Hamburger Beiträge
zur Angewandten Mathematik

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Klaus Deckelnick and Michael Hinze

Nr. 2009-04
March 2009

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Klaus Deckelnick*& Michael Hinze†

Abstract: We consider a parabolic optimal control problem with pointwise state constraints. The optimization problem is approximated by a discrete control problem based on a discretization of the state equation by linear finite elements in space and a discontinuous Galerkin scheme in time. Error bounds for control and state are obtained both in two and three space dimensions. These bounds follow from uniform estimates for the discretization error of the state under natural regularity requirements.

Mathematics Subject Classification (2000): 49J20, 49K20, 35B37

Keywords: Parabolic optimal control problem, state constraints, error estimates

1 Introduction

Optimal control of time-dependent production processes plays an important role in many practical applications such as crystal growth [8, 12, 13] and cooling of glass melts [4, 16]. These processes are frequently described by systems of partial differential equations involving the temperature as a system variable. A need to avoid overheating of the device or to prevent solidification/melting at the wrong places then naturally leads to pointwise bounds on the temperature variable. The introduction of pointwise state conditions however yields adjoint variables and multipliers which only admit low regularity complicating the analysis of the necessary first order conditions. These problems need to be taken into account in the numerical approximation and necessitate the development of tailored discrete concepts.
In the present work we consider an optimal control problem for the heat equation and with pointwise bounds on the state. The optimization problem is approximated using variational discretization [10] combined with linear finite elements in space and a discontinuous Galerkin scheme in time for the discretization of the state equation, compare [11, Chapter 3]. Our main result are L^2–error estimates for the optimal state and the optimal control. To derive these bounds, uniform estimates for the discretization error of the state under natural regularity requirements are proved. For the numerical analysis of the optimal control problem we use an approach which avoids error estimates for the adjoint state and which was developed in [5], [11, Chapter 3] for the analysis of elliptic optimal control problems with state and gradient constraints.
To the best of the authors knowledge numerical analysis of parabolic optimal control problems with pointwise bounds in space-time for the state has not yet been considered in the literature. In this work we present the numerical analysis for our result of Theorem 4.1 which we already

*Institut für Analysis und Numerik, Otto–von–Guericke–Universität Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany

†Schwerpunkt Optimierung und Approximation, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany.

announced in [9]. However, there are some contributions on the analysis of related control problems. In [14] Lavrentiev regularization of state constrained parabolic control problems is investigated, optimal control problems with pointwise state constraints in time and averaged state constraints in space are considered in [1]. We note that numerical analysis for this particular setting is announced by Vexler in [9]. Optimality conditions for parabolic optimal control problems in the presence of state constraints are investigated in [6], where further references on analysis aspects of state constrained parabolic control problems can be found.

2 The optimal control problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a bounded convex polygonal domain, $T > 0$, $\Omega_T := \Omega \times (0, T)$ and $\Gamma_T := \partial\Omega \times (0, T)$. Let us consider the initial boundary value problem

$$y_t - \Delta y = f \quad \text{in } \Omega_T \tag{2.1}$$

$$\frac{\partial y}{\partial \nu} = 0 \quad \text{on } \Gamma_T \tag{2.2}$$

$$y(\cdot, 0) = y_0 \quad \text{in } \Omega. \tag{2.3}$$

It is well–known that for a given $f \in L^2(0, T; L^2(\Omega)), y_0 \in H^1(\Omega)$ problem (2.1)–(2.3) has a unique solution $y \in C^0([0, T]; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega))$. In what follows we shall keep the initial datum y_0 fixed and denote by $\hat{y}$ the solution of (2.1)–(2.3) corresponding to $f \equiv 0$. This allows us to write the solution of (2.1)–(2.3) in the form

$$y = \mathcal{G}(f) = \hat{y} + \mathcal{G}_0(f), \tag{2.4}$$

where $\mathcal{G}_0(f)$ is the linear operator that assigns to f the solution of (2.1)–(2.3) for $y_0 \equiv 0$. Note that if $f \in L^2(0, T; H^1(\Omega))$ and

$$y_0 \in H^2(\Omega) \quad \text{with } \frac{\partial y_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \tag{2.5}$$

then we have

$$y \in W := \{w \in C^0([0, T]; H^2(\Omega)) \mid w_t \in L^2(0, T; H^1(\Omega))\}$$

and

$$\max_{0 \le t \le T} \|y(t)\|^2_{H^2} + \int_0^T \|y_t(t)\|^2_{H^1} dt \le C\big(\|y_0\|^2_{H^2} + \int_0^T \|f(t)\|^2_{H^1} dt\big). \tag{2.6}$$

We remark that $W \subset C^0(\overline{\Omega_T})$ since we have the continuous embedding $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$ for $d = 2, 3$.

Next, suppose that the functions $f_1, \ldots, f_m \in H^1(\Omega) \cap L^\infty(\Omega)$ are given and define $U := L^2(0, T; \mathbb{R}^m)$ as well as $B : U \to L^2(0, T; H^1(\Omega))$ by

$$(Bu)(x, t) := \sum_{i=1}^m u_i(t) f_i(x), \quad (x, t) \in \Omega_T.$$

This parametrization of the control is motivated by practical considerations. The functions f_i represent given practical control actuations, whose impact is controlled through the time-dependent amplitudes u_i which in our context play the role of control functions.
Note that (2.6) implies that $y = \mathcal{G}(Bu) \in W$ for $u \in U$ with

$$\max_{0 \le t \le T} \|y(t)\|^2_{H^2} + \int_0^T \|y_t(t)\|^2_{H^1} dt \le C\big(\|y_0\|^2_{H^2} + \int_0^T |u(t)|^2 dt\big), \tag{2.7}$$

