^h) + a(y^h - \tilde{y}_h, \tilde{\Pi}_h y^h - \tilde{y}_h) \\ &= a(y^h - \tilde{y}_h, y^h - \tilde{\Pi}_h y^h) + [a_h(y_h, (\tilde{\Pi}_h y^h) \circ G_h - y_h) - a(\tilde{y}_h, \tilde{\Pi}_h y^h - \tilde{y}_h)] \\ &\equiv I + II. \end{aligned}$$

In view of (3.4), (3.5) and Poincaré's inequality we have

$$\begin{aligned}
|II| &\leq Ch\|\tilde{y}_h\|_{1,\Omega}\|\tilde{\Pi}_h y^h - \tilde{y}_h\|_{1,\Omega} \\
&\leq Ch\big(\|y^h\|_{1,\Omega} + \|y^h - \tilde{y}_h\|_{1,\Omega}\big)\big(\|y^h - \tilde{\Pi}_h y^h\|_{1,\Omega} + \|y^h - \tilde{y}_h\|_{1,\Omega}\big) \\
&\leq (\epsilon + Ch)\|\nabla(y^h - \tilde{y}_h)\|^2_{0,\Omega} + C_\epsilon h^{\frac{3}{2}}\|y^h\|^2_{\frac{3}{2},\Omega}.
\end{aligned}$$

Estimating $I$ with the help of (3.5) and Young's inequality we obtain for $0 < h \leq h_0$, $h_0$ sufficiently small

$$\|y^h - \tilde{y}_h\|_{1,\Omega} \leq C\sqrt{h}\|y^h\|_{\frac{3}{2},\Omega}. \tag{3.12}$$

In order to estimate the $L^2$–norm of $y - \tilde{y}_h$ we employ the usual duality argument, namely denote by $\psi \in H^2(\Omega)$ the solution of

$$\mathcal{A}\psi = y - \tilde{y}_h \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \Gamma. \tag{3.13}$$

Then, (2.2) and integration by parts imply that

$$\int_\Omega |y - \tilde{y}_h|^2 = \int_\Omega (y - \tilde{y}_h)\mathcal{A}\psi = -a(\tilde{y}_h, \psi) - \int_\Gamma (u - \tilde{P}_h u)\partial_{\nu_\mathcal{A}}\psi = I + II.$$

Observing that $\psi, \tilde{I}_h\psi \in H^1_0(\Omega), I_h(\psi \circ G_h) \in X_{h0}$ we infer from (3.3) and (3.10)

$$\begin{aligned}
I &= a(y^h - \tilde{y}_h, \psi - \tilde{I}_h\psi) + [-a\big(\tilde{y}_h, I_h\widetilde{(\psi \circ G_h)}\big) + a_h\big(y_h, I_h(\psi \circ G_h)\big)] \\
&\leq Ch^{\frac{3}{2}}\|y^h\|_{\frac{3}{2},\Omega}\|\psi\|_{2,\Omega} + Ch\|\tilde{y}_h\|_{1,A_h}\|\tilde{I}_h\psi\|_{1,A_h}
\end{aligned}$$

by (3.12), (3.4) and an interpolation estimate. Next, using the continuous embeddings $H^{\frac{1}{2}}(\Omega) \hookrightarrow L^3(\Omega)$, $H^1(\Omega) \hookrightarrow L^6(\Omega)$ as well as (3.12) we obtain

$$\begin{aligned}
\|\tilde{y}_h\|_{1,A_h} &\leq \|y^h\|_{1,A_h} + \|y^h - \tilde{y}_h\|_{1,A_h} \leq C|A_h|^{\frac{1}{6}}\|y^h\|_{1,3,A_h} + C\sqrt{h}\|y^h\|_{\frac{3}{2},\Omega} \leq Ch^{\frac{1}{6}}\|y^h\|_{\frac{3}{2},\Omega}, \\
\|\tilde{I}_h\psi\|_{1,A_h} &\leq \|\psi\|_{1,A_h} + \|\psi - \tilde{I}_h\psi\|_{1,A_h} \leq C|A_h|^{\frac{1}{3}}\|\psi\|_{1,6,A_h} + Ch\|\psi\|_{2,\Omega} \leq Ch^{\frac{1}{3}}\|\psi\|_{2,\Omega}.
\end{aligned}$$

Thus,

$$|I| \leq Ch^{\frac{3}{2}}\|y^h\|_{\frac{3}{2},\Omega}\|\psi\|_{2,\Omega} \leq Ch^{\frac{3}{2}}\|\tilde{P}_h u\|_{1,\Gamma}\|\psi\|_{2,\Omega} \tag{3.14}$$

in view of (3.11). For $II$ we obtain with the help of (3.7)

$$\begin{aligned}
II &= -\int_\Gamma (u - \tilde{P}_h u)\partial_{\nu_\mathcal{A}}\psi\, d_h + \int_\Gamma (u - \tilde{P}_h u)\partial_{\nu_\mathcal{A}}\psi(d_h - 1) \qquad (3.15) \\
&= -\int_\Gamma (u - \tilde{P}_h u)(\partial_{\nu_\mathcal{A}}\psi - \tilde{P}_h\partial_{\nu_\mathcal{A}}\psi)d_h + \int_\Gamma (u - \tilde{P}_h u)\partial_{\nu_\mathcal{A}}\psi(d_h - 1)
\end{aligned}$$

and hence using (3.8) and (3.1)

$$|II| \leq Ch^{\frac{3}{2}}\|u\|_{1,\Gamma}\|\partial_{\nu_\mathcal{A}}\psi\|_{\frac{1}{2},\Gamma} + Ch^2\|u\|_{1,\Gamma}\|\partial_{\nu_\mathcal{A}}\psi\|_{0,\Gamma} \leq Ch^{\frac{3}{2}}\|u\|_{1,\Gamma}\|\psi\|_{2,\Omega}.$$

Combining this bound with (3.14), the stability of $\tilde{P}_h$ in $H^1(\Gamma)$ and a standard elliptic regularity result we deduce that

$$\|y - y^h\|_{0,\Omega} \leq Ch^{\frac{3}{2}}\|u\|_{1,\Gamma}. \tag{3.16}$$

Let us next look at the case $s = 0$ and define $\psi \in H^2(\Omega) \cap H^1_0(\Omega)$ again via (3.13). As above we obtain

$$\int_\Omega |y - \tilde{y}_h|^2 = I + II.$$

Using (3.14) together with an inverse inequality we have

$$|I| \le Ch^{\frac{3}{2}} \|\tilde{P}_h u\|_{1,\Gamma} \|\psi\|_{2,\Omega} \le Ch^{\frac{1}{2}} \|\tilde{P}_h u\|_{0,\Gamma} \|\psi\|_{2,\Omega}.$$

Returning to (3.15) we infer for the second term

$$|II| \le C\big(\|u\|_{0,\Gamma} + \|\tilde{P}_h u\|_{0,\Gamma}\big)\big(h^{\frac{1}{2}} \|\partial_{\nu_{\mathcal{A}}}\psi\|_{\frac{1}{2},\Gamma} + h\|\partial_{\nu_{\mathcal{A}}}\psi\|_{0,\Gamma}\big) \le Ch^{\frac{1}{2}} \|u\|_{0,\Gamma} \|\psi\|_{2,\Omega}.$$

Combining the above two bounds we deduce that

$$\|y - \tilde{y}_h\|_{0,\Omega} \le Ch^{\frac{1}{2}} \|u\|_{0,\Gamma}. \tag{3.17}$$

The case $0 < s < 1$ then follows from (3.16) and (3.17) by interpolation.
If $u \equiv 0$, $f \in L^2(\Omega)$ we can proceed in a similar way as above, starting with a bound of the form $\|y - \tilde{y}_h\|_{1,\Omega} \le Ch\|f\|_{0,\Omega}$ followed by a duality argument to give

$$\|y - \tilde{y}_h\|_{0,\Omega} \le Ch^{\frac{3}{2}} \|f\|_{0,\Omega}.$$

Since our primary interest lies on the boundary values we leave the details to the reader. ∎

Our next aim is to bound the discrete solution corresponding to $f \equiv 0$ in terms of $\|u\|_{0,\Gamma}$. In order to formulate the result we introduce the distance function $d_\Gamma(x) := \mathrm{dist}(x, \Gamma)$. It follows from [7], 14.6 that there exists $\delta > 0$ such that $d_\Gamma \in C^3(\Omega_\delta)$, where $\Omega_r := \{x \in \bar{\Omega} \,|\, d_\Gamma(x) < r\}$ for $r > 0$. Choose a function $\eta \in C^3(\bar{\Omega})$ such that $0 \le \eta \le 1$, $\eta(x) = 1, x \in \Omega_{\frac{\delta}{2}}$ and $\eta(x) = 0, x \in \bar{\Omega} \setminus \Omega_{\frac{2\delta}{3}}$. Then, $\rho(x) := \eta(x) d_\Gamma(x) + (1 - \eta(x))\frac{\delta}{2}$, $x \in \bar{\Omega}$ belongs to $C^3(\bar{\Omega})$ and satisfies

$$\rho(x) = d_\Gamma(x),\ x \in \Omega_{\frac{\delta}{2}}, \quad \rho(x) \ge \frac{\delta}{2},\ x \in \bar{\Omega} \setminus \Omega_{\frac{\delta}{2}}. \tag{3.18}$$

Furthermore, let

$$\omega(x) := \rho(x) + h, \quad x \in \bar{\Omega}.$$

**Lemma 3.2.** *Let $u \in L^2(\Gamma)$ and suppose that $z_h \in X_h$ is the solution of*

$$a_h(z_h, \phi_h) = 0 \quad \forall \phi_h \in X_{h0}, \quad z_h = P_h(u \circ g_h) \text{ on } \Gamma_h. \tag{3.19}$$

*Then*

$$\int_\Omega \big(|\tilde{z}_h|^2 + \omega |\nabla \tilde{z}_h|^2\big) \le C\|u\|_{0,\Gamma}^2. \tag{3.20}$$

*Proof.* Let $y^h$ be again the solution of (3.10). Since $(\tilde{P}_h u) \circ g_h = P_h(u \circ g_h)$ and $P_h^2 = P_h$, Lemma 3.1 for $s = 0$ implies that

$$\|y^h - \tilde{z}_h\|_{0,\Omega} \le C\sqrt{h} \|\tilde{P}_h u\|_{0,\Gamma} \le C\sqrt{h}\|u\|_{0,\Gamma}. \tag{3.21}$$

Combining this estimate with (3.11) we deduce

$$\|\tilde{z}_h\|_{0,\Omega} \le \|y^h\|_{0,\Omega} + C\sqrt{h}\|u\|_{0,\Gamma} \le C\|u\|_{0,\Gamma}. \tag{3.22}$$

On the other hand, an inverse estimate, (3.5), (3.11) and (3.21) yield

$$\begin{aligned} \|\nabla \tilde{z}_h\|_{0,\Omega} &\le \|\nabla(\tilde{z}_h - \tilde{\Pi}_h y^h)\|_{0,\Omega} + \|\nabla \tilde{\Pi}_h y^h\|_{0,\Omega} \le Ch^{-1}\|\tilde{z}_h - \tilde{\Pi}_h y^h\|_{0,\Omega} + C\|y^h\|_{1,\Omega} \\ &\le Ch^{-1}\|\tilde{z}_h - y^h\|_{0,\Omega} + C\|y^h\|_{1,\Omega} \le Ch^{-\frac{1}{2}}\|u\|_{0,\Gamma} + C\|\tilde{P}_h u\|_{\frac{1}{2},\Gamma} \le Ch^{-\frac{1}{2}}\|u\|_{0,\Gamma}. \end{aligned} \tag{3.23}$$

It remains to bound $\int_\Omega \rho\,|\nabla \tilde z_h|^2$. The ellipticity of $\mathcal{A}$ and the fact that $a_0 \geq 0$ imply

$$c_0 \int_\Omega \rho\,|\nabla \tilde z_h|^2 \leq \sum_{i,j=1}^d \int_\Omega \rho a_{ij} \tilde z_{h,x_i} \tilde z_{h,x_j} \leq a(\tilde z_h, \rho \tilde z_h) - \frac{1}{2} \sum_{i,j=1}^d \int_\Omega a_{ij} \rho_{x_i} (\tilde z_h^2)_{x_j} \equiv I + II.$$

Since $\rho(x) = d_\Gamma(x) = 0, x \in \Gamma$, we have that $\phi_h := I_h\big((\rho \circ G_h) z_h\big) \in X_{h0}$. Hence, (3.19) and (3.4) yield

$$\begin{aligned} I &= a(\tilde z_h, \rho \tilde z_h - \tilde I_h(\rho \tilde z_h)) + [a(\tilde z_h, \tilde I_h(\rho \tilde z_h)) - a_h\big(z_h, I_h((\rho \circ G_h) z_h)\big)] \\ &\leq C\|\tilde z_h\|_{1,\Omega} \|\rho \tilde z_h - \tilde I_h(\rho \tilde z_h)\|_{1,\Omega} + Ch\|\tilde z_h\|_{1,A_h} \|\tilde I_h(\rho \tilde z_h)\|_{1,A_h}. \end{aligned} \tag{3.24}$$

For fixed $\tilde T \in \tilde{\mathcal{T}}_h$ we have observing (3.1) together with the fact that $z_h \in P_1(T)$

$$\begin{aligned} &\|\rho \tilde z_h - \tilde I_h(\rho \tilde z_h)\|_{1,\tilde T} \\ &\quad\leq C\|(\rho \circ G_h) z_h - I_h\big((\rho \circ G_h) z_h\big)\|_{1,T} \leq Ch\|D^2[(\rho \circ G_h) z_h]\|_{0,T} \\ &\quad\leq Ch\big(\|z_h D^2(\rho \circ G_h)\|_{0,T} + \|\nabla(\rho \circ G_h) \otimes \nabla z_h\|_{0,T} + \|\nabla z_h \otimes \nabla(\rho \circ G_h)\|_{0,T}\big) \\ &\quad\leq Ch\|z_h\|_{1,T} \leq Ch\|\tilde z_h\|_{1,\tilde T}, \end{aligned} \tag{3.25}$$

where $\otimes$ denotes the dyadic product of two vectors. In particular

$$\|\tilde I_h(\rho \tilde z_h)\|_{1,\tilde T} \leq \|\rho \tilde z_h - \tilde I_h(\rho \tilde z_h)\|_{1,\tilde T} + \|\rho \tilde z_h\|_{1,\tilde T} \leq C\|\tilde z_h\|_{1,\tilde T}. \tag{3.26}$$

Inserting (3.25) and (3.26) into (3.24) we deduce with the help of (3.22) and (3.23)

$$I \leq Ch\|\tilde z_h\|_{1,\Omega}^2 \leq C\|u\|_{0,\Gamma}^2. \tag{3.27}$$

Finally, integration by parts and (3.22) imply

$$\begin{aligned} II &= \frac{1}{2} \sum_{i,j=1}^d \int_\Omega \big(a_{ij,x_j} \rho_{x_i} + a_{ij} \rho_{x_i x_j}\big) \tilde z_h^2 - \frac{1}{2} \sum_{i,j=1}^d \int_\Gamma \partial_{\nu_{\mathcal{A}}} \rho\, \tilde z_h^2 \\ &\leq C\big(\|\tilde z_h\|_{0,\Omega}^2 + \|\tilde z_h\|_{0,\Gamma}^2\big) \leq C\big(\|u\|_{0,\Gamma}^2 + \|\tilde P_h u\|_{0,\Gamma}^2\big) \leq C\|u\|_{0,\Gamma}^2. \end{aligned}$$

Combining this estimate with (3.27) completes the proof. ∎

## 4 Error analysis for the control problem

We approximate (2.3) using the variational discretization from [8]. This leads to the following control problem depending on $h$:

$$\begin{aligned} &\min_{u_h \in U_{h,\text{ad}}} J_h(u_h) = \frac{1}{2} \int_{\Omega_h} |y_h - y_{h,0}|^2 + \frac{\alpha}{2} \int_{\Gamma_h} |u_h|^2 \\ &\text{subject to } y_h = \mathcal{G}_h(u_h), \end{aligned} \tag{4.1}$$

where $U_{h,\text{ad}} = \{u_h \in L^2(\Gamma_h) \,|\, a \leq u_h \leq b \text{ a.e. on } \Gamma_h\}$ and $y_{h,0} = y_0 \circ G_h$. It is not difficult to see that (4.1) has a unique solution $u_h \in U_{h,\text{ad}}$ and that this solution is characterized by the variational inequality

$$\int_{\Omega_h} (y_h - y_{h,0})(z_h - y_h) + \alpha \int_{\Gamma_h} u_h(v_h - u_h) \geq 0 \qquad \forall v_h \in U_{h,\text{ad}}. \tag{4.2}$$

Here $z_h = \mathcal{G}_h(v_h) \in X_h$. It is easy to show that (compare (2.6))

$$u_h = P_{[a,b]}(\frac{1}{\alpha}\partial^h_{\nu_{\mathcal{A}}} p_h), \tag{4.3}$$

where $p_h \in X_{h0}$ and $\partial^h_{\nu_{\mathcal{A}}} p_h \in \gamma X_h$ are defined by

$$a_h(\phi_h, p_h) = \int_{\Omega_h} (y_h - y_{h,0})\phi_h \qquad \forall\, \phi_h \in X_{h0}$$

and

$$\int_{\Gamma_h} (\partial^h_{\nu_{\mathcal{A}}} p_h) w_h = a_h(w_h, p_h) - \int_{\Omega_h} (y_h - y_{h,0}) w_h \qquad \forall\, w_h \in X_h.$$

**Theorem 4.1.** *Let $u$ and $u_h$ be the solutions of (2.3) and (4.1) with corresponding states $y$ and $y_h$ respectively. Then*

$$\|u - \tilde{u}_h\|_{0,\Gamma} + \|y - \tilde{y}_h\|_{0,\Omega} \le Ch\sqrt{|\log h|}$$