where the constant C depends in addition on the H^1–norms of $f_1, \ldots, f_m$.
Let us denote by $\mathcal{M}(\overline{\Omega_T})$ the space of regular Borel measures on $\overline{\Omega_T}$. Given $\mu \in \mathcal{M}(\overline{\Omega_T})$ we consider the following backward parabolic problem

$$-\varphi_t - \Delta\varphi = \mu_{\Omega_T} \quad \text{in } \Omega_T \tag{2.8}$$

$$\frac{\partial\varphi}{\partial\nu} = \mu_{\Gamma_T} \quad \text{on } \Gamma_T \tag{2.9}$$

$$\varphi(\cdot, T) = \mu_T \quad \text{in } \Omega. \tag{2.10}$$

Here, $\mu_{\Omega_T} := \mu_{|\Omega_T}$, $\mu_{\Gamma_T} := \mu_{|\Gamma_T}$ and $\mu_T := \mu_{|\bar{\Omega}\times\{T\}}$.

Theorem 2.1. *There exists a unique function φ that belongs to $L^s(0,T;W^{1,\sigma}(\Omega))$ for all $s, \sigma \in [1,2)$ with $\frac{2}{s} + \frac{d}{\sigma} > d+1$ and which solves (2.8)–(2.10) in the sense that*

$$\int_0^T (w_t - \Delta w, \varphi) + \int_0^T \int_{\partial\Omega} \frac{\partial w}{\partial\nu}\varphi = \int_{\overline{\Omega_T}} w d\mu \qquad \forall w \in W_0^\infty, \tag{2.11}$$

where $W_0^\infty := \{w \in W \,|\, w(\cdot,0) = 0 \text{ in } \bar{\Omega}, w_t - \Delta w \in L^\infty(\Omega_T), \frac{\partial w}{\partial\nu} \in L^\infty(\Gamma_T)\}$. Here, $(.,.)$ denotes the inner product in $L^2(\Omega)$.

Proof. See [6], Theorem 6.3. ∎

Note that $\varphi \in L^1(0,T;W^{1,1}(\Omega))$ so that all integrals in (2.11) exist.
We consider the optimization problem

$$(TP) \quad \begin{cases} \min_{u\in U} J(u) := \dfrac{1}{2}\displaystyle\int_0^T \|y(\cdot,t) - \bar{y}(\cdot,t)\|^2 dt + \dfrac{\alpha}{2}\displaystyle\int_0^T |u(t)|^2 dt \\ \text{s.t. } y = \mathcal{G}(Bu), \text{ and } y(x,t) \geq 0, (x,t) \in \overline{\Omega_T}, \end{cases} \tag{2.12}$$

where $\bar{y} \in H^1(0,T;L^2(\Omega))$ is given. From now on we shall assume (2.5) and that $\min_{x\in\bar{\Omega}} y_0(x) > 0$. It is not difficult to verify with the help of a comparison argument that the function $\hat{y}$ in (2.4) satisfies

$$\hat{y}(x,t) > 0, (x,t) \in \overline{\Omega_T}. \tag{2.13}$$

Since the state constraints form a convex set and the set of admissible controls is closed and convex one obtains the existence of a unique solution $u \in U$ to problem (2.12) by standard arguments.

Theorem 2.2. *Let $u \in U$ denote the unique solution of* (2.12)*. Then there exist $\mu \in \mathcal{M}(\overline{\Omega_T})$ and a function $p \in L^s(0,T;W^{1,\sigma}(\Omega))$ for all $s, \sigma \in [1,2)$ with $\frac{2}{s} + \frac{d}{\sigma} > d+1$, such that with $y = \mathcal{G}(Bu)$ there holds*

$$\int_0^T (w_t - \Delta w, p) + \int_0^T \int_{\partial\Omega} \frac{\partial w}{\partial\nu} p = \int_0^T (y - \bar{y}, w) + \int_{\overline{\Omega_T}} w d\mu \qquad \forall w \in W_0^\infty, \tag{2.14}$$

$$\alpha u(t) + \big((p(\cdot,t), f_i)\big)_{i=1,\ldots,m} = 0 \qquad \textit{a.e. in } (0,T), \tag{2.15}$$

$$\mu \leq 0, \ y(x,t) \geq 0, (x,t) \in \overline{\Omega_T} \textit{ and } \int_{\overline{\Omega_T}} y d\mu = 0. \tag{2.16}$$

Proof. We apply Theorem 5.2 in [3] (compare also [2, Theorem 2]) with the choices $U = L^2(0,T;\mathbb{R}^m)$, $Z = C^0(\overline{\Omega_T})$, $K = U$ and

$$C = \{z \in Z \,|\, z(x,t) \geq 0 \quad \forall (x,t) \in \overline{\Omega_T}\}.$$

Furthermore, let $G : U \to Z, G(v) := \mathcal{G}(Bv) = \hat{y} + \mathcal{G}_0(Bv)$. Clearly, $DG(u)v = \mathcal{G}_0(Bv)$ so that we obtain in particular with the choice $u_0 = 0$

$$G(u) + DG(u)(u_0 - u) = \hat{y} + \mathcal{G}_0(Bu) - \mathcal{G}_0(Bu) = \hat{y} \in \mathrm{int}(C)$$

by (2.13). According to Theorem 5.2 in [3] there exists $\mu \in \big(C^0(\overline{\Omega_T})\big)' = \mathcal{M}(\overline{\Omega_T})$ such that

$$\begin{aligned} \int_{\overline{\Omega_T}} (z - y) d\mu &\leq 0 \quad \forall z \in C, & (2.17)\\ J'(u)v + \langle DG(u)^* \mu, v \rangle_U &= 0 \quad \forall v \in U. & (2.18) \end{aligned}$$