*for all $0 < h \le h_0$. Here, $\tilde{u}_h = u_h \circ g_h^{-1}$.*

*Proof.* Using $v = \tilde{u}_h \in U_{\text{ad}}$ in (2.4) and $v_h = u \circ g_h \in U_{h,\text{ad}}$ in (4.2) we obtain

$$\int_\Omega (y - y_0)(y^h - y) + \alpha \int_\Gamma u(\tilde{u}_h - u) \;\ge\; 0 \tag{4.4}$$

$$\int_{\Omega_h} (y_h - y_{h,0})(z_h - y_h) + \alpha \int_{\Gamma_h} u_h(u \circ g_h - u_h) \;\ge\; 0 \tag{4.5}$$

where $y^h = \mathcal{G}(u_h \circ g_h^{-1})$ and $z_h = \mathcal{G}_h(u \circ g_h)$. Transforming (4.5) to $\Omega$ and $\Gamma$ respectively we obtain

$$\int_\Omega (\tilde{y}_h - y_0)(\tilde{z}_h - \tilde{y}_h)|\text{det} DG_h^{-1}| + \alpha \int_\Gamma \tilde{u}_h(u - \tilde{u}_h) d_h \ge 0$$

or equivalently

$$\int_\Omega (\tilde{y}_h - y_0)(\tilde{z}_h - \tilde{y}_h) + \alpha \int_\Gamma \tilde{u}_h(u - \tilde{u}_h) + \delta_h \ge 0 \tag{4.6}$$

where, using (3.1),

$$\begin{aligned} |\delta_h| &\le Ch\big(\|\tilde{z}_h - \tilde{y}_h\|_{0,\Omega} + \|u - \tilde{u}_h\|_{0,\Gamma}\big) \le Ch\big(\|y - \tilde{y}_h\|_{0,\Omega} + \|y - \tilde{z}_h\|_{0,\Omega} + \|u - \tilde{u}_h\|_{0,\Gamma}\big) \\ &\le \epsilon\big(\|y - \tilde{y}_h\|^2_{0,\Omega} + \|u - \tilde{u}_h\|^2_{0,\Gamma}\big) + C_\epsilon h^2 + C\|y - \tilde{z}_h\|^2_{0,\Omega}. \end{aligned} \tag{4.7}$$

Combining (4.4), (4.6) and (4.7) we deduce

$$\begin{aligned} \alpha\|u - \tilde{u}_h\|^2_{0,\Gamma} &\le \int_\Omega (y - y_0)(y^h - y) + \int_\Omega (\tilde{y}_h - y_0)(\tilde{z}_h - \tilde{y}_h) + \delta_h \\ &= -\int_\Omega (y - \tilde{y}_h)^2 + \int_\Omega (y - \tilde{y}_h)(y - \tilde{z}_h) - \int_\Omega (y - y_0)\big((y - y^h) - (\tilde{z}_h - \tilde{y}_h)\big) + \delta_h \\ &\le -\frac{1}{2}\|y - \tilde{y}_h\|^2_{0,\Omega} - \int_\Omega (y - y_0)\big((y - y^h) - (\tilde{z}_h - \tilde{y}_h)\big) \\ &\quad + \epsilon\big(\|y - \tilde{y}_h\|^2_{0,\Omega} + \|u - \tilde{u}_h\|^2_{0,\Gamma}\big) + C_\epsilon h^2 + C_\epsilon\|y - \tilde{z}_h\|^2_{0,\Omega} \end{aligned}$$

and hence after choosing $\epsilon > 0$ small enough and recalling Lemma 3.1

$$\frac{\alpha}{2}\|u - \tilde{u}_h\|^2_{0,\Gamma} + \frac{1}{4}\|y - \tilde{y}_h\|^2_{0,\Omega} \le Ch^2 - \int_\Omega (y - y_0)\big((y - y^h) - (\tilde{z}_h - \tilde{y}_h)\big). \tag{4.8}$$

Using (2.5), (2.2), integration by parts, the definition of $\tilde{P}_h$ and the fact that $a_h(z_h - y_h, \phi_h) = 0$ for $\phi_h \in X_{h0}$ we obtain

$$\begin{aligned}
\int_\Omega (y-y_0)\big((y-y^h)-(\tilde{z}_h-\tilde{y}_h)\big) &= \int_\Omega (y-y^h)\mathcal{A}p - \int_\Omega (\tilde{z}_h-\tilde{y}_h)\mathcal{A}p \\
&= -\int_\Gamma (u-\tilde{u}_h)\partial_{\nu_\mathcal{A}}p - a(p,\tilde{z}_h-\tilde{y}_h) + \int_\Gamma \tilde{P}_h(u-\tilde{u}_h)\partial_{\nu_\mathcal{A}}p \\
&= -a(p-\tilde{I}_hp,\tilde{z}_h-\tilde{y}_h) - \int_\Gamma \big((u-\tilde{u}_h)-\tilde{P}_h(u-\tilde{u}_h)\big)\partial_{\nu_\mathcal{A}}p \\
&\quad + [a_h(I_h(p\circ G_h), z_h-y_h) - a(\tilde{I}_hp,\tilde{z}_h-\tilde{y}_h)] \\
&\equiv I + II + III. \qquad (4.9)
\end{aligned}$$

The first integral is estimated with the help of an interpolation inequality and Lemma 3.2:

$$\begin{aligned}
|I| &\leq \big(\int_\Omega \omega^{-1}|\nabla(p-\tilde{I}_hp)|^2\big)^{\frac{1}{2}}\big(\int_\Omega \omega|\nabla(\tilde{z}_h-\tilde{y}_h)|^2\big)^{\frac{1}{2}} \\
&\leq Ch\|p\|_{2,\infty,\Omega}\big(\int_\Omega \omega^{-1}\big)^{\frac{1}{2}}\|u-\tilde{u}_h\|_{0,\Gamma} \leq Ch\|p\|_{3,r,\Omega}\big(\int_\Omega \omega^{-1}\big)^{\frac{1}{2}}\|u-\tilde{u}_h\|_{0,\Gamma}.
\end{aligned}$$

In view of (3.18) and the coarea formula we have

$$\int_\Omega \omega^{-1} \leq \int_{\Omega_{\frac{\delta}{2}}} \frac{1}{d_\Gamma + h} + \int_{\Omega\setminus\Omega_{\frac{\delta}{2}}} \frac{2}{\delta} \leq C\int_0^{\frac{\delta}{2}}\int_{\{d_\Gamma=\tau\}} \frac{1}{\tau+h}dAd\tau + C \leq C|\log h|$$

so that

$$|I| \leq \epsilon\|u-\tilde{u}_h\|_{0,\Gamma}^2 + C_\epsilon h^2|\log h|. \qquad (4.10)$$

Next, $II = II_1 + II_2$ where

$$\begin{aligned}
II_1 &= -\int_\Gamma \big((u-\tilde{u}_h)-\tilde{P}_h(u-\tilde{u}_h)\big)\partial_{\nu_\mathcal{A}}p\, d_h \\
II_2 &= \int_\Gamma \big((u-\tilde{u}_h)-\tilde{P}_h(u-\tilde{u}_h)\big)\partial_{\nu_\mathcal{A}}p(d_h-1).
\end{aligned}$$

We infer from (3.7) and (3.8) that

$$\begin{aligned}
|II_1| &= \big| -\int_\Gamma (u-\tilde{u}_h)\big(\partial_{\nu_\mathcal{A}}p - \tilde{P}_h\partial_{\nu_\mathcal{A}}p\big)d_h \big| \\
&\leq Ch^{\frac{3}{2}}\|\partial_{\nu_\mathcal{A}}p\|_{\frac{3}{2},\Gamma}\|u-\tilde{u}_h\|_{0,\Gamma} \leq \epsilon\|u-\tilde{u}_h\|_{0,\Gamma}^2 + C_\epsilon h^3.
\end{aligned}$$

On the other hand, (3.1) implies

$$|II_2| \leq Ch\|u-\tilde{u}_h\|_{0,\Gamma}\|\partial_{\nu_\mathcal{A}}p\|_{0,\Gamma} \leq \epsilon\|u-\tilde{u}_h\|_{0,\Gamma}^2 + C_\epsilon h^2$$

so that in conclusion

$$|II| \leq \epsilon\|u-\tilde{u}_h\|_{0,\Gamma}^2 + C_\epsilon h^2. \qquad (4.11)$$

Finally, recalling (3.4) we have

$$\begin{aligned}
|III| &\leq Ch\|\tilde{I}_hp\|_{1,A_h}\|\tilde{z}_h-\tilde{y}_h\|_{1,\Omega} \leq Ch|A_h|^{\frac{1}{2}}\|p\|_{1,\infty,A_h}\|\tilde{z}_h-\tilde{y}_h\|_{1,\Omega} \\
&\leq Ch\|p\|_{1,\infty,\Omega}\|u-\tilde{u}_h\|_{0,\Gamma} \leq \epsilon\|u-\tilde{u}\|_{0,\Gamma}^2 + C_\epsilon h^2 \qquad (4.12)
\end{aligned}$$

in view of Lemma 3.2. Inserting (4.10), (4.11) and (4.12) into (4.8) and choosing $\epsilon$ small enough yields the result. ∎

## 5 Superconvergence

In the following section we demonstrate that it is possible to improve the order of convergence under additional conditions on the underlying mesh. We assume from now on that $d = 2$ making use of the theory developed in [1], where the following definition can be found:

**Definition 5.1.** The triangulation $\mathcal{T}_h$ is called $O(h^{2\sigma})$ irregular if the following holds:
a) The set of interior edges of $\mathcal{T}_h$ can be decomposed into two disjoint sets $\mathcal{E}_1$ and $\mathcal{E}_2$ with the following properties:

- For each $e \in \mathcal{E}_1$ let $T, T' \in \mathcal{T}_h$ with $T \cap T' = e$. Then in the parallelogram formed by $T \cup T'$ the lengths of any two opposite edges only differ by $O(h^2)$.
- $\sum_{e\in\mathcal{E}_2}(|T| + |T'|) = O(h^{2\sigma})$.

b) The set of boundary vertices $\mathcal{P}$ can be decomposed into two disjoint sets $\mathcal{P}_1$ and $\mathcal{P}_2$ with the following properties:

- For each vertex $x \in \mathcal{P}_1$ denote by $e \subset T, e' \subset T'$ the two boundary edges sharing $x$ and let $t$, $t'$ be the unit tangents. Also denote by $e, f, g$ and $e', f', g'$ the edges obtained by making a clockwise traversal of $\partial T, \partial T'$ respectively. Then
$$|t - t'| = O(h), |e| - |e'| = O(h^2), |f| - |f'| = O(h^2), |g| - |g'| = O(h^2).$$
- $|\mathcal{P}_2| \le C$ where $C$ is independent of $h$.

The following result is essentially proved in [1], Lemma 2.5 for functions $f$ belonging to $W^{3,\infty}(\Omega)$. Since we would like to use a corresponding estimate for the solution of the adjoint problem which only belongs to $W^{3,r}(\Omega)$ for some $r > 2$ we require a suitable modification allowing a boundary term of the discrete test function $\phi_h$.

**Lemma 5.2.** *Suppose that the triangulation $\mathcal{T}_h$ is $O(h^{2\sigma})$ irregular and let $f \in W^{3,r}(\Omega_h)$ for some $r > 2$. Then*
$$|\int_{\Omega_h} \nabla(f - I_h f)\cdot\nabla\phi_h| \le C\|f\|_{3,r,\Omega_h}\big(h^{1+\min(1,\sigma)}\|\phi_h\|_{1,\Omega_h} + h^{\frac{3}{2}}\|\phi_h\|_{0,\Gamma_h}\big) \quad \forall\phi_h \in X_h.$$

*Proof.* Lemma 2.3 in [1] gives
$$\begin{aligned}\int_{\Omega_h} \nabla(f - I_h f)\cdot\nabla\phi_h &= \sum_{T\in\mathcal{T}_h}\int_T \nabla(f - I_h f)\cdot\nabla\phi_h \\ &= \sum_{T\in\mathcal{T}_h}\sum_{e\subset\partial T}\int_e q_e\big(\alpha_e\frac{\partial^2 f}{\partial t^2} + \beta_e\frac{\partial^2 f}{\partial t\partial n}\big)\frac{\partial\phi_h}{\partial t} - \sum_{T\in\mathcal{T}_h}\int_T \sum_{|\lambda|=3,|\mu|=1}\gamma_{T,\lambda\mu}\,\partial^\lambda f\,\partial^\mu\phi_h \\ &\equiv I_1 + I_2. \end{aligned} \tag{5.1}$$

Here, $q_e$ is the quadratic function vanishing at the endpoints of $e$ and being equal to $\frac{1}{4}$ at the midpoint. Furthermore, $n$ is the unit normal to $e$ pointing away from $T$ while $t$ denotes the unit tangent with the tangents on $\partial T$ being oriented counterclockwise. The numbers $\alpha_e, \beta_e$ and $\gamma_{T,\lambda\mu}$ depend on the geometry of $T$ and their precise form can be found in [1]. For our purposes it is sufficient to note that the conditions in Definition 5.1 imply
$$|\alpha_e|,\ |\beta_e|,\ |\gamma_{T,\lambda\mu}| \le Ch^2, \quad e \in \mathcal{E}_1 \cup \mathcal{E}_2, \tag{5.2}$$
$$|\alpha_e - \alpha'_e|,\ |\beta_e - \beta'_e| \le Ch^3, \quad T \cap T' = e \in \mathcal{E}_1, \tag{5.3}$$
$$|\alpha_e - \alpha_{e'}|,\ |\beta_e - \beta_{e'}| \le Ch^3, \quad e, e' \subset \Gamma_h, e \cap e' = \{x\}, x \in \mathcal{P}_1. \tag{5.4}$$

In view of (5.2) we have

$$|I_2| \le Ch^2 \|f\|_{3,\Omega_h} \|\phi_h\|_{1,\Omega_h}. \tag{5.5}$$

Next, we write as in [1]

$$I_1 = I_{11} + I_{12} + I_{13},$$

where

$$\begin{aligned} I_{1j} &= \sum_{e\in\mathcal{E}_j} \int_e q_e \big\{(\alpha_e - \alpha_e')\frac{\partial^2 f}{\partial t^2} + (\beta_e - \beta_e')\frac{\partial^2 f}{\partial t \partial n}\big\}\frac{\partial \phi_h}{\partial t}, \quad j = 1, 2, \\ I_{13} &= \sum_{e\subset\Gamma_h} \int_e q_e \big\{\alpha_e \frac{\partial^2 f}{\partial t^2} + \beta_e \frac{\partial^2 f}{\partial t\partial n}\big\}\frac{\partial \phi_h}{\partial t}. \end{aligned}$$

Arguing as in [1] we have

$$|I_{11}| + |I_{12}| \le C(h^2 + h^{1+\sigma})\|f\|_{2,\infty,\Omega_h}\|\phi_h\|_{1,\Omega_h}. \tag{5.6}$$

In order to treat $I_{13}$ we proceed in a slightly different manner compared to [1]. Let us set

$$B_e(f) := \alpha_e \frac{\partial^2 f}{\partial t^2} + \beta_e \frac{\partial^2 f}{\partial t \partial n}, \ e \subset \Gamma_h \quad \text{as well as } \bar{B}_e(f) := \frac{1}{|e|}\int_e B_e(f).$$

Then we can write

$$\sum_{e\subset\Gamma_h}\int_e q_e B_e(f)\frac{\partial\phi_h}{\partial t} = \sum_{e\subset\Gamma_h}\int_e q_e\big(B_e(f) - \bar{B}_e(f)\big)\frac{\partial \phi_h}{\partial t} + \sum_{e\subset\Gamma_h}\int_e q_e \bar{B}_e(f)\frac{\partial\phi_h}{\partial t}.$$

A Poincaré type inequality along with a scaling argument yields for $g \in H^{1,\tilde{q}}(T)$

$$\|g - \frac{1}{|e|}\int_e g\|_{0,q,e} \le Ch^{1+\frac{1}{q}-\frac{2}{\tilde{q}}}\|\nabla g\|_{0,\tilde{q},T}, \quad e \subset \partial T, 1 + \frac{1}{q} - \frac{2}{\tilde{q}} > 0. \tag{5.7}$$

Applying this estimate with $q = \tilde{q} = 2$ and using (5.2) as well as an inverse inequality we deduce

$$\begin{aligned} &|\sum_{e\subset\Gamma_h}\int_e q_e\big(B_e(f) - \bar{B}_e(f)\big)\frac{\partial\phi_h}{\partial t}| \\ &\le C\sum_{e\subset\Gamma_h}\|B_e(f) - \bar{B}_e(f)\|_{0,e}\|\nabla\phi_h\|_{0,e} \le Ch^2\|f\|_{3,\Omega_h}\|\nabla\phi_h\|_{0,\Omega_h}. \end{aligned}$$

For the second term we write as in [1]

$$\begin{aligned} \sum_{e\subset\Gamma_h}\int_e q_e\bar{B}_e(f)\frac{\partial\phi_h}{\partial t} &= \sum_{e\subset\Gamma_h}\bar{B}_e(f)\frac{\partial\phi_h}{\partial t}\int_e q_e = \sum_{e\subset\Gamma_h}\bar{B}_e(f)\frac{\partial\phi_h}{\partial t}\frac{|e|}{6} \\ &= \frac{1}{6}\sum_{x\in\mathcal{P}_1}\big(\bar{B}_e(f) - \bar{B}_{e'}(f)\big)\phi_h(x) + \frac{1}{6}\sum_{x\in\mathcal{P}_2}\big(\bar{B}_e(f) - \bar{B}_{e'}(f)\big)\phi_h(x), \end{aligned}$$

where $e$ and $e'$ are the edges sharing $x$. Using (5.4) as well as $|t - t'| = O(h)$ for $e \cap e' = \{x\}$ we have for $x \in \mathcal{P}_1$

$$\begin{aligned} |\bar{B}_e(f) - \bar{B}_{e'}(f)| &\le |\bar{B}_e(f) - B_e(f)(x)| + |\bar{B}_{e'}(f) - B_{e'}(f)(x)| + |B_e(f)(x) - B_{e'}(f)(x)| \\ &\le C\big(\|B_e(f) - \bar{B}_e(f)\|_{0,\infty,e} + \|B_{e'}(f) - \bar{B}_{e'}(f)\|_{0,\infty,e'}\big) + Ch^3|D^2 f(x)| \\ &\le Ch^{3-\frac{2}{r}}\|f\|_{3,r,T\cup T'} \end{aligned}$$

by (5.7) with $q = \infty, \tilde{q} = r$. On the other hand we have for $x \in e \subset T$

$$|\phi_h(x)| \le \|\phi_h\|_{\infty,e} \le Ch^{-\frac{1}{2}}\|\phi_h\|_{0,e} + Ch^{1-\frac{2}{r'}}\|\nabla\phi_h\|_{0,r',T}.$$