Standard measure theoretic arguments imply that $\mu \leq 0$ and that $\mathrm{supp}\mu \subset \{(x,t) \in \overline{\Omega_T} \,|\, y(x,t) = 0\}$ giving (2.16). Since $y_0 > 0$ in $\bar{\Omega}$ this yields in particular that $\mathrm{supp}\mu \subset \bar{\Omega} \times (0,T]$. Next, we calculate

$$J'(u)v = \int_0^T (y - \bar{y}, y_v) + \alpha \int_0^T u \cdot v, \quad v \in U, \text{ where } y_v = \mathcal{G}_0(Bv). \tag{2.19}$$

Furthermore, since $DG(u)v = y_v$ we have

$$\langle DG(u)^* \mu, v \rangle_U = \int_{\overline{\Omega_T}} y_v d\mu.$$

Hence, combining (2.18) and (2.19) we derive

$$\int_0^T (y - \bar{y}, y_v) + \alpha \int_0^T u \cdot v + \int_{\overline{\Omega_T}} y_v d\mu = 0 \quad \forall v \in U. \tag{2.20}$$

In view of Theorem 2.1 there exists a unique solution $p \in L^s(0,T; W^{1,\sigma}(\Omega))$ ($s, \sigma \in [1,2)$ with $\frac{2}{s} + \frac{d}{\sigma} > d + 1$) of the backward parabolic problem

$$\begin{aligned} -p_t - \Delta p &= y - \bar{y} + \mu_{\Omega_T} \quad \text{in } \Omega_T & (2.21)\\ \frac{\partial p}{\partial \nu} &= \mu_{\Gamma_T} \quad \text{on } \Gamma_T & (2.22)\\ p(\cdot, T) &= \mu_T \quad \text{in } \Omega, & (2.23) \end{aligned}$$

so that

$$\int_0^T (w_t - \Delta w, p) + \int_0^T \int_{\partial\Omega} \frac{\partial w}{\partial \nu} p = \int_0^T (y - \bar{y}, w) + \int_{\overline{\Omega_T}} w d\mu \quad \forall w \in W_0^\infty. \tag{2.24}$$

It remains to verify (2.15). If $v \in C_0^\infty(0,T;\mathbb{R}^m)$ then $y_v = \mathcal{G}_0(Bv)$ belongs to W_0^∞ because we have assumed that $f_i \in L^\infty(\Omega), i = 1, \ldots, m$. Hence we deduce from (2.20), (2.24) and the definition of y_v that

$$\begin{aligned} 0 &= \int_0^T (y - \bar{y}, y_v) + \alpha \int_0^T u \cdot v + \int_{\overline{\Omega_T}} y_v d\mu = \int_0^T (y_{v,t} - \Delta y_v, p) + \alpha \int_0^T u \cdot v \\ &= \sum_{i=1}^m \int_0^T v_i \big\{ (p(\cdot, t), f_i) + \alpha u_i \big\}. \end{aligned}$$

Since $v \in C_0^\infty(0,T;\mathbb{R}^m)$ is arbitrary we obtain (2.15). ∎

3 Discretization

Let $\mathcal{T}_h$ be a quasi–uniform triangulation of Ω with maximum mesh size $h := \max_{S\in\mathcal{T}_h} \operatorname{diam}(S)$. Let us denote by $x_1, \ldots, x_J$ the set of nodes of $\mathcal{T}_h$. We consider the space of linear finite elements

$$X_h := \{\phi_h \in C^0(\bar{\Omega}) \,|\, \phi_h \text{ is a linear polynomial on each } S \in \mathcal{T}_h\}.$$

We denote by I_h the usual Lagrange interpolation operator and by $P_h : L^2(\Omega) \to X_h$ the L^2–projection, i.e.

$$(z, \phi_h) = (P_h z, \phi_h) \qquad \forall \phi_h \in X_h.$$

Furthermore, let $R_h : H^1(\Omega) \to X_h$ be the Ritz–projection, defined by the relation

$$(\nabla R_h z, \nabla \phi_h) + (R_h z, \phi_h) = (\nabla z, \nabla \phi_h) + (z, \phi_h) \quad \forall \phi_h \in X_h. \tag{3.1}$$

It is well–known that

$$\|z - R_h z\| + h\|\nabla(z - R_h z)\| \le C h^m \|z\|_{H^m} \quad \forall z \in H^m(\Omega),\ m = 1, 2. \tag{3.2}$$

We shall also require a uniform bound on $z - R_h z$. Using interpolation and inverse estimates together with (3.2) we find for $z \in H^2(\Omega)$ that

$$\begin{aligned} \|z - R_h z\|_{L^\infty} &\le \|z - I_h z\|_{L^\infty} + \|I_h z - R_h z\|_{L^\infty} \\ &\le C h^{2-\frac{d}{2}} \|z\|_{H^2} + C h^{-\frac{d}{2}} \|I_h z - R_h z\| \le C h^{2-\frac{d}{2}} \|z\|_{H^2}. \end{aligned} \tag{3.3}$$

Furthermore, we have the following estimate for functions $\phi_h \in X_h$,

$$\|\phi_h\|_{L^\infty} \le C \rho(d, h) \|\phi_h\|_{H^1} \tag{3.4}$$

where

$$\rho(d, h) = \begin{cases} \sqrt{|\log h|}, & d = 2, \\ h^{-\frac{1}{2}}, & d = 3. \end{cases}$$