Thus,

$$\begin{aligned}
&|\sum_{x\in\mathcal{P}_1}\big(\bar{B}_e(f) - \bar{B}_{e'}(f)\big)\phi_h(x)| \\
&\le Ch^{3-\frac{2}{r}}\sum_{x\in\mathcal{P}_1}\|f\|_{3,r,T\cup T'}\big(h^{-\frac{1}{2}}\|\phi_h\|_{0,e} + h^{1-\frac{2}{r'}}\|\nabla\phi_h\|_{0,r',T}\big) \\
&\le Ch^{\frac{5}{2}-\frac{2}{r}}\Big(\sum_{T\in\mathcal{T}}\|f\|^r_{3,r,T}\Big)^{\frac{1}{r}}\Big(\sum_{e\subset\Gamma_h}\|\phi_h\|^2_{0,e}\Big)^{\frac{1}{2}}\Big(\sum_{x\in\mathcal{P}}1\Big)^{\frac{1}{2}-\frac{1}{r}} \\
&\quad + Ch^{4-\frac{2}{r}-\frac{2}{r'}}\Big(\sum_{T\in\mathcal{T}_h}\|f\|^r_{3,r,T}\Big)^{\frac{1}{r}}\Big(\sum_{T\in\mathcal{T}_h}\|\nabla\phi_h\|^{r'}_{0,r',T}\Big)^{\frac{1}{r'}} \\
&\le Ch^{2-\frac{1}{r}}\|f\|_{3,r,\Omega_h}\|\phi_h\|_{0,\Gamma_h} + Ch^2\|f\|_{3,r,\Omega_h}\|\nabla\phi_h\|_{0,\Omega_h},
\end{aligned}$$

since $\sum_{x\in\mathcal{P}} 1 \le Ch^{-1}$ and $r' < 2$. Furthermore, recalling that $|\mathcal{P}_2| \le C$,

$$|\sum_{x\in\mathcal{P}_2}\big(\bar{B}_e(f) - \bar{B}_{e'}(f)\big)\phi_h(x)| \le Ch^2\|D^2 f\|_{0,\infty,\Gamma_h}\|\phi_h\|_{0,\infty,\Gamma_h} \le Ch^{\frac{3}{2}}\|f\|_{3,r,\Omega_h}\|\phi_h\|_{0,\Gamma_h}.$$

In conclusion,

$$|I_{13}| \le Ch^{\frac{3}{2}}\|f\|_{3,r,\Omega_h}\|\phi_h\|_{0,\Gamma_h} + Ch^2\|f\|_{3,r,\Omega_h}\|\nabla\phi_h\|_{0,\Omega_h}. \tag{5.8}$$

Combining (5.8) with (5.6) and (5.5) we finish the proof of the lemma. ∎

**Remark 5.3.** Lemma 5.2 continues to hold if the triangulation $\mathcal{T}_h$ is piecewise $O(h^{2\sigma})$ irregular, that is, if $\Omega_h$ can be written as the union of a bounded number of polygonal subdomains each of which is $O(h^{2\sigma})$ irregular (cf. Theorem 4.4 in [1]).

In order to simplify the subsequent analysis we assume from now on that $\Omega \subset \mathbb{R}^2$ is convex and that $\mathcal{A} = -\Delta$. As a consequence, $\Omega_h \subset \Omega$ and $y_h = \mathcal{G}_h(u_h)$ is defined by

$$\int_{\Omega_h}\nabla y_h\cdot\nabla\phi_h = \int_{\Omega_h} f\phi_h, \quad \forall\phi_h \in X_{h0}, \quad y_h = P_h(u_h) \text{ on } \Gamma_h, \tag{5.9}$$

where $P_h$ is again given by (3.2). We extend a function $\phi_h \in X_h$ to $\bar{\Omega}$ as follows: if $\Omega_e$ is the subset of $\Omega \setminus \Omega_h$ bounded by the boundary edge $e \subset T \cap \Gamma_h$ and the curved segment $\tilde{e} \subset \Gamma$, then $\tilde{\phi}_{h|\Omega_e}$ is given by the linear extension of $\phi_h$ from $T$. Furthermore, let $g_h : \Gamma_h \to \Gamma$ be defined by

$$g_h(x) := x + \delta(x)\nu_h(x), \quad x \in e \subset \Gamma_h,$$

where $\nu_h$ is the constant normal to $\Gamma_h$ on $e$ and $\delta(x)$ is chosen in such a way that $g_h(x) \in \Gamma$. Note that $g_h$ is bijective for small $h$. Given $u \in H^s(\Gamma), 0 \le s \le 1$, it follows from Theorem 1 in [4] that

$$\|y - \tilde{y}_h\|_{0,\Omega} \le C\big(h^2\|f\|_{0,\Omega} + h^{s+\frac{1}{2}}\|u\|_{s,\Gamma}\big), \quad y = \mathcal{G}(u), y_h = \mathcal{G}_h(u\circ g_h). \tag{5.10}$$

We are now in position to prove the main result of this section.

**Theorem 5.4.** *Suppose that the triangulation $\mathcal{T}_h$ is piecewise $O(h^2)$ irregular. Let $u$ and $u_h$ be the solutions of (2.3) and (4.1) with corresponding states $y$ and $y_h$ respectively. Then*

$$\|u - \tilde{u}_h\|_{0,\Gamma} + \|y - \tilde{y}_h\|_{0,\Omega} \le Ch^{\frac{3}{2}}$$

*for all $0 < h \le h_0$. Here, $\tilde{u}_h = u_h \circ g_h^{-1}$.*

*Proof.* As in the proof of Theorem 4.1 let $y^h = \mathcal{G}(u_h \circ g_h^{-1}), z_h = \mathcal{G}_h(u \circ g_h)$. We again have

$$\int_\Omega (\tilde{y}_h - y_0)(\tilde{z}_h - \tilde{y}_h) + \alpha \int_\Gamma \tilde{u}_h(u - \tilde{u}_h) + \delta_h \ge 0 \tag{5.11}$$

where now

$$\delta_h = -\int_{\Omega\setminus\Omega_h} (\tilde{y}_h - y_0)(\tilde{z}_h - \tilde{y}_h) + \alpha \int_\Gamma \tilde{u}_h(u - \tilde{u}_h)(d_h - 1).$$

Since $|d_h - 1| \le Ch^2$ in our situation we obtain

$$|\delta_h| \le \big(\|y_0\|_{0,\Omega\setminus\Omega_h} + \|\tilde{y}_h\|_{0,\Omega\setminus\Omega_h}\big)\|\tilde{z}_h - \tilde{y}_h\|_{0,\Omega\setminus\Omega_h} + Ch^2\|u - \tilde{u}_h\|_{0,\Gamma}. \tag{5.12}$$

Using Lemma 2 in [4] we infer that

$$\|y_0\|_{0,\Omega\setminus\Omega_h} \le C\big(h\|y_0\|_{0,\Gamma} + h^2\|y_0\|_{1,\Omega}\big) \le Ch.$$

On the other hand it follows from (2.10) in [4] that for $\phi_h \in X_h$

$$\|\tilde{\phi}_h\|_{0,\Omega\setminus\Omega_h} \le C\big(h\|\phi_h\|_{0,\Gamma_h} + h^2\|\tilde{\phi}_h\|_{1,\Omega}\big) \le C\big(h\|\phi_h\|_{0,\Gamma_h} + h^2\|\phi_h\|_{1,\Omega_h}\big). \tag{5.13}$$

Combining the bounds

$$\|y_h\|_{1,\Omega_h} \le C\big(h^{-\frac{1}{2}}\|u_h\|_{0,\Gamma_h} + \|f\|_{0,\Omega_h}\big) \le Ch^{-\frac{1}{2}}, \quad \|z_h - y_h\|_{1,\Omega_h} \le Ch^{-\frac{1}{2}}\|u \circ g_h - u_h\|_{0,\Gamma_h}$$

with (5.13) we deduce from (5.12)

$$|\delta_h| \le Ch^2\big(\|u \circ g_h - u_h\|_{0,\Gamma_h} + \|u - \tilde{u}_h\|_{0,\Gamma}\big) \le Ch^2\|u - \tilde{u}_h\|_{0,\Gamma}. \tag{5.14}$$

Thus, we deduce from (4.4), (5.11) and (5.14) similarly as in the proof of Theorem 4.1

$$\begin{aligned}\alpha\|u - \tilde{u}_h\|^2_{0,\Gamma} \le& -\frac{1}{2}\|y - \tilde{y}_h\|^2_{0,\Omega} - \int_\Omega (y - y_0)\big((y - y^h) - (\tilde{z}_h - \tilde{y}_h)\big)\\ &+\epsilon\big(\|y - \tilde{y}_h\|^2_{0,\Omega} + \|u - \tilde{u}_h\|^2_{0,\Gamma}\big) + C_\epsilon h^4 + C\|y - \tilde{z}_h\|^2_{0,\Omega}\end{aligned}$$

and hence after choosing $\epsilon$ sufficiently small and applying (5.10)

$$\frac{\alpha}{2}\|u - \tilde{u}_h\|^2_{0,\Gamma} + \frac{1}{4}\|y - \tilde{y}_h\|^2_{0,\Omega} \le Ch^3 - \int_\Omega (y - y_0)\big((y - y^h) - (\tilde{z}_h - \tilde{y}_h)\big). \tag{5.15}$$

Using (2.5) for our case $\mathcal{A} = -\Delta$ as well as integration by parts we have

$$\begin{aligned}&\int_\Omega (y - y_0)\big((y - y^h) - (\tilde{z}_h - \tilde{y}_h)\big) \qquad (5.16)\\ &= -\int_\Omega (y - y^h)\Delta p + \int_{\Omega_h} (z_h - y_h)\Delta p + \int_{\Omega\setminus\Omega_h} (\tilde{z}_h - \tilde{y}_h)\Delta p\\ &= -\int_\Gamma (u - \tilde{u}_h)\partial_\nu p - \int_{\Omega_h} \nabla(z_h - y_h)\cdot\nabla p + \int_{\Gamma_h} P_h\big((u \circ g_h) - u_h\big)\partial_{\nu_h} p\\ &\quad + \int_{\Omega\setminus\Omega_h} (\tilde{z}_h - \tilde{y}_h)\Delta p \equiv S_1 + S_2 + S_3 + S_4.\end{aligned}$$

Taking into account (5.9) we infer with the help of Lemma 5.2

$$\begin{aligned}|S_2| &= |\int_{\Omega_h} \nabla(z_h - y_h)\cdot\nabla(p - I_h p)| \le C\|p\|_{3,r,\Omega_h}\big(h^2\|z_h - y_h\|_{1,\Omega_h} + h^{\frac{3}{2}}\|z_h - y_h\|_{0,\Gamma_h}\big)\\ &\le Ch^{\frac{3}{2}}\|u\circ g_h - u_h\|_{0,\Gamma_h} \le Ch^{\frac{3}{2}}\|u - \tilde{u}_h\|_{0,\Gamma}.\end{aligned}$$

Since $p \in H^3(\Omega)$ we deduce similarly as above that

$$|S_4| \le Ch^2\|u - \tilde{u}_h\|_{0,\Gamma}.$$

Next, recalling the relation $[\tilde{P}_h v]\circ g_h = P_h(v\circ g_h)$ as well as (3.6) we have

$$\begin{aligned}S_3 &= \int_\Gamma \tilde{P}_h(u - \tilde{u}_h)[\nabla p\cdot\nu_h]\circ g_h^{-1}\, d_h\\ &= \int_\Gamma \tilde{P}_h(u-\tilde{u}_h)\partial_\nu p\, d_h + \int_\Gamma \tilde{P}_h(u-\tilde{u}_h)\big([\nabla p\cdot\nu_h]\circ g_h^{-1} - \nabla p\cdot\nu\big)d_h.\end{aligned}$$

In order to deal with the second term we let $y = g_h(x) \in \Gamma$. Since $p = 0$ on $\Gamma$ we have that $\nabla p = \partial_\nu p\, \nu$ on $\Gamma$. Hence

$$\begin{aligned}&[\nabla p\cdot\nu_h](g_h^{-1}(y)) - (\nabla p\cdot\nu)(y) = \nabla p(x)\cdot\nu_h(x) - \nabla p(g_h(x))\cdot\nu(g_h(x))\\ &= \big(\nabla p(x) - \nabla p(g_h(x))\big)\cdot\nu_h(x) + \partial_\nu p(g_h(x))\,\nu(g_h(x))\cdot(\nu_h(x) - \nu(g_h(x)))\\ &= \big(\nabla p(x) - \nabla p(g_h(x))\big)\cdot\nu_h(x) - \frac{1}{2}\partial_\nu p(g_h(x))|\nu(g_h(x)) - \nu_h(x)|^2,\end{aligned}$$

so that

$$|[\nabla p\cdot\nu_h]\circ g_h^{-1} - \nabla p\cdot\nu| \le Ch^2 \qquad \text{on } \Gamma$$

since $|g_h(x) - x| \le Ch^2$, $|\nu(g_h(x)) - \nu_h(x)| \le Ch$. As a result we may write

$$S_1 + S_3 = -\int_\Gamma \big((u-\tilde{u}_h) - \tilde{P}_h(u-\tilde{u}_h)\big)\partial_\nu p d_h + r_h = -\int_\Gamma (u-\tilde{u}_h)\big(\partial_\nu p - \tilde{P}_h\partial_\nu p\big)\, d_h + r_h$$

where $|r_h| \le Ch^2\|u-\tilde{u}_h\|_{0,\Gamma}$. Now, (3.8) implies

$$|S_1 + S_3| \le Ch^{\frac{3}{2}}\|\partial_\nu p\|_{\frac{3}{2},\Gamma}\|u - \tilde{u}_h\|_{0,\Gamma} + |r_h| \le Ch^{\frac{3}{2}}\|u-\tilde{u}_h\|_{0,\Gamma}.$$

Returning to (5.16) we finally obtain

$$|\int_\Omega (y - y_0)\big((y - y^h) - (z_h - y_h)\big)| \le Ch^{\frac{3}{2}}\|u - \tilde{u}_h\|_{0,\Gamma}$$

and the result follows after inserting this estimate into (5.15). ∎

## 6 Numerical examples

For our numerical experiments we consider the variational discretization (4.1) of problem (2.3) with the unit circle $\Omega = B_1(0)$ as domain and $\mathcal{A} = -\Delta$ as differential operator. We set $\alpha = 1$, $a = 0$ and $b = 1$. For the numerical solution of the optimal control problem (4.1) we apply the fixpoint iteration

- $v \in U_{h,\text{ad}}$ given
- $v^+ := P_{[a,b]}(\frac{1}{\alpha}\partial^h_{\nu_\mathcal{A}} p_h(v))$

- $v := v^+$.

Here, for given $v \in U_{h,\mathrm{ad}}$ the function $\partial^h_{\nu_{\mathcal{A}}} p_h(v)$ is defined by (4.3) with $y_h = \mathcal{G}_h(v)$. We note that the variational discrete solution $u_h$ may admit active sets whose boundaries do not coincide with finite element nodes, compare Fig. 1(a), where the boundary control $u_h$ (bold) is depicted on a coarse mesh together with function $\frac{1}{\alpha}\partial^h_{\nu_{\mathcal{A}}} p_h(u_h)$ (dotted).

We consider two examples and investigate the error functionals

$$E^0_u(h) = \|u - \tilde{u}_h\|_{0,\Gamma}, \qquad E^0_y(h) = \|y - y_h\|_{0,\Omega_h}, \qquad E^1_y(h) = \|y - y_h\|_{1,\Omega_h},$$
$$E^0_p(h) = \|p - p_h\|_{0,\Omega_h}, \qquad E^1_p(h) = \|p - p_h\|_{1,\Omega_h},$$

both on a sequence of arbitrary meshes and on a sequence of congruently refined, piecewise $O(h^2)$ irregular meshes. Fig. 1(b) shows an arbitrary mesh while Fig. 1(c) depicts a grid of the type which we use to numerically confirm our superconvergence result of Theorem 5.4.

**Remark 6.1.** The triangulation in Fig. 1(c) is piecewise $O(h^2)$ irregular, but only $O(h)$ irregular. It is automatically constructed by congruent refinement from the initial grid formed by the 8 bold sector boarders together with the corresponding sector secants. Here we note that new boundary nodes are projected onto the unit circle. The resulting triangulation in each of the 8 sectors then is $O(h^2)$ irregular.

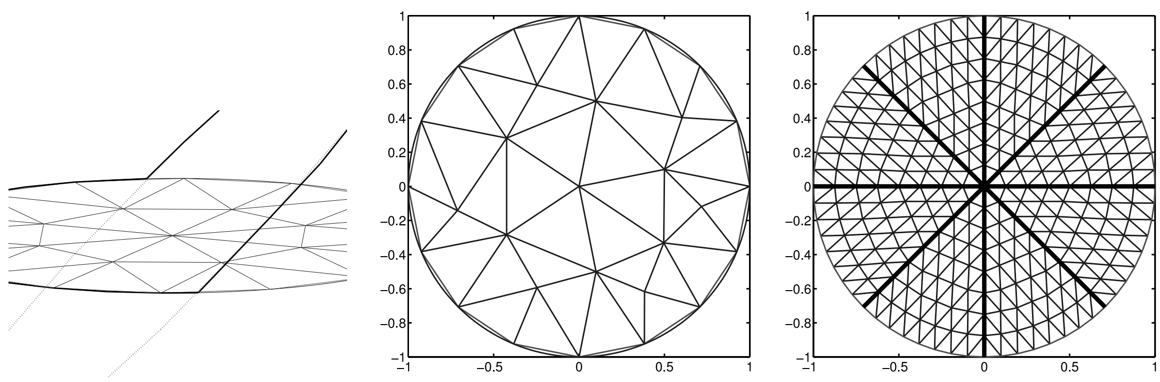
Piecewise $O(h^2)$ irregular meshes are often generated automatically by congruent refinement, say from an initial grid $\mathcal{T}_0$ containing finitely many triangles $T$ combined with projecting boundary nodes onto smooth domain boundaries. Every sub-triangulation obtained in this way from some $T \in \mathcal{T}_0$ then is $O(h^2)$ irregular. This in view of Theorem 5.4 explains why in practice one often observes better rates of convergence than expected from the general theory, compare the discussion in [1].