Next, let $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ a time grid with $\tau_n := t_n - t_{n-1}, n = 1, \ldots, N$ and $\tau := \max_{1\le n\le N} \tau_n$. We set

$$W_{h,\tau} := \{\Phi : \bar{\Omega} \times [0, T] \,|\, \Phi(\cdot, t) \in X_h \text{ is constant in } t \in (t_{n-1}, t_n), 1 \le t \le N\}.$$

For $Y, \Phi \in W_{h,\tau}$ we let

$$A(Y, \Phi) := \sum_{n=1}^{N} \tau_n (\nabla Y^n, \nabla \Phi^n) + \sum_{n=2}^{N} (Y^n - Y^{n-1}, \Phi^n) + (Y^0_+, \Phi^0_+),$$

where $\Phi^n := \Phi^n_-, \Phi^n_\pm = \lim_{s\to 0^\pm} \Phi(t_n + s)$. Given $u \in U$, our approximation $Y \in W_{h,\tau}$ of the solution y of (2.1)–(2.3) is obtained by the following discontinuous Galerkin scheme:

$$A(Y, \Phi) = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (Bu(t), \Phi^n) + (y_0, \Phi^0_+) \qquad \forall \Phi \in W_{h,\tau}. \tag{3.5}$$

The above solution will be denoted by $Y = \mathcal{G}_{h,\tau}(Bu)$. We have the following uniform error estimate.

Theorem 3.1. *Let $u \in U, y = \mathcal{G}(Bu), Y = \mathcal{G}_{h,\tau}(Bu)$. Then*

$$\max_{1 \le n \le N} \|y(\cdot, t_n) - Y^n\|_{L^\infty} \le C\rho(d,h)(h + \sqrt{\tau})\big(\|y_0\|_{H^2} + \|u\|_U\big).$$

Proof. We begin by deriving an error relation using standard arguments. Take $\Phi \in W_{h,\tau}$, multiply (2.1) by $\Phi^n \in X_h$ and integrate over $\Omega \times (t_{n-1}, t_n)$: Abbreviating $y^n := y(\cdot, t_n)$ we have

$$(y^n - y^{n-1}, \Phi^n) + \int_{t_{n-1}}^{t_n} (\nabla y, \nabla \Phi^n) dt = \int_{t_{n-1}}^{t_n} (Bu(t), \Phi^n) dt, \qquad 1 \le n \le N. \tag{3.6}$$

Next, let us introduce $\tilde{Y} \in W_{h,\tau}$ by

$$\tilde{Y}(\cdot, t) := R_h y^n, \qquad t \in (t_{n-1}, t_n), 1 \le n \le N. \tag{3.7}$$

Using (3.6) along with (3.1) we derive by straightforward calculation

$$A(\tilde{Y}, \Phi) = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (Bu(t), \Phi^n) + (y_0, \Phi_+^0) + r(\Phi) \qquad \forall \Phi \in W_{h,\tau},$$

where

$$\begin{aligned} r(\Phi) \;\; = \;\; & \sum_{n=1}^N \tau_n \big(\nabla y^n - \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \nabla y dt, \nabla \Phi^n\big) + \sum_{n=1}^N \tau_n (y^n - R_h y^n, \Phi^n) \\ & + \sum_{n=2}^N \big(R_h(y^n - y^{n-1}) - (y^n - y^{n-1}), \Phi^n\big) + (R_h y^1 - y^1, \Phi^1) \equiv: \sum_{j=1}^4 r_j(\Phi). \end{aligned} \tag{3.8}$$

As a consequence, the error $E := \tilde{Y} - Y \in W_{h,\tau}$ satisfies

$$A(E, \Phi) = r(\Phi) \qquad \forall \Phi \in W_{h,\tau}. \tag{3.9}$$

Let us fix $l \in \{2, \dots, N\}$ and define $\Phi \in W_{h,\tau}$ by

$$\Phi^n := \begin{cases} 0, & n = 1 \text{ or } n > l, \\ \dfrac{E^n - E^{n-1}}{\tau_n}, & 2 \le n \le l. \end{cases}$$

Inserting Φ into (3.9) yields

$$\sum_{n=2}^l \tau_n \|\frac{E^n - E^{n-1}}{\tau_n}\|^2 + \frac{1}{2}\|\nabla E^l\|^2 - \frac{1}{2}\|\nabla E^1\|^2 + \frac{1}{2}\sum_{n=2}^l \|\nabla(E^n - E^{n-1})\|^2 = r(\Phi).$$

Let us estimate the integrals in the remainder term $r(\Phi)$. To begin,

$$\begin{aligned} |r_1(\Phi)| \;\; \le \;\; & \sum_{n=2}^l \|\nabla y^n - \frac{1}{\tau_n}\int_{t_{n-1}}^{t_n} \nabla y(t) dt\| \, \|\nabla(E^n - E^{n-1})\| \\ \le \;\; & \frac{1}{4}\sum_{n=2}^l \|\nabla(E^n - E^{n-1})\|^2 + C\tau \int_0^T \|\nabla y_t\|^2 dt. \end{aligned}$$

We infer from Young's inequality and (3.2) that

$$|r_2(\Phi)| \leq \sum_{n=2}^{l} \|y^n - R_h y^n\| \, \|E^n - E^{n-1}\| \leq \frac{1}{4}\sum_{n=2}^{l} \tau_n \|\frac{E^n - E^{n-1}}{\tau_n}\|^2 + Ch^4 \max_{1\leq n\leq N} \|y^n\|_{H^2}^2.$$