Tables 1 and 2 summarize the mesh-properties in terms of the number of triangles $nt$, the number of nodes $np$ and the mesh parameter $h$.

For an error functional we define the experimental order of convergence by

$$\mathrm{EOC} = \frac{\log E(h_1) - \log E(h_2)}{\log h_1 - \log h_2}.$$

Finally for an arbitrary function $g : B_1(0) \to \mathbb{R}$ we abbreviate $\hat{g}(r,\phi) := g(r\cos\phi, r\sin\phi)$, where $(r,\phi) \in (0,1] \times [0,2\pi)$.




(a) Variational discretization. (b) Arbitrary mesh ($i = 2$). (c) Superconvergence mesh ($j = 4$).

Figure 1: Variational discretization and considered triangulations.

| $i$ | $nt$ | $np$ | $h$ |
|---|---|---|---|
| 1 | 8 | 9 | 1.000000 |
| 2 | 40 | 29 | 0.596568 |
| 3 | 170 | 102 | 0.298819 |
| 4 | 684 | 371 | 0.149721 |
| 5 | 2680 | 1393 | 0.074921 |
| 6 | 10812 | 5511 | 0.037497 |
| 7 | 44568 | 22489 | 0.018749 |
| 8 | 179292 | 90051 | 0.009375 |
| 9 | 701964 | 351791 | 0.004687 |

Table 1: Mesh parameters for the sequence of arbitrary meshes.

| $j$ | $nt$ | $np$ | $h$ |
|---|---|---|---|
| 1 | 8 | 9 | 1.000000 |
| 2 | 32 | 25 | 0.571070 |
| 3 | 128 | 81 | 0.302195 |
| 4 | 512 | 289 | 0.155086 |
| 5 | 2048 | 1089 | 0.078516 |
| 6 | 8192 | 4225 | 0.039498 |
| 7 | 32768 | 16641 | 0.019809 |
| 8 | 131072 | 66049 | 0.009919 |
| 9 | 524288 | 263169 | 0.004963 |

Table 2: Mesh parameters for the sequence of piecewise $O(h^2)$ irregular meshes.

In our first example we consider problem (2.3) with contiuous data $f$ and smooth data $y_0$. For this purpose we set

$$\begin{aligned} \hat{y}(r,\phi) &= r^3 \max(0, \cos^3\phi) \\ \widehat{y_0}(r,\phi) &= (7r^2\cos^2\phi + 6r^2 - 6r)\cos\phi + \hat{y}(r,\phi) \quad \text{and} \\ \hat{f}(r,\phi) &= -6r\max(0,\cos\phi). \end{aligned}$$

Then it is easy to check that $\hat{u}(1,\phi) = \hat{u}(\phi) = \max(0,\cos^3\phi)$ solves (2.3) and the associated adjoint variable is given by $\hat{p}(r,\phi) = r^3(r-1)\cos^3\phi$. In the present example we deal with classical solutions in the sense that $y, p \in C^2(\bar{\Omega})$ and $u \in C^2(\Gamma)$, see Figs. 2(a) and 2(b).

Table 3 summarizes the numerical results for the sequence of arbitrary meshes from Table 1. In addition to the EOCs for two consecutive meshes also the average and the EOC between coarsest and finest grid is computed in the rows $\varnothing$ and $\frac{1}{9}$. The EOC for $E_u^0$ behaves as predicted by Theorem 4.1, whereas the $L^2$-error of the state $E_y^0$ converges with a rate of 1.5 faster than predicted. In Table 4 we present the numerical results for our sequence of $O(h^2)$ irregular meshes. One clearly observes the superconvergence effect for piecewise $O(h^2)$ irregular grids predicted by Theorem 5.4. Again the rate of convergence for $E_u^0$ behaves as expected whereas the EOC for $E_y^0$ is nearly quadratic.

Next, let us construct an analytical solution to problem (2.3) in the same way as in the previous example but with less regular data and hence less regular optimal control. We choose

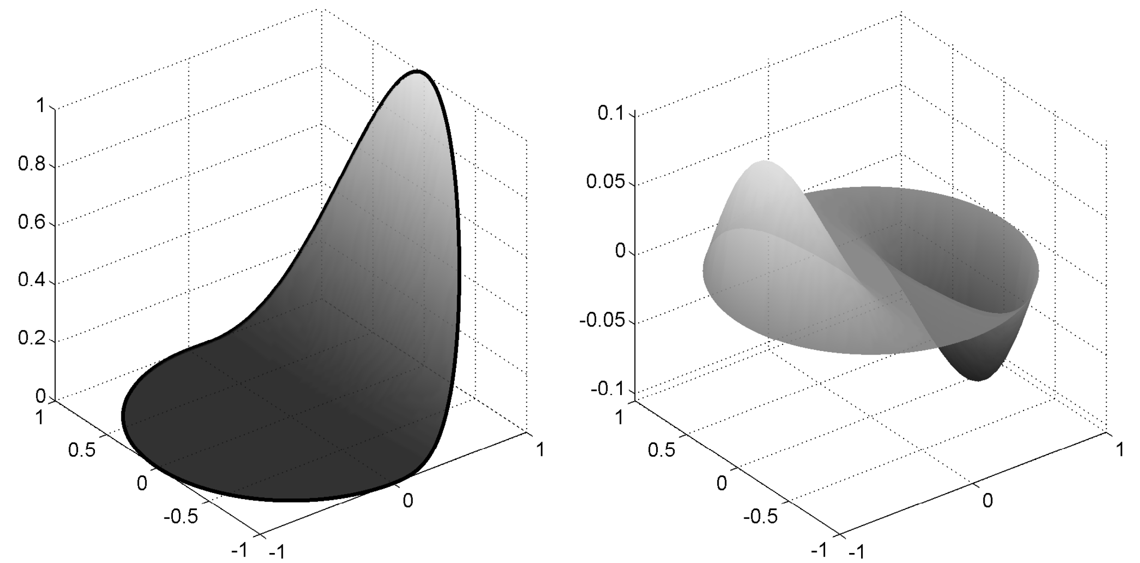
$$\begin{aligned} \hat{y}(r,\phi) &= r^3 \max(0,\cos\phi), \\ \widehat{y_0}(r,\phi) &= (15r^2 - 8r)\cos\phi + \hat{y}(r,\phi) \end{aligned}$$

and set $f := -\Delta y$. Then $\hat{u}(1,\phi) = \hat{u}(\phi) = \max(0,\cos\phi)$ solves (2.3) and the associated adjoint variable is given by $\hat{p}(r,\phi) = r^3(r-1)\cos\phi$. Let us note that $f = -\Delta y$ has to be understood in the distributional sense, i.e.

$$\langle f, \zeta\rangle = -\int_{\mathfrak{D}} 8r(x_1,x_2)\cos(\phi(x_1,x_2))\zeta(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 - \int_{-1}^{1} x_2^2 \zeta(0,x_2)\,\mathrm{d}x_2 \qquad \forall\, \zeta \in C_0^\infty(\Omega),$$

where $\mathfrak{D} = \{(x_1,x_2) \in \bar{\Omega} \,|\, x_1 > 0\}$.

Fig. 3(a) shows the optimal state $y$ with the optimal boundary control $u$ and Fig. 3(b) presents the associated adjoint state $p$. The convergence behaviour of our error functionals is similar to that observed in the previous example. For arbitrary meshes $E_u^0$ converges linearly as is shown in Table 5. On our sequence of piecewise $O(h^2)$ irregular meshes the convergence rate of this error functional improves to 1.5 as displayed in Table 6. Again in both cases the behaviour of $E_y^0$ is better than predicted and the convergence rate on our sequence of piecewise $O(h^2)$ irregular meshes is higher than on the sequence of arbitrary meshes.




(a) Optimal state $y$ with boundary control $u$.

(b) Adjoint state $p$.

Figure 2: Analytical solution of Example 1.

| $i$ | $E_u^0$ | EOC | $E_y^0$ | EOC | $E_y^1$ | EOC | $E_p^0$ | EOC | $E_p^1$ | EOC |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.277414 | - | 0.149239 | - | 0.983167 | - | 0.073546 | - | 0.313464 | - |
| 2 | 0.071514 | 2.624 | 0.040577 | 2.521 | 0.441360 | 1.550 | 0.039436 | 1.207 | 0.287896 | 0.165 |
| 3 | 0.070380 | 0.023 | 0.023135 | 0.813 | 0.407958 | 0.114 | 0.012772 | 1.631 | 0.175988 | 0.712 |
| 4 | 0.018892 | 1.903 | 0.005006 | 2.215 | 0.158316 | 1.370 | 0.003133 | 2.034 | 0.085166 | 1.050 |
| 5 | 0.011166 | 0.760 | 0.001868 | 1.424 | 0.104513 | 0.600 | 0.000771 | 2.024 | 0.041827 | 1.027 |
| 6 | 0.006742 | 0.729 | 0.000762 | 1.295 | 0.081769 | 0.355 | 0.000197 | 1.970 | 0.021083 | 0.990 |
| 7 | 0.004180 | 0.690 | 0.000341 | 1.159 | 0.078123 | 0.066 | 0.000050 | 1.978 | 0.010630 | 0.988 |
| 8 | 0.002040 | 1.035 | 0.000124 | 1.456 | 0.050939 | 0.617 | 0.000012 | 2.013 | 0.005287 | 1.008 |
| 9 | 0.000994 | 1.037 | 0.000044 | 1.513 | 0.033625 | 0.599 | 0.000003 | 2.004 | 0.002635 | 1.005 |
| $\frac{1}{9}$ | | 1.050 | | 1.518 | | 0.629 | | 1.879 | | 0.891 |
| ∅ | | 1.100 | | 1.550 | | 0.659 | | 1.858 | | 0.868 |

Table 3: Errors and EOCs for arbitrary meshes of Example 1.

## Acknowledgements

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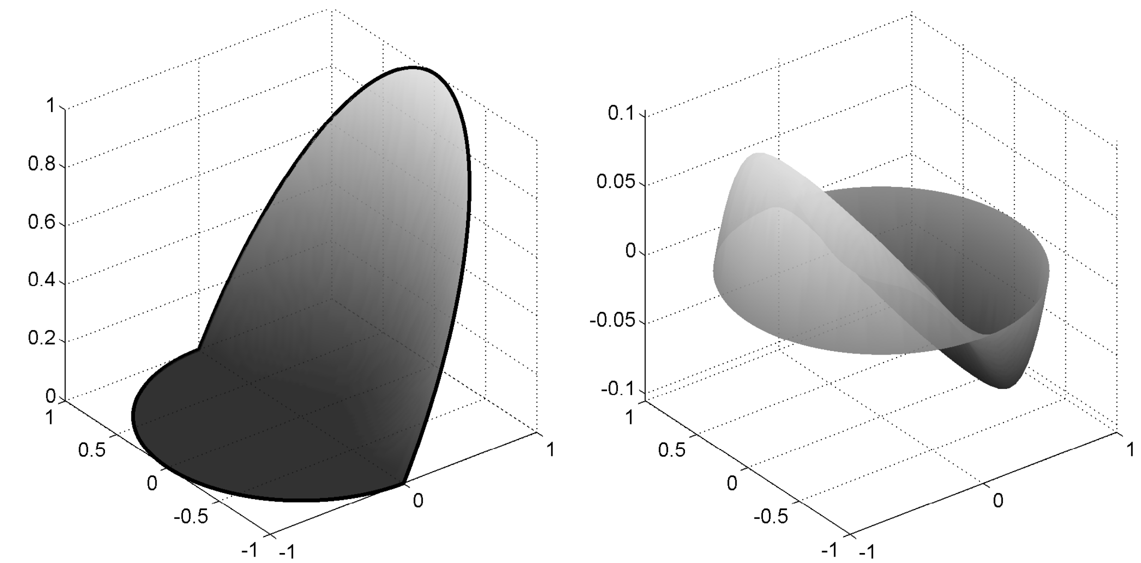
| $j$ | $E_u^0$ | EOC | $E_y^0$ | EOC | $E_y^1$ | EOC | $E_p^0$ | EOC | $E_p^1$ | EOC |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.277414 | - | 0.149239 | - | 0.983167 | - | 0.073546 | - | 0.313464 | - |
| 2 | 0.170809 | 0.866 | 0.099800 | 0.718 | 0.904301 | 0.149 | 0.050445 | 0.673 | 0.330714 | -0.096 |
| 3 | 0.096494 | 0.897 | 0.033170 | 1.731 | 0.587454 | 0.678 | 0.017067 | 1.703 | 0.207877 | 0.730 |
| 4 | 0.044380 | 1.164 | 0.010026 | 1.794 | 0.336040 | 0.837 | 0.004614 | 1.961 | 0.110293 | 0.950 |
| 5 | 0.018420 | 1.292 | 0.003010 | 1.768 | 0.184591 | 0.880 | 0.001177 | 2.007 | 0.056021 | 0.995 |
| 6 | 0.007163 | 1.375 | 0.000878 | 1.794 | 0.098095 | 0.920 | 0.000296 | 2.010 | 0.028126 | 1.003 |
| 7 | 0.002676 | 1.427 | 0.000248 | 1.833 | 0.050908 | 0.950 | 0.000074 | 2.007 | 0.014078 | 1.003 |
| 8 | 0.000976 | 1.458 | 0.000068 | 1.864 | 0.026013 | 0.971 | 0.000019 | 2.004 | 0.007041 | 1.002 |
| 9 | 0.000351 | 1.477 | 0.000019 | 1.884 | 0.013161 | 0.984 | 0.000005 | 2.002 | 0.003521 | 1.001 |

Table 4: Errors and EOCs for piecewise $O(h^2)$ irregular meshes of Example 1.




(a) Optimal state $y$ with boundary control $u$.

(b) Adjoint state $p$.

Figure 3: Analytical solution of Example 2.

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| $i$ | $E_u^0$ | EOC | $E_y^0$ | EOC | $E_y^1$ | EOC | $E_p^0$ | EOC | $E_p^1$ | EOC |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.297512 | - | 0.179552 | - | 0.904504 | - | 0.092648 | - | 0.376590 | - |
| 2 | 0.138275 | 1.483 | 0.057686 | 2.198 | 0.639030 | 0.673 | 0.049097 | 1.229 | 0.353687 | 0.121 |
| 3 | 0.098899 | 0.485 | 0.029068 | 0.991 | 0.557107 | 0.198 | 0.015616 | 1.657 | 0.214636 | 0.722 |
| 4 | 0.019660 | 2.338 | 0.005318 | 2.458 | 0.181005 | 1.627 | 0.003832 | 2.033 | 0.103822 | 1.051 |
| 5 | 0.016497 | 0.253 | 0.002845 | 0.904 | 0.167813 | 0.109 | 0.000959 | 2.000 | 0.051436 | 1.014 |
| 6 | 0.008651 | 0.932 | 0.001009 | 1.498 | 0.113754 | 0.562 | 0.000244 | 1.979 | 0.025931 | 0.989 |
| 7 | 0.005008 | 0.789 | 0.000422 | 1.256 | 0.097192 | 0.227 | 0.000061 | 1.987 | 0.013014 | 0.995 |
| 8 | 0.002531 | 0.985 | 0.000158 | 1.421 | 0.066351 | 0.551 | 0.000015 | 2.003 | 0.006483 | 1.005 |
| 9 | 0.001241 | 1.028 | 0.000057 | 1.469 | 0.045102 | 0.557 | 0.000004 | 1.993 | 0.003234 | 1.003 |
| $\frac{1}{9}$ | | 1.022 | | 1.502 | | 0.559 | | 1.881 | | 0.887 |
| ∅ | | 1.037 | | 1.524 | | 0.563 | | 1.860 | | 0.863 |

Table 5: Errors and EOCs for arbitrary meshes of Example 2.

| $j$ | $E_u^0$ | EOC | $E_y^0$ | EOC | $E_y^1$ | EOC | $E_p^0$ | EOC | $E_p^1$ | EOC |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0.325180 | - | 0.138937 | - | 0.856030 | - | 0.093237 | - | 0.377756 | - |
| 2 | 0.208354 | 0.795 | 0.105716 | 0.488 | 1.047727 | -0.361 | 0.062975 | 0.700 | 0.405792 | -0.128 |
| 3 | 0.121702 | 0.845 | 0.038899 | 1.571 | 0.720985 | 0.587 | 0.021157 | 1.714 | 0.256137 | 0.723 |
| 4 | 0.057121 | 1.134 | 0.012582 | 1.692 | 0.435196 | 0.757 | 0.005715 | 1.962 | 0.136028 | 0.949 |
| 5 | 0.023779 | 1.287 | 0.003810 | 1.755 | 0.245236 | 0.843 | 0.001459 | 2.006 | 0.069111 | 0.995 |
| 6 | 0.009233 | 1.377 | 0.001096 | 1.813 | 0.131543 | 0.907 | 0.000367 | 2.010 | 0.034699 | 1.003 |
| 7 | 0.003442 | 1.430 | 0.000305 | 1.853 | 0.068408 | 0.947 | 0.000092 | 2.007 | 0.017368 | 1.003 |
| 8 | 0.001254 | 1.460 | 0.000083 | 1.877 | 0.034941 | 0.971 | 0.000023 | 2.004 | 0.008686 | 1.002 |
| 9 | 0.000451 | 1.478 | 0.000022 | 1.895 | 0.017675 | 0.984 | 0.000006 | 2.002 | 0.004344 | 1.001 |

Table 6: Errors and EOCs for piecewise $O(h^2)$ irregular meshes of Example 2.


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