Finally, again by (3.2),

$$\begin{aligned} |r_3(\Phi)| &\leq Ch\sum_{n=2}^{l} \frac{1}{\tau_n}\|y^n - y^{n-1}\|_{H^1}\|E^n - E^{n-1}\| \\ &\leq Ch\sum_{n=2}^{l} \frac{1}{\sqrt{\tau_n}}\big(\int_{t_{n-1}}^{t_n} \|y_t\|_{H^1}^2\big)^{\frac{1}{2}}\|E^n - E^{n-1}\| \leq \frac{1}{4}\sum_{n=2}^{l}\tau_n\|\frac{E^n - E^{n-1}}{\tau_n}\|^2 + Ch^2\int_0^T \|y_t\|_{H^1}^2. \end{aligned}$$

Since $r_4(\Phi) = 0$ we obtain upon combining the above inequalities and recalling (2.7) that

$$\frac{1}{2}\sum_{n=2}^{l}\tau_n\|\frac{E^n - E^{n-1}}{\tau_n}\|^2 + \frac{1}{2}\|\nabla E^l\|^2 \leq \frac{1}{2}\|\nabla E^1\|^2 + C(h^2+\tau)\big(\|y_0\|_{H^2}^2 + \|u\|_U^2\big). \tag{3.10}$$

It remains to estimate $\|\nabla E^1\|^2$. We insert $\Phi \in W_{h,\tau}$ with $\Phi^1 = \phi_h \in X_h, \Phi^n = 0, n \geq 2$ into the error relation (3.9). After straightforward calculations we obtain

$$\tau_1(\nabla E^1, \nabla\phi_h) + (P_h y^1 - Y^1, \phi_h) = \tau_1\big(\nabla y^1 - \frac{1}{\tau_1}\int_{t_0}^{t_1}\nabla y dt, \nabla\phi_h\big) + \tau_1(y^1 - R_h y^1, \phi_h)$$

for all $\phi_h \in X_h$. Choosing $\phi_h = P_h y^1 - Y^1 = E^1 + (P_h y^1 - R_h y^1)$ we have

$$\begin{aligned} \tau_1\|\nabla E^1\|^2 + \|P_h y^1 - Y^1\|^2 &= \tau_1\big(\nabla y^1 - \frac{1}{\tau_1}\int_{t_0}^{t_1}\nabla y dt, \nabla E^1 + \nabla(P_h y^1 - R_h y^1)\big) \\ &\quad + \tau_1\big(\nabla E^1, \nabla(R_h y^1 - P_h y^1)\big) + \tau_1(y^1 - R_h y^1, P_h y^1 - Y^1) \\ &\leq \tau_1^{\frac{3}{2}}\big(\int_{t_0}^{t_1}\|\nabla y_t\|^2 dt\big)^{\frac{1}{2}}\big(\|\nabla E^1\| + \|\nabla(R_h y^1 - P_h y^1)\|\big) + \tau_1\|\nabla E^1\|\,\|\nabla(R_h y^1 - P_h y^1)\| \\ &\quad + C\tau_1\|y^1 - R_h y^1\|\,\|P_h y^1 - Y^1\| \\ &\leq \frac{1}{2}\big(\tau_1\|\nabla E^1\|^2 + \|P_h y^1 - Y^1\|^2\big) + C\tau_1 h^2\|y^1\|_{H^2}^2 + C\tau_1^2\int_{t_0}^{t_1}\|\nabla y_t\|^2 dt. \end{aligned}$$

Hence, recalling (2.7),

$$\|\nabla E^1\|^2 + \frac{1}{\tau_1}\|P_h y^1 - Y^1\|^2 \leq C(h^2+\tau)\big(\|y_0\|_{H^2}^2 + \|u\|_U^2\big). \tag{3.11}$$

Inserting (3.11) into (3.10) we deduce that

$$\sum_{n=2}^{N}\tau_n\|\frac{E^n - E^{n-1}}{\tau_n}\|^2 + \max_{1\leq n\leq N}\|\nabla E^n\|^2 \leq C(h^2+\tau)\big(\|y_0\|_{H^2}^2 + \|u\|_U^2\big). \tag{3.12}$$

Furthermore, we infer from (3.11) and (3.12) for $1 \leq n \leq N$,

$$\begin{aligned} \|E^n\| &\leq \|E^1\| + \sum_{i=2}^{n}\|E^i - E^{i-1}\| \\ &\leq \|P_h y^1 - Y^1\| + \|R_h y^1 - P_h y^1\| + \big(\sum_{i=2}^{n}\tau_i\|\frac{E^i - E^{i-1}}{\tau_i}\|^2\big)^{\frac{1}{2}}\big(\sum_{i=1}^{n}\tau_i\big)^{\frac{1}{2}} \\ &\leq C(h+\sqrt{\tau})\big(\|y_0\|_{H^2} + \|u\|_U\big). \end{aligned}$$

Combining this estimate with (3.12) we obtain

$$\max_{n=1,\dots,N} \|E^n\|_{H^1} \le C\big(h+\sqrt{\tau}\big)\big(\|y_0\|_{H^2} + \|u\|_U\big). \tag{3.13}$$

Finally, we infer with the help of (3.3), (3.4), (2.7) and (3.13) that

$$\begin{aligned}\|y^n - Y^n\|_{L^\infty} &\le \|y^n - R_h y^n\|_{L^\infty} + \|E^n\|_{L^\infty} \le Ch^{2-\frac{d}{2}}\|y^n\|_{H^2} + C\rho(d,h)\|E^n\|_{H^1} \\ &\le C\rho(d,h)(h+\sqrt{\tau})\big(\|y_0\|_{H^2} + \|u\|_U\big),\end{aligned}$$

which completes the proof. ∎

Remark 3.2. Note that the error bound in Theorem 3.1 is derived under the condition that the right hand side and hence the time derivative of the solution is only square integrable in time. Classical results known from the literature require higher regularity requirements, see e.g. [7, Theorem 1.2] and thus are not applicable in our case. A situation that is comparable to ours in that the time derivative only belongs to some L^p–space is considered in [15]. For a function

$$y \in W_p^{2,1}(\Omega_T) := \{z \in L^p(0,T;W^{2,p}(\Omega)), z_t \in L^p(0,T;L^p(\Omega))\} \quad (p>2)$$

with $y=0$ on Γ_T the following parabolic projection is analyzed: $Y^0 = I_h y_0$ and

$$\frac{1}{\tau}(Y^n - Y^{n-1}, \phi_h) + (\nabla Y^n, \nabla\phi_h) = \frac{1}{\tau}(y^n - y^{n-1}, \phi_h) + (\nabla \bar{y}^n, \nabla\phi_h)$$

for all $\phi_h \in X_{h0} := X_h \cap H_0^1(\Omega), 1 \le n \le N$. Here, $y^n = y(\cdot,t_n)$ and $\bar{y}^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} y(\cdot,t)dt$. It is shown in [15, Theorem 4.1] that

$$\max_{1\le n\le N} \|y^n - Y^n\|_{L^\infty} \le Cp^2|\log h|^2\big(h^{2-4/p} + \tau^{1-2/p}\big)\|y\|_{W_p^{2,1}} \tag{3.14}$$

provided that $d=2$ and $\tau \ge C^*|\log h|^3 h^2$. We expect that the techniques in [15] can be applied to the scheme (3.5) and Neumann boundary conditions provided that the solution has the necessary regularity.

Remark 3.3. In what follows we shall assume that the time step is coupled to the spatial grid size h in such a way that $\tau = o(\rho(d,h)^{-2})$ as $h\to 0$. With this choice we infer from Theorem 3.1 that

$$\max_{1\le n\le N} \|y(\cdot,t_n) - Y^n\|_{L^\infty} \to 0, \quad h\to 0.$$

We use the variational approach of [10] in order to discretize our optimal control problem as follows:

$$(TP)_h \quad \begin{cases} \min_{u\in U} J_{h,\tau}(u) := \dfrac{1}{2}\displaystyle\sum_{n=1}^{N} \tau_n\|Y^n - \bar{y}^n\|^2 + \dfrac{\alpha}{2}\int_0^T |u(t)|^2 dt \\ \text{s.t. } Y = \mathcal{G}_{h,\tau}(Bu) \text{ and } Y^n(x_j) \ge 0, 1\le j\le J, 1\le n\le N. \end{cases} \tag{3.15}$$

As a minimization problem for a quadratic functional over a closed and convex domain, $(TP)_h$ has a unique solution $u_h \in U$. Furthermore, using [3, Theorem 5.2] again, we conclude that there exist $\mu_j^n \in \mathbb{R}, 1 \le n \le N, 1 \le j \le J$ as well as $P \in W_{h,\tau}$ such that

$$A(\Phi, P) = \sum_{n=1}^{N} \tau_n (Y^n - \bar{y}^n, \Phi^n) + \sum_{n=1}^{N}\sum_{j=1}^{J} \Phi^n(x_j)\mu_j^n \qquad \forall \Phi \in W_{h,\tau}, \tag{3.16}$$

$$\alpha u_h(t) + \big((P^n, f_i)\big)_{i=1,\dots,m} = 0 \qquad a.e. \text{ in } (t_{n-1}, t_n), \tag{3.17}$$

$$\mu_j^n \le 0,\ Y^n(x_j) \ge 0, \text{ and } \sum_{n=1}^{N}\sum_{j=1}^{J} Y^n(x_j)\mu_j^n = 0. \tag{3.18}$$

Let us define the measure $\mu_{h,\tau} \in \mathcal{M}(\overline{\Omega_T})$ by

$$\int_{\overline{\Omega_T}} f d\mu_{h,\tau} := \sum_{n=1}^{N}\sum_{j=1}^{J} f(x_j, t_n)\mu_j^n, \qquad f \in C^0(\overline{\Omega_T}).$$

As a first result for (3.15) we prove that the sequence of optimal controls, states and measures $\mu_{h,\tau}$ are uniformly bounded.

Lemma 3.4. *Let $u_h \in U$ be the optimal solution of (3.15) with corresponding state $Y = \mathcal{G}_{h,\tau}(Bu_h)$ and adjoint variables $P \in W_{h,\tau}$ and $\mu_{h,\tau} \in \mathcal{M}(\overline{\Omega_T})$. Then there exists $h_0 > 0$ such that*

$$\sum_{n=1}^{N} \tau_n \|Y^n\|^2 + \int_0^T |u_h(t)|^2 dt + \sum_{n=1}^{N}\sum_{j=1}^{J} |\mu_j^n| \le C \qquad \textit{for all } 0 < h \le h_0.$$

Proof. We infer from (2.13) that there exists $\delta > 0$ such that

$$\hat{y} = \mathcal{G}(0) \ge \delta \qquad \text{in } \overline{\Omega_T}.$$

Theorem 3.1 then implies that $\hat{Y} := \mathcal{G}_{h,\tau}(0) \in W_{h,\tau}$ satisfies

$$\hat{Y}^n(x_j) \ge \frac{\delta}{2}, \quad 1 \le n \le N, 1 \le j \le J, 0 < h \le h_0. \tag{3.19}$$

Using (3.18) and (3.16) we obtain

$$\begin{aligned}
\sum_{n=1}^{N}\sum_{j=1}^{J} \hat{Y}^n(x_j)\,|\mu_j^n| &= \sum_{n=1}^{N}\sum_{j=1}^{J} (Y^n(x_j) - \hat{Y}^n(x_j))\mu_j^n \\
&= -\sum_{n=1}^{N} \tau_n (Y^n - \bar{y}^n, Y^n - \hat{Y}^n) + A(Y - \hat{Y}, P) \\
&= \sum_{n=1}^{N} \tau_n \int_\Omega \big(-(Y^n)^2 + Y^n\hat{Y}^n + \bar{y}^n Y^n - \bar{y}^n \hat{Y}^n\big) + \sum_{n=1}^{N}\sum_{i=1}^{m} \tau_n\, u_{h,i|(t_{n-1},t_n)}(P^n, f_i) \\
&\le -\frac{1}{2}\sum_{n=1}^{N} \tau_n \|Y^n\|^2 - \alpha \int_0^T |u_h(t)|^2 dt + C
\end{aligned}$$

recalling that $Y = \mathcal{G}_{h,\tau}(Bu_h), \hat{Y} = \mathcal{G}_{h,\tau}(0)$ as well as (3.17). Combining this estimate with (3.19) implies the result. ∎

4 Error estimate

Theorem 4.1. *Let u be the solution of (TP), u_h the solution of $(TP)_h$ with corresponding states $y = \mathcal{G}(Bu)$ and $Y = \mathcal{G}_{h,\tau}(Bu_h)$. Then*

$$\sum_{n=1}^{N} \tau_n \|y(\cdot, t_n) - Y^n\|^2 + \int_0^T |u(t) - u_h(t)|^2 dt \le C\rho(d,h)\big(h + \sqrt{\tau}\big) + C\tau.$$

Proof. Let us write

$$\begin{aligned}\alpha \int_0^T |u(t) - u_h(t)|^2 dt &= \alpha \int_0^T u(t) \cdot (u(t) - u_h(t))dt - \alpha \int_0^T u_h(t) \cdot (u(t) - u_h(t))dt \\ &\equiv I + II. \qquad (4.1)\end{aligned}$$

In order to deal with I we choose a sequence $(v_k)_{k\in\mathbb{N}}, v_k \in C_0^\infty(0,T;\mathbb{R}^m)$ such that $v_k \to u - u_h$ in $L^2(0,T;\mathbb{R}^m)$ as $k \to \infty$. Furthermore, let $y^h := \mathcal{G}(Bu_h)$ and $z_k := \mathcal{G}_0(Bv_k)$. Note that $z_k \in W_0^\infty$ in view of the smoothness of v_k and the fact that $f_i \in L^\infty(\Omega), i = 1, \dots, m$. In addition, (2.7) yields

$$\begin{aligned}\|(y - y^h) - z_k\|_{C^0(\overline{\Omega_T})} &\le C \max_{0\le t\le T} \|(y - y^h)(t) - z_k(t)\|_{H^2} \qquad (4.2)\\ &\le C\Big(\int_0^T |(u - u_h)(t) - v_k(t)|^2 dt\Big)^{\frac{1}{2}} \to 0, k \to \infty.\end{aligned}$$

Hence, we infer from (2.15), the definition of z_k, (2.14) and (4.2) that

$$\begin{aligned} I &= \alpha \lim_{k\to\infty} \int_0^T u(t) \cdot v_k(t) dt = -\lim_{k\to\infty} \int_0^T \sum_{i=1}^m v_{k,i}(t)\,(p(\cdot,t), f_i) dt \\ &= -\lim_{k\to\infty} \int_0^T (Bv_k, p)dt = -\lim_{k\to\infty} \int_0^T (z_{k,t} - \Delta z_k, p) \\ &= -\lim_{k\to\infty} \Big\{ \int_0^T (y - \bar{y}, z_k) + \int_{\overline{\Omega_T}} z_k d\mu \Big\} = \int_0^T (y - \bar{y}, y^h - y) + \int_{\overline{\Omega_T}} (y^h - y) d\mu. \end{aligned}$$

Recalling (2.16) we may continue

$$\begin{aligned} I &= \sum_{n=1}^N \tau_n (y^n - \bar{y}^n, y^{h,n} - y^n) + \int_{\overline{\Omega_T}} (y^h)^- d\mu \\ &\quad + \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \big\{ (y - \bar{y}, y^h - y) - (y^n - \bar{y}^n, y^{h,n} - y^n) \big\} \equiv I_1 + I_2 + I_3. \end{aligned}$$

Here, we have abbreviated $y^- = \min(y, 0)$. Let us start with the second term. For $(x,t) \in \bar{\Omega} \times (t_{n-1}, t_n)$ we deduce upon recalling (3.3), (3.4), Theorem 3.1 and the fact that $Y^n(x) \ge$

$0, x \in \bar\Omega$

$$
\begin{aligned}
|(y^h)^-(x,t)| &\le |(y^h)^-(x,t) - (y^h)^-(x,t_n)| + |(y^h)^-(x,t_n) - (Y^n)^-(x)| \\
&\le |y^h(x,t) - y^h(x,t_n)| + |y^h(x,t_n) - Y^n(x)| \\
&\le 2 \max_{0\le s\le T} \|y^h(\cdot,s) - R_h y^h(\cdot,s)\|_{L^\infty} + \|R_h y^h(\cdot,t) - R_h y^h(\cdot,t_n)\|_{L^\infty} + \|y^{h,n} - Y^n\|_{L^\infty} \\
&\le Ch^{2-\frac{d}{2}} \max_{0\le s\le T} \|y^h(\cdot,s)\|_{H^2} + C\rho(d,h)\|R_h y^h(\cdot,t) - R_h y^h(\cdot,t_n)\|_{H^1} \\
&\quad + C\rho(d,h)(h+\sqrt{\tau})\big(\|y_0\|_{H^2} + \|u_h\|_U\big) \qquad (4.3) \\
&\le C\rho(d,h)(h+\sqrt{\tau})\big(\|y_0\|_{H^2} + \|u_h\|_U\big) + C\rho(d,h)\sqrt{\tau_n}\Big(\int_{t_{n-1}}^{t_n} \|R_h y_t^h\|_{H^1}^2 dt\Big)^{\frac{1}{2}} \\
&\le C\rho(d,h)(h+\sqrt{\tau})\big(\|y_0\|_{H^2} + \|u_h\|_U\big) \le C\rho(d,h)(h+\sqrt{\tau})
\end{aligned}
$$

in view of (2.7) and Lemma 3.4. By continuity this estimate also holds at the points $t = t_n, n = 0, \dots, N$. An elementary calculation shows that

$$
\begin{aligned}
|I_3| &\le C\tau \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \big(\|y_t\| + \|y_t^h\| + \|\bar y_t\|\big)\big(\|y\| + \|y^h\| + \|\bar y\|\big) dt \\
&\le C\tau. \qquad (4.4)
\end{aligned}
$$

Inserting the estimates (4.3) and (4.4) into our formula for I we have

$$
I \le \sum_{n=1}^N \tau_n (y^n - \bar y^n, y^{h,n} - y^n) + C\rho(d,h)(h+\sqrt{\tau}) + C\tau. \qquad (4.5)
$$

Next, let us introduce $\tilde Y = \mathcal{G}_{h,\tau}(Bu)$. Then, (3.17), (3.16) and (3.18) imply that

$$
\begin{aligned}
II &= \sum_{n=1}^N \sum_{i=1}^m (P^n, f_i) \int_{t_{n-1}}^{t_n} (u_i - u_{h,i})(t) dt = \sum_{n=1}^N \int_{t_{n-1}}^{t_n} (P^n, B(u-u_h)) = A(\tilde Y - Y, P) \\
&= \sum_{n=1}^N \tau_n (Y^n - \bar y^n, \tilde Y^n - Y^n) + \sum_{n=1}^N \sum_{j=1}^J (\tilde Y^n(x_j) - Y^n(x_j))\mu_j^n \\
&\le \sum_{n=1}^N \tau_n (Y^n - \bar y^n, \tilde Y^n - Y^n) + \max_{1\le n\le N, 1\le j\le J} |(\tilde Y^n)^-(x_j)| \sum_{n=1}^N \sum_{j=1}^J |\mu_j^n|.
\end{aligned}
$$

Recalling that $y \ge 0$ in $\overline{\Omega_T}$ we have for $1 \le n \le N, 1 \le j \le J$

$$
\begin{aligned}
|(\tilde Y^n)^-(x_j)| &= |(\tilde Y^n)^-(x_j) - y^-(x_j,t_n)| \le |\tilde Y^n(x_j) - y(x_j,t_n)| \\
&\le \|\tilde Y^n - y(\cdot,t_n)\|_{L^\infty} \le C\rho(d,h)(h+\sqrt{\tau})\big(\|y_0\|_{H^2} + \|u\|_U\big) \\
&\le C\rho(d,h)(h+\sqrt{\tau})
\end{aligned}
$$

again by Theorem 3.1. As a result,

$$
II \le \sum_{n=1}^N \tau_n (Y^n - \bar y^n, \tilde Y^n - Y^n) + C\rho(d,h)(h+\sqrt{\tau}). \qquad (4.6)
$$

Inserting (4.5) and (4.6) into (4.1) we have

$$
\begin{aligned}
&\alpha \int_0^T |u(t)-u_h(t)|^2 dt \\
\le\ & \sum_{n=1}^N \tau_n (y^n - \bar{y}^n, y^{h,n} - y^n) + \sum_{n=1}^N \tau_n (Y^n - \bar{y}^n, \tilde{Y}^n - Y^n) + C\rho(d,h)(h+\sqrt{\tau}) + C\tau \\
=\ & \sum_{n=1}^N \tau_n \int_\Omega \big\{ -(y^n - Y^n)^2 + (Y^n - \bar{y}^n)(\tilde{Y}^n - y^n) + (y^n - \bar{y}^n)(y^{h,n} - Y^n) \big\} \\
& + C\rho(d,h)(h+\sqrt{\tau}) + C\tau \\
\le\ & -\sum_{n=1}^N \tau_n \|y^n - Y^n\|^2 + + C\rho(d,h)(h+\sqrt{\tau}) + C\tau
\end{aligned}
$$

where we once more used Theorem 3.1. The proof is complete. ∎

The order of convergence obtained in Theorem 4.1 is essentially determined by the error bound in Theorem 3.1, which in turn is limited by our regularity assumptions on the control u. A situation in which one can expect better error bounds occurs when additional control constraints of the form

$$a \le u_i(t) \le b \quad \text{a.e. in } (0,T), i = 1, \ldots, m$$

are prescribed. Here, $a < b$ are given constants. Then, $Bu \in L^\infty(\Omega_T)$ so that parabolic regularity theory implies that $y \in W_p^{2,1}(\Omega_T)$ for all $p < \infty$. Recalling Remark 3.2 it seems possible to us that an error bound of the form

$$\sum_{n=1}^N \tau_n \|y(\cdot, t_n) - Y^n\|^2 + \int_0^T |u(t) - u_h(t)|^2 dt \le C_\epsilon \big(h^{2-\epsilon} + \tau^{1-\epsilon}\big) \quad (\epsilon > 0)$$

can be proved in this situation.

Acknowledgements

The authors gratefully acknowledge the support of the DFG Priority Program 1253 entitled Optimization With Partial Differential Equations.

